The mathematics of ocean waves

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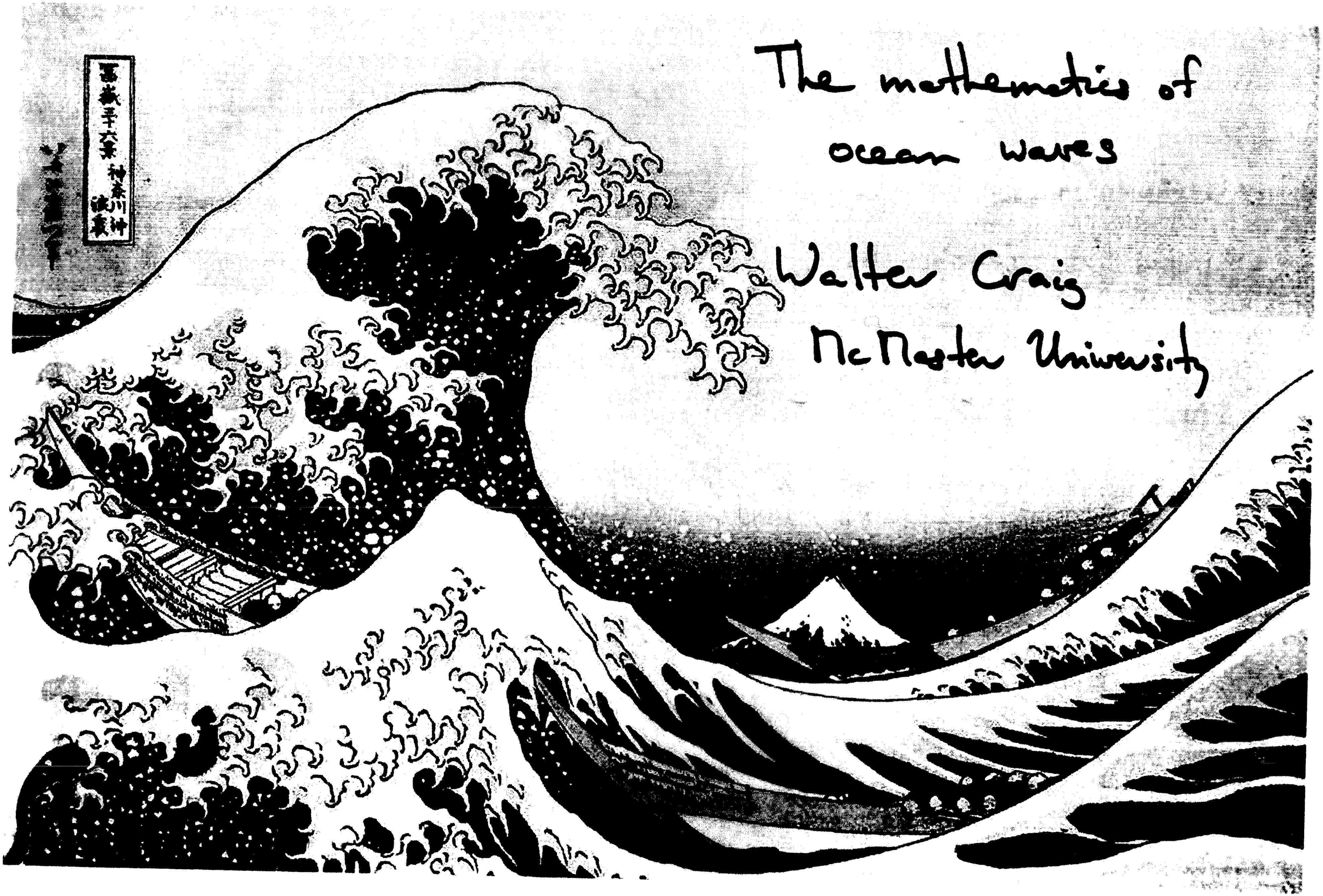
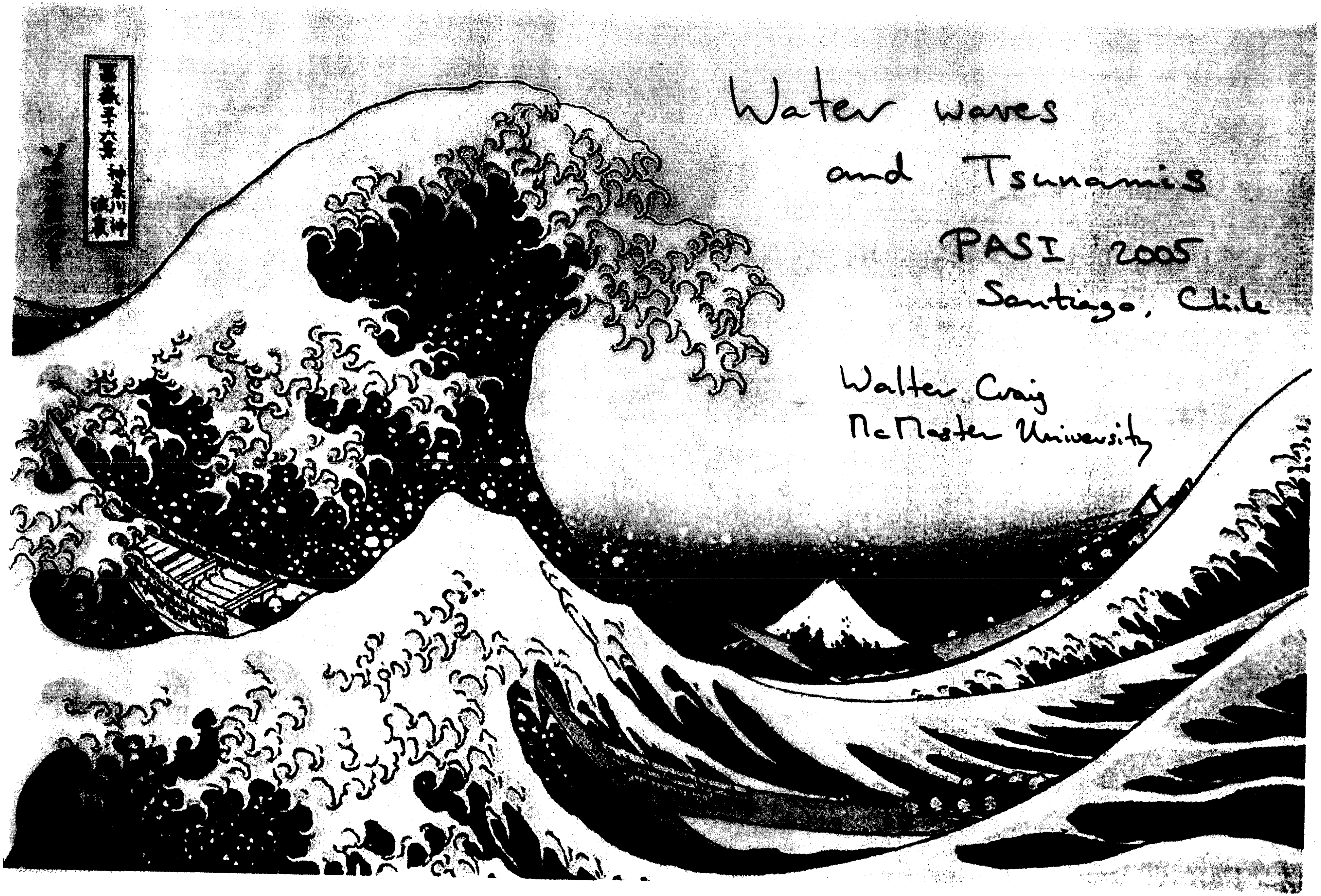


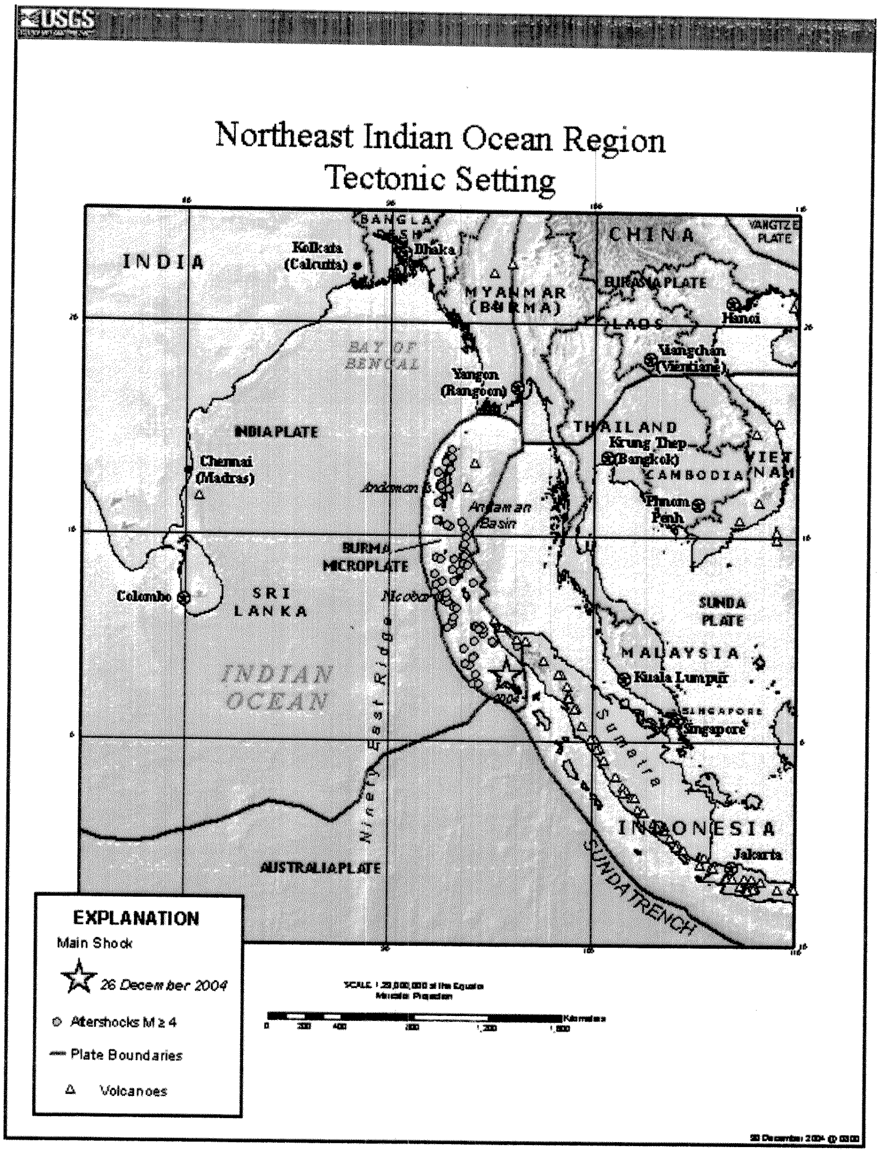

Water waves
and Tsunamis
PASI 2005
Santiago, Chile
Walter Craig
McMaster University

Magnitude 9.0 - SUMATRA-ANDAMAN ISLANDS EARTHQUAKE
OFF THE WEST COAST OF NORTHERN SUMATRA
2004 December 26 00:58:53 UTC

Magnitude	9.0
Date-Time	Sunday, December 26, 2004 at 00:58:53 (UTC) = Coordinated Universal Time Sunday, December 26, 2004 at 7:58:53 AM = local time at epicenter Time of Earthquake in other Time Zones
Location	3.316°N, 95.854°E
Depth	30 km (18.6 miles) set by location program
Region	OFF THE WEST COAST OF NORTHERN SUMATRA
Distances	250 km (155 miles) SSE of Banda Aceh, Sumatra, Indonesia 310 km (195 miles) W of Medan, Sumatra, Indonesia 1260 km (780 miles) SSW of BANGKOK, Thailand 1605 km (990 miles) NW of JAKARTA, Java, Indonesia
Location Uncertainty	horizontal +/- 5.6 km (3.5 miles); depth fixed by location program
Parameters	Nst=276, Nph=276, Dmin=654.9 km, Rmss=1.04 sec, Gp= 29°, M-type=teleseismic moment magnitude (Mw), Version=U
Source	USGS NEIC (WDCS-D)
Event ID	usslav

Felt Reports: At least 79,900 people were killed by the earthquake and tsunami in Indonesia. Tsunamis killed at least 41,000 people in Sri Lanka, 10,000 in India, 4,000 in Thailand, 120 in Somalia, 90 in Myanmar, 66 in Malaysia, 46 in Maldives, 10 in Tanzania, 2 in Bangladesh, 1 in Seychelles and 1 in Kenya. Tsunamis caused damage in Madagascar and Mauritius and also occurred on Cocos Island and Reunion. The tsunami crossed into the Pacific Ocean and was recorded in New Zealand and along the west coast of South and North America. The earthquake was felt (VIII) at Banda Aceh and (V) at Medan, Sumatra and (II-IV) in parts of Bangladesh, India, Malaysia, Maldives, Myanmar, Singapore, Sri Lanka and Thailand. A mud volcano near Baratang, Andaman Islands began erupting on December 28. This is the fourth largest earthquake in the world since 1900 and is the largest since the 1964 Prince William Sound, Alaska earthquake. (last updated 12/30/04)

The devastating megathrust earthquake of December 26th, 2004 occurred on the interface of the India and Burma plates and was caused by the release of stresses that develop as the India plate subducts beneath the overriding Burma plate. The India plate begins its descent into the mantle at the Sunda trench which lies to the west of the earthquake's epicenter. The trench is the surface expression of the plate interface



- depth 30 km
- horizontal dislocation 20 m at this depth.
  10 m on sea bed
- vertical dislocation 2 m on sea bed
- major rupture 400 km length; Sunda fault
  total rupture 1500 km length.

TSUNAMI LABORATORY

December 26, 2004 North Sumatra earthquake and tsunami

00:58:51 GMT Lat. 3.298S Long. 95.779E mb 8.5 Ms 8.9 Mw 9.0 Mt ? I=4 Hmax=15m

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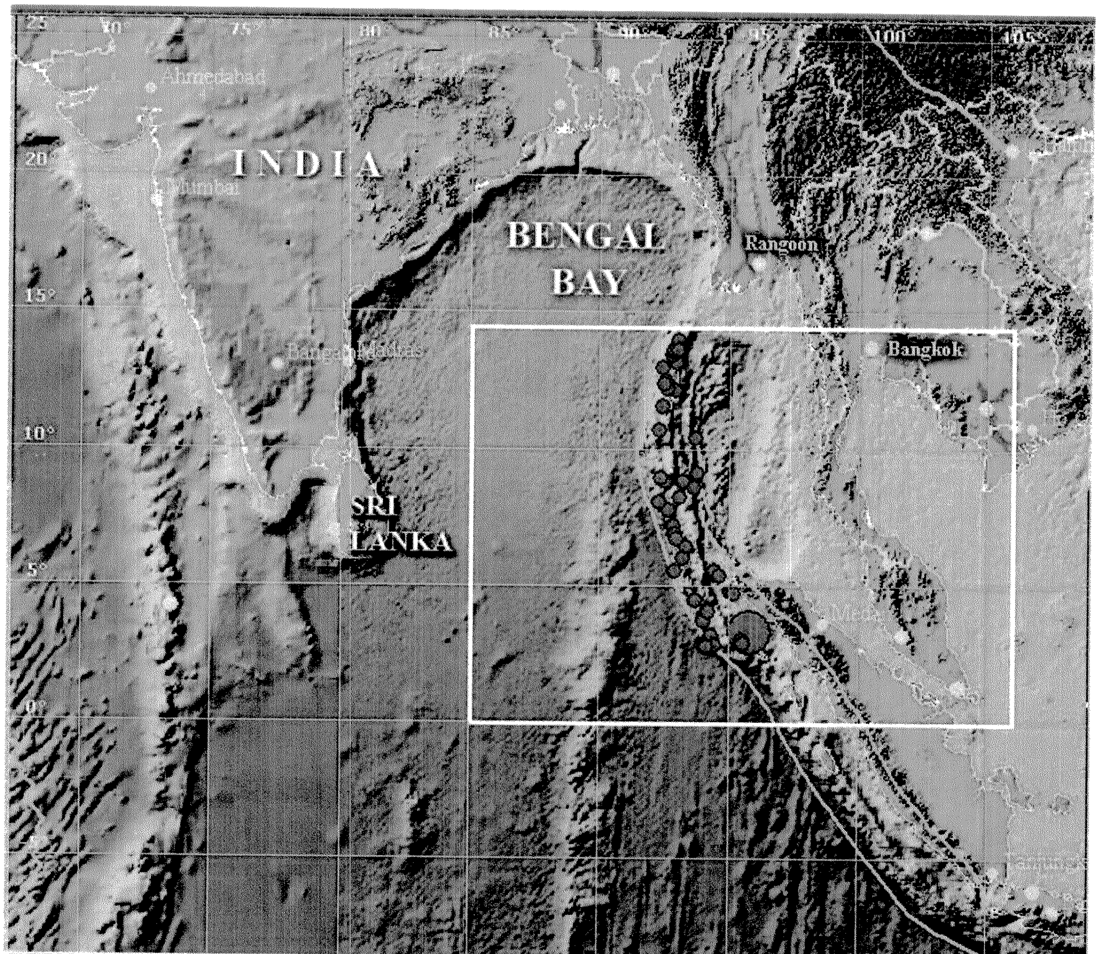
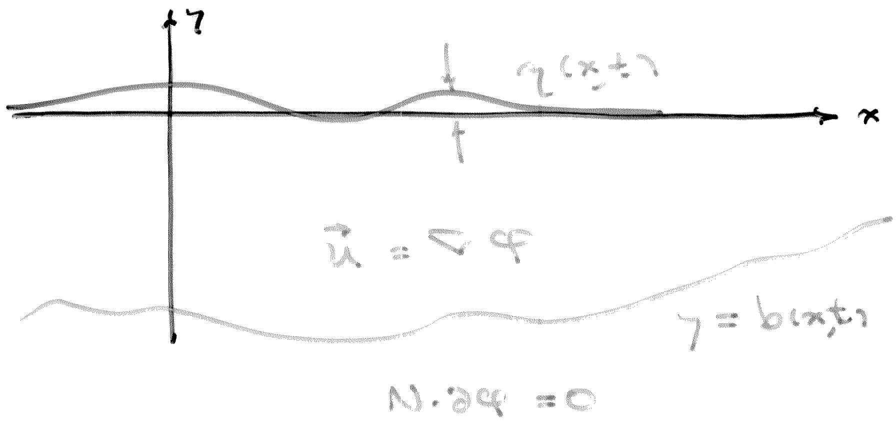


The location map of the December 26, 2004 earthquake. The large circle shows the positions of the main shock as determined by NEIS along with the first day aftershocks that roughly outline the earthquake source area. The main plate boundaries are shown in green. White rectangle outlines the area shown in the section "Source area"

- depth of Bay of Bengal ~ 3000 - 4000 m
- depth of Andaman Basin ~ 1000 m
- depth at the Gulf of Thailand ~ 500 m

1)

§1. Equations of motion



(1) $$\nabla \cdot u = 0$$

$$\nabla \times u = 0 \qquad\qquad u = \nabla\varphi$$

conditions for potential flow

(2) $$\Delta\varphi = 0 .$$

bottom boundary conditions

$$N \cdot \nabla\varphi = -\partial_t b(x,t) \qquad y = b(x,t)$$

Top boundary conditions, on $\{y = \eta(x,t)\}$

(3) $$\vec{n} = (-\partial_t\eta, -\partial_x\eta, 1)$$

space-time normal

$$\vec{\tau} = (1, \partial_x\varphi, \partial_y\varphi)$$

tangent vector to fluid particle path

then

$$\vec{n} \cdot \vec{\tau} = 0$$

2)

That is, we obtain the <u>kinematic condition</u>

(4) $$\partial_t \eta - \partial_x \eta \cdot \partial_x \varphi + \partial_y \varphi = 0 \, .$$

A second boundary condition on the free surface

(5) $$\partial_t \varphi + g\eta + \tfrac{1}{2}|\nabla\varphi|^2 = 0$$

the <u>dynamic</u> or <u>Bernoulli condition</u>.

Compare (5) with the Euler equations

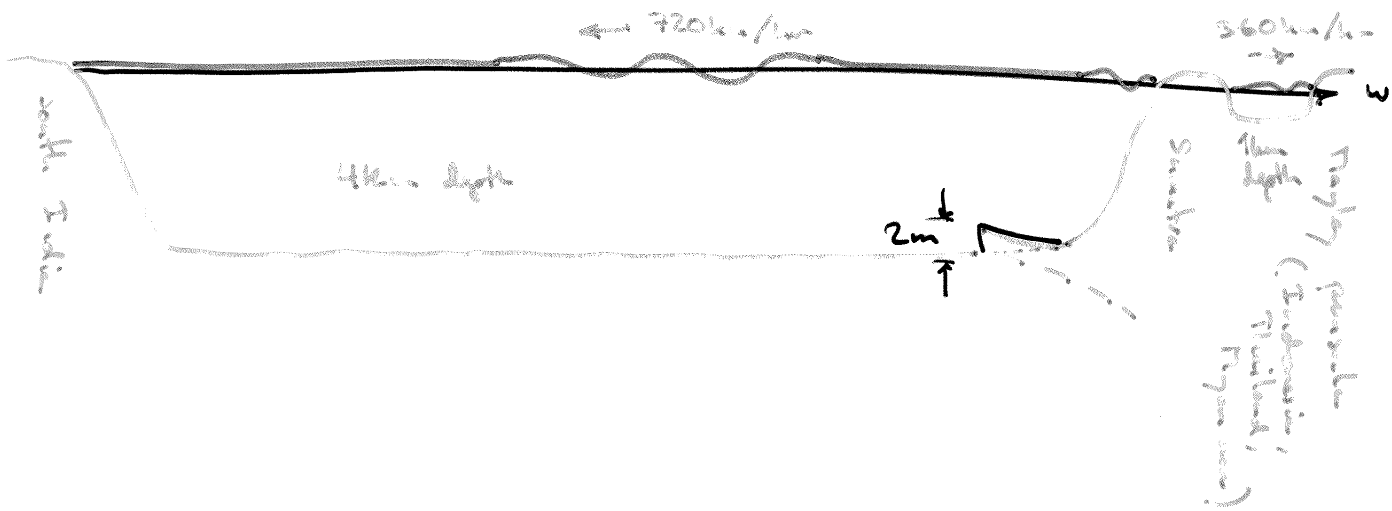
(6) $$\partial_t u + (u \cdot \nabla u) = -\nabla p - g\begin{pmatrix}0\\1\end{pmatrix}$$

and integrate once for a condition on φ, where $\nabla\varphi = u$.

Questions: how can an earthquake create waves of large mass, momentum and energy, which travel at great velocity in a coherent form?

3)

What happened in the Dec 26, 2004 earthquake and the resulting tsunami from north east Sumatra?



* earthquake 8h58 local time
  ~100 km offshore north Sumatra
* soon afterwards, impact of tsunami on Sumatra coast
  – Banda Aceh
* tsunami waves travel at ~360 km/hr east through the Andaman Basin, arriving in just over one hour on the Malay peninsula coast.

* tsunami waves travel at ~720 km/hr west through the lower Bay of Bengal to impact the west coast of Sri Lanka and the west coast of southern India in approximately 2½-3 hours (10h local time).

Basic explanations for:

(1) wave speed, and thus the travel time
(2) wave form of the coherent nonlinear wave packet which forms the tsunami phenomenon.

4)

§2 linearized equations

The wave speed of free surface water waves is

$$c = \sqrt{gh} \qquad \text{[Stokes, 1847]}$$

Task: describe this

* The hydrodynamic equations, linearized about $\eta(x,t) = 0$, $\varphi(x,y,t) = 0$

$$(6) \qquad \Delta \Phi = 0 \quad \text{for} \quad -b(x) < y < 0,$$

$$N \cdot \nabla \Phi = 0 \quad \text{for} \quad y = -b(x)$$

and linearized free surface conditions

$$\partial_t \eta = \partial_y \Phi \qquad \text{on the linearized free surface}$$

$$\partial_t \Phi = -g\eta \qquad y = 0$$

* Describe $\partial_y \Phi$ using the Dirichlet-Neumann operator

(7) $\Phi(x,0)$ boundary data

$\longmapsto \Phi(x,y)$ such that $\begin{cases} \Delta \Phi = 0 \\ N \cdot \nabla \Phi = 0 \quad y = -b(x) \end{cases}$

Poisson extension

$$\longmapsto \partial_y \Phi(x,0) = G\, \Phi(x,0)$$

normal derivative at free surface

The operator $G = G(b(x))$, linear in Φ, coefficients in $b(x)$.

Re writing the linearized equations

$$\partial_t^2 \eta = -g\, \partial_y \Phi$$

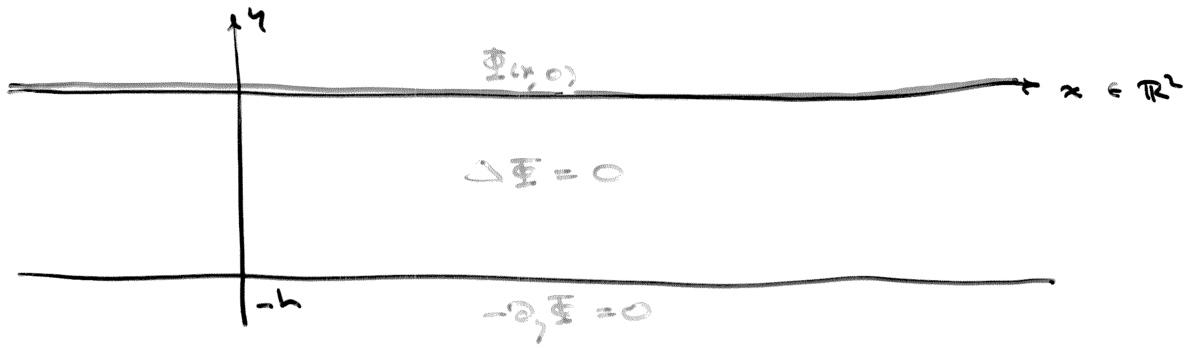
$$= -g\, G(b(x))\, \eta .$$

Separate variables $\eta(x,t) = e^{i\omega t} w(x)$, then

(8) $$g\, G(b)\, w = \omega^2 w$$

spectral problem for the Dirichlet - Neumann operator.

Case of a flat bottom $b(x) = -h$.



- Data $\qquad \Phi(x,0) = e^{ik\cdot x} \qquad k \in \mathbb{R}^2$
- Poisson extension $\qquad \Phi(x,y) = (a\, e^{|k|y} + b\, e^{-|k|y})\, e^{ik\cdot x}$
- bottom bndry condition $\qquad \Phi(x,y) = \dfrac{\cosh(|k|(y+h))}{\cosh(|k|h)}\, e^{ik\cdot x}$
- gradient on the free surface $\qquad G\, e^{ik\cdot x} = |k| \dfrac{\sinh(|k|h)}{\cosh(|k|h)}\, e^{ik\cdot x}$

$$= |k| \tanh(h|k|)\, e^{ik\cdot x}$$

For general data, a linear superposition $\xi(x) = \int e^{ik\cdot x} \hat{\xi}(k) \frac{dk}{\sqrt{2\pi}}$

$$(G\,\xi)(x) = \frac{1}{\sqrt{2\pi}^{n-1}} \int e^{ik\cdot x} \, |k| \tanh(h|k|) \, \hat{\xi}(k) \, dk$$

Using the notation of Fourier multipliers

$$(9) \qquad (G\,\xi)(x) = |D| \tanh(h|D|)\, \xi(x)$$

where

$$D = \frac{1}{i}\partial_x \, .$$

* Back to the spectral problem

$$(10) \qquad g\, G\, w = \omega^2 w$$

one has

$$\omega^2(k) = g|k| \tanh(h|k|)$$

Solution of the wave equation (flat bottom)

$$\partial_t^2 \eta = -g\, G(D)\eta$$

given by

$$(11) \qquad \eta(x,t) = \frac{1}{\sqrt{2\pi}^{n-1}} \int e^{ik\cdot x} \Big[\cos(\omega(k)t)\, \hat{\eta}_0(k) + \frac{\sin(\omega(k)t)}{\omega(k)} \, \widehat{\partial_t \eta_0}(k) \Big] dk$$

with of course

$$\omega(k) = \sqrt{g|k| \tanh(h|k|)}$$

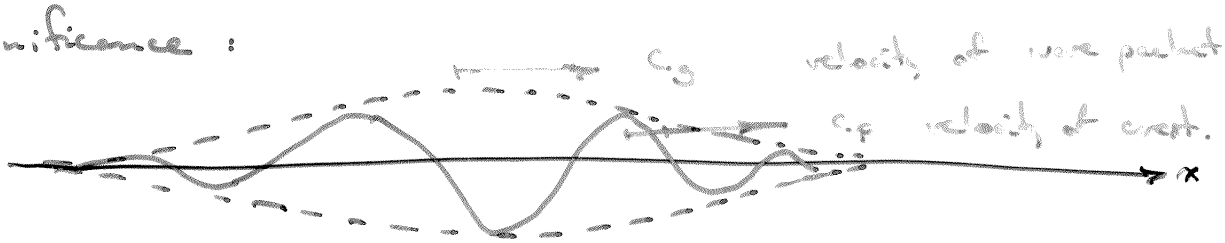
7)

* Analysis of (11) by the method of stationary phase

phase velocity $c_p = c_p(k) = \frac{\omega(k)}{|k|}\left(\frac{k}{|k|}\right)$

group velocity $c_g(k) = \partial_k \omega(k)$

Significance :



Water waves case

$$\omega(k) = \sqrt{g|k| \tanh(h|k|)}$$

phase velocity $\frac{\omega(k)}{|k|} \sim \begin{cases} \sqrt{\frac{g}{|k|}} & |k| \to \infty \\ \sqrt{gh} & |k| \to 0 \end{cases}$

group velocity $\partial_k \omega(k) \sim \begin{cases} \frac{1}{2}\sqrt{\frac{g}{|k|}} & |k| \to \infty \\ \sqrt{gh} & |k| \to 0 \end{cases}$

Conclude: wave speed for long waves is $\sqrt{gh}$.

singularities (short waves) do not propagate.

8)

Sample scratch calculations

(i) Sumatra coast → south India 2,500 km

Bay of Bengal ~ 3K - 4K m depth

$$g = 9.8 \text{ m/sec}^2$$

$$c = \sqrt{gh} \sim \sqrt{10 \text{ m/sec}^2 \times 4 \cdot 10^3 \text{ m}}$$

$$= 2 \times 10^2 \text{ m/sec}$$

$$= 720 \text{ km/hr}$$

To travel 2,500 km it would take approximately 3.5 hrs

(ii) Sumatra coast → Malay peninsula 500 km

Andaman Basin ~ 1K m depth

$$c = \sqrt{gh} \sim \sqrt{10 \text{ m/sec}^2 \times 10^3 \text{ m}}$$

$$= 10^2 \text{ m/sec}$$

$$= 360 \text{ km/hr}$$

Travel time for 500 km

1.8 hrs

These somewhat over estimate the time of travel on Dec 26.

8 1/2)

Hypothetical situation:

(iii) If the source of the tsunami wave is the Gulf of Thailand.

- Malay peninsula → coast of central Thailand, 500km
- Gulf of Thailand depth ~ 200m

$$\sqrt{gh} = \sqrt{10\,m/sec^2 \times 200m}$$

$$\sim 45\ m/sec$$

$$= 160\ km/hr$$

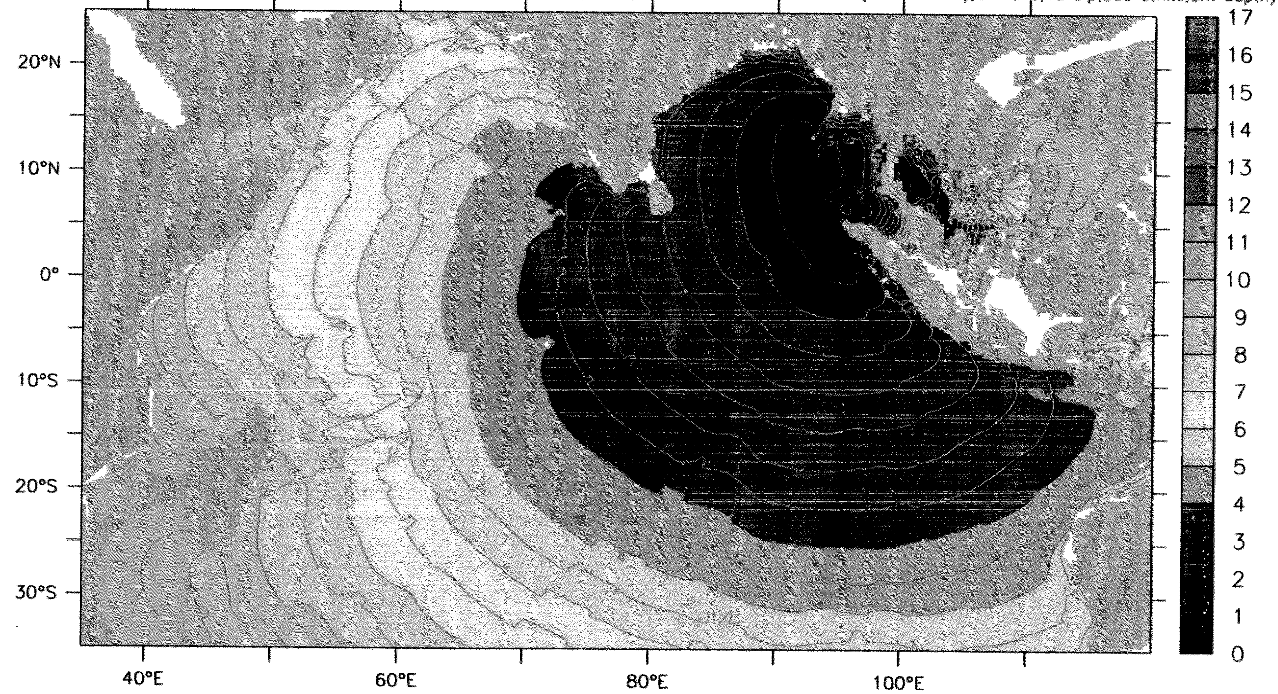
more than 4 hrs travel time

Facility for the Analysis and Comparison of Tsunami Simulations (FACTS)

Arrival Time of First Wave(hours) – 2004.12.26 Indonesian Tsunami

T (SECONDS) : -30 to 36030

Source: Mw 9.0 (4°N,95.7°E–20m*(200x150km),90°rake,13°dip,300°strike,5m depth)+
(7.3559°N,94.1393°E–20m*(670x150km),90°rake,13°dip,345°strike,5m depth)+(11.605°N,93.4723°E–20m*(300x150km),90°rake,13°dip,365°strike,5m depth)



7)

* method of stationary phase

phase velocity:

$$e^{i[k\cdot x - \omega(k)t]} = e^{it[k\cdot(\frac{x}{t} - \frac{\omega(k)}{|k|}\cdot\frac{k}{|k|})]}$$

the phase is

$$it[k\cdot(\frac{x}{t} - \underbrace{\frac{\omega(k)}{|k|}\frac{k}{|k|}}_{c_p})]$$

group velocity:

$$u(x,t) = \int e^{i[k\cdot x - \omega(k)t]}\,\hat{u}_0(k)\,dk$$

suppose (x,t) is such that for all $k\in\operatorname{supp}\hat{u}_0$

$$|x - \partial_k\omega(k)t| > R$$

then

$$e^{i[k\cdot x - \omega(k)t]} = \frac{x-\partial_k\omega(k)t}{|x-\partial_k\omega(k)t|^2}\frac{1}{i}\partial_k\left(e^{i(k\cdot x-\omega t)}\right)$$

In the integral

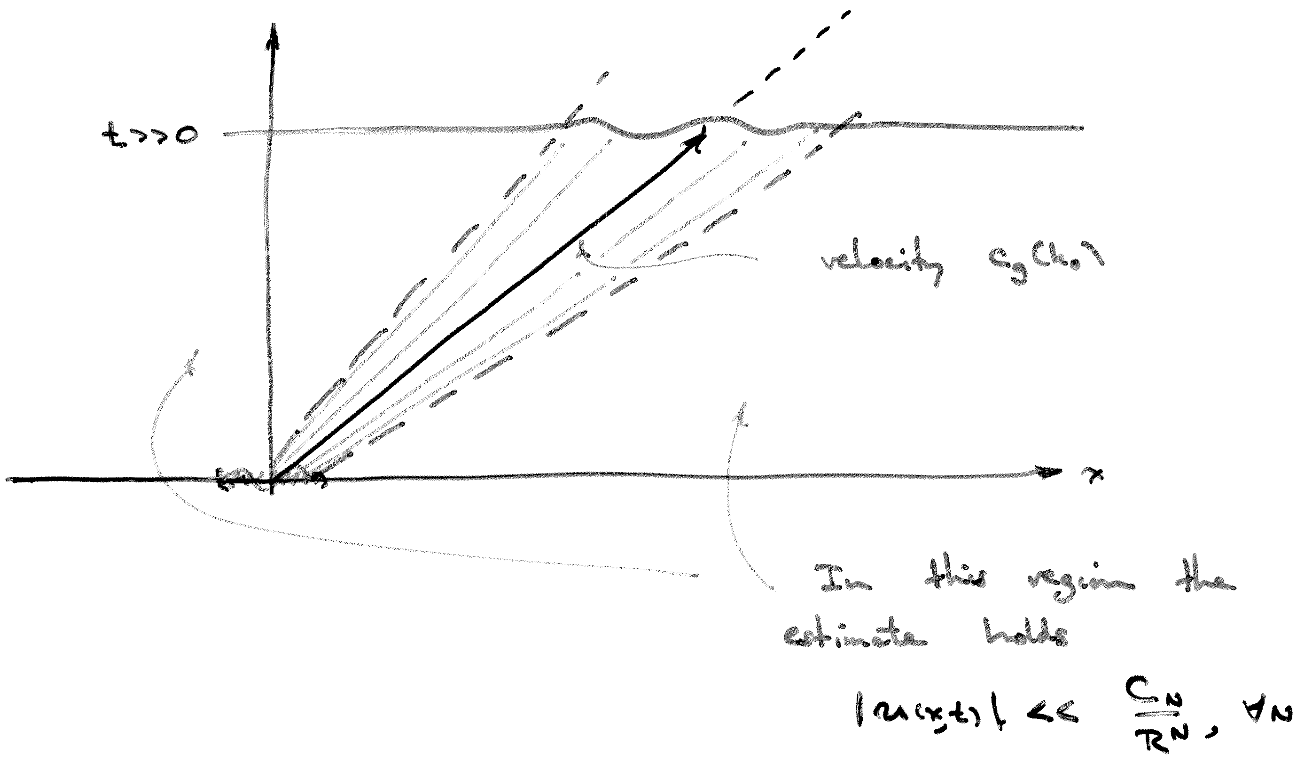
$$u(x,t) = \int e^{i[k\cdot x - \omega(k)t]}\left(\frac{1}{i}\partial_k\cdot\left(\frac{x-\partial_k\omega t}{|x-\omega_k t|^2}\right)\right)^N\hat{u}_0(k)\,dk$$

$$\leq O\left(\frac{1}{R^N}\right) \qquad \forall N$$

For the present purpose we define a wave packet as having Fourier transform localised near a given wave number, say k_0.

$$\hat{u}_0(k) = e^{ik_0 \cdot x} \hat{v}(k \qquad \hat{v} \in C_0^\infty$$

The previous calculation shows that for wave packets

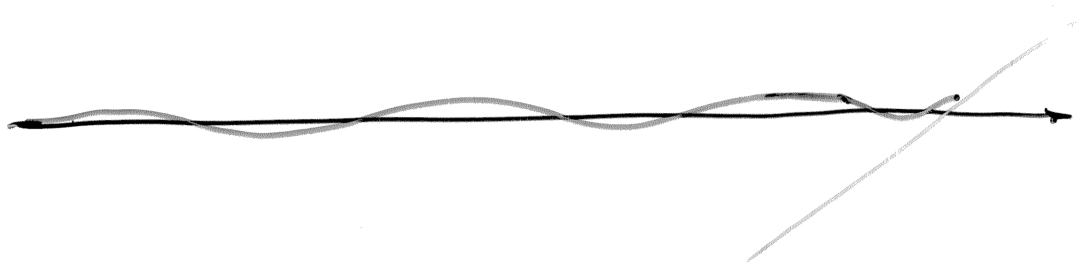


Question: linear waves exhibit dispersion, resulting in temporal decay. The nonlinearity of the water wave problem accounts for the propagation of coherent wave packets at velocities $\sim c_g(k_0)$ over long distances.

§3. coherent waves

Regarding the 26 Dec 2004 Sumatra earthquake and tsunami, there is almost no published data.

Most of my statements come from listening to firsthand reports in the news, photos, and from M. Krishna (IMSC – Chennai).



Certain facts:

* • 6 or 7 waves observed (crests)?, in some cases (Thai coast).
* temporal period • $T = 10 \sim 15$ min
* wave amplitude • 1m – 2m at sea
  • 1m – 3m on shore

Q: From this, deduce some aspects of tsunami wave form.

12)

Spatial period (and hence slope)

time period $\quad T = 15 \text{ min}$

phase velocity $\quad c_p = \sqrt{gk \tanh(hk)} / k$

Deduce wave number $\quad k = 2\pi / X$:

$$X = \frac{2\pi}{k} = c_p T$$

$$= \sqrt{gk \tanh(hk)} \, \frac{T}{k}$$

that is

$$(2\pi)^2 = gk \tanh(hk) \; T^2$$

Since k will be small

$$(2\pi)^2 \sim gh \, k^2 \, T^2$$

$$= 10 \frac{m}{sec} \times (4 \times 10^3 \, m) \times (900 \, sec)^2 \; k^2$$

15 min = 900 sec

$$k = \frac{2\pi}{18} \times 10^{-4} \, \frac{1}{m} \sim 4 \times 10^{-5} \, \frac{1}{m}$$

corresponding to spatial period

$$X = \frac{2\pi}{k} = \underline{\underline{180 \text{ km}}}$$

Slope: $\quad \partial_x \eta(x,t) \simeq \frac{2}{X} |\eta|_\infty \sim 2 \times \frac{2m}{18 \times 10^4 \, m}$

$$\sim 2 \times 10^{-5} = \varepsilon.$$