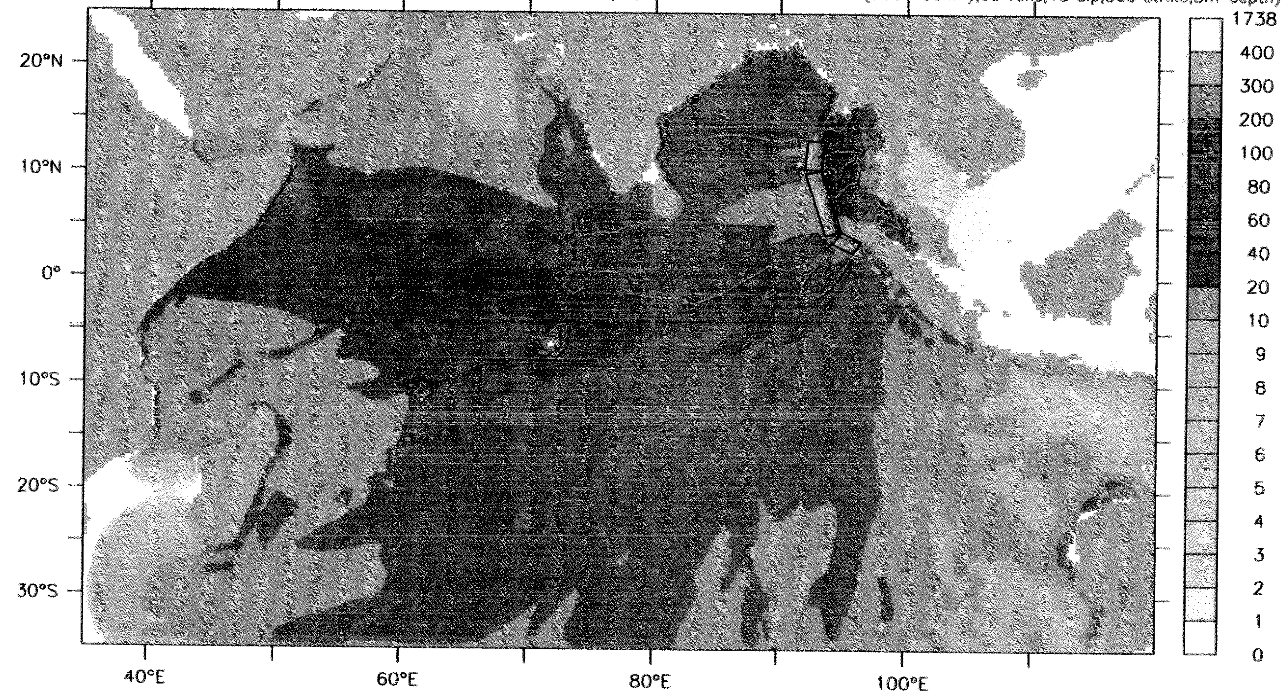
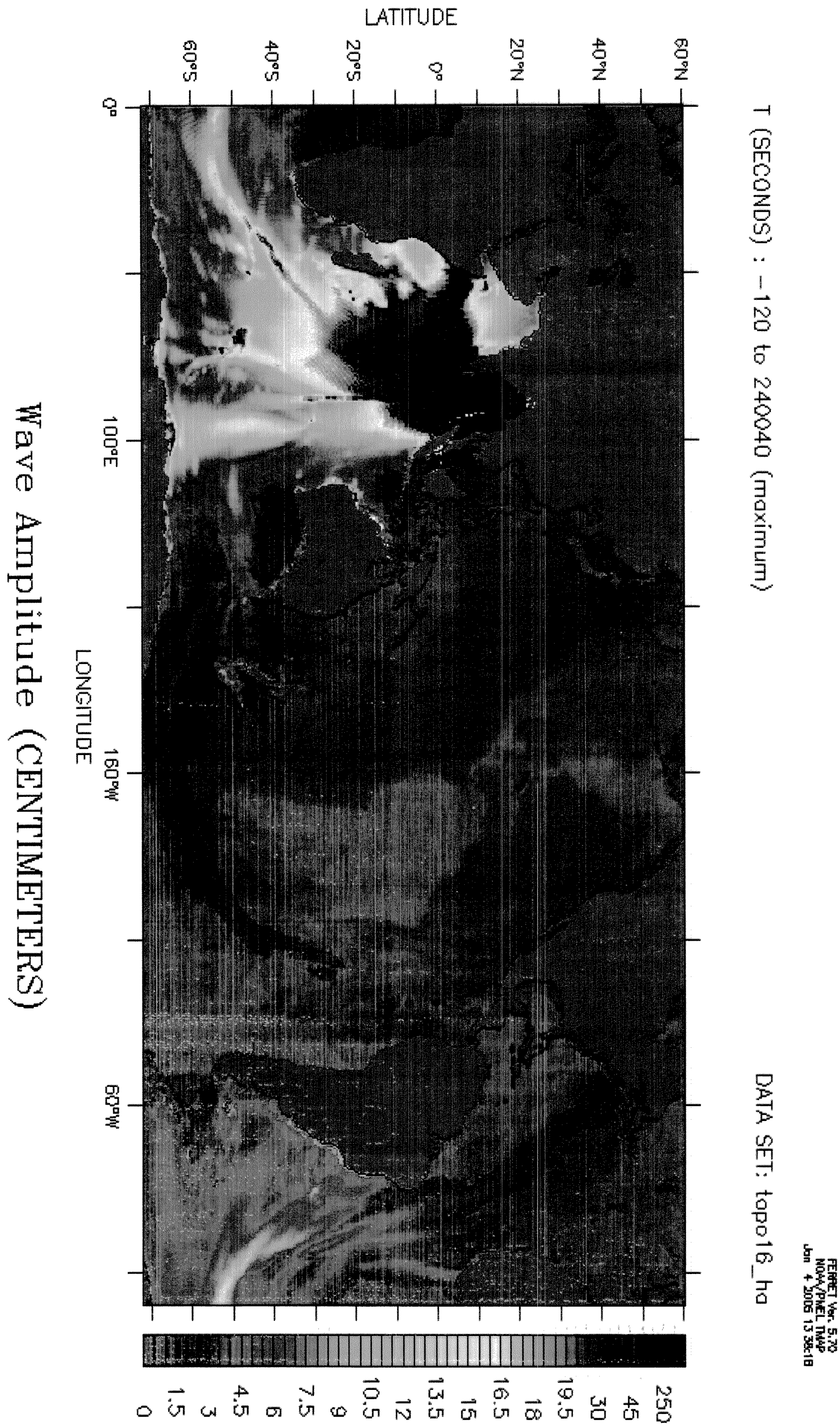
Facility for the Analysis and Comparison of Tsunami Simulations (FACTS)

Maximum Wave Height(cm) – 2004.12.26 Indonesian Tsunami

T (SECONDS) : –30 to 36030

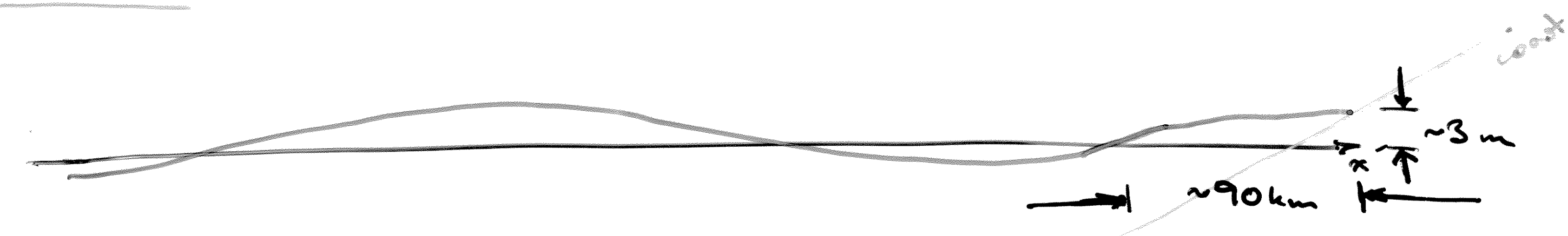
Source: Mw 9.0 (4°N,95.7°E–20m*(200x150km),90°rake,13°dip,300°strike,5m depth)+
(7.3559°N,94.1393°E–20m*(670x150km),90°rake,13°dip,345°strike,5m depth)+(11.605°N,93.4723°E–20m*(300x150km),90°rake,13°dip,365°strike,5m depth)



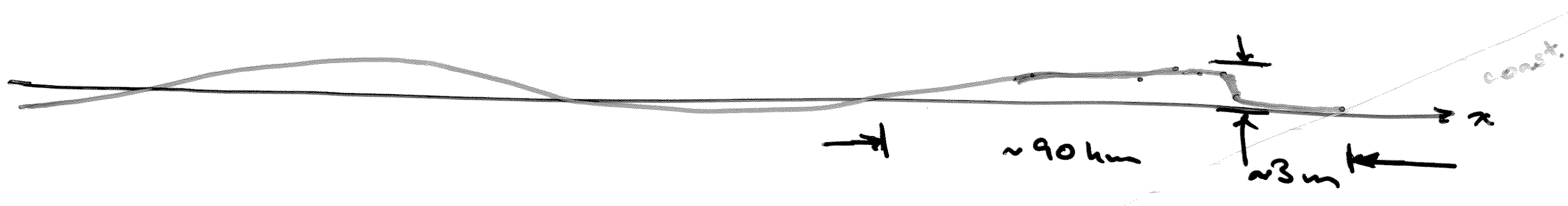


Wave Amplitude (CENTIMETERS)

Speculation: interaction with shoreline



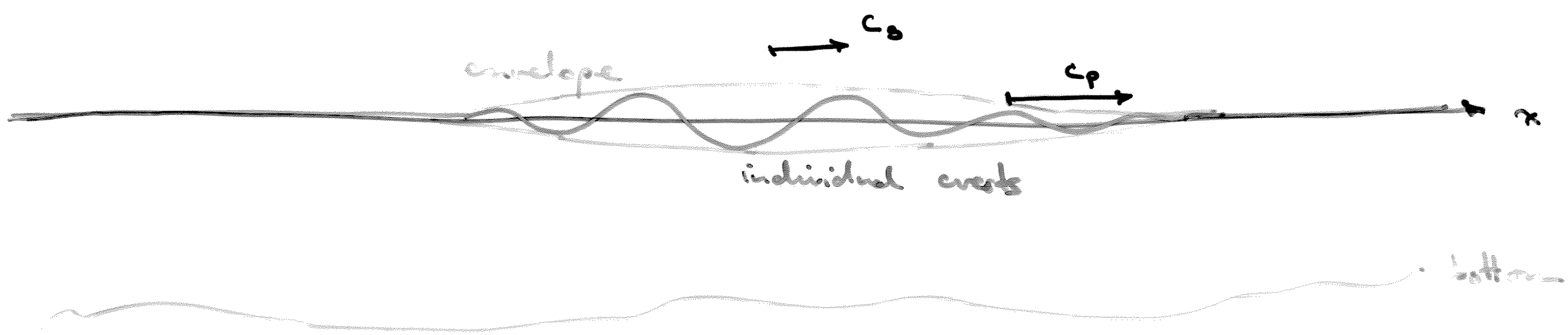
smooth case:



breaking case: hydraulic jump

- smooth case: water level simply rises over ~10 min to 2-3 meters higher than normal.
- breaking case: impact of a hydraulic jump (speed $\lesssim$ 30 km/hr, similar to breaking wave),
- backwash: returning water, or undertow very dangerous.

Furthermore, the tsunami travels as a wave packet:

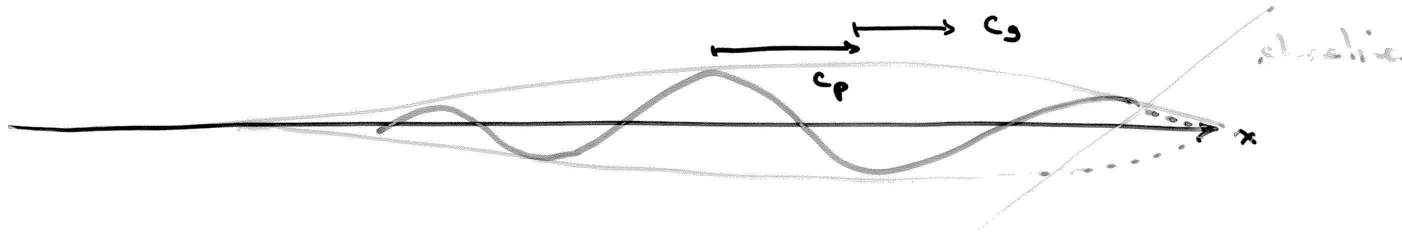


* Observers of the Dec 26, 2004 tsunami recall experiencing up to 7 crests, at roughly 15 minute intervals.

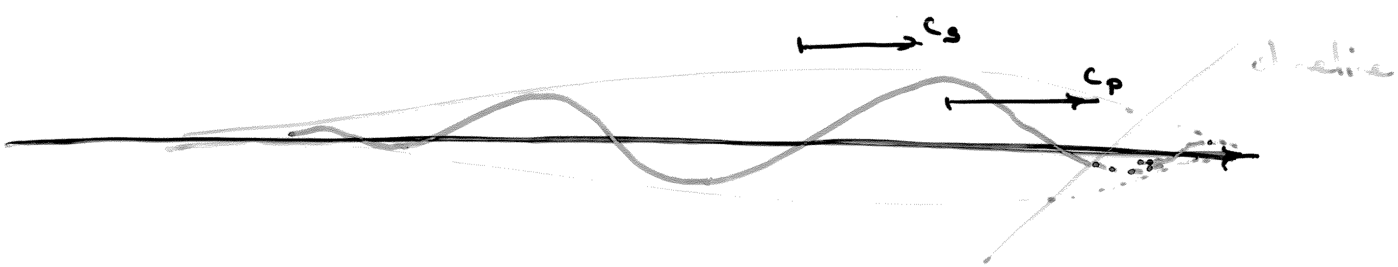
- In a wave packet, the first crest or first few crests will be smaller than later crests.

- When a wave packet impinges on the shoreline, the relative placement of the first significant oscillation determines whether the water level initially recedes, or rises.

13 ter



case: Water level initially rises



second case: Water level initially recedes

* Nonlinear equations of motion:

Euler's equations

$$(1) \qquad \Delta \varphi = 0 \qquad\qquad u = \nabla \varphi$$

$$\begin{cases} \partial_t \eta = \partial_y \varphi - \partial_x \eta \cdot \partial_x \varphi \\ \partial_t \varphi = -g\eta - \frac{1}{2}|\nabla \varphi|^2 \end{cases} \quad \text{on } \{ y = \eta(x,t) \}$$

In cases of a flat bottom at least:

Korteweg-deVries equation

$$(12) \qquad \partial_t Q = \mp \left(\tfrac{1}{6} \partial_x^3 Q + \tfrac{3}{2} Q \partial_x Q \right)$$

nonlinear Schrödinger equation

$$(13) \qquad \partial_t \Psi = -\frac{\partial_k^2 \omega(k_0)}{2} \partial_x^2 \Psi + \lambda |\Psi|^2 \Psi$$

All of these equations admit localized solutions.

$$(14) \qquad \begin{aligned} \partial_t \eta &= -\partial_{x_1} \left(\left(\bar{h}(x) + \varepsilon^2 (1 + \bar{c}_1) \eta \right) \partial_{x_1} \xi \right. \\ &\quad - \varepsilon^2 \partial_{x_1}^2 \left(\left(\bar{c}_2 + \tfrac{1}{3} h^3 + \bar{c}_3 \right) \partial_{x_1}^2 \xi \right) \\ &\quad - \varepsilon^2 \partial_{x_2} \left((h + \bar{c}_1) \partial_{x_2} \xi \right) \\ \partial_t \xi &= -g\eta - \tfrac{1}{2} \varepsilon^2 (1 + \bar{c}_1)(\partial_{x_1} \xi)^2 \end{aligned}$$

[Papanicolaou Rosales 1983].
[WC, P. Guyenne, D. Nicholls & C. Sulem 2004]. Coeffs $\bar{c}_j(x)$ by homogenizati

* Hamiltonian systems of PDE

I want to base my derivation of these various equations of motion on Hamiltonian perturbation theory.

\+ Euler equations [Zakharov, 1968]

$$H = \int\int_{-b(x)}^{\eta(x,t)} \tfrac{1}{2} |\nabla \varphi|^2 \, dy\, dx + \int \tfrac{g}{2} \eta^2 \, dx$$

kinetic potential

variables, canonical coordinates

$$\left(\eta(x),\ \xi(x) = \varphi(x, \eta(x)) \right)$$

$$(15) \qquad H(\eta, \xi) = \int \tfrac{1}{2} \xi \, G(\eta) \xi + \tfrac{g}{2} \eta^2 \, dx$$

Dirichlet-Neumann operator

$$\int\int \tfrac{1}{2} |\nabla \varphi|^2 \, dy\, dx = -\tfrac{1}{2} \int\int \varphi \underbrace{\Delta \varphi}_{=0} \, dy\, dx$$

$$+ \tfrac{1}{2} \int \underbrace{\varphi \, N \cdot \nabla \varphi \, dS_{\text{bottom}}}_{=0}$$

$$+ \tfrac{1}{2} \int \varphi \, N \cdot \nabla \varphi \, dS_{\text{top}}$$

Define the operator

Dirichlet data $\xi(x) \longmapsto \varphi(x, y)$ harmonic extension to $\{-b(x) < y < \eta\}$

$$\longmapsto N \cdot \nabla \varphi \, dS_{\text{top}} := \underline{G(\eta) \xi \, dx}$$

16)

Theorem The Dirichlet-Neumann operator $G(\eta)$ is:

(i) bounded $G(\eta) : H^1 \to L^2$

(ii) positive (semi) definite $G(\eta) \geq 0$

(iii) symmetric $G(\eta)^T = G(\eta)$

(iv) [Coifman & Meyer, 1985] analytic in its dependence on $\eta \in Lip$ (2-dim case)

[Christ & Journé, 1987][[illegible], [illegible] & Sulem 1997] analytic in $\eta \in C^1$ (n-dim case)

Taylor expansion:

$$G(\eta) = G_0 + G_1(\eta) + G_2(\eta) + \cdots \qquad G_j(\alpha\eta) = \alpha^j G_j(\eta)$$

examples

$$G_0 = |D| \tanh(h|D|) \qquad D_j = \frac{1}{i}\partial_{x_j}$$

$$G_1(\eta) = D \cdot \eta D - G_0 \eta G_0$$

$$G_2(\eta) = \frac{1}{2}\left(|D|^2 \eta^2 G_0 + G_0 \eta^2 |D|^2 - 2 G_0 \eta G_0 \eta G_0\right)$$

* Transformation theory:

Zakharov's form for equations (1) are

(16) $$\partial_t \eta = \delta_\xi H = G(\eta)\,\xi$$

$$\partial_t \xi = -\delta_\eta H = -g\,\eta - \tfrac{1}{2}\,\delta_\eta \langle \xi, G(\eta)\,\xi \rangle$$

This is in Hamiltonian form (Darboux coordinates).

(17) $$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = J \begin{pmatrix} \partial_\eta H \\ \partial_\xi H \end{pmatrix} \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Under transformations $w = f(v)$ with $v = \begin{pmatrix} \eta \\ \xi \end{pmatrix}$, the canonical form (17) changes to

(18) $$\partial_t w = (\partial_v f)\, J\, (\partial_v f)^T\, \delta_w K(w)$$

transformed Hamiltonian $H(v) = K(w)$.

Definition: The transformation $w = f(v)$ is canonical if $(\partial_v f)\, J\, (\partial_v f)^T = J$.

In general case, a new symplectic form $J_1 = (\partial_v f)\, J\, (\partial_v f)^T$

(8)

Hamiltonian perturbation theory: (KdV regime)

(1) amplitude scaling

$$\varepsilon^2 \eta' = \eta \qquad \varepsilon \xi' = \xi \qquad J' = \varepsilon^3 J$$

(2) spatial scaling

$$x \longmapsto \varepsilon x = x' \qquad J'' = \varepsilon J'$$
$$D = \varepsilon D'$$

Fourier multipliers transform

$$|D| \tanh(h|D|) \longrightarrow \varepsilon |D'| \tanh(\varepsilon h |D'|)$$
$$\sim \varepsilon^2 h |D'|^2 - \frac{\varepsilon^4 h^3}{3} |D'|^4 + \cdots$$

(3) moving reference frame

$$v'(x,t) = v(x - \sqrt{gh}\, t, t)$$

add an integral of momentum to the Hamiltonian

$$H \longrightarrow H - \sqrt{gh}\, I$$

where the momentum is

$$I(\eta, \xi) = \int \xi \, \partial_x \eta \, dx .$$

It is a conserved quantity

$$\{H, I\} = \int \delta_v H \, J \, \delta_v I \, dx = 0 .$$

(4) characteristic coordinates

(i) $$(\eta, \xi) \longmapsto (\eta, u = \partial_x \xi) \qquad J''' = \varepsilon^4 \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}$$

(ii) $$(\eta, u) \longmapsto \begin{pmatrix} Q_+ \\ Q_- \end{pmatrix} = \begin{pmatrix} \sqrt[4]{\frac{g}{4h}} & \sqrt[4]{\frac{h}{4g}} \\ \sqrt[4]{\frac{g}{4h}} & -\sqrt[4]{\frac{h}{4g}} \end{pmatrix} \begin{pmatrix} \eta \\ u \end{pmatrix} \qquad J^{(4)} = \varepsilon^4 \begin{pmatrix} -\partial_x & 0 \\ 0 & \partial_x \end{pmatrix}$$

19)

Applying the above transformation to the water wave Hamiltonian of Zakharov:

$$H \longmapsto H_1 = \frac{1}{2}\int \sqrt{gh}\,(Q_+^2 + Q_-^2)\,\varepsilon^3 dx$$

$$+ \frac{1}{2}\int -\frac{1}{6}\sqrt{g}\,h^{\frac{5}{2}}\left((\partial_x Q_+)^2 - 2\partial_x Q_+ \partial_x Q_- + (\partial_x Q_-)^2\right)\varepsilon^5$$

$$+ \frac{1}{2}\int \frac{1}{2\sqrt{2}}\left(Q_+^3 - Q_+^2 Q_- - Q_+ Q_-^2 + Q_-^3\right)\varepsilon^5 dx$$

$$+ O(\varepsilon^7)$$

In a neighborhood in H^m where $\|Q_-\|_{H^m} < O(\varepsilon^2)$

$$H_1 = \frac{\varepsilon^3}{2}\int \sqrt{gh}\,Q_+^2 - \frac{c_1}{6}(\partial_x Q_+)^2 + c_2 Q_+^3\,dx \;\cancel{+ O(\varepsilon^4)}$$

$$I_1 = \varepsilon^3 \int Q_+^2\,dx \;\cancel{+ O(\varepsilon^4)}$$

truncate

This is the Hamiltonian for the KdV equation

$$\tau = \varepsilon^2 t$$

$$(19) \qquad \partial_\tau Q_\pm = \mp\left(\frac{c_1}{6}\partial_x^3 Q_\pm + 6c_2 Q_\pm \partial_x Q_\pm\right)$$