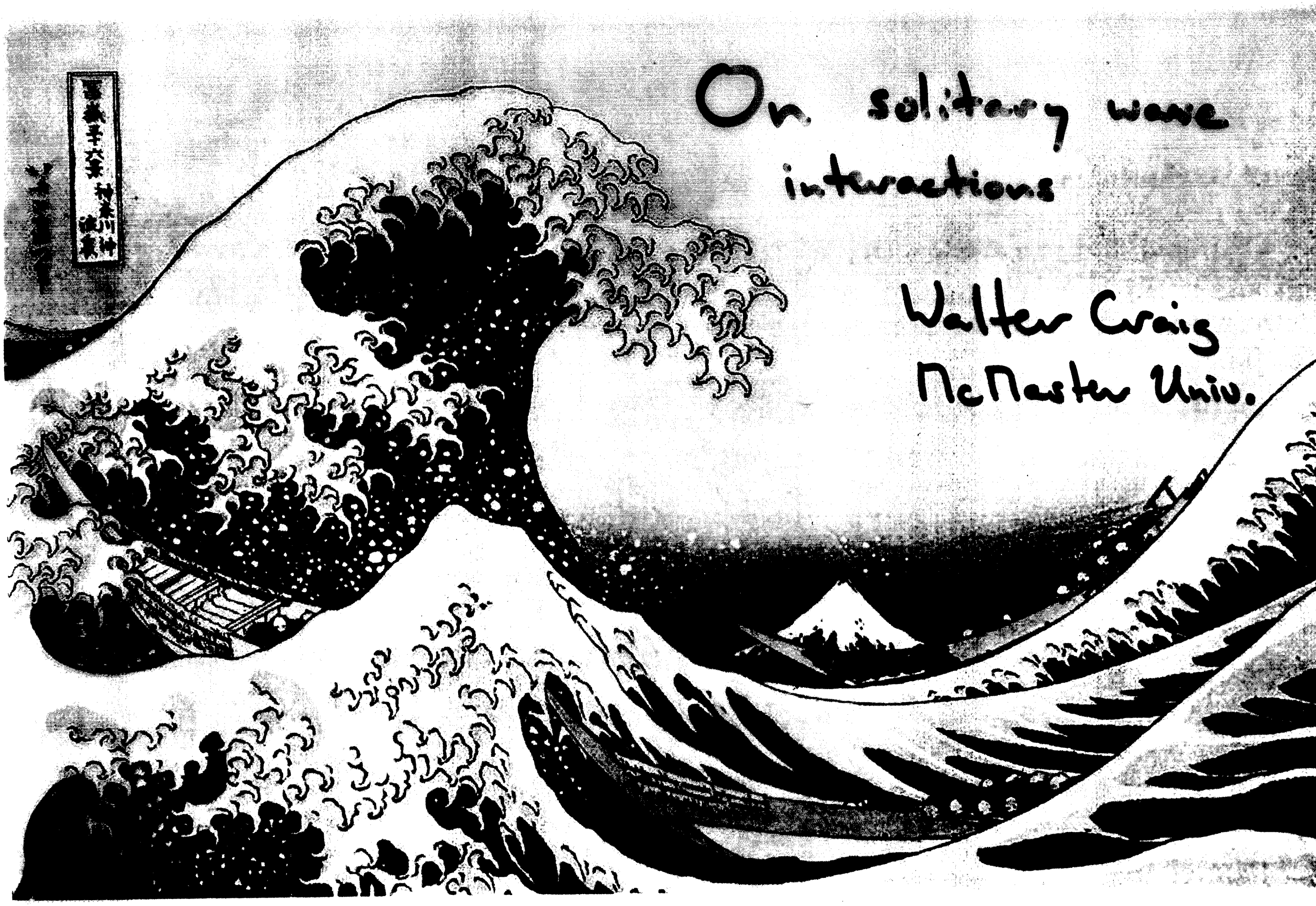
On solitary wave interactions
Walter Craig
McMaster Univ.

2.)

Focused Research Group:

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Bernard Deconinck	University of Washington
* Philippe Guyenne	McMaster University & Fields Institute
* Joe Hammack	Penn State University
* Diane Henderson	Penn State University
David Nicholls	University of Notre Dame
Harvey Segur	University of Colorado
* Catherine Sulem	University of Toronto
[illegible]	McMaster University & Fields Institute

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- Fields Institute
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- SHARCNET

3)

1) Surface water waves

$$\begin{cases} \nabla \cdot \vec{u} = 0 \\ \nabla \wedge \vec{u} = 0 \end{cases}$$

(1) $\Delta \varphi = 0$ within the fluid domain

$\partial_N \varphi = 0$ on $y = -h$

$$\partial_t \eta = \partial_y \varphi - \partial_x \eta \cdot \partial_x \varphi$$
$$\partial_t \varphi = -g\eta - \frac{1}{2} |\nabla \varphi|^2 \qquad \text{on} \quad y = \eta(x,t)$$

Conserved quantities

(2) (i) $\displaystyle\int_{-\infty}^{+\infty} \eta \, dx = m_0$

(ii) $\displaystyle\int_{-\infty}^{\infty} \int_{-h}^{\eta(x)} \partial_x \varphi \, dy\, dx = \mu_0$

(iii) $\displaystyle\int_{-\infty}^{\infty} \int_{-h}^{\eta(x)} \frac{1}{2} |\nabla \varphi|^2 \, dy\, dx + \int_{-\infty}^{\infty} \frac{g}{2} \eta^2 \, dx = e_0$

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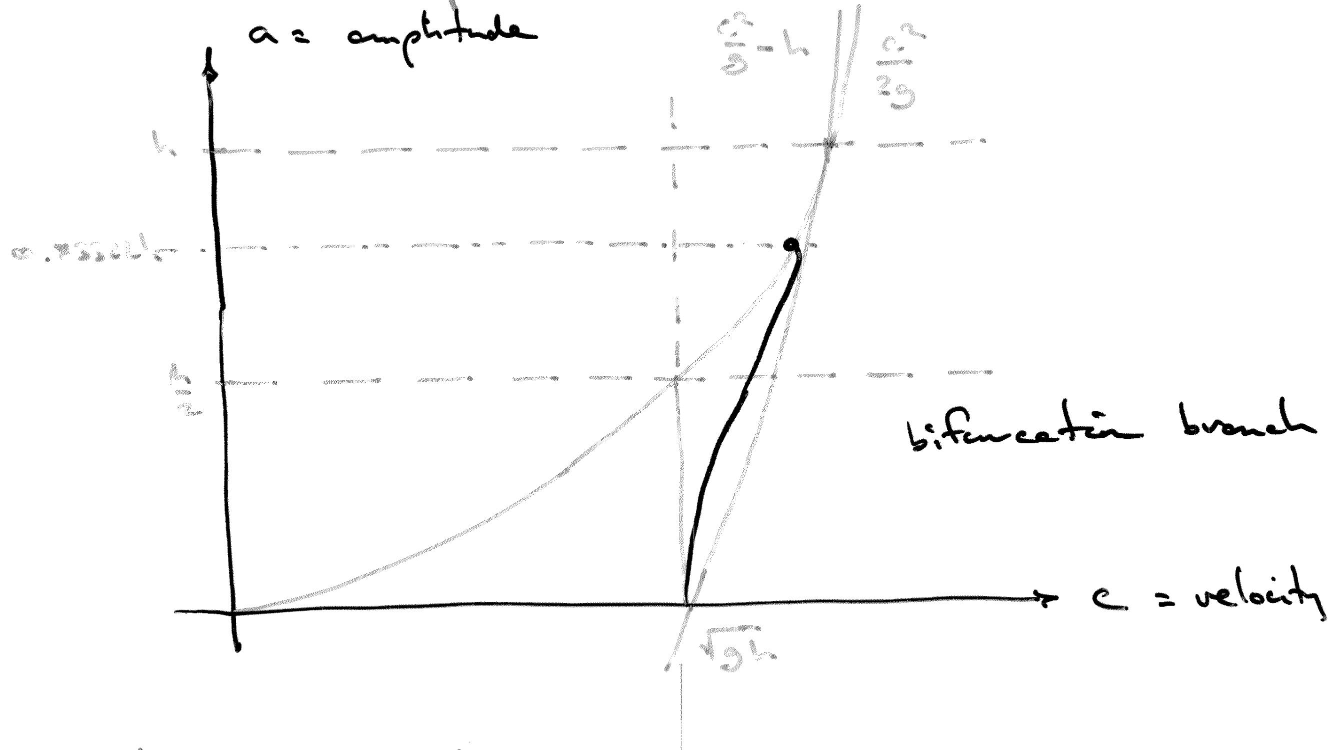
Solitary waves in the free surface

$$\eta(x,t) = \eta(x-tc), \quad \varphi(x,y,t) = \varphi(x-tc,y)$$

bifurcation problem (from continuous spectrum)



Description: Stokes (1847)(1880)

Existence theory: Lavrentiev (1947)
Friedrichs & Hyers (1954)
Beale (1979)
Kirchgässner & Mielke (1980's)

Extremal form: Amick & Toland (1981)

Numerical simulations: Byatt-Smith & Longuet-Higgins (1976)
Hunter & Vanden Broeck (1983)
* Tanaka (1986)

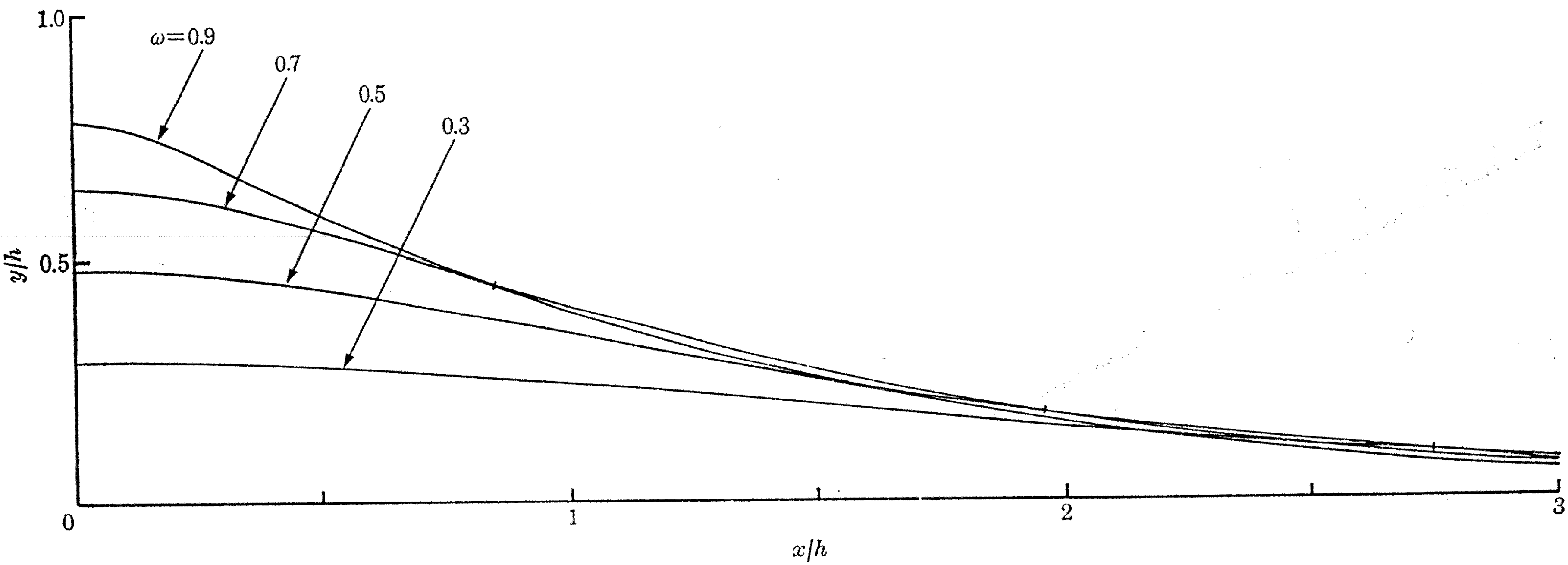


FIGURE 2. Surface profiles of solitary waves of different amplitudes, in water of constant depth.

Byatt-Smith & Longuet-Higgins

Comparison with the Korteweg-de Vries (KdV) equation

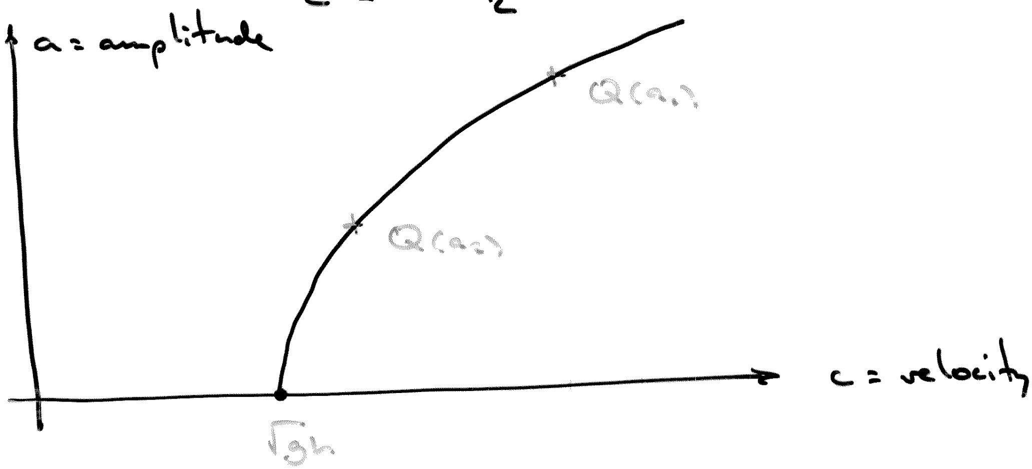
$$(3) \qquad \partial_\tau Q = \mp\left(\frac{1}{6}\partial_X^3 Q + \frac{3}{4}\partial_X(Q^2)\right)$$

with the change of scales $X = \varepsilon(x \mp \sqrt{gh}\,t)$, $\tau = \varepsilon^3 t$

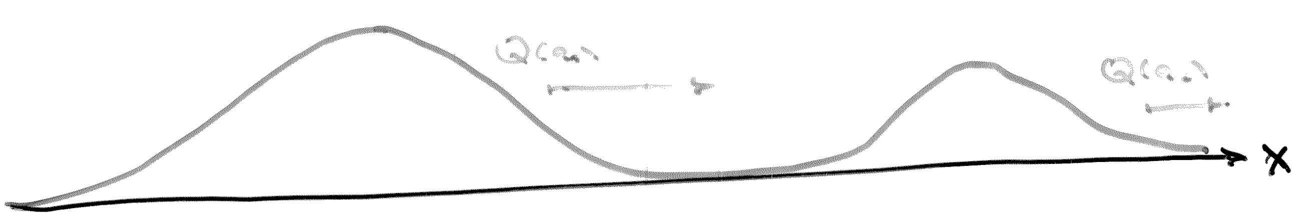
The family of KdV solitons

$$Q_{s_1}(X - c\tau) = a^2 \operatorname{sech}^2\left(\sqrt{\tfrac{3}{4}}\, a\,(X \mp c\tau)\right)$$

$$c = \pm \frac{a^2}{2}$$



Multi solitons: $Q_{s_2}(X,\tau)$



- Overtaking collisions
- Clean ('elastic') interactions

4)

2) binary collisions of solitary waves

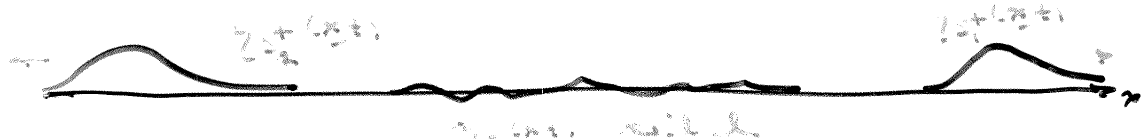
<u>counter-propagating</u>:

Initial data

$$\eta_0(x) = \eta_{s_1}(x) + \eta_{s_2}(x)$$

where each is a Stokes solitary wave.

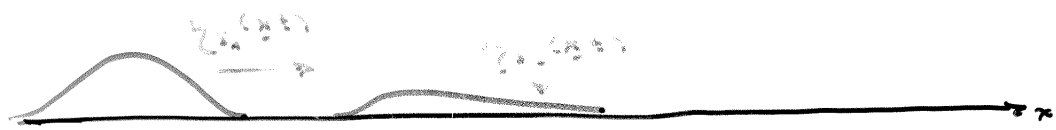
After collision the solution should look like



nb: this happens on a fast time scale.

<u>co-propagating</u>:

Initial data



After collision

nb: this happens on a substantially longer time scale, modeled by the KdV equation.

* Basic issues arising in head-on solitary wave collisions

(i) phase shift – small and negative

(ii) run-up (super linear amplitude superposition)

$$\sup_{x\in\mathbb{R}} (\eta_{s_1}) + \sup_{x\in\mathbb{R}} (\eta_{s_2}) < \sup_{x,t} (\eta(x,t))$$

and/or 'wall residence time'.

(iii) degree to which collisions are inelastic

$$\eta(x,t) \quad \text{a solution of (1)}$$

$$\lim_{t\to-\infty} \left[\eta(x,t) - (\eta_{s_1}(x-c_1t) + \eta_{s_2}(x-c_2t)) \right] = 0$$

Define the residual for $t \gg 1$

$$\eta(x,t) = (\eta_{s_1^+}(x-c_1^+t+a_1) + \eta_{s_2^+}(x-c_2^+t+a_2)) + \eta_R(x,t)$$

Conclusions:

(i) the residual is very small but always present.

(ii) there is a characteristic Fourier signature of the residual.

(iii) comparison with PSU experiments.

7) alt

3) Prior results: counter-propagating interactions

* Experimental

  - Maxworthy (1976)
    - measurements of phase shift (negative) and run-up.
  - Renouard, Seabra Santos & Temperville (1985)
    - negative phase shift and run-up.

  My conclusion: experiments have not really pinned this down yet.

  - nb: R. Pierce (unpublished, PSU)

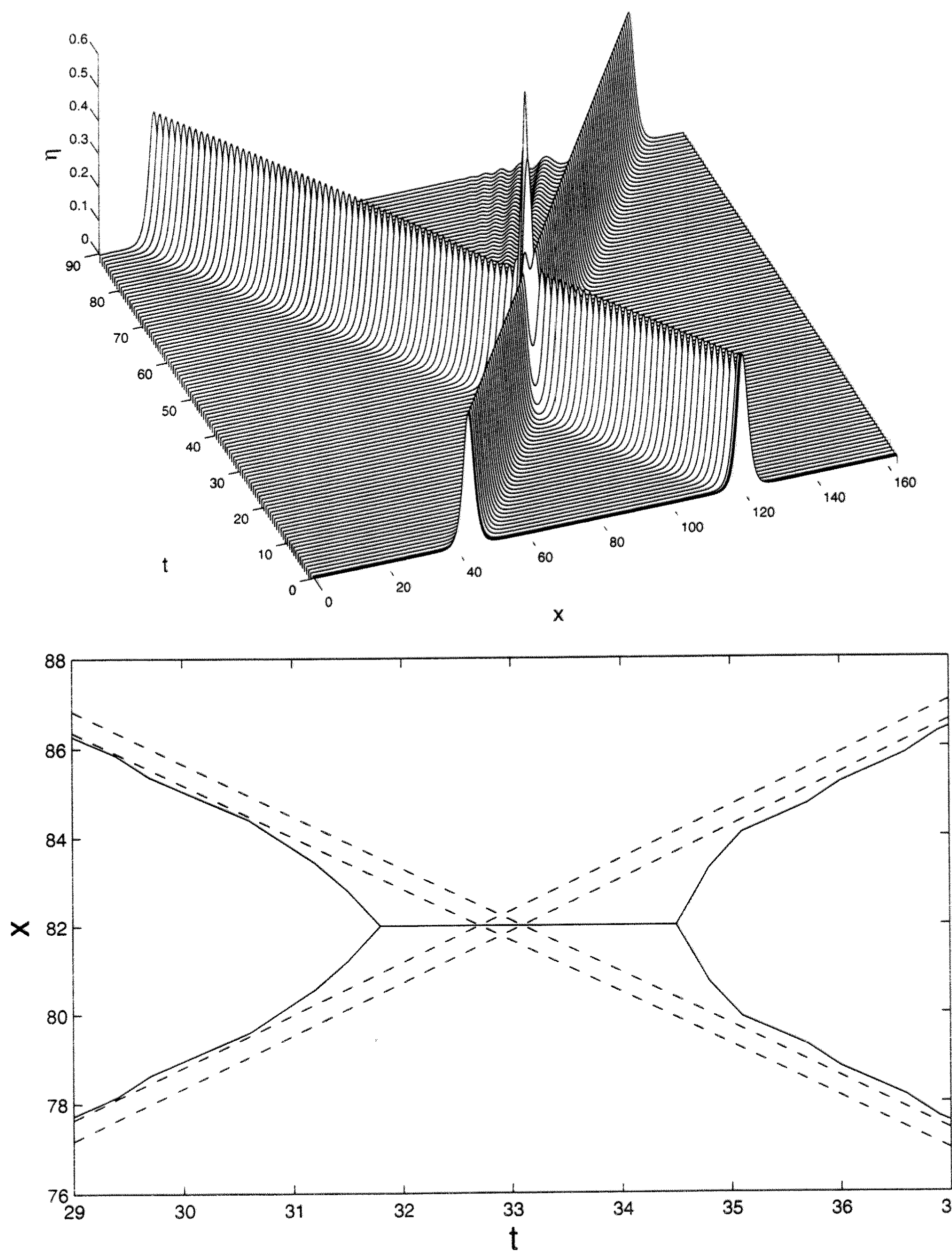
* Asymptotic analysis and numerical modeling

  - Chan & Street (1970)
  - Mirie & Su (1980), (1982)
  - Fenton & Rienecker (1982)
  - Byatt-Smith (1988)
  - Cooker, Weidman & Bale (1997)

Issues:
- predict the phase shift
- compute the run-up
- discuss the character of the dispersing residual
- understand the speed of the reflected waves

Numerical results

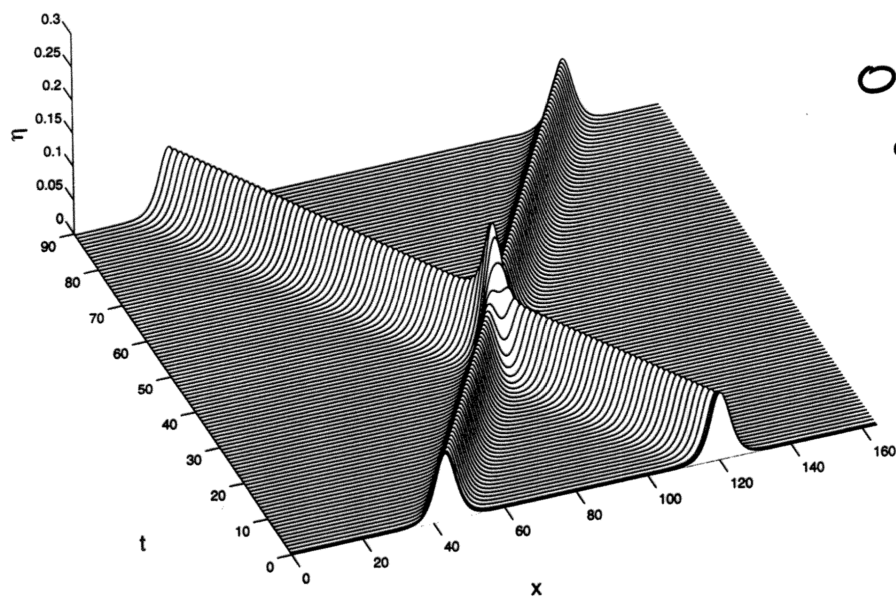
- Symmetric head-on collisions



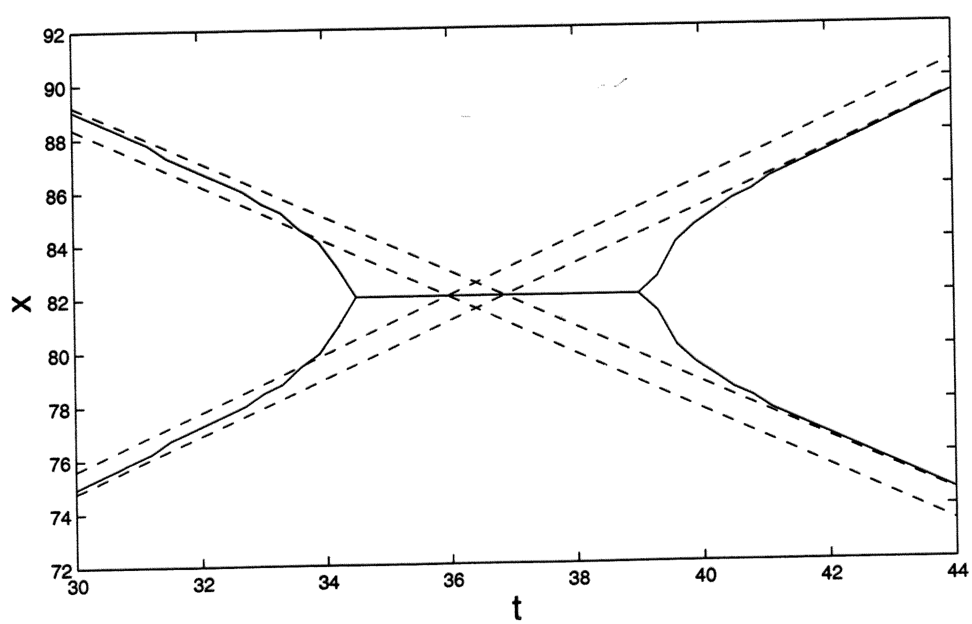
Amplitudes
$a_1 = 0.5h$
$a_2 = 0.5h$
initially

Numerical results (Euler)

- Symmetric head-on collisions

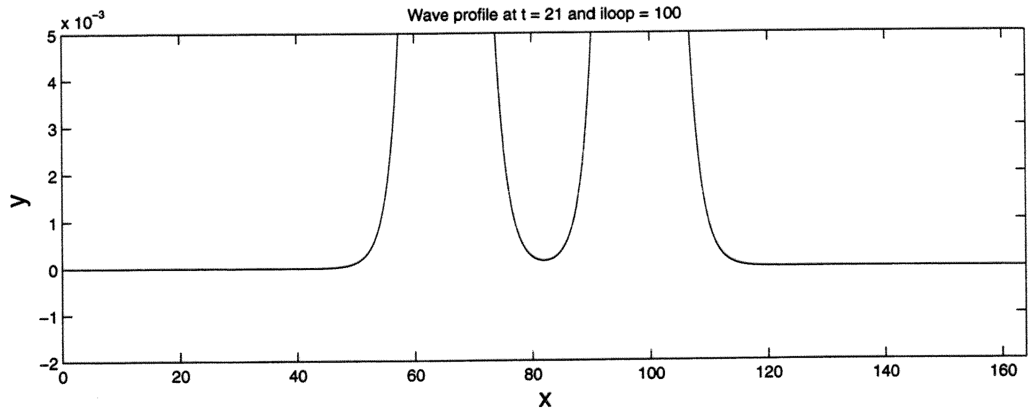


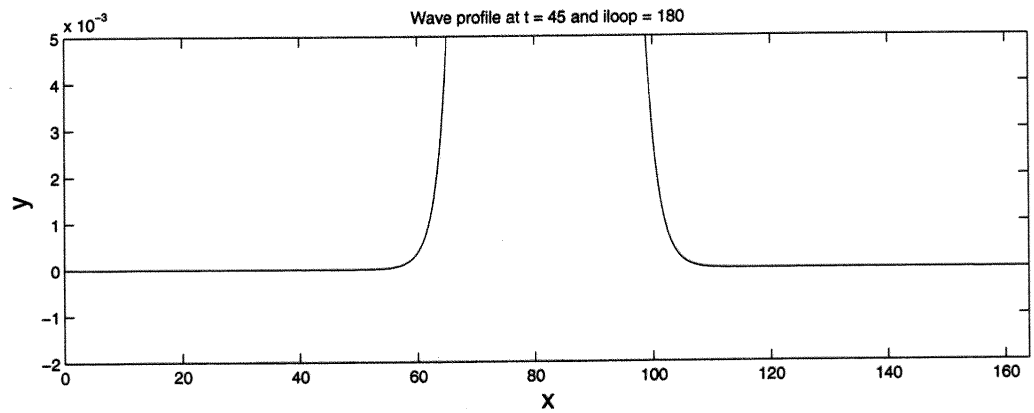
Original amplitudes
$a_1 = 0.1\ h$
$a_2 = 0.1\ h$

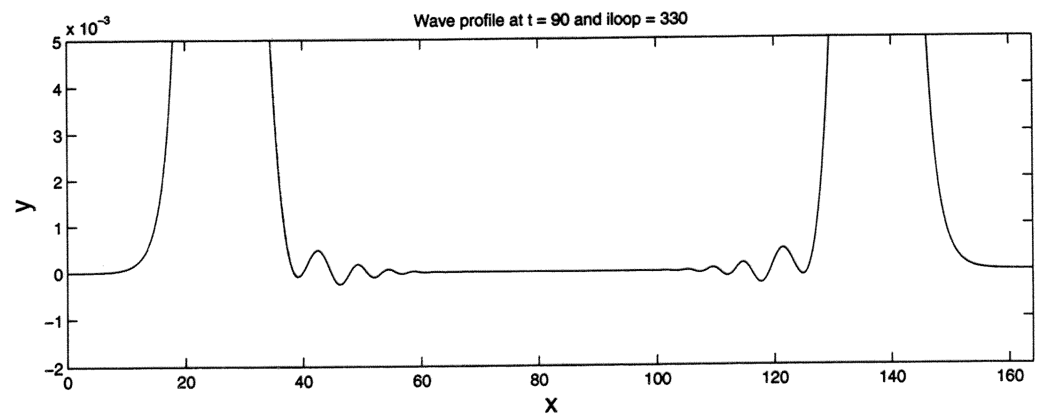


Numerical results

- Symmetric head-on collisions





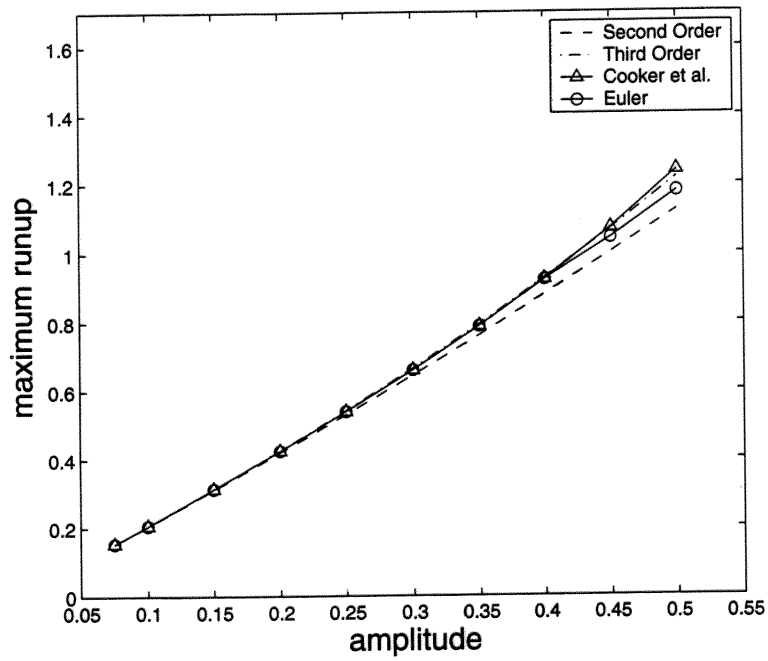


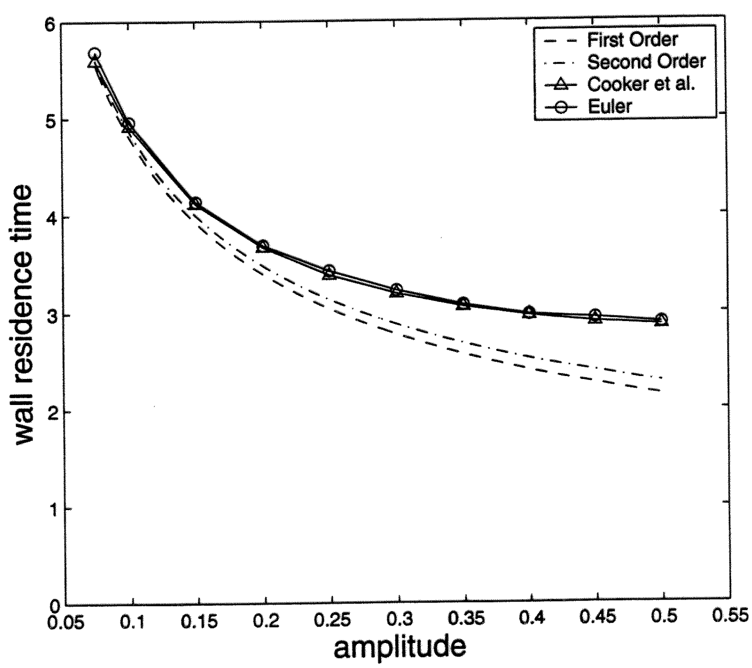
Initial amplitudes $a_1 = 0.1\,h$, $a_2 = 0.1\,h$

The residual is small but present;

Numerical results

- Symmetric head-on collisions





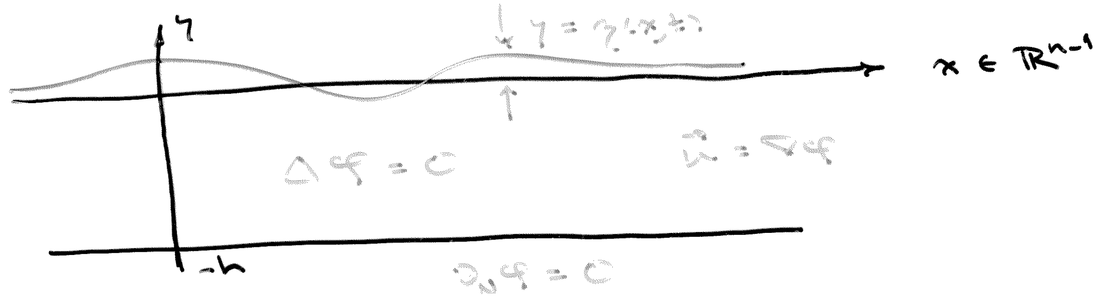
Su & Mirie (1980)

Cooker, Wiedman & Bale (1997)

Guyenne et al

5)

The numerical method:



* Zakharov's Hamiltonian

(4) $$H = K + \mathcal{U} = \text{kinetic} + \text{potential energy}$$

$$\mathcal{U} = \int_{-\infty}^{\infty} \frac{g}{2}\eta^2 \, dx$$

$$K = \int_{-\infty}^{\infty}\int_{-h}^{\eta(x)} \frac{1}{2}|\nabla\varphi|^2 \, dy \, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2}\xi \, G(\eta)\xi \, dx$$

where $G(\eta)$ is the Dirichlet-Neumann operator:

(5) $$\xi(x) \longmapsto \varphi(x,y) \longmapsto \nabla\varphi(x,y)\cdot N \, dS(\eta) := G(\eta)\xi \, dx$$

The equations of motion are expressed as

(6) $$\partial_t \eta = \delta_\xi H = G(\eta)\xi$$

$$\partial_t \xi = -\delta_\eta H = -g\eta - \frac{1}{2}\delta_\eta \langle \xi, G(\eta)\xi \rangle .$$

This is in the Hamiltonian form $\dot{z} = J \operatorname{grad} H(z)$ for $z = (\eta, \xi)$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

6)

<u>**Theorem**</u>: The Dirichlet - Neumann operator $G(\eta)$ is an analytic operator-valued function in domains $\{\eta \in Lip\}$.

[Coifman & Meyer (1985), Christ & Journé (1987), Craig, Schanz & Sulem (1997)]

+ This is to say that the Taylor expansion

$$(7) \qquad G(\eta)\xi = G_0\xi + G_1(\eta)\xi + G_2(\eta)\xi + \cdots$$

converges in $\mathcal{L}(H^1 : L^2)$ for sufficiently small $|\eta(x)|_{Lip} < \varepsilon$;

where $G_j(\alpha\eta) = \alpha^j G_j(\eta)$ for $\alpha \in \mathbb{R}$.

+ Similarly, the Hamiltonian is

$$(8) \qquad H(\eta, \xi) = \int \tfrac{1}{2}\xi G(\eta)\xi + \tfrac{g}{2}\eta^2 \, dx$$

$$= \int \tfrac{1}{2}\xi G_0 \xi + \tfrac{g}{2}\eta^2 \, dx$$

$$+ \sum_{j\geq 1} \int \tfrac{1}{2}\xi G_j(\eta)\xi \, dx$$

$$= H^{(2)}(\eta, \xi) + \sum_{j\geq 1} H^{(j+2)}(\eta, \xi)$$

Explicitly

$$G_0\xi(x) = D\tanh(hD)\xi, \qquad D = \tfrac{1}{i}\partial_x.$$

implemented using the Fourier transform.

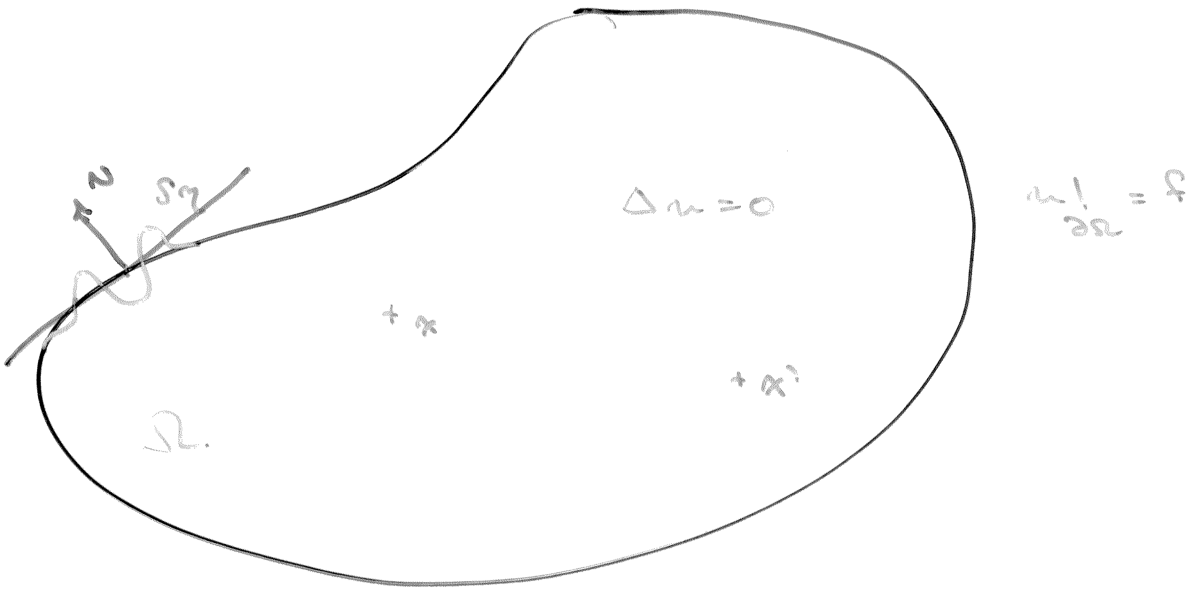
The numerical scheme is a <u>surface spectral method</u>

$$(9) \qquad \partial_t \eta^N = \delta_\xi \Big(H^{(2)}(\eta^N, \xi^N) + \sum_{1\leq j\leq N} H^{(j+2)}(\eta^N, \xi^N) \Big)$$

$$\partial_t \xi^N = -\delta_\eta \Big(H^{(2)}(\eta^N, \xi^N) + \sum_{1\leq j\leq N} H^{(j+2)}(\eta^N, \xi^N) \Big)$$

ref: W. Craig & C. Sulem (1993), [illegible]

Connection with the Hadamard variational formula



The Green's function for a domain $\Omega \subseteq \mathbb{R}^n$,

$$g(x, x'; \Omega)$$

Variations with respect to domain [Hadamard 1914]

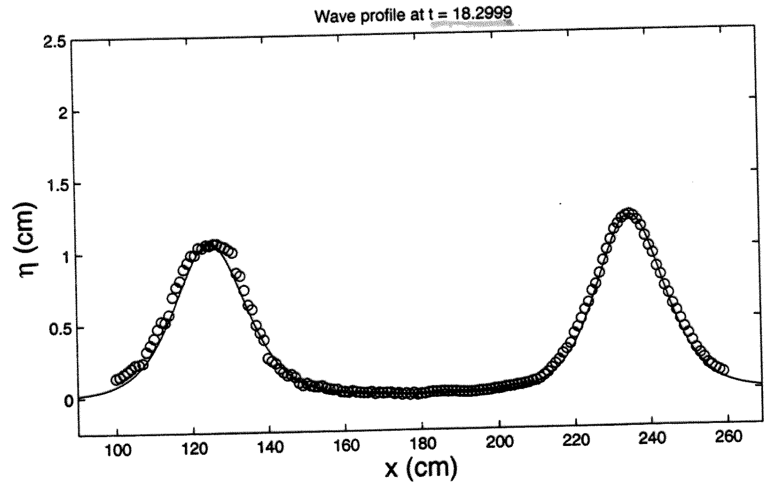
$$(10) \qquad \delta g \cdot \delta\eta\,(x, x') = \int_{\partial\Omega} N_z \cdot \nabla_z g(x, z)\; N_z \cdot \nabla_z g(z, x')\; \delta\eta(z)\, dS$$

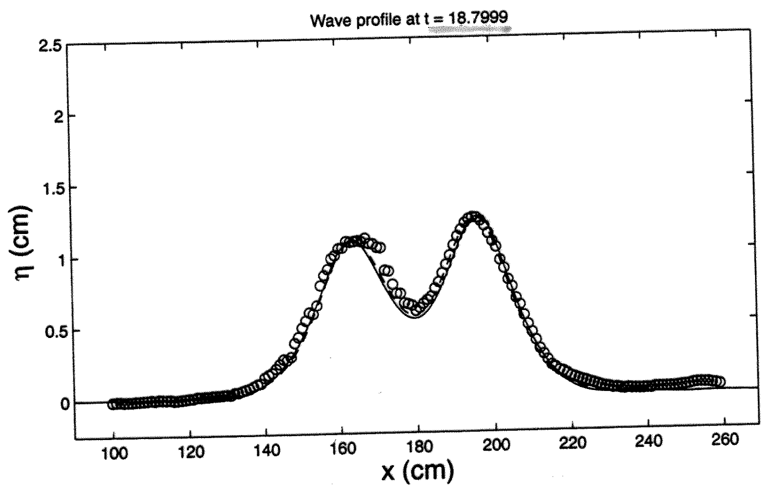
Recall the kernel of the Dirichlet - Neumann operator

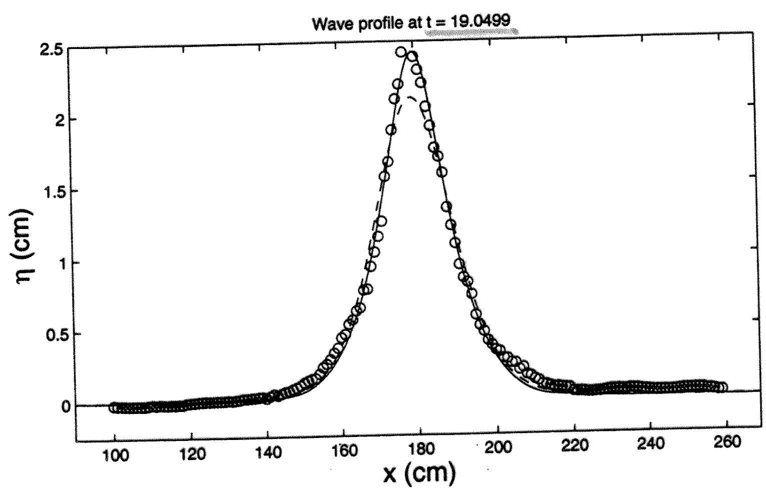
$$(11) \qquad G(\Omega)(x, x') = N_x \cdot \nabla_x\; N_{x'} \cdot \nabla_{x'}\; g(x, x') \,.$$

Comparison with experiment

- Asymmetric head-on collision



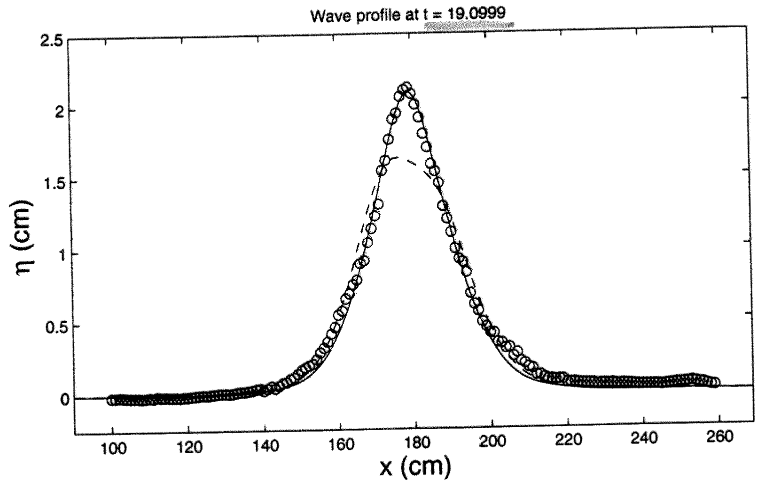


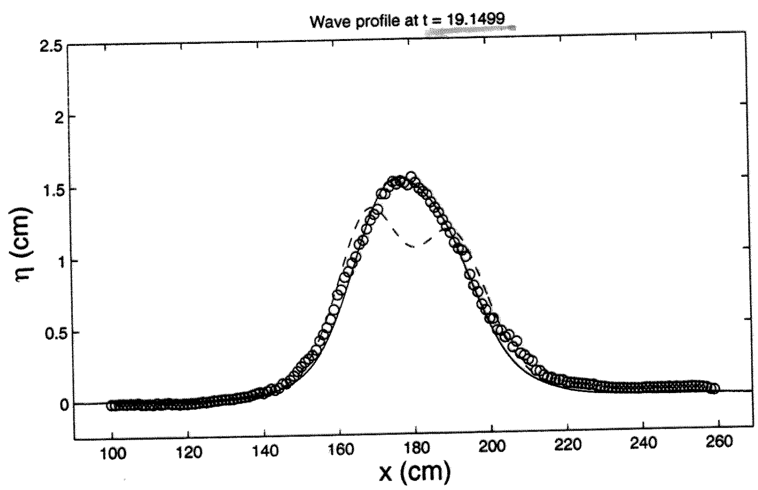


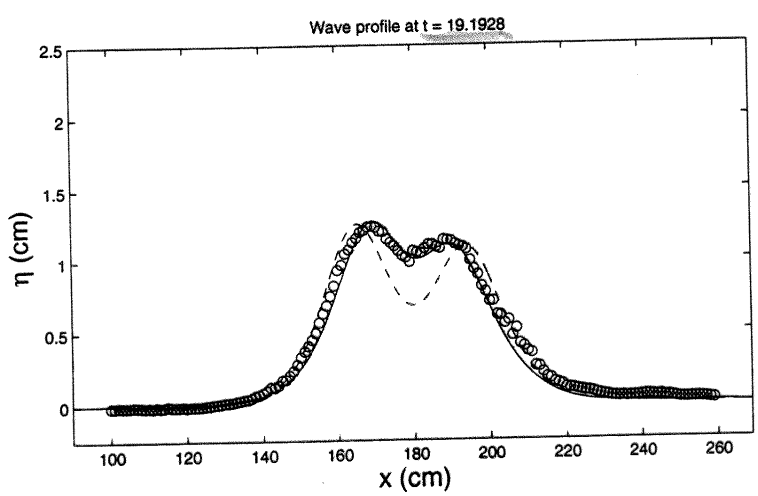
- PSU wavetank (2004)
- Euler's equations, numerical
- two KdV solitons

Comparison with experiment

- Asymmetric head-on collision

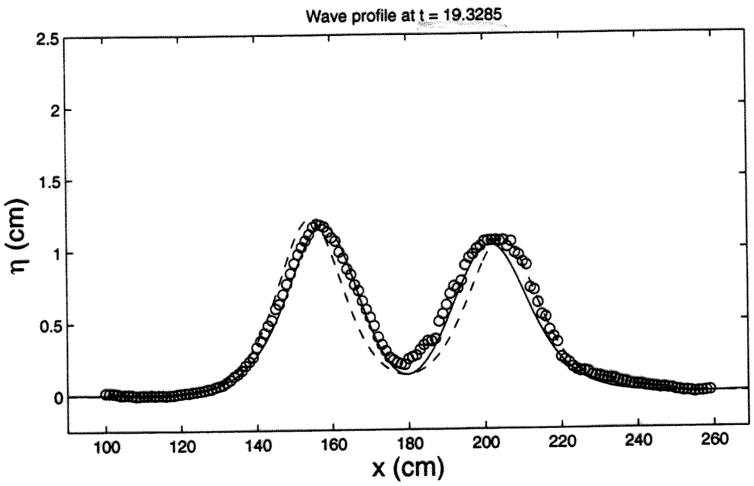


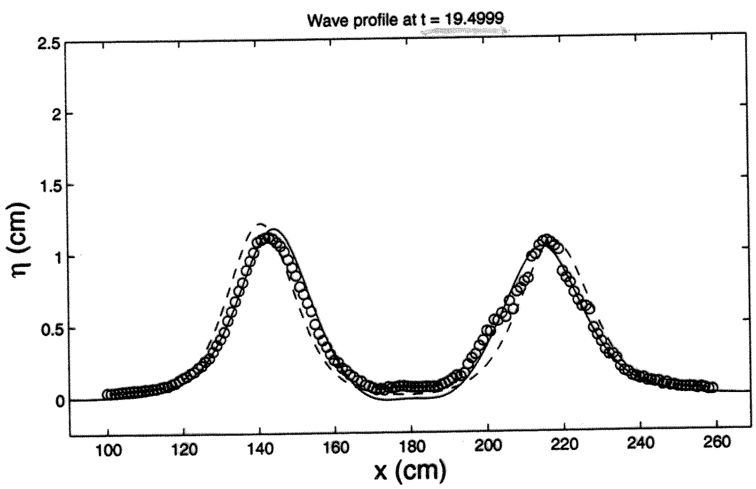


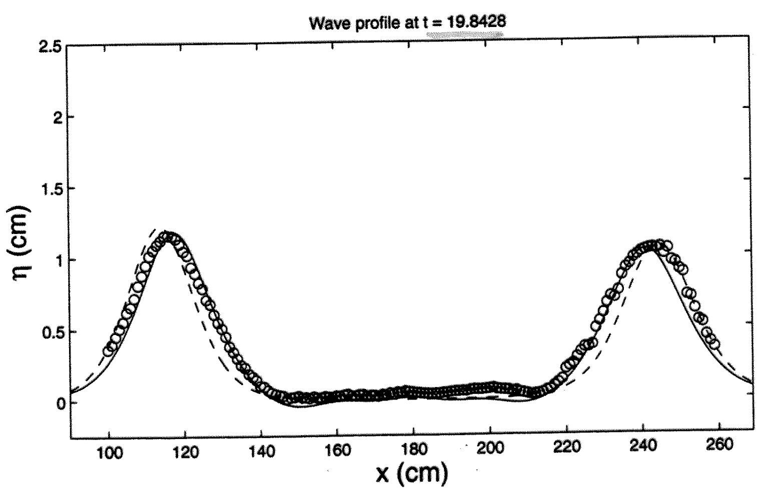


Comparison with experiment

- Asymmetric head-on collision







7)

Outline of our results

(1) numerical simulations - counter propagating case

(2) description of the numerical method

(3) comparison with experiments (Penn State wavetank)

(4) several rigorous results

(5) numerical simulations - co propagating case and comparison with experiments

(6) the criterion of P. Lax (1968)

Collaborative effort :

- J. Hammack, D. Henderson & team (PSU)
  Experiments in their linear wavetank
- P. Guyenne (McMaster Univ.)
  Numerical simulations
- P. Guyenne, C. Sulem (Univ. Toronto) & D. Wright (Minn.)
  Analysis

8)

Rigorous results :

Theorem 2. [W.C. 1985] Prepare the initial data for the water wave problem for the long wave regime:

$$\eta_0(\cdot) = \eta_0(\varepsilon x) \qquad \partial_t \eta_0(\cdot) = \nu_0(\varepsilon x)$$

$$\|(\eta_0, \nu_0)\|_{H^s} \le \varepsilon^2 C_0 \qquad \text{recall } X = \varepsilon x$$

Then (i) solutions of equation (1) exist over long time intervals

$$t \in (-T_\varepsilon, T_\varepsilon), \text{ where } T_\varepsilon = C_1/\varepsilon^3 .$$

and (ii) when data is prepared for a unidirectional case

(12) $$\| KdV_+(\eta_0, \partial_t \eta_0)\|_{H^{s_0+1}} < \varepsilon^2 C_2$$

then the solution $\eta(x,t)$ of (1) converges with $\varepsilon \to 0$ to a solution $Q(X,\tau)$ of the KdV equation (3_+).

[Schneider & Wayne, 2000]

(iii) Replace condition (12) by the condition of localization of the initial data

$$(1+|x|^2)^2 (\eta_0(x), \nu_0(x)) \in H_{s_0} .$$

Then there exist two solutions of KdV $(3\pm)$ such that

$$\sup_{[-T_\varepsilon, T_\varepsilon]} \| \eta(x,t) - \varepsilon^2 (Q_+(Y_+,\tau) + Q_-(Y_-,\tau))\|_{H_{s_0-1}} < \varepsilon^{5/2} C_3$$

recall $\tau = \varepsilon^3 t, \quad Y_\pm = \varepsilon(x \mp \sqrt{gh}\, t)$

9)

Theorem 3 [D. Wright 2004] Under the conditions of theorem 2 (iii) then there is an estimate of solutions of (1) as follows:

$$\| \eta(x,t) - \varepsilon^2 (Q_+ + Q_-) - \varepsilon^4 P(X,T) - \varepsilon^4 (F_+(Y_+,\tau) + F_-(Y_-,\tau)) \|_{H^s_{\sigma,\gamma}} \le \varepsilon^{11/2} C_4$$

where the various functions satisfy

$X = \varepsilon x \qquad T = \varepsilon t$ • long space & time region

$Y_\pm = \varepsilon(x \mp \sqrt{gh}\, t)$ • moving frames

$\tau = \varepsilon^3 t$ • slow time modulation

(13) $$(\partial_T^2 - \partial_X^2) P = 3 \partial_X^2 (Q_+(Y_+,\tau) Q_-(Y_-,\tau))$$

wave eq'n

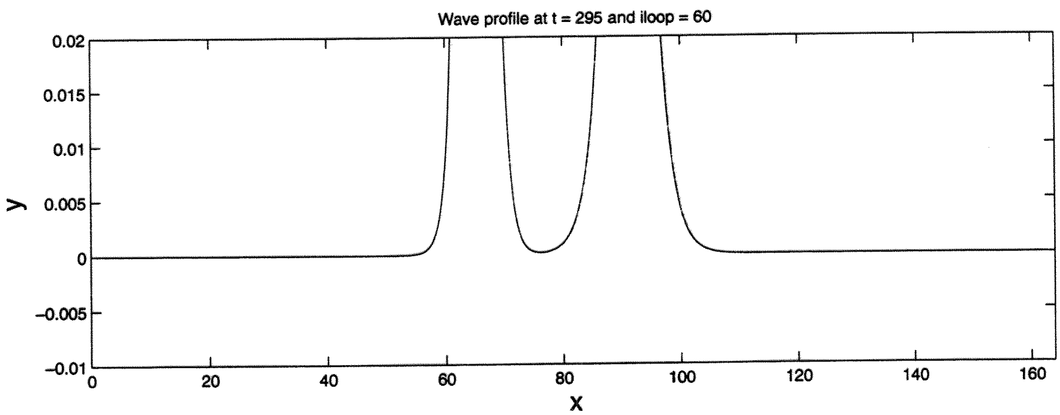
$$\partial_\tau F_\pm = \mp \left(\tfrac{1}{6} \partial_{Y_\pm}^3 F_\pm + \tfrac{3}{2} \partial_{Y_\pm} (Q_\pm F_\pm) \right) + J^\pm$$

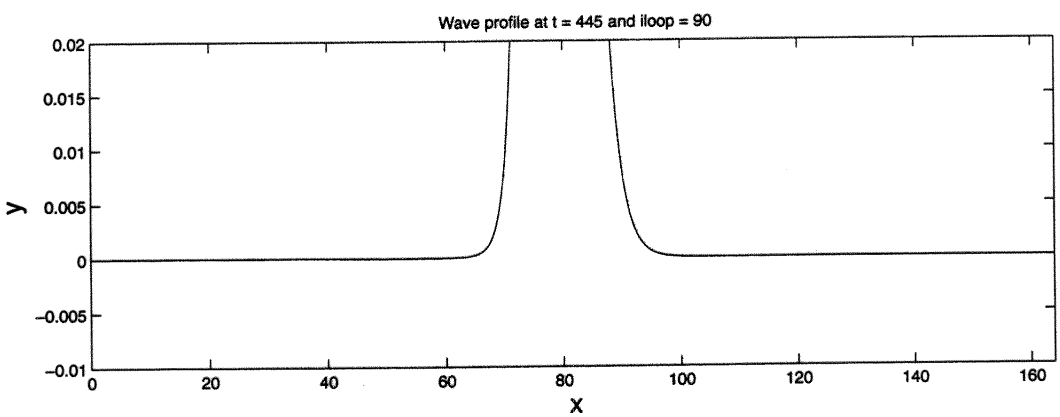
linearized KdV eq'n

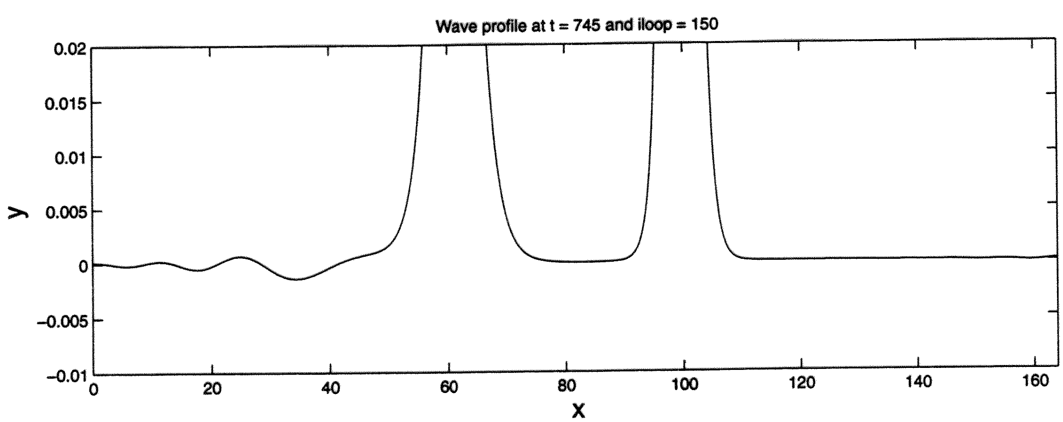
The inhomogeneous terms $J_\pm$ are explicit polynomials in $Q_\pm$, P.

Numerical results

- Overtaking collisions







$a_1 = 0.4$

$a_2 = 0.3$

Note the presence of a nonzero residual.

8)

* Basic issues arising in overtaking solitary wave interactions

(i) phase shift - Large and positive

(ii) amplitude of the interaction

$$\forall t \qquad \sup_{x\in\mathbb{R}} (\eta_{S_2}) < \sup_{x\in\mathbb{R}} (\eta(x,t)) < \sup_{x\in\mathbb{R}} (\eta_{S_1})$$

(where we chose $|\eta_{S_2}|_{L^\infty} < |\eta_{S_1}|_{L^\infty}$).

(iii) the residual is very small, but always nonzero

$$\eta(x,t) \simeq \eta_{S_1}(x - c_1 t) + \eta_{S_2}(x - c_2 t), \quad t \ll 0$$

$$\eta(x,t) = \eta_{S_1^+}(x - c_1^+ t - a_1) + \eta_{S_2^+}(x - c_2^+ t - a_2) + \eta_R(x,t), \quad t \gg 0$$

Typically $|\eta_R(x,t)|_{L^\infty} < 0.002\,|\eta_{S_1}|_{L^\infty}$.

(iv) the interaction type classification of Lax:

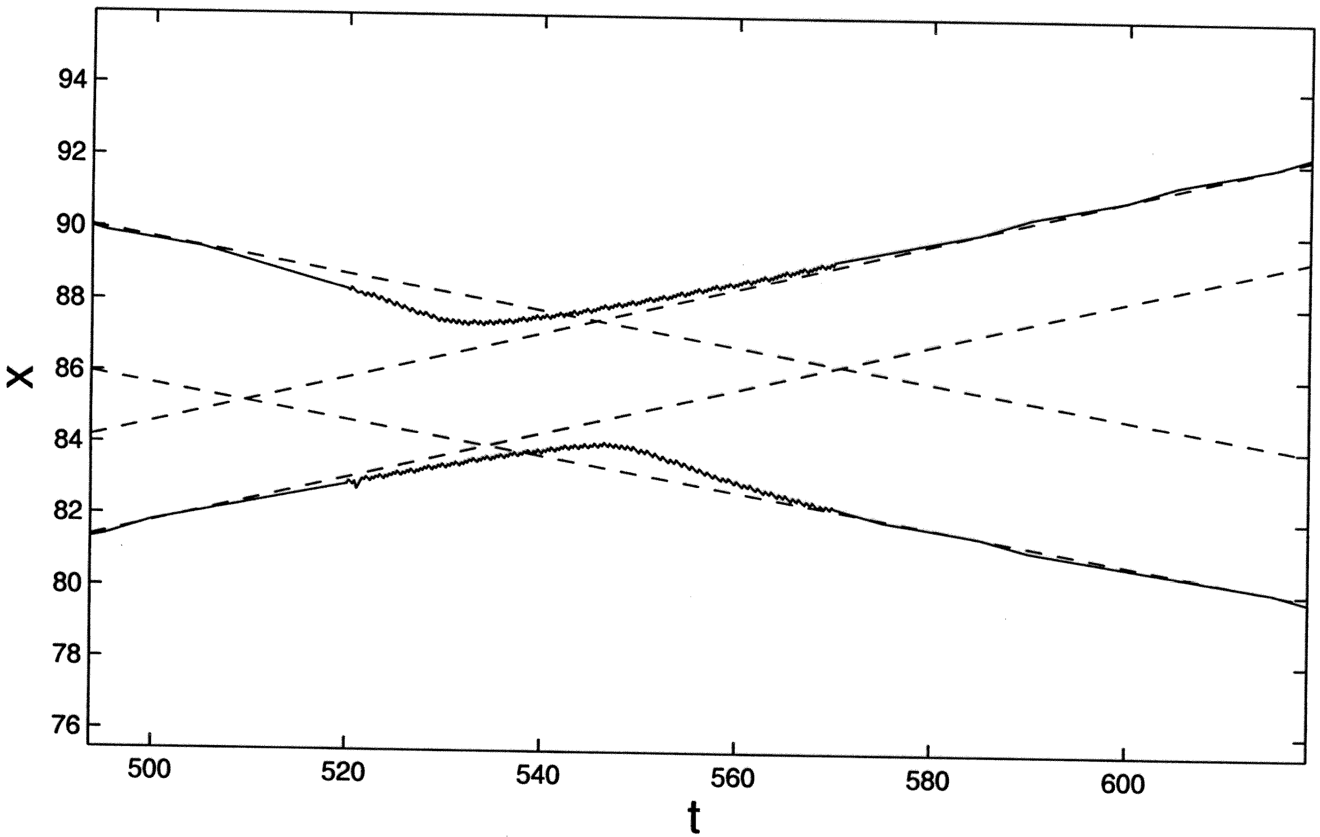
class (a) class (b) class (c)

$$\left(\frac{a_1}{a_2}\right) < 2.90 < \left(\frac{a_1}{a_2}\right) < 3.52 < \left(\frac{a_1}{a_2}\right)$$

nb: For the KdV, the classes are as follows:

class (a) class (b)

$$\left(\frac{a_1}{a_2}\right) < \frac{3+\sqrt{5}}{2} \quad \ldots \quad < \left(\frac{a_1}{a_2}\right) < 3 < \left(\frac{a_1}{a_2}\right)$$



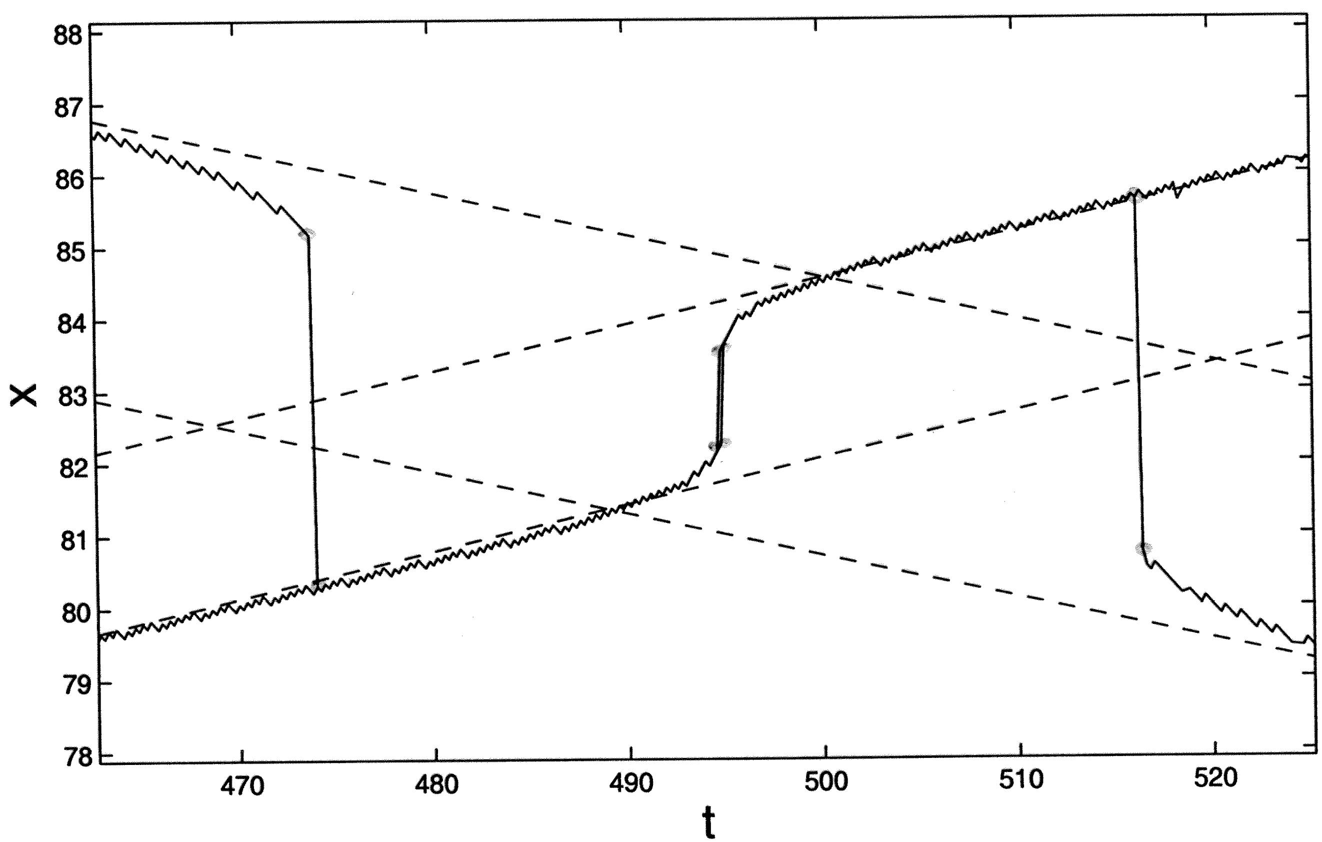
Euler equations

$a_1 = 0.4$

$a_2 = 0.135$

$\frac{a_1}{a_2} = 2.90..$

case (a), just prior to the transition to case (b).



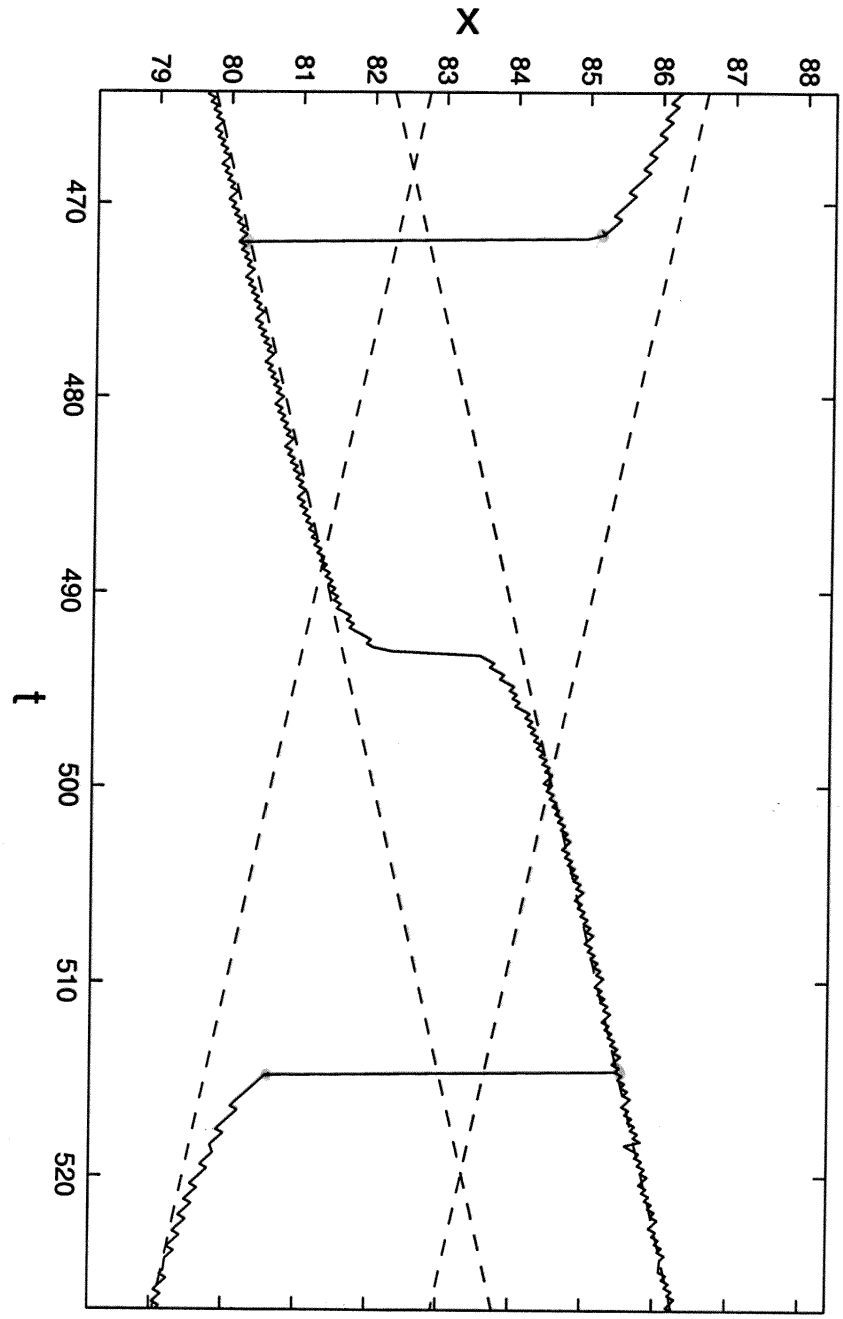
Euler equation

$a_1 = 0.4$

$a_2 = 0.114$

$\frac{a_1}{a_2} = 3.509$

case (ii), just prior to the transition to case (i).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