Chapter 1

Introduction

1.1. Overview of the subject

It is the irony of taking university courses that one doesn't understand the real reason for studying a subject until one knows it already and has been steeped in its culture. With this paradox in mind, I will attempt to give an introduction that will motivate the material we are going to address in this course, so we can at least start with a sense of its content.

The first questions are possibly 'Where do PDEs arise, and why are they useful?'. In fact, the language of the sciences is mathematics (the joke has it that the language of the sciences is English with an accent). Many if not most statements in the physical sciences are in the form of mathematical equations, and the vast majority of these are differential equations, quantifying the change of one quantity in terms of others. Indeed, equations physical, chemical and sometimes biological phenomena are for the majority PDEs, and the same statement holds for the engineering sciences.

Secondly, disciplines of mathematics such as geometry and dynamical systems also give rise to PDEs. Conditions such as (i) a surface is of minimal area, (ii) a submanifold is invariant under a flow, (iii) a mapping is conformal, or (iv) the curvature tensor satisfies a particular property, are very often PDEs. For example, the statement that "the tensor $(g_{\mu\nu})$ is an Einstein metric for a manifold M" is a system of PDEs.

The course material will discuss the most commonly occurring PDEs, and the implications that it has for a function u to be a solution. We are particularly interested in knowing whether solutions exist, whether thay are unique, and what their properties are of smoothness, positivity, etc.

Figure 1. A conformal mapping between $\mathbb{R}^n$ and its image as a submanifold of $\mathbb{R}^N$

1.2. Examples

Some common situations where mathematical properties of an object are described by PDEs.

Conformal mappings. A mapping between two Euclidian spaces $x \in \mathbb{R}^n \mapsto u(x) \in \mathbb{R}^N$ can be explicitely locally represented as

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mapsto u(x) = (u_1(x), \ldots, u_N(x)) \in \mathbb{R}^N$$
$$u : x \mapsto u(x) \ .$$

A conformal mapping is one that preserves angles. Set $n = N = 2$ for our example. In the x-variables, write $\partial_x := (\partial_{x_1}, \partial_{x_2})$. The vector fields $\partial_{x_1} = (1,0) \cdot \partial_x$ and $\partial_{x_2} = (0,1) \cdot \partial_x$ are orthogonal and have the same length in the domain $\mathbb{R}^2$, and a conformal mapping must preserve this property for the two tangent vectors $\partial_{x_1} u$ and $\partial_{x_2} u$ in the image;

$$\partial_{x_1} u \cdot \partial_{x_2} u = 0 \ , \quad |\partial_{x_1} u|^2 = |\partial_{x_2} u|^2 \ . \tag{1.1}$$

The equations (1.1) are equivalent to the Cauchy – Riemann equations for either $u = (u_1(x_1,x_2), u_2(x_1,x_2))$ or for $\overline{u} = (u_1(x_1,x_2), -u_2(x_1,x_2))$. That is, either u is holomorphic (analytic) in $x_1 + ix_2$

$$\partial_{x_1} u_1 = \partial_{x_2} u_2 \ , \quad \partial_{x_2} u_1 = -\partial_{x_1} u_2 \tag{1.2}$$

or else u is anti-holomorphic

$$\partial_{x_1} u_1 = -\partial_{x_2} u_2 \ , \quad \partial_{x_2} u_1 = \partial_{x_1} u_2 \ . \tag{1.3}$$

The study of such mapping is thus a central topic of complex analysis. Problem 1.2 of this chapter is to show that (1.1) implies either (1.2) or (1.3).

Laplace's equation. Starting with the Cauchy Riemann equations (1.2) and differentiating again, we have

$$\partial_{x_1}^2 u_1 = \partial_{x_1}(\partial_{x_2} u_2) = \partial_{x_2}(\partial_{x_1} u_2) = -\partial_{x_2}^2 u_1 \ ,$$

thus

$$(\partial_{x_1}^2 + \partial_{x_2}^2) u_1 = 0 \ , \tag{1.4}$$

which is that u_1 is *harmonic*. The same goes for u_2 of course. The operator $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ is called the *Laplacian*; its higher dimensional version is

$$\Delta u := (\partial_{x_1}^2 + \cdots + \partial_{x_n}^2)u(x) = (\sum_{j=1}^{n} \partial_{x_j}^2)u(x) \ . \tag{1.5}$$

Laplace's equations is

$$\Delta u = 0 \ , \tag{1.6}$$

and in general solutions of (1.6) are called *harmonic* as well. It is a principal example of an elliptic equation. The Laplacian operator appears very often in mathematics, partially because it is the only second order linear differential operator which is invariant under the symmetries of Euclidian space, that is translations and rotations;

$$x \mapsto y = x + c \ , \qquad\qquad x \mapsto y = Rx \ , \tag{1.7}$$

with the rotation R represented by an orthogonal matrix $R^T = R^{-1}$. This is one elementary manifestation of the principle of relativity, which is that equations which describe physical phenomena should be invariant under symmetries of the underlying space.

The heat equation. Given coordinates of time and space as $(t, x) = (t, x_1, \ldots x_n) \in \mathbb{R}^{1+n}$, the heat equation for a function $u(t, x)$ is

$$\partial_t u = \Delta u \ ; \tag{1.8}$$

it is the principal example of a parabolic equation, and it occurs very often in probability and in problems of gradient flow, among other places.

The wave equation. Again in coordinates $(t, x) \in \mathbb{R}^{1+n}$, the wave equation for a function $u(t, x)$ is

$$\partial_t^2 u = \Delta u \ . \tag{1.9}$$

It is a principal example of a hyperbolic equation, we will study a number of related equations in the next chapter, including the case of the wave equation in one space dimension.

Maxwell's equations. The propagation of electromagnetic radiation is governed by Maxwell's equations, the derivation of which is a towering achievement of 19^{th} century science. This is a system of equations coupling two vector quantities in $\mathbb{R}^3$, the electric and the magnetic vector fields;

$$\begin{aligned} E(t, x) &= (E_1(t, x), E_2(t, x), E_3(t, x)) \\ B(t, x) &= (B_1(t, x), B_2(t, x), B_3(t, x)) \ . \end{aligned} \tag{1.10}$$

These six functions satisfy the coupled system of equations

$$\begin{aligned} \partial_t E &= \nabla \times B - 4\pi j \ , \\ \partial_t B &= -\nabla \times E \ , \\ \nabla \cdot E &= 4\pi\rho \ , \\ \nabla \cdot B &= 0 \ . \end{aligned} \tag{1.11}$$

This is a system of 8 equations for the six components of (E, B). The vector calculus notation is that the divergence operator $\nabla \cdot E$ is

$$\nabla \cdot E := \partial_{x_1} E_1 + \partial_{x_2} E_2 + \partial_{x_3} E_3 = \sum_{j=1}^{3} \partial_{x_j} E_j$$

and the curl is given by

$$\nabla \times B := (\partial_{x_2} B_3 - \partial_{x_3} B_2, \partial_{x_3} B_1 - \partial_{x_1} B_3, \partial_{x_1} B_2 - \partial_{x_2} B_1) \ .$$

The function $\rho(t, x)$ of the third line of system (1.11) represents the electric charge density and vector function $j(t, x) = (j_1(t, x), j_2(t, x), j_3(t, x))$ of the first line is the electric current density. If there were magnetic monopoles, as there are electrons, protons and other electrically charged particles, then the other two equation components would have the analogous magnetic charge and current densities.

Proposition 1.1. In the case that $\rho = 0$ and $j = 0$ (the conditions for electromagnetic wave propagation in a vacuum) Maxwell's equations (1.11) are equivalent to wave equations for components of the electric and the magnetic fields.

Proof. In the case $\rho = 0$ and $j = 0$, one differentiates the equations (1.11) to find;

$$\begin{aligned} \partial_t^2 E &= \partial_t(\nabla \times B) = \nabla \times (\partial_t B) \\ &= -\nabla \times (\nabla \times E) = -(-\Delta E + \nabla(\nabla \cdot E)) \\ &= \Delta E \ . \end{aligned}$$

This is to say that each component of the electric field $E_j, j = 1, 2, 3$ satisfies individually the wave equation (1.9). The second line of this calculation uses a vector calculus identity and the fact that in a vacuum $\nabla \cdot E = 0$. There is a similar calculation for the components of the magnetic field. □

The standard three equations above, namely the wave equation, the heat equation and Laplace's equation are linear constant coefficient and of second order, however this short list of equations of interest is far from being exhaustive. Many other forms of equations, both linear and nonlinear, occur frequently.

Monge – Ampère equations. This equation arises when a mapping $f(x) : \mathbb{R}^n \mapsto \mathbb{R}^n$ is constructed as the gradient of a potential function $f(x) = \nabla u(x)$. If the mapping is required to be volume preserving, then $u(x)$ satisfies the Monge – Ampère equation

$$\det(\partial_{x_j}\partial_{x_\ell}u(x)) = 1 \ . \tag{1.12}$$

Similar equations play a role in the construction of Calabi – Yau metrics on manifolds. The matrix of second partial derivatives of a function $u(x)$ is known as the *Hessian* of u. We note in this context that the Laplacian is the trace of the Hessian

$$\Delta u = \mathrm{tr}\,(\partial_{x_j}\partial_{x_\ell}u) \ .$$

Schrödinger's equation. The complex valued function $\psi(t,x) = X(t,x) + iY(t,x)$ is called the *wave function* of quantum mechanics when it solves the equation

$$i\partial_t\psi = -\tfrac{1}{2}\Delta\psi + V(x)\psi \ . \tag{1.13}$$

The lower order coefficient $V(x)$ is called the *potential term*; it serves to represent the environment in which the quantum particle evolves. A commonly occurring nonlinear Schrödinger equations is

$$i\partial_t\Psi = -\tfrac{1}{2}\Delta\Psi + c|\Psi|^2\Psi \ , \tag{1.14}$$

where the local intensity $|\Psi|^2$ of the wave function creates its own potential. This nonlinearly self-interacting potential has the effect of focusing the wave function when $c = -1$ and defocusing it when $c = +1$.

Einstein's equations. In general relativity our space-time is a manifold M^4 with a metric tensor $g = (g_{\mu\nu}(x))$. The Riemann curvature tensor $(R_{ijk\ell})$ is a certain nonlinear function of the metric coefficients and their derivatives, $R_{ijk\ell}(g, \partial_{x_j}g, \partial_{x_j x_\ell}g)$, as is the Ricci curvature tensor $(R_{\mu\nu})$. Einstein's equations in a vacuum consist of the system of partial differential equations

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 0 \ ,$$

where $R = \mathrm{tr}\,R_{\mu\nu} = \sum_{\mu\nu} R_{\mu\nu}g^{\mu\nu}$ is the scalar curvature tensor. The Einstein summation convention allows us to omit the summation symbol over repeated indices in the latter expression, so it can be represented as $R = R_{\mu\nu}g^{\mu\nu}$. When energy and/or matter are present, Einstein's equations become

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} \ , \tag{1.15}$$

where T is the energy - momentum tensor.

Figure 2. A minimal surface that is a graph in $\mathbb{R}^3$ whose boundary lies on a specified curve.

Minimal surfaces. Given a mapping $u : D \to \mathbb{R}^N$ defined on a domain D, the surface area of the image is

$$A(u) := \int_D \sqrt{\det(\partial_{x_j} u \cdot \partial_{x_\ell} u)}\, dx \ .$$

In the case that the mapping u is given as a graph of a function $f(x)$ over $D \subseteq \mathbb{R}^n$ (where $N = n+1$), namely that $u(x) = (x_1, \dots, x_n, f(x))$, then

$$A(f) = \int_D \sqrt{1 + (\partial_x f)^2}\, dx \ .$$

A *minimal surface* is the result of a mapping for which the area functional $A(u)$ is minimal (or at least a critical point), generally with given additional constraints such as boundary conditions. That is to say, the variations of A about u all vanish, namely

$$0 = \delta A(u) \cdot v := \frac{d}{d\tau}|_{\tau=0} A(u + \tau v)$$

for all variations $v(x)$ such that $u + \tau v$ continues to satisfy the constraints. For the case where $n = 2$, and for a graph, if f is such a critical point then it must satisfy

$$(1 + (\partial_{x_2} f)^2)\partial_{x_1}^2 f + (1 + (\partial_{x_1} f)^2)\partial_{x_2}^2 f - 2\partial_{x_1} f \partial_{x_2} f \partial_{x_1}\partial_{x_2} f = 0 \ . \tag{1.16}$$

The Navier – Stokes equations. In Eulerian coordinates on describes a fluid in motion in a region $D \subseteq \mathbb{R}^n$ by its velocity vector field $u(t,x) = (u_j(t,x))_{j=1}^n$ at each point of D and at eact time t. An incompressible but viscous fluid will satisfy the system of equations

$$\begin{aligned} &\partial_t u + (u \cdot \nabla) u + \nabla p = \nu \Delta u \ , \\ &\nabla \cdot u = 0 \ . \end{aligned} \tag{1.17}$$

The *pressure* $p(t,x)$ can be thought of as the extra degrees of freedom that allow the flow determined by (1.17) to satisfy the constraint of incompressibility $\nabla \cdot u = 0$. The vorticity of the fluid is given by $\omega(t,x) := \nabla \times u$. It is an open question whether every solution of (1.17), even given smooth initial data, will remain smooth at all future times.

Exercises: Chapter 1

Exercise 1.2. Prove that statement (1.1) characterizing conformal mappings implies that either (1.2) or (1.3) holds.

Exercise 1.3 (*)**.** Determine the class of conformal mappings in the case that $n = N \geq 3$. The conditions (1.1) are much more rigid than in two dimensions.

Exercise 1.4. Prove that the Laplacian (1.6) is invariant under Euclidian symmetries (1.7).

Exercise 1.5. In dimension $n = 2$ some elementary harmonic functions are polynomials, they include:

$$u(x) = x_j \ , \quad j = 1, 2 \ ,$$
$$u(x) = x_1^2 - x_2^2 \ , \quad u(x) = x_1 x_2 \ .$$

How many independent harmonic polynomials are there of general degree ℓ?

When the dimension $n \geq 3$ how many harmonic polynomials of degree ℓ are there?

Exercise 1.6. Prove the vector calculus identity

$$\nabla \times (\nabla \times V) = -\Delta V + \nabla(\nabla \cdot V) \ ,$$

for a vector fields $V(x)$, where in our notation the gradient of a function F is

$$\nabla F = (\partial_{x_1} F, \ldots, \partial_{x_n} F) \ .$$