Math 4FT /Math 6 FT
Problem Set #5

Problem 1. (Focusing singularity of solutions of the wave equation in $\mathbb{R}^3$):
(i) Suppose that the initial data for the wave equation in three dimensions has spherically symmetric data;

$$f(x) = f(r) \ , \quad g(x) = g(r) \ , \qquad r^2 = x_1^2 + x_2^2 + x_3^2 \ .$$

Show that the general solution can be expressed as

$$u(t, r) = \frac{1}{r}\big(F(r + t) + G(r - t)\big) \ ,$$

that is, it consists of an incoming wave and an outgoing wave.
(ii) With the special initial data $u(0, r) = 0$, $\partial_t u(r) = g(r)$ with $g(r)$ an even function of r, then

$$u(t, r) = \frac{1}{2r} \int_{r-t}^{r+t} \rho g(\rho) \, d\rho \ .$$

(iii) Set the initial data to be

$$g(r) = 1 \ , \quad 0 \le r < 1 \ , \quad g(r) = 0 \ , 1 \le r \ ,$$

show that $u(t, r)$ is continuous for $|t| < 1$ but at time $t = 1$ it exhibits a jump discontinuity. This is due to the focusing of the singularity in $\partial_t u(0, r)$ given at $t = 0$.

Problem 2. This problem addresses the decay rate of solutions of the wave equation in $\mathbb{R}^3$. Suppose that the initial data $(f(x), g(x)) \in C^1(\mathbb{R}^3) \times C(\mathbb{R}^3)$ and that it is supported in the bounded set $B_1(0)$. By inspecting the Kirchhoff formula for the solution $u(t, x)$, show that

$$|u(t, x)| \le \frac{C}{|t|}$$

for some constant C, which can be quantified using $\|f\|_{C^1(\mathbb{R}^3)}$ and $\|g\|_{C(\mathbb{R}^3)}$.

Problem 3. (Global existence with small initial data for certain nonlinear wave equations): This question is to show that certain nonlinear wave equations possess smooth solutions for all $t \in \mathbb{R}$. This contrasts with other cases where solutions form singularities in finite time. Consider the equation

$$\begin{aligned} &\partial_t^2 v - \Delta v + (\partial_t v)^2 - |\nabla v|^2 = 0 \\ &v(0, x) = f(x) \quad \partial_t v(0, x) = g(x) \qquad (f(x), g(x)) \in C^1(\mathbb{R}^3) \times C(\mathbb{R}^3) \ , \end{aligned} \tag{1}$$

with f, g supported in a compact set.

(i) Setting $u = e^v - 1$, show that $u(t,x)$ satisfies the wave equation

$$\begin{aligned}
&\partial_t^2 u - \Delta u = 0 \\
&u(0,x) = e^{f(x)} - 1 := F(x) \quad \partial_t u(0,x) = g(x)e^{f(x)} := G(x) \\
&\quad (F(x), G(x)) \in C^1(\mathbb{R}^3) \times C(\mathbb{R}^3) \ .
\end{aligned}$$

Explain why (F,G) has compact suport.
(ii) Show that for sufficiently small $\|(F, \partial_x F, G)\|_C$, the solution $u(t,x)$ is bounded by

$$|u(t,x)| < 1 \ ,$$

using the result of Problem 2.
In this case the transformation $v \mapsto u$ is invertible for all $(t,x) \in \mathbb{R}_t^1 \times \mathbb{R}_x^3$, giving rise to a global solution $v(t,x)$ of the equation (1).

Problem 4. (method of decent for the wave equation for $x \in \mathbb{R}^2$)
(i) Show that if $x \in \mathbb{R}^3$ but the Cauchy data for the wave equation only depends upon (x_1, x_2), namely

$$f = f(x_1, x_2) \ , \qquad g = g(x_1, x_2) \ , \tag{2}$$

then the solution of the wave equation in $\mathbb{R}_t^1 \times \mathbb{R}_x^3$ is also independent of x_3;

$$u = u(t, x_1, x_2) \ ,$$

and therefore $u(t, x_1, x_2)$ satisfies the wave equation in two space dimensions;

$$\partial_t^2 u - (\partial_{x_1}^2 + \partial_{x_2}^2) u = 0 \ .$$

(ii) Use the Kirchhoff formula to express the solution to the wave equation in $\mathbb{R}^3$ for data satisfying (2).
(iii) In the expression in (ii) reparametrize the spherical integrals by their projection onto the (x_1, x_2)-plane; *e.g.*

$$\int_{\mathbb{S}^2 : |x-y|=t} f(y)\, dS_y = \iint_{|(x_1-y_1, x_2-y_2)|<t} f(y_1, y_2) \sqrt{t^2 - ((x_1-y_1)^2 + (x_2-y_2)^2)}\, dy_1 dy_2 \ ,$$

which gives a general formula in $\mathbb{R}^2$ for the solution of the wave equation

$$\Box u = 0 \ .$$

(iv) Describe the nature of this solution in the case that the support of f and g as functions on $\mathbb{R}^2$ is compact, say supported in the ball $B_R(0) = \{|(x_1, x_2)| < R\}$. In particular comment on the Huygen's principle. Does the solution satisfy the sharp Huygens principle, and why? Describe what an observer sees as time progresses when they are situated farther than R from the origin.