Problem 1. Equations with variable dispersion
Consider the linear evolution equation
\[ \partial_t u + x \partial_x^3 u = 0 , \]
\[ u(0, x) = h(x) \in L^2(\mathbb{R}^1) , \]
which is related to the KdV equation.
(a) Give an explicit expression for the solution, using the Fourier transform and the method of characteristics.
(b) For initial data \( h(x) \) such that \( \text{supp}(\hat{h}(\xi)) \subseteq B_R(0) \), what is the (future) lifespan \([0, T^*)\) of the solution? What happens to the solution as \( t \to T^* \)?
(c) Is the solution unique?
(d) For \( t < 0 \) what is the lifespan of the solution?

Problem 2. Convolutions and the central limit theorem
Consider a function \( h(x) \in L^1(\mathbb{R}^1) \) which satisfies
\[ \int h(x) \, dx = 1 , \quad \int xh(x) \, dx = 0 , \quad \int x^2 h(x) \, dx = \sigma^2 < +\infty . \]
(a) Show that \( \hat{h}(0) = \frac{1}{\sqrt{2\pi}} \). Furthermore show that the multiple convolutions \( h^{(n)}(x) := h * h \ldots (n \times) \ldots h(x) \) also satisfy
\[ \int h^{(n)} \, dx = 1 , \]
and therefore \( \hat{h}^{(n)}(0) = \frac{1}{\sqrt{2\pi}} \).
(b) Explain why \( \hat{h} \) is twice continuously differentiable at \( \xi = 0 \). It follows that in a neighborhood of \( \xi = 0 \) we have
\[ \hat{h}(\xi) = \frac{1}{\sqrt{2\pi}} \left( 1 + i\xi \hat{h}'(0) - \frac{\xi^2}{2} \hat{h}''(0) + O(\xi^3) \right) . \]
Furthermore \( \hat{h}'(0) = 0 \).

(c) Rescaling by \( \sqrt{n} \) and taking the Fourier transform, show that

\[
\mathcal{F}(h^{(n)})(\frac{\xi}{\sqrt{n}}) = (\hat{h}(\frac{\xi}{\sqrt{n}}))^n = \frac{1}{\sqrt{2\pi}} \left( 1 - \frac{\xi^2}{2n} \hat{h}''(0) \right)^n + o(\frac{1}{n}).
\]

In the limit \( n \to +\infty \) this quantity converges to

\[
\frac{1}{\sqrt{2\pi}} e^{-(\xi^2/2\sigma^2)}.
\]

This shows that repeated convolution, in the (appropriately rescaled) limit converges to the Gaussian

\[
\lim_{n \to +\infty} h^{(n)}(\sqrt{n}) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)},
\]

which is one way to state the central limit theorem.

**Problem 3. Laplace’s equation with a reentrant corner.** Consider domains \( \Omega \) in \( \mathbb{R}^2 \) consisting of a disk \( B_1(0) \) intersect the conic neighborhood \( \{ (r,\theta) : 0 < \theta < \pi/\alpha \} \), where \( 1/2 < \alpha < 1 \) is a constant. When \( 1/2 < \alpha < 1 \) this is a domain with a corner removed.

(a) Show that \( u(x,y) = \text{im}(z^\alpha) \) is a harmonic function on \( \Omega \) (where we are using complex notation). Show that \( u \) satisfies Dirichlet boundary conditions on the two boundary components \( \{ \theta = 0, 0 < r < 1 \} \) and \( \{ \theta = \pi/\alpha, 0 < r < 1 \} \), and that \( u \) is bounded on the third boundary component consisting of the arc \( \{ r = 1, 0 < \theta < \pi/\alpha \} \).

(b) Show that the gradient \( \nabla u(x,y) \) is not in \( L^p(\Omega) \) for some range of \( 2 < p \leq +\infty \).

**Problem 4. The Cauchy problem for the wave equation, and Duhamel’s principle**

Consider the wave equation on \( \mathbb{R}^1_t \times \mathbb{R}^n_x \),

\[
\Box u = \partial^2_t u - \Delta u = 0,
\]

with the special initial data

\[
u(0,x) = 0, \quad \partial_t u(0,x) = g(x).
\]

Define the solution operator for this problem to be

\[
W(g)(t,x) = u(t,x),
\]

so that

\[
W(g)(0,x) = 0, \quad \partial_t W(g)(0,x) = g(x).
\]
(a) Show that the solution with general initial data \( u(0, x) = f(x) \) and \( \partial_t u(0, x) = g(x) \) can be expressed in terms of the superposition

\[
u(t, x) = \partial_t W(f)(t, x) + W(g)(t, x) .
\]

(b) In the case \( n = 1 \) give an expression for the operator \( W(g) \) in terms of the d’Alembert formula.

In the case \( n = 3 \) give an expression for the operator \( W(g) \) in terms of the Kirchhoff formula and spherical means.

(c) Consider the inhomogeneous problem for the wave equation

\[
\square u = h(t, x) ,
\]

where without loss of generality we may set \( f = g = 0 \). Show that the solution can be expressed in terms of \( W(g)(t, x) \) as follows:

\[
u(t, x) = \int_0^t W(h(s, \cdot))(t - s, x) \, ds .
\]

(1)

This is the content of the Duhamel principle in the case of the wave equation. Give an explicit expression for (1) in the case that \( n = 1 \).

**Problem 5. Gaussian wave packets**

(a) Express the solutions of the free Schrödinger equation

\[
\frac{1}{i} \partial_t \psi = -\frac{1}{2} \partial_x^2 \psi , \quad x \in \mathbb{R}^1 ,
\]

with the initial data

\[
\psi_0(x) = e^{-Ax^2/2} e^{ikx} .
\]

(b) Calculate the first several moments of the solution

\[
m_0(\psi(t, \cdot)) , \quad m_1(\psi(t, \cdot)) , \quad \hat{m}_1(\psi(t, \cdot)) , \quad m_2(\psi(t, \cdot))
\]

Describe the trajectory of the solution.
**Problem 6. Soliton solutions of the KdV**

The KdV equation is

\[ \partial_t q = -\frac{1}{6} \partial_x^3 q + 2q \partial_x q . \]

Soliton solutions (in the case of single solitons) are traveling waves, taking the form \( q(t, x) = q(x - ct) \) for some velocity \( c \).

(a) Show that such solutions satisfy

\[ \frac{1}{6} \frac{d^2}{dx^2} q - q^2 = cq + \text{Const.} \]

(b) The *energy* of KdV solutions is defined as

\[ E(q) = \int_{-\infty}^{+\infty} \frac{1}{12} (\partial_x q)^2 + \frac{1}{3} q^3 \, dx \]

and the *momentum* is

\[ I(q) = \int_{-\infty}^{+\infty} \frac{1}{2} q^2 \, dx . \]

Show that single soliton solutions of the KdV are critical points of the energy \( E(q) \) for fixed momentum \( I(q) \). What is the associated Lagrange multiplier?

(c) Solve the equations to find the one parameter family of single soliton solutions.