Math 4FT: Problem Set 2.

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Due Monday March 11, 2013.

Problem 1. Derive expressions in the sine (and respectively, the cosine) series for the Schrödinger kernel with Dirichlet (respectively Neumann) boundary conditions on the interval $[0, 2\pi]$.

Problem 2. Consider a sequence of functions $\{f_j(x)\}_{j=1}^{\infty}$ for $x \in [0, 2\pi]$. There are many different ways in which the sequence could converge to a limit $f(x)$.

(a.) Pointwise, when $\lim_{j\to\infty} f_j(x) = f(x)$ for each $x \in [0, 2\pi]$.

(b.) Uniform, when for all $\varepsilon > 0$ there is a cutoff $J = J(\varepsilon)$ such that for all $j \geq J$ and all $x \in [0, 2\pi]$ then $|f_j(x) = f(x)| < \varepsilon$.

(c.) L^2 convergence, when $\lim_{j\to\infty} \|f_j(x) - f(x)\|_{L^2} = 0$.

(d.) L^1 convergence, when $\lim_{j\to\infty} \|f_j(x) - f(x)\|_{L^1} = 0$.

Show the following relationships between these different modes of convergence.

(i) If $\{f_j(x)\}_{j=1}^{\infty}$ converges to $f(x)$ in L^2 then $\{f_j(x)\}_{j=1}^{\infty}$ also converges to $f(x)$ in L^1.

(ii) If $\{f_j(x)\}_{j=1}^{\infty}$ converges uniformly to $f(x)$, then $\{f_j(x)\}_{j=1}^{\infty}$ converges to $f(x)$ in L^2 (and hence also in L^1 by Question (i)).

(iii) If $\{f_j(x)\}_{j=1}^{\infty}$ converges to $f(x)$ in L^1 and $\{f_j(x)\}_{j=1}^{\infty}$ converges pointwise to $g(x)$ (or converges pointwise *a.e.* to $g(x)$), then $f(x) = g(x)$ *a.e.*

(iv) Show by counterexample that it is possible for $\{f_j(x)\}_{j=1}^{\infty}$ to converge to $f(x)$ pointwise, but not uniformly.

(v) Show by counterexample that it is possible for $\{f_j(x)\}_{j=1}^{\infty}$ to converge to $f(x)$ in L^1 but not in L^2.

Problem 3. The functions $f_n(x) = x^n$, $n \geq 0$ span the Hilbert space $L^2[-1, +1]$. Using the Gram – Schmidt orthonormalization procedure, compute the first five orthogonal polynomials, those spanning the subspace generated by $\{x^n : 0 \leq n \leq 4\}$. These are the first five Legendre polynomials.

Problem 4. Show that any (complex) linear functional $\ell : L^2 \to \mathbb{C}$ can be written as an inner product. That is,

$$\ell(f) = (f, g)$$

for some $g = g_\ell$. This is the content of the Riesz representation theorem.