Math 4FT / 6FT
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Mathematics 4FT / 6FT Topics in differential equations.

'Applications of the Fourier transform in mathematical physics'

- Walter Craig Instructor
  - Office HH 418, x23422, craig@math
  - Office hours Thurs 11:00–12:30 or by appointment — typo on web.
  - Course meetings M 14:30–16:20 HH217 and F 14:30–15:20 HH217
    change? Try: Th 14:30, Th 13:30, Th 11:30, Tu times, F. AM
- Syllabus – hand out
  – describe on blackboard
  1) Introduction
  2) Mathematical physics
  3) Geometry of Hilbert space
  4) Applications of Fourier series to geometry and number theory
  5) Convergence properties of Fourier series
  6) Fourier integrals
  7) Advanced mathematical physics ← typo on webpage
  8) Applications of Fourier integrals to geometry and number theory

- Course structure: Introduction, then 7 micro-modules.
- Expectations: 4FT / 6FT two populations in this course
  (1) interest in the material, (2) attendance in class
- Course requirements.
  - Problem sets (planning for 4 of them) grade = 2/3
  - Final take-home exam grade = 1/3.

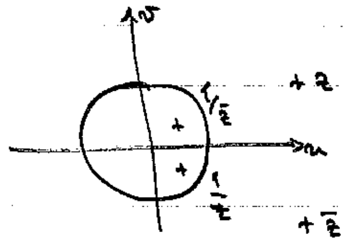
- Texts:
  - Dym & McKean: Fourier series and Integrals
  - Stein & Shakarchi: Fourier Analysis – I
  - Rauch: PDEs
- Questions / Information sheet.

1) Introduction

We shall use the standard notation for real and complex variables

- $x \in \mathbb{R}$ a real number $-\infty < x < +\infty$
- $z = u + iv \in \mathbb{C}$ a complex number $(u, v) \in \mathbb{R}^2 \simeq \mathbb{C}$.

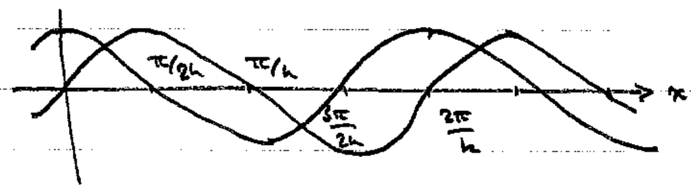
  - Complex conjugation $\bar{z} = u - iv$
  - Inversion $\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{u - iv}{u^2 + v^2}$



Trigonometric functions

$u(x) = \cos(kx)$ $\quad x \in \mathbb{R}$

$v(x) = \sin(kx)$



Complex notation for trigonometric functions (Euler's formula)

$$f(x) = \cos(kx) + i \sin(kx)$$
$$= e^{ikx}.$$

NB: we are used to seeing Euler's formula with the polar coordinate angle θ in place of x. Here, and for Fourier series, it will be played by space variables x, or time variables $t \in \mathbb{R}$.

Example 1: Harmonic motion

$$\dot{u} = -k v$$
$$\dot{v} = +k u$$

(motion w.a linear restoring force)

Initial data $\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in \mathbb{R}^2$ a point of phase space.

Introduce complex notation $z = u + iv$

$$\dot{z} = \dot{u} + i\dot{v}$$
$$= -kv + iku$$
$$= ik(u+iv)$$
$$\dot{z} = ikz.$$

The solution of the harmonic oscillator is

$$z(t) = e^{ikt} z(0) \qquad z(0) = z_0 = u_0 + iv_0$$

Back to real notation

$$u(t) = \cos(kt) u_0 - \sin(kt) v_0$$
$$v(t) = \sin(kt) u_0 + \cos(kt) v_0$$

or

$$\begin{pmatrix} u \\ v \end{pmatrix}(t) = \begin{pmatrix} \cos(kt) & -\sin(kt) \\ \sin(kt) & \cos(kt) \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

Moral: we are led naturally to study superpositions of trigonometric functions.

Problem set: problem 1 Describe the motion of a harmonic oscillator with two degrees of freedom

$$\dot{z}_1 = i z_1$$
$$\dot{z}_2 = i \omega z_2$$

(i) when $\omega \in \mathbb{Q}$ (ii) when $\omega \notin \mathbb{Q}$.

Fourier's problem:

Physical laws governing changes of quantities over time or space (indeed space-time) are very often stated in terms of differential equations. Fourier's problem concerns heat flow.

$u(x,t)$ heat (measured as temperature) at (x,t).

(1) $\partial_t u = \varkappa \partial_x^2 u$ Fourier's law of heat conduction

Suppose the space domain is $0 \le x \le \pi$, and suppose boundary conditions are given as $u(0,t) = 0 = u(\pi,t)$.

Particular solutions (by separation of variables)

$u_1(x,t) = e^{-\varkappa t}\sin(x)$ $\qquad u_1(x,0) = \sin(x)$

$u_2(x,t) = e^{-4\varkappa t}\sin(2x)$ $\qquad u_2(x,0) = \sin(2x)$

...

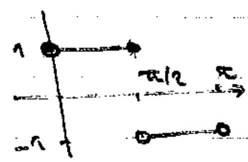
It is a linear equation, so that the linear superposition of two solutions is again a solution

$$u(x,t) = \sum_{j=1} a_j \sin(jx)\, e^{-j^2 t}.$$

Fourier's question: can any solution of (1) with Dirichlet b.c. be represented as a sum (possibly infinite) of such basic solutions.

Rephrased - can any function $u_0(x)$, considered as initial data for the heat equation (1), be decomposed into a sum of trigonometric functions?

For instance, can $u_0(x) = \begin{cases} +1 & 0 < x \le \pi/2 \\ -1 & \frac{\pi}{2} < x < \pi \end{cases}$ be so represented.



The answer, which was surprising to many at the time of Fourier, is yes. The subtle questions over the manner in which the infinite sum of trigonometric functions converges to $u_0(x)$;

$$u_0(x) = \sum_{j=1}^{\infty} a_j \sin(jx) ,$$

is indeed YES. This forms the basis of the topic of Fourier analysis.

Outline of the seven micro-modules of this course

1) mathematical physics

~~(i) classical mechanics: harmonic oscillators, periodic and quasiperiodic motion~~

Fourier synthesis of the solution operator for many basic equations of mathematical physics.

(i) classical harmonic oscillator

(ii) the heat equation and Fourier's law of heat flux

(iii) wave equation

(iv) Schrödinger's equation

We will prove Dirichlet's approximation theorem, for $f(x) \in C^2(\mathbb{T}^1)$

2) The geometry of Hilbert space

Euclidean space $\mathbb{R}^d$ can be represented in coordinates as d-tuples of real numbers

$$x = (x_1, x_2 \cdots x_d) = \sum_{j=1}^{d} x_j \vec{e}_j \qquad \vec{e}_j \text{ basis vectors}$$

where $x_j = (x, e_j)$.

Pythagorean theorem: $|x|^2 = x_1^2 + x_2^2 + \cdots + x_d^2$

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Hilbert space is the (possibly) infinite dimensional) analog, and is a natural setting for Fourier analysis in many ways.

Dictionary: · basis vector $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ of H

· a point in Hilbert space $f(x)$

· coordinates of $f(x)$ in the Fourier basis

$$f_n = (f, e_n) = \int_0^{2\pi} f(x) \frac{e^{-inx}}{\sqrt{2\pi}} dx$$

Pythagorean theorem (the Plancherel identity)

$$\int_0^{2\pi} |f(x)|^2 dx = \|f\|^2 = \sum_{n=-\infty}^{+\infty} |f_n|^2 .$$

If one uses the Lebesgue integral, the limits of the series $\sum_{n=-\infty}^{+\infty} f_n \frac{e^{inx}}{\sqrt{2\pi}}$ converge to a point of H, whenever $\sum_{n=-\infty}^{+\infty} |f_n|^2 < +\infty$.

(3) Applications of Fourier series to geometry and number theory

(i) geometry: what shape minimizes perimeter for fixed area.

(ii) ergodic theory: Consider the sequence

$\alpha, 2\alpha, 3\alpha, \ldots, n\alpha$ as real numbers $\mathrm{mod}(2\pi) = \mathbb{T}^1$

If $\frac{\alpha}{2\pi} \in \mathbb{Q}$ a finite set

If $\frac{\alpha}{2\pi} \in \mathbb{R} \setminus \mathbb{Q}$ the set is dense and equidistributed.

(4) Convergence properties of Fourier series

· The geometry of $H = L^2(\mathbb{T})$ implies L^2-convergence of Fourier series. Question: when is it better, when worse?

· For $f \in C(\mathbb{T})$, it is not necessarily true that the Fourier series for $f(x)$, $f^{(N)}(x) = \sum_{n=-N}^{+N} f_n(x) \frac{e^{inx}}{\sqrt{2\pi}}$ converges (pointwise) to $f(x)$.

Sometimes, resummation methods do better, such as Cesaro sums.

. In case $f(x)$ has a jump discontinuity (say at $x=\pi/2$) then Gibb's phenomenon occurs, and the series $f_N^{(N)}(x)$ cannot converge uniformly.

(5) Fourier integrals: $x \in \mathbb{R}$ $f(x)$

In analogy, define $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$, $k \in \mathbb{R}$.

When $\int_{-\infty}^{\infty} |f(x)|^2\, dx < +\infty$ $\quad (f \in L^2(\mathbb{R}_x))$

then $\hat{f}(k)$ is well defined, and is $\hat{f} \in L^2(\mathbb{R}_k)$.

The analog of the Pythagorean theorem holds

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}(k)|^2\, dk = \|\hat{f}\|^2_{L^2}.$$

In fact, the Fourier transform can be seen as an unitary transform of L^2.

(6) Advanced mathematical physics

use the Fourier transform to describe

- Heisenberg uncertainty principle
- heat kernel with potential
- central limit theorem
- Dyson expansion and the Feynman-Kac formula

(7) Application of Fourier integrals to geometry and number theory.

Define $\#(n)$:= number of primes $2 \le p \le n$. From numerical data, Gauss conjectured that

$$\lim_{n \to \infty} \frac{\#(n)}{n/\log(n)} = 1 .$$

This was proved by Hadamard and de la Vallée Poussin at the end of the 19^{th} c, and by 'elementary means' by Erdös & Selberg (1949), via use of Fourier analysis.