Math4FT
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Heat flow on an interval

$$\partial_t u = \tfrac{1}{2}\partial_x^2 u \qquad t>0 \qquad 0\le x\le\pi$$
$$u(0,t) = 0 = u(\pi,t) \qquad \text{Dirichlet boundary conditions}$$
$$u(x,0) = f(x)$$

Compatibility conditions are that $f(0) = 0 = f(\pi)$.

Expression for the heat kernel $h_D(x,y,t)$ given by the sine series

$$u(x,t) = \sum_{k=1}^{\infty} e^{-k^2 t/2} a_k(t) \frac{\sin(kx)}{\sqrt{\pi/2}}$$

where $a_k = \frac{1}{\sqrt{\pi/2}}\int_0^\pi f(y)\sin(ky)\,dy$

$$= \sum_{k=1}^{\infty} e^{-k^2t/2}\frac{1}{(\pi/2)}\int_0^\pi \sin(kx)\sin(ky)f(y)\,dy$$

$$= \int_0^\pi f(y)\underbrace{\left(\frac{2}{\pi}\sum_{k=1}^{\infty} e^{-\frac{k^2t}{2}}\sin(kx)\sin(ky)\right)}_{h_D(x,y,t)}dy$$

N.B. Not a convolution product, because the Dirichlet boundary conditions are not invariant under translation.

A second expression for the heat kernel can be found from the periodic case, recognizing that we may take $f(x+2\pi)=f(x)$ periodic, but also $f(-x) = -f(x) = -f(-x)$ ...

$$u(x,t) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{-(x-y)^2/2t} f(y)\,dy$$

$$= \frac{1}{\sqrt{2\pi t}}\sum_{n=-\infty}^{+\infty}\int_0^{2\pi} e^{-(x-y-2\pi n)^2/2t} f(y)\,dy$$

$$= \frac{1}{\sqrt{2\pi t}}\sum_{n=-\infty}^{+\infty}\int_0^{\pi} e^{-(x-y-2\pi n)^2/2t} f(y)\,dy + \int_0^\pi e^{-(x-y-2\pi(n+\frac{1}{2}))^2/2t} f(y+\pi)\,dy$$

$$= \frac{1}{\sqrt{2\pi t}} \int_0^{\pi} \sum_{m=-\infty}^{\infty} \left(e^{-(x-y-2\pi m)^2/2t} - e^{-(x+y'-2\pi m)^2/2t} \right) f(y)\, dy$$

where we took $y' = -y + \pi$

$(x-y-2\pi m)^2 \to (x+y'-2\pi m)^2$

Again it is clear from this expression that for $x \in (0,\pi)$ and $y \in (0,\pi)$, $(x-y-2\pi m)^2 < (x+y-2\pi m)^2$

thus

$$0 < h_D(x,y,t) < h(x-y,t)$$

A similar exercise appears in the problem set, for Neumann boundary conditions.

This is an example of the Kelvin method of images.

(vii) The wave equation

Maxwell posed that the phenomenon of electromagnetic radiation is described by two interrelated fields.

$$E = (E_1(x,t),\ E_2(x,t),\ E_3(x,t))$$
$$B = (B_1(x,t),\ B_2(x,t),\ B_3(x,t))$$

for

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3 \quad \text{and time } t.$$

Different forms of radiation (radio, radar, heat (infrared), light, ultraviolet, x-rays, cosmic rays) are all solutions of Maxwell's equations, whose properties are distinguished by being different spatial frequencies.

The two fields satisfy Maxwell's equations.

(1) $\nabla \cdot E = 4\pi \rho$ $\qquad$ ρ = electric charge density (a scalar)

$\nabla \cdot B = 0$ $\qquad$ no magnetic monopoles.

$\partial_t E = \frac{1}{\varepsilon} \nabla \times B - 4\pi j$ $\qquad$ j = electric current density (a vector)

$\partial_t B = -\frac{1}{\mu} \nabla \times E$

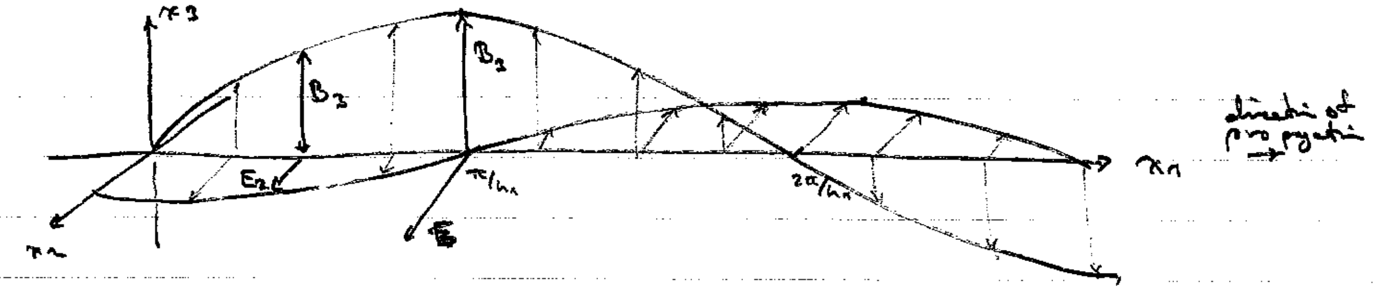
where ε = electric permittivity, μ = magnetic permittivity. The speed of light will turn out to be $c^2 = \frac{1}{\varepsilon\mu}$.

Light waves exist even in a vacuum, with $(\rho, j) = 0$.

$$E = (0,\ E_2 \cos(k_1 x_1 - \omega t),\ E_3 \sin(k_1 x_1 - \omega t))$$
$$B = (0,\ B_2 \sin(k_1 x_1 - \omega t),\ B_3 \cos(k_1 x_1 - \omega t))$$

$\left\{ B_2 = \frac{-k_1}{\mu\omega} E_3,\ B_3 = \frac{+k_1}{\mu\omega} E_2 \right.$

which solves (1) if $\omega^2 = k_1^2/\varepsilon\mu$. This is a plane wave solution whe



Since the equations are linear, we may superpose solutions of this form. In fact, all solutions can be described this way, which we will show (later) in our discussion of the Fourier transform.

Proposition 1: In the vacuum case ($(\rho, j) = 0$) each component $E_j(x,t)$, $B_j(x,t)$ of a solution of Maxwell's equations satisfies the (scalar) wave equation.

(2) $$\partial_t^2 u = c^2 \Delta u .$$

proof: Differentiate Maxwell's equations

$$\partial_t^2 E = \frac{1}{\varepsilon} \nabla \times \partial_t B = -\frac{1}{\varepsilon\mu} \nabla \times (\nabla \times E)$$

A vector calculus identity

$$\nabla \times (\nabla \times E) = -\Delta E + \nabla(\nabla \cdot E),$$

therefore

(3) $$\partial_t^2 E = \frac{1}{\varepsilon\mu} \Delta E - \frac{1}{\varepsilon\mu} \nabla(\nabla \cdot E), \qquad \nabla \cdot E = 4\pi\rho = 0 .$$

Similarly

$$\partial_t^2 B = -\frac{1}{\mu} \nabla \times (\partial_t E) = -\frac{1}{\mu\varepsilon} \nabla \times (\nabla \times B) = \frac{1}{\varepsilon\mu} \Delta B .$$

These equations are now reduced to diagonal, that is a scalar equation in each component. ∎

The D'Alembert formula: pertains to the one-dimensional wave equation.

Suppose that a solution of (2) depends only upon (x_1, t) (as does our plane wave solution).

$$\partial_t^2 u = c^2 \partial_{x_1}^2 u \qquad x_1 \in \mathbb{R}^1$$

In this special case the equations ~~...~~ are factored as

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x) u = 0$$

Characteristic coordinates re-interpret this; set

$$r = x + ct \qquad s = x - ct,$$

for which this reads

$$\partial_s (\partial_r u) = 0.$$

Therefore

$$\partial_r u = g = g(r) \qquad \text{independent of } s,$$

and thus

$$u(r,s) = \int^r g(r')\,dr' + H(s)$$
$$= G(r) + H(s)$$
$$= G(x+ct) + H(x-ct),$$

a superposition of left moving and right moving components.

Initial conditions

$$u(x,0) = f(x) \qquad \text{initial displacement}$$
$$\partial_t u(x,0) = g(x) \qquad \text{initial velocity}$$

Using the form of the solution

$$u(x,0) = f(x) = G(x) + H(x)$$
$$\partial_t u(x,0) = g(x) = cG'(x) - cH'(x)$$

Solving for G' and H'.

$$G'(x) = \frac{1}{2c}(cf'(x) + g(x))$$
$$H'(x) = \frac{1}{2c}(cf'(x) - g(x))$$

therefore upon integrating

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(x')dx' + c$$
$$H(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(x')dx' + d.$$

Adding the two, we find

$$G(x) + H(x) = f(x) + (c+d) \qquad \text{so } (c+d) = 0.$$

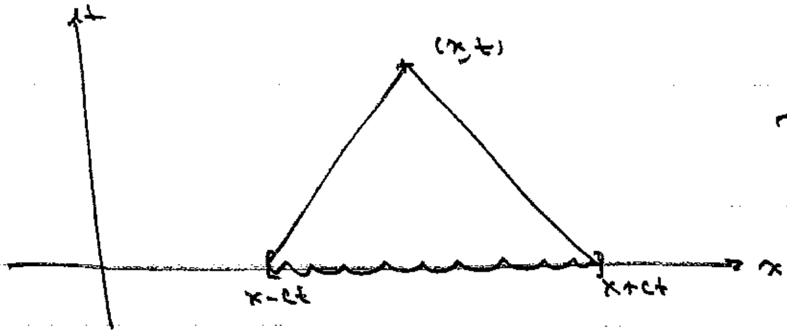
Theorem 2 The solution of (2) is given by D'Alembert's formula

$$(4) \qquad u(x,t) = G(x+ct) + H(x-ct)$$
$$= \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x')dx'$$

If $f \in C^2$ and $g \in C^1$ this gives an expression for a 'classical solution' $u \in C^2$. However with less stringent demands on f and g, we still receive a reasonable "weak" solution from the D'Alembert formula (4).

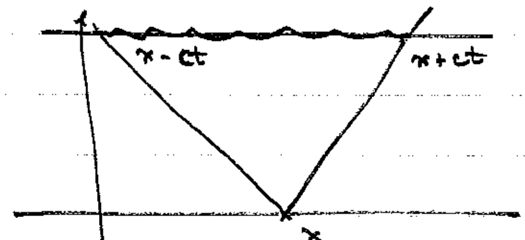
Theorem 3 (finite propagation speed). A solution $u(x,t)$ for which $f(x) = 0$ and $g(x) = 0$ outside a ball $B_R(0)$ will also vanish at time t outside a ball $B_{R+ct}(0)$.

proof: The solution value at space-time point (x,t) depends upon the initial data $f(x)$ at $x-ct$ and $x+ct$, and upon $g(x)$ over the interval $[x-ct, x+ct]$.

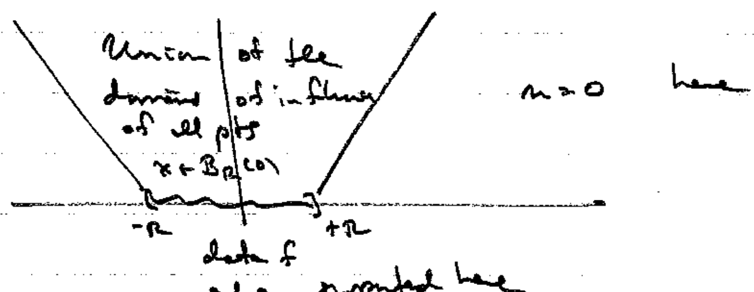


The domain of dependence.

Similarly, the domain of influence of the initial data at space point x is the open light cone region $(x-ct,\ x+ct)$



The proof of theorem 3 is again by a diagram



The method of images:

A wave incident upon a boundary will reflect in a manner dictated by the boundary conditions. The two most common are Dirichlet and Neumann boundary conditions.

Dirichlet:

(5)
$$\partial_t^2 u - c^2 \partial_x^2 u = 0 \quad \text{in } x > 0$$
$$u(0,t) = 0 \quad \text{on } x = 0 \quad \text{Dirichlet boundary condition}$$
$$u(x,0) = f(x)$$
$$\partial_t u(x,0) = g(x) \ ; \quad \text{for } x > 0.$$

And Neumann

(6) $\partial_t^2 u - c^2 \partial_x^2 u = 0$

$-\partial_x u(0,t) = 0$ Neuman boundary condition

$u(x,0) = f(x)$ for $x > 0$.

$\partial_t u(x,0) = g(x)$

The method of images

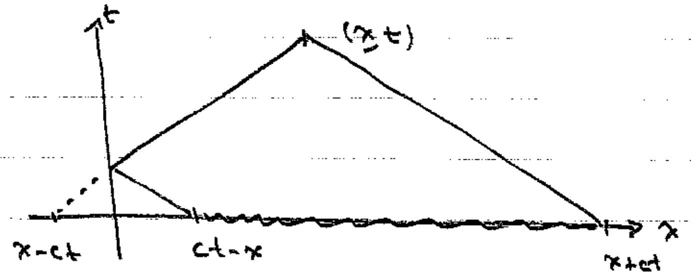
Dirichlet condition: extend $f(x)$ $x > 0$ to be an odd function of x;

$f(x) = -f(-x)$ for $x < 0$.

Similarly extend $g(x)$ to be odd.

Use D'Alembert's formula

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x')\,dx'$$

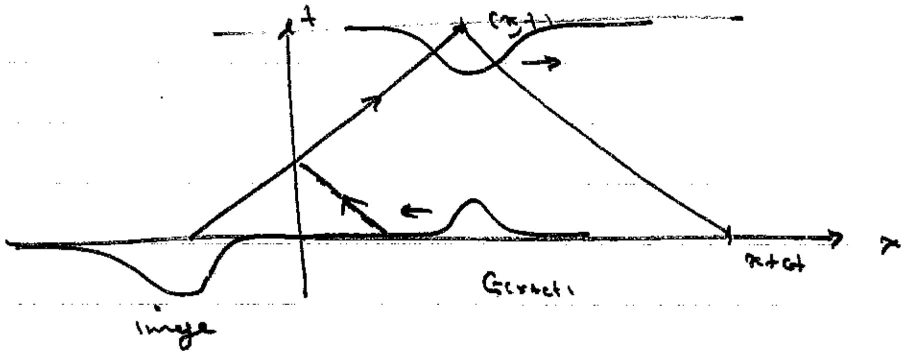


For $0 < x < ct$ the domain of dependence includes an interval $[x-ct, 0]$ where we have created a reverse image of our given data.

Using the property that $f(x)$ $g(x)$ are odd extensions,

$$u(x,t) = \frac{1}{2}\left(f(x+ct) - f(ct-x)\right) + \frac{1}{2c}\int_{ct-x}^{x+ct} g(x')\,dx'$$

In particular the reflected component of the wave comes out inverted.



Under Neumann boundary conditions, we reflect $f(x), g(x)$ to the negative x-axis as even functions

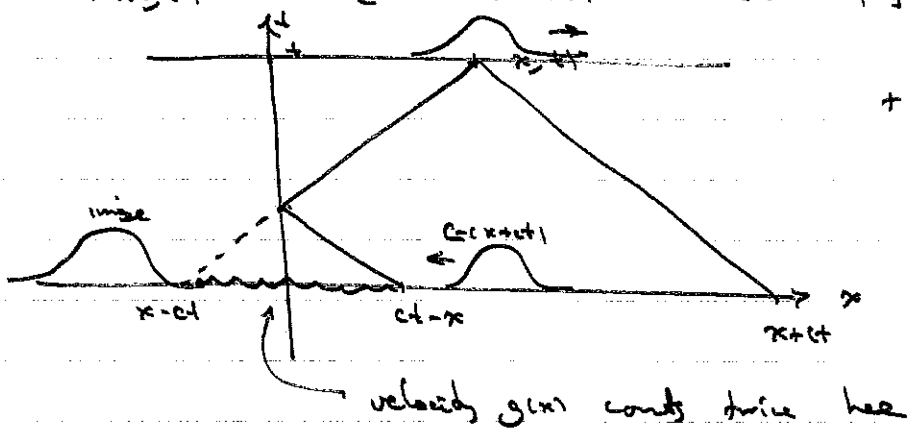
$$f(-x) = f(x)$$

$$g(-x) = g(x).$$

The D'Alembert formula gives, for $0<x<ct$

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(ct-x)\right) + \frac{1}{2c}\int_{ct-x}^{x+ct} g(x')\,dx'$$

$$+ \frac{1}{c}\int_{x-ct}^{ct-x} g(x')\,dx'$$



The reflection is <u>not</u> inverted.

(viii) musical instruments ref: R. Beals, Analysis, an Introduction, CUP.

The wave equation comes into play in the theory of musical instruments. We will discuss

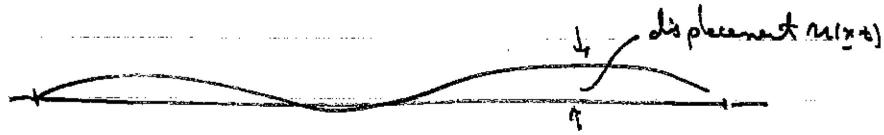
(1) strings: guitar, violin, piano

(2) winds and brass: flute, clarinet, oboe

(3) drums, and other higher dimensional instruments

The basic equation - the wave equation. Derivation

(1) String dynamics.



A displacement experiences a restoring force proportional to the curvature of the string; using $F = ma$, m = string density

$$m\partial_t^2 = \tau \partial_x^2 u$$

The speed of ~~sound~~ displacements of the string is $c = \sqrt{\frac{\tau}{m}}$; choose units in which this is $c = 1$.

(2) Winds and brass:

The equations satisfied by the air column are linear Euler's equations

$$\partial_t v = -\partial_x p$$

$$\partial_t p = -\gamma \partial_x v$$

Therefore

$$\partial_t^2 v = -\partial_t \partial_x p = -(-\gamma \partial_x^2 v)$$

and similarly

$$\partial_t^2 p = \gamma \partial_x^2 p.$$

Choose units so that $c = \sqrt{\gamma} = 1$.

Solution of the initial value problem for a string, with its boundary conditions $u(0,t) = u(2\pi,t) = 0$.

Separation of variables gives special solutions:

$$u_n(x,t) = \sin\left(\tfrac{k}{2}x\right)\left[\alpha_k \cos\left(\tfrac{k}{2}t\right) + \beta_k \frac{\sin\left(\tfrac{k}{2}t\right)}{k/2}\right]$$

$$= \sin\left(\tfrac{k}{2}x\right) a_n e^{ik/2\, t}$$

$a_n \in \mathbb{C}$ complex amplitude

<u>Proposition 1</u>: The temporal frequencies for the string (with Dirichlet boundary conditions) are

$$\omega_n = \frac{n}{2} \qquad n = 1, 2, 3 \dots$$

A typical solution is not a 'pure' overtone' $a_n \sin(\frac{n}{2}x)$, but rather a linear superposition of such tones

$$u(x,t) = \sum_n a_n e^{i\frac{n}{2}t} \sin\left(\tfrac{n}{2}x\right).$$

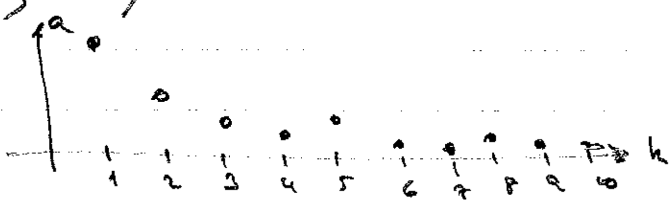
The temporal frequencies $\omega_n = \frac{n}{2}$, as well as the ~~weights~~ amplitudes a_n, give the characteristic of the sound of the instrument.

The fundamental is $\omega_1 = \frac{1}{2}$, and a_1 is usually the largest coefficient in magnitude.

The overtone frequencies are

$$\omega_2 = 2\omega_1, \quad \omega_3 = 3\omega_1, \quad \omega_4 = 4\omega_1 \quad \dots \text{etc},$$

and enter a musical tone in a subsidiary way. The <u>timbre</u> of the instrument (its characteristic sound) is determined by the overtone frequencies and by the ~~weights~~ amplitudes a_n, normally maybe the first 9 or 10 of them.

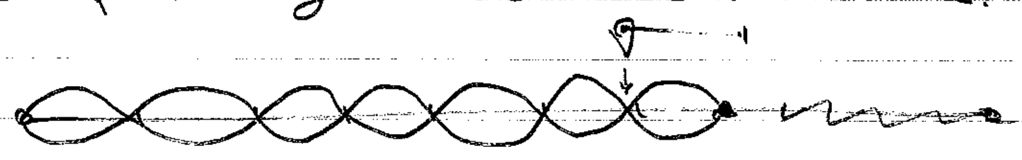


In musical nomenclature, the first several overtones fit into the diatonic scale

ω_1 = fundamental, or root.
$\omega_2 = 2\omega_1$ = octave
$\omega_3 = 3\omega_1$ = $\Pi 12^{th}$ = octave + $\Pi 5$
$\omega_4 = 4\omega_1 = 2\omega_2$ = 2^{nd} octave
$\omega_5 = 5\omega_1$ = 2 octaves + $\Pi 3$
$\omega_6 = 6\omega_1$ = $\Pi 5$ above ω_3 by one octave
$\omega_7 = 7\omega_1$ = anharmonic, between m7 and $\Pi 7$
$\omega_8 = 8\omega_1$ = 3^{rd} octave above ω_1.
ω_9 = ? = 9^{th} = $\Pi 2$ above ω_8.

Note the role of the prime numbers. Also note the first anharmonic (and thus the strongest) overtone ω_7.

Remark: A piano hammer strikes the piano strings at $1/7$ of the distance from the end, in order to de-emphasize generation of the 7^{th} overtone.



(2) wind instruments

(a) flute, operates much the same way as a string

$$\partial_t^2 v = \partial_x^2 v$$

with boundary condition for v

$$-\partial_x v(0,t) = 0 = \partial_x v(2\pi, t)$$

The pure tones (eigenmodes) are

$$v_k(x,t) = \cos\left(\tfrac{k}{2}x\right) b_k \, e^{i\frac{k}{2}t}$$

$$k = 0, 1, 2, 3 \ldots$$

b_k complex amplitude

Proposition 2 The temporal frequencies for the flute

$$\omega_k = \frac{k}{2}, \quad k = 0, 1, 2, \ldots$$

Remarks: The mode $k=0$ corresponds to a constant flow through the flute, that is not vibrating. The tone $\omega_1 = \frac{1}{2}$, $k=1$, is the fundamental frequency.

Then $\omega_2 = 2\omega_1$, $\omega_3 = 3\omega_1$ etc, as a stringed instrument.

(b) clarinet

Again the air column satisfies

$$\partial_t^2 v = \partial_x^2 v$$

The boundary conditions are however different.

$v(0,t) = 0$ at the reed

$\partial_x v(2\pi, t) = 0$ at the opening.

The pure tones are $v_k(x,t) = d_k \sin\left(\frac{2k-1}{4}x\right) e^{i\frac{2k-1}{4}t}$

$\omega_1 = \frac{1}{4}$ fundamental

$\omega_2 = \frac{3}{4} = 3\omega_1$

$\omega_3 = \frac{5}{4} = 5\omega_1$

$\omega_4 =$ etc. all odd multiples of ω_1.

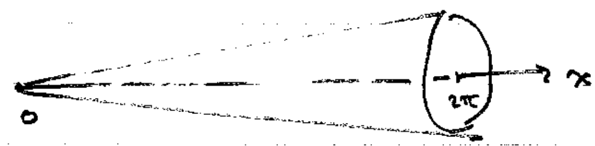
Proposition 3 The temporal frequencies of the clarinet are

$$\omega_1 = \frac{1}{4} \quad \text{and} \quad \omega_k = (2k-1)\omega_1$$

Note: Flutes overblow by one octave. The clarinet overblows by one 12th (octave + P5).

(c) oboe's (to think about)

The oboe (and the saxophone) are conical bore instruments



With conical cross-section, the effective mass density of the air column in x is $\rho(x) = \rho_0 \pi x^2$.

$$\partial_t^2 u = \partial_x^2 u + \frac{2}{x}\partial_x u .$$

Set $w(x,t) = x\, u(x,t)$ to find the wave eqn for w.

(d) Drums.

The equation for a vibrating membrane of two dimensions is

$$\partial_t^2 u = \Delta u \qquad \Delta u = (\partial_{x_1}^2 + \partial_{x_2}^2)u$$

A drum is a disk (generally) $\{x_1^2 + x_2^2 \leq 1\}$,
with boundary condition $u(x,t) = 0$ for $(x_1^2 + x_2^2) = 1$.

Special separation of variables solutions are derived as follows

$$u(x,t) = e^{i\lambda t}\, v_\lambda(x)$$

where

$$\Delta v_\lambda(x) = \lambda v$$

Furthermore use polar coordinates

$$(\partial_r^2 + \tfrac{1}{r}\partial_r)F(r)H(\theta) + \tfrac{1}{r^2}\partial_\theta^2 H(\theta)F(r) = \lambda H(\theta)F(r)$$

giving $\partial_\theta^2 H(\theta) = n^2 H(\theta)$ $\quad H(\theta + 2\pi) = H(\theta)$ $\quad n^{th}$ harmonic.

$$(\partial_r^2 + \tfrac{1}{r}\partial_r)F(r) = \frac{n^2}{r^2}F(r);$$ solutions in Bessel functions.

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(ix) Schrödinger's equation

The basic function in quantum mechanics is the <u>wave function</u> $\psi(x,t)$, normalized so that

$$\int |\psi(x,t)|^2 \, dx = 1 .$$

The interpretation of ψ is that the probability of a quantum particle q being in a set A (a space region) is given by at time t

$$P[q \in A] = \int_A |\psi(x,t)|^2 \, dx$$

Since $\int |\psi(x,t)|^2 dx = 1$, the total probability is one. Note however that $\psi(x,t)$ is allowed to take on values in $\mathbb{C}$.

We will study the motion of quantum particles $q \in \mathbb{T}^1$, using Fourier series.

The energy of a particle is defined as

$$H = \int_{-\pi}^{\pi} \tfrac{1}{2} |\partial_x \psi(x,t)|^2 \, dx \qquad \text{(kinetic energy)}$$

To derive the equations of evolution for ψ, the rules governing conservation of probability are similar to the Newton-Fourier derivation of the heat equation. We suppose that $\psi(x,t)$ evolves in time, with the principle of conservation of probability being

$$\frac{d}{dt} \int_a^b |\psi(x,t)|^2 \, dx = - F(b) + F(a),$$

where $F(a)$ is the quantum flux through the point a.

In the case of Schrödinger's equation, the choice of quantum flux is that

$$F(x) = 2\,\mathrm{re}(i\bar{\psi}\,\partial_x\psi) = i(\psi\overline{\partial_x\psi} - \bar{\psi}\,\partial_x\psi).$$

Using this in the above conservation law, for all $[a,b]$,

$$\frac{d}{dt}\int_a^b |\psi(x,t)|^2\,dx = \int_a^b (\bar{\psi}\,\partial_t\psi + \psi\,\overline{\partial_t\psi})\,dx$$

$$= i\left[\bar{\psi}\,\partial_x\psi - \overline{\partial_x\psi}\,\psi\right]_{x=a}^{b}$$

This is a boundary term, which arises from integration by parts.

$$= i\int_a^b (\overline{\partial_x\psi}\,\partial_x\psi + \bar{\psi}\,\partial_x^2\psi) - (\overline{\partial_x\psi}\,\partial_x\psi + \overline{\partial_x^2\psi}\,\psi)\,dx$$

$$= \int_a^b \bar{\psi}\,(i\partial_x^2\psi) + (-i\overline{\partial_x^2\psi})\,\psi\,dx$$

For this to hold for any arbitrary interval $[a,b]$, we need to have

$$\partial_t\psi = i\partial_x^2\psi \quad ,$$

which is to say

$$i\,\partial_t\psi = -\partial_x^2\psi, \qquad \text{Schrödinger's equation.}$$

It's an initial value problem, with given initial data $\psi(x,0)=\psi_0(x)$.

<u>Theorem 1</u> Solutions of Schrödinger's equation conserve

1) probability $\quad \frac{d}{dt}\int_0^{2\pi} |\psi(x,t)|^2\,dx = 0$

2) energy $\quad \frac{d}{dt}\int_0^{2\pi} \frac{1}{2}|\partial_x\psi(x,t)|^2\,dx = 0.$

<u>proof</u>: (1) $\frac{d}{dt}\int_0^{2\pi}|\psi(x,t)|^2\,dx = \int_0^{2\pi}(\bar\psi\,\partial_t\psi + \overline{\partial_t\psi}\,\psi)\,dx$

$$= \int_0^{2\pi}\left(\bar\psi\,(i\partial_x^2\psi) + \overline{(i\partial_x^2\psi)}\,\psi\right)dx$$

$$= i\int_0^{2\pi}\bar\psi\,(\partial_x^2\psi) - (\partial_x^2\bar\psi)\,\psi\,dx.$$

Observe that ∂_x^2 is a symmetric operator, so that $i\partial_x^2$ is skew-symmetric, and this expression has the form

$$= \langle\psi, A\psi\rangle + \langle A^*\psi,\psi\rangle = \langle\psi, A\psi\rangle - \langle A\psi,\psi\rangle = 0.$$

(2) $\frac{d}{dt}\int_0^{2\pi}|\psi_x(x,t)|^2\,dx = 2\,\mathrm{re}\int_0^{2\pi}\overline{\partial_x\psi}\,\partial_t\partial_x\psi\,dx$

$$= -2\,\mathrm{re}\int_0^{2\pi}\overline{\partial_x^2\psi}\,\partial_t\psi\,dx$$

$$= +2\,\mathrm{re}\int_0^{2\pi}\overline{\partial_x^2\psi}\,i\partial_x^2\psi\,dx$$

$$= 2\,\mathrm{im}\int_0^{2\pi}|\partial_x^2\psi|^2\,dx = 0.$$

As long as our solution is sufficiently differentiable so as to perform the above manipulations, the proof is complete. ∎

Compare the heat equation and the Schrödinger equation

(i) $\int_a^b u\,dx$ heat quantity in the set $[a,b]$

$\int_a^b|\psi|^2\,dx$ quantum probability in the set $[a,b]$.

(ii) $\partial_t\int_0^{2\pi}u\,dx = 0$

$\partial_t\int_0^{2\pi}|\psi(x,t)|^2\,dx = 0$

(iii) $\partial_t\int_0^{2\pi}u^2\,dx \le 0$ not reversible in time

$u(x,t) \to \bar u(x,-t)$ reversing time also solves same equation.

Construction of the Schrödinger propagator by Fourier synthesis

Write the wave function $\psi(x,t) = \sum_{k\in\mathbb{Z}} \hat{\psi}_k(t) \frac{e^{ikx}}{\sqrt{2\pi}}$, being 2π periodic, and use this Ansatz in the equation

$$i\partial_t\psi = \sum_{k\in\mathbb{Z}} i\frac{d\psi_k(t)}{dt} \frac{e^{ikx}}{\sqrt{2\pi}}$$

$$\partial_x^2\psi = \sum_{k\in\mathbb{Z}} -k^2\psi_k(t) \frac{e^{ikx}}{\sqrt{2\pi}}$$

hence if $i\partial_t\psi = -\partial_x^2\psi$, each Fourier mode obeys

$$i\frac{d}{dt}\psi_k(t) = k^2\psi_k(t)$$

This is our harmonic oscillator in complex coordinates, whose solution is

$$\psi_k(t) = e^{-ik^2t}\psi_k(0) .$$

Therefore the full solution can be read off

$$\psi(x,t) = \sum_{k\in\mathbb{Z}} \psi_k(0)\, e^{-ik^2t} \frac{e^{ikx}}{\sqrt{2\pi}}$$

Use the definition of the Fourier coefficients $\psi_k(0) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\psi_0(y)e^{-iky}dy$

$$\psi(x,t) = \sum_{k\in\mathbb{Z}} \frac{1}{2\pi}\int_0^{2\pi}\psi_0(y)\, e^{ik(x-y)}e^{-ik^2t}\, dy$$

$$= \int_0^{2\pi}\Big(\sum_{k\in\mathbb{Z}} \frac{1}{2\pi} e^{i[k(x-y)-k^2t]}\Big)\psi_0(y)\, dy$$

$$= \int_0^{2\pi} S(x-y,t)\,\psi_0(y)\, dy$$

the Schrödinger kernel for the circle $\mathbb{T}^1$. Note that the solution operator is again given by a convolution operator.

<u>Theorem 2</u> For initial data $\psi_0(x) \in C^2(\mathbb{T}^1)$ the solution to Schrödinger's equation is given by

$$\psi(x,t) = \int_0^{2\pi} S(x-y,t)\,\psi_0(y)\,dy$$

$$= \sum_{k\in\mathbb{Z}} e^{-ik^2t} \int_0^{2\pi} \frac{e^{ik(x-y)}}{2\pi}\,\psi_0(y)\,dy ,$$

and this infinite series converges uniformly.

proof: All that is left for us to prove is the uniform convergence. For any time $t \in \mathbb{R}$, observe that the k^{th} Fourier coefficient of $\psi(x,t)$ is given by

$$e^{-ik^2t}(\psi_0)_k ,$$

and that

$$|e^{-ik^2t}(\psi_0)_k| = |(\psi_0)_k| ,$$

which gives a ~~converge~~ uniformly convergent Fourier series, again by Dirichlet's theorem ∎

The fact is that the Schrödinger kernel, while being straightforward to express, is not so simple. Here is an example of this.

<u>Theorem 3</u> (M. Berry, L. Kapitanski) The level sets $S(x,t) = $ const. are fractal, ~~in the cylinder~~ $(x,t) \in \mathbb{T}^1 \times \mathbb{R}$, at least when t is irrational.

A final question has to do with the solution operator as given to us in the form of the Schrödinger propagation. That is, given $\psi_0(x)$, define $\psi_1(x) = \int_0^{2\pi} S(x-y,t_1)\psi_0(y)\,dy$. Does the propagator satisfy

$$\psi(x,t_2) = \int_0^{2\pi} S(x-y,t_2)\psi_0(y)\,dy$$
$$= \int_0^{2\pi} S(x-y,t_2-t_1)\psi_1(y)\,dy \ ?$$

Phrase this in terms of an evolution operator

$$(S(t)\psi_0)(x) := \psi(x,t) = \int_0^{2\pi} S(x-y,t)\psi_0(y)\,dy.$$

The question is restated as to whether $S(t_2-t_1)S(t_1) = S(t_2)$ for all $t_1, t_2 \in \mathbb{R}$ (could be negative). If so, then $S(t)S(-t) = id$.

Try this out using the Schrödinger kernel

$$\int_0^{2\pi} S(t_2-t_1,x-z)\int_0^{2\pi} S(t_1,z-y)\psi_0(y)\,dy\,dz$$
$$\stackrel{?}{=} \int_0^{2\pi} S(t_2,x-y)\psi_0(y)\,dy$$

In Fourier representation this becomes more transparent,

$$\psi(x,t_2) = \sum_{k\in\mathbb{Z}} e^{-ik^2t_2}\frac{e^{ikx}}{\sqrt{2\pi}}(\psi_0)_k$$
$$= \sum_{k\in\mathbb{Z}} e^{-ik^2(t_2-t_1)}e^{-ik^2t_1}\frac{e^{ikx}}{\sqrt{2\pi}}(\psi_0)_k$$
$$= \sum_{k\in\mathbb{Z}} e^{-ik^2(t_2-t_1)}\underbrace{\left(e^{-ik^2t_1}(\psi_0)_k\right)}_{\text{Fourier coefficient of } \psi_1(x)=S(t_1)\psi_0}\frac{e^{ikx}}{\sqrt{2\pi}}$$
$$= S(t_2-t_1)\psi_1 = S(t_2-t_1)S(t_1)\psi_0.$$

<u>Theorem 4</u> The Schrödinger propagator defines a one parameter group (of unitary operators), giving a flow on solution space.