Math 4FT
Thurs Jan 17 2013
W. Craig

Set class space-time:

Mon $14^{h30} - 16^{h20}$ HH217

Th $11^{30} - 12^{h20}$ HH410.

sorry about the confusion, and about missing class Monday due to the flu.

(2) Mathematical physics.

(time oscillations and their decomposition)

(i) classical harmonic oscillator.

On day one we studied briefly the simple harmonic oscillator

$$\dot{x} = -kv \qquad\qquad \text{solution} \quad z(t) = x(t) + i\,v(t)$$
$$\dot{v} = +kx \qquad\qquad , \quad \dot{z} = ikz$$
$$\text{thus} \quad z(t) = \Phi_t(z(0)) = e^{ikt} z(0).$$

Revisit this simplest of problems, to study various linear ODEs.

Integration reminder

(1) $\dot{x} = g(t)$ $\qquad x(t) = x(0) + \int_0^t g(s)\,ds$

(2) $\dot{x} = bx$ $\qquad x(t) = e^{tb} x(0)$

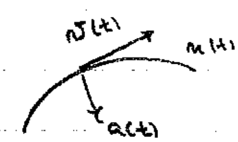
(3) $\dot{x} = b(t)x$ $\qquad x(t) = \exp\left(\int_0^t b(s)\,ds\right) x(0).$

Newton's laws

assume a curve $x(t) \in \mathbb{R}^d$

x = position $\qquad v = \dot{x}$ = velocity

$a = \ddot{x}$ = acceleration



Newton's law is that $m\,a = F$ force

(4) no force $m\ddot{u} = 0$

hence $\dot{u} = v_0$ constant velocity

$u(t) = u_0 + t\,v_0$ linear motion

(5) One spring (Hooke's law for restoring force)

- $d=1$ dimension
- $F = -\varkappa u$ force opposite to and proportional to displacement

$\ddot{u} = -\frac{\varkappa}{m} u$ classical harmonic oscillator

Write this as a matrix equation

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{\varkappa}{m} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = B\begin{pmatrix} u \\ v \end{pmatrix}$$

This is the ODE we solved in the introductory lecture (in slightly different coordinates)

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \Phi_t \begin{pmatrix} u(0) \\ v(0) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\left(\sqrt{\frac{\varkappa}{m}}\,t\right) & \frac{\sin\left(\sqrt{\frac{\varkappa}{m}}\,t\right)}{\sqrt{\varkappa/m}} \\ -\sqrt{\frac{\varkappa}{m}}\sin\left(\sqrt{\frac{\varkappa}{m}}\,t\right) & \cos\left(\sqrt{\frac{\varkappa}{m}}\,t\right) \end{pmatrix}\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}$$

Another way to write the fundamental solution matrix using linear algebra is $\Phi_t = e^{tB}$. That is

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \Phi_t \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = e^{tB}\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}.$$

All this is indeed very simple. However to show you that we are actually at the frontier of something, consider solving the matrix eqn

$\dot{w} = B(t)\,w$, $w \in \mathbb{R}^d$ (and not $\mathbb{R}^1$ this time).

(c) Many springs (the wave equation can sometimes be thought of as the case of ∞-many springs)

$$
\begin{aligned}
m_1 \ddot{u}_1 &= -\varkappa_{11} u_1 - \varkappa_{12} u_2 \cdots - \varkappa_{1d} u_d \\
m_2 \ddot{u}_2 &= -\varkappa_{21} u_1 - \cdots - \varkappa_{2d} u_d \\
&\vdots \\
m_d \ddot{u}_d &= -\varkappa_{d1} u_1 - \cdots - \varkappa_{dd} u_d .
\end{aligned}
$$

a $d \times d$ system of linear equations. Write this in matrix notation

$$M = \operatorname{diag}(m_j)_{j=1}^d \qquad K = (\varkappa_{jk})_{j,k=1}^d ,$$

then

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} \text{ satisfies } \ddot{u} = -Cu \quad \text{with } C = +M^{-1}K$$

The case of a harmonic oscillator is when $C^T = C > 0$ (equal and opposite reaction, restoring forces).

Solution method: Rotate coordinates $u = Ry$, with

$$R^T C R = D = \text{diagonal} = \operatorname{diag}(\omega_j^2)_{j=1}^d$$

Then ω_j^2 = e-values of C.

$$\ddot{y} = -Dy \quad \text{is diagonal:} \quad \ddot{y}_j = -\omega_j^2 y_j$$

Normal modes

$$\begin{pmatrix} y_j \\ \dot{y}_j \end{pmatrix} = \begin{pmatrix} \cos(\omega_j t) & \frac{\sin(\omega_j t)}{\omega_j} \\ -\omega_j \sin(\omega_j t) & \cos(\omega_j t) \end{pmatrix} \begin{pmatrix} y_j(0) \\ \dot{y}_j(0) \end{pmatrix}$$

Question: If all ω_j = integers, this represents periodic motion also if $\omega_j = \omega_0 n_j$ $\qquad \omega_0$ base frequency

However what if $\omega_1 = 1$ and $\omega_2 = \sqrt{2}$?
In general, motion takes place on tori, and is Quasi-Periodic.

(ii) Path expansions – alternate expressions

Write the harmonic oscillator as a first order system

$$\dot u = v \qquad\qquad w := (u, v) \in \mathbb{R}^{2d}$$
$$\dot v = -Cu \qquad\qquad C = C^T > 0$$

Then $\dot w = Bw$ $\qquad B = \begin{pmatrix} 0 & I \\ -C & 0 \end{pmatrix}$, $2d \times 2d$.

Linear algebra gives us the solution expression $\Phi_t w = e^{tB} w$.

$$\Phi_t = e^{tB} = \sum_{m=0}^{\infty} \left(\frac{t^m}{m!} B^m \right)$$

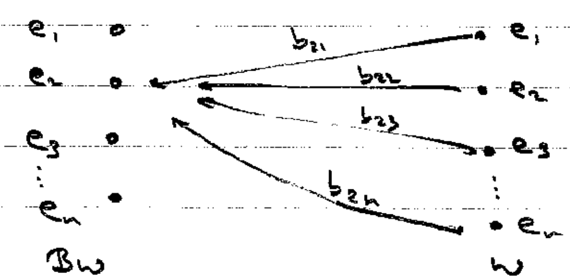
Exercise: Show $\Phi_{t_1+t_2} = \Phi_{t_1} \circ \Phi_{t_2}$, $\Phi_0 = I$, and that $\dot\Phi_t = B\Phi_t$.

My goal, to explain an interpretation of this expression that has a lot of physical content. Set $n = 2d$, B $n \times n$ matrix.

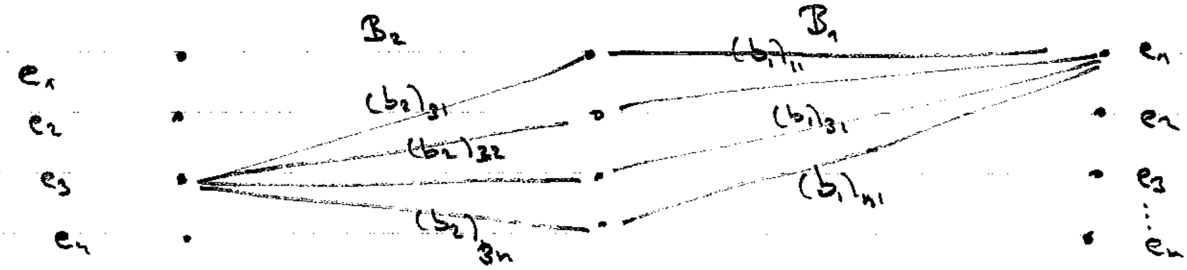
Let $e_1, \dots e_n$ denote abstract points; it is sometimes useful to think of $w \in \mathbb{R}^n$ as a field over the set $\{e_j\}_{j=1}^n$ with values $(w_1, \dots w_n)$.

A matrix B can be interpreted as transformation of field which takes into account the field's values at different points.

$$B = (b_{jk})_{j,k=1}^n \qquad (Bw)_j = \sum_{k=1}^n b_{jk} w_k$$



Multiplying matrices is thought of as a composition of such transformations

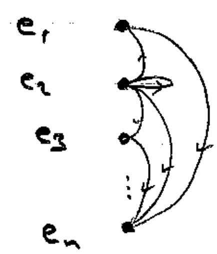


For example $(\mathcal{B}_2\mathcal{B}_1)_{31}$ matrix element

$$= \sum_{\ell} (b_2)_{3\ell}(b_1)_{\ell 1}.$$

Powers of a matrix are expressed in this way

$$(\mathcal{B}^2)_{jh} = \sum_{\ell} b_{j\ell}\, b_{\ell h}$$



$(\mathcal{B}^2)_{n2}$

$$= \sum_{\substack{\text{2-step paths } \beta \text{ among} \\ n\text{-many points} \\ \beta(0)=h \\ \beta(2)=j}} b_{\beta(2)\beta(1)}\, b_{\beta(1)\beta(0)}$$

$$(\mathcal{B}^m)_{jh} = \sum_{\substack{m\text{-step paths } \beta: \\ \beta(0)=h \\ \beta(m)=j}} \left(\prod_{\ell=1}^{m} b_{\beta(\ell)}\, b_{\beta(\ell-1)} \right)$$

From this we can further write down the exponential

$$(e^{t\mathcal{B}})_{jh} = \sum_{m=0}^{\infty} \frac{t^m}{m!} (\mathcal{B}^m)_{jh}$$

$$= \sum_{m=0}^{\infty} \frac{t^m}{m!} \left(\sum_{\substack{m\text{-step paths } \beta \\ \beta(0)=h \\ \beta(m)=j}} \prod_{\ell=1}^{m} b_{\beta(\ell)\beta(\ell-1)} \right)$$

$$= \sum_{\text{all paths } \beta: h \to j} \left[\left(\frac{t^m}{m!} \right)_{m=\#\text{steps}} \prod_{\ell=1}^{\#(\beta)} b_{\beta(\ell)\beta(\ell-1)} \right].$$

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(iii) Fourier's law and the heat equation

The heat equation describes the evolution of a function $u(x,t)$ (the temperature) at points x, as a function of time.

$$\partial_t u = \kappa \partial_x^2 u \qquad 0 \le x < 2\pi \quad \text{space domain}$$

We are given additionally that
(1) boundary conditions for $u(\cdot,t)$ at $x=0$ and $x=2\pi$
(2) an initial temperature distribution $u(x,0) = u_0(x)$.

Boundary conditions come in several types:
(1) Dirichlet conditions

$$u(0,t) = 0 \qquad u(2\pi,t) = 0 \qquad \forall t$$

Heat conducting ends – the endpoints are held at a constant ($u=0$) temperature.

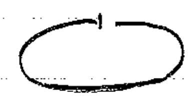
(2) Neumann conditions

$$-\partial_x u(0,t) = 0, \qquad \partial_x u(2\pi,t) = 0 \qquad \forall t$$

Insulators: there is no flux of heat across the domain ends at $x = 0, 2\pi$.

(3) periodic boundary conditions

$$u(x+2\pi,t) = u(x,t)$$

The space domain is a circle 

(2)

Separation of variables solutions.

Assume the very special form for a solution $u(x,t) = \rho(t)\varphi(x)$ and use the heat equation.

$$\partial_t \rho(t)\varphi(x) = \varkappa\, \rho(t)\,\partial_x^2 \varphi(x)$$

Therefore (at least away from zeros of $\rho(t)$ and/or of $\varphi(x)$),

$$\frac{\partial_t \rho(t)}{\rho(t)} = \varkappa \frac{\partial_x^2 \varphi(x)}{\varphi(x)},$$

where the LHS depends only upon t, while the RHS depends only upon x. Therefore they must individually equal a mutual constant λ:

$$\partial_t \rho = \varkappa \lambda \rho \qquad \text{solution} \quad \rho(t) = e^{\varkappa\lambda t}\rho(0).$$

$$\partial_x^2 \varphi = \lambda \varphi .$$

The boundary conditions also imply that $\varphi(x)$ satisfies:

(1) Dirichlet case: $\varphi(0) = 0 = \varphi(2\pi)$

(2) Neumann case: $-\partial_x\varphi(0) = 0 = \partial_x \varphi(0)$

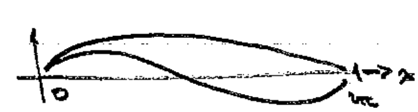
(3) periodic case: $\varphi(x+2\pi) = \varphi(x)$.

NB: You could say that the boundary conditions have 'quantized' the problem.

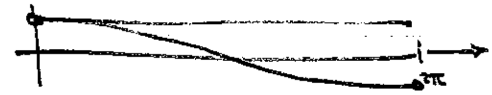
Solutions are given by trigonometric functions, with associated eigenvalues:

(1) $\varphi_n(x) = a_n \sin(\frac{n}{2}x)$ $\qquad \lambda_k^D = -(\frac{k}{2})^2, \; k=1,2\ldots$

$n = 1,2,3\ldots$



(2) $\phi_k(x) = b_k \cos\left(\frac{k}{2}x\right)$ $\qquad k = 0, 1, 2 \dots$ $\qquad \lambda_k^N = -\left(\frac{k}{2}\right)^2 \quad k = 0, 1, \dots$



(3) $e_k(x) = c_k e^{ikx}$ $\qquad -\infty < k < +\infty$ integer $\qquad \lambda_k^P = -k^2, \ k \in \mathbb{Z}.$

These are special solutions, found by classical undergraduate techniques. However remark that the heat equation is linear, so that linear superpositions are also solutions (modulo questions of convergence). Furthermore the boundary conditions are homogeneous, which is compatible with such superpositions.

(1) Dirichlet case

$$u(x,t) = \sum_{k=1}^{+\infty} a_k \sin\left(\frac{k}{2}x\right) e^{-\varkappa\left(\frac{k}{2}\right)^2 t}$$

(2) Neumann case

$$u(x,t) = \sum_{k=0}^{\infty} b_k \cos\left(\frac{k}{2}x\right) e^{-\varkappa\left(\frac{k}{2}\right)^2 t}$$

(3) Periodic case

$$u(x,t) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} e^{-\varkappa k^2 t}$$

Initial data: we are asked to specify initial heat (temperature) distribution at $t = 0$. Using the above superposition

(1) $u(x,0) = u_0(x) = \sum_{k=1}^{+\infty} a_k \sin\left(\frac{k}{2}x\right)$ $\qquad$ sine series

(2) $u(x,0) = u_0(x) = \sum_{k=0}^{+\infty} b_k \cos\left(\frac{k}{2}x\right)$ $\qquad$ cosine series

(3) $u(x,0) = \sum_{-\infty < k < +\infty} c_k e^{ikx}$ $\qquad$ Fourier series

Fourier's question: How general is this class of initial data, that which is given by a superposition of separation of variables solutions?

Namely, which function $u_0(x) = f(x)$ can be written = this way, as a superposition of elementary trig. functions.

The answer - we will discover - in a very good sense all reasonable function $f(x)$ can be represented by a (generally infinite) linear superposition of

(1) sine series
or (2) cosine series
or (3) Fourier series.

First exercise - to deduce from a given $f(x)$ what the coefficients (a_n, b_n and respectively c_n) should be, in order to have this representation.

(3) periodic case first:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Integrate against a complex exponential $e^{-ilx} = \overline{e_l}(x)$

$$\int_0^{2\pi} f(x)\, e^{-ilx}\, dx = \int_0^{2\pi} \Big(\sum_{n=-\infty}^{\infty} c_n e^{inx} \Big) e^{-ilx}\, dx$$

$$= \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{i(n-l)x}\, dx$$

A side calculation shows that $\int_0^{2\pi} e^{i(n-l)x}\, dx = \begin{cases} 0 & n \neq l \\ 2\pi & n = l \end{cases}$

Therefore

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-inx}\, dx = c_n .$$

Readjusting the constants, we write the Fourier coefficients

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)\, e^{-inx}\, dx \qquad \text{and} \qquad c_n = \frac{1}{\sqrt{2\pi}} \hat{f}_n .$$

(1) Dirichlet case: $\quad f(x) = \sum_{n=1}^{\infty} a_n \sin(\frac{n}{2}x)$

$$\int_0^{2\pi} f(x) \sin(\tfrac{\ell}{2}x)\, dx = \int_0^{2\pi} \Big(\sum_{n=1}^{\infty} a_n \sin(\tfrac{n}{2}x) \Big) \sin(\tfrac{\ell}{2}x)\, dx$$

$$= a_\ell \int_0^{2\pi} \sin^2(\tfrac{\ell}{2})\, dx = a_\ell\, \pi$$

Note: The integrals $\int_0^{2\pi} \sin(\frac{n}{2}x) \sin(\frac{\ell}{2}x)\, dx = \begin{cases} 0 & n \neq \ell \\ \pi & n = \ell. \end{cases}$
proof of this calculation

$$\int_0^{2\pi} \sin(\tfrac{n}{2}x) \sin(\tfrac{\ell}{2}x)\, dx = -\left(\tfrac{2}{n}\right)^2 \int_0^{2\pi} \partial_x^2 \sin(\tfrac{n}{2}x) \sin(\tfrac{\ell}{2}x)\, dx$$

$$= +\left(\tfrac{2}{n}\right)^2 \int_0^{2\pi} \partial_x \sin(\tfrac{n}{2}x)\, (\tfrac{\ell}{2}) \cos(\tfrac{\ell}{2}x)\, dx$$

$$= +\left(\tfrac{2}{n}\right)^2 \left(\tfrac{\ell}{2}\right)^2 \int_0^{2\pi} \sin(\tfrac{n}{2}x) \sin(\tfrac{\ell}{2}x)\, dx$$

Hence $= 0$ unless $n = \ell$.

(2) Neumann case: $\quad f(x) = \sum_{n=0}^{\infty} b_n \cos(\frac{n}{2}x)$

$$\int_0^{2\pi} f(x) \cos(\tfrac{\ell}{2}x)\, dx = \sum_{n=0}^{\infty} b_n \int_0^{2\pi} \cos(\tfrac{\ell}{2}x) \cos(\tfrac{n}{2}x)\, dx$$

$$= \begin{cases} b_\ell\, \pi & \text{if } \ell \neq 0 \\ b_0 \times 2\pi & \text{if } \ell = 0. \end{cases}$$

Table: Fourier coefficients: (3) $\hat{f}_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)\, e^{-inx}\, dx$

Sine series coefficients (1) $\quad a_n = \frac{1}{\sqrt{\pi}} \int_0^{\pi} f(x) \sin(\frac{n}{2}x)\, dx$

$b_0 = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x)\, dx \quad$ cosine series coefficients (2) $\quad b_n = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \cos(\frac{n}{2}x)\, dx \quad n \neq 0,$

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(iv) derivation of the heat equation

Our goal is to derive the heat equation from more basic principles. We will do this in two ways,

(1) physical principle of conservation of energy (heat energy in this case)

(2) geometrical principle - gradient flow in Hilbert space.

The heat equation again

$$\partial_t u = \kappa \Delta u \qquad \Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_d}^2$$

posed on a region $D \subseteq \mathbb{R}^d$ in which heat is to be conducted. The amount of heat energy of any subset $A \subseteq D$ is defined to be

$$\sigma \int_A u(x,t)\, dx$$

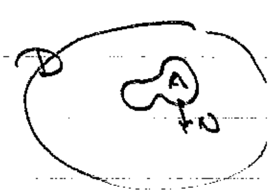
where $\sigma > 0$ is a constant, the specific heat of the material.

The amount of heat in A can change over time, it is given by

$$\partial_t \left(\sigma \int_A u(x,t)\, dx \right) = \sigma \int_A \partial_t u(x,t)\, dx .$$

Newton's law of heat flux:

(i) the only way for heat energy to enter or leave A is through heat flux across the boundary



$$\partial_t \left(\sigma \int u(x,t)\, dx \right) = - \int_{\partial A} \vec{F}(u) \cdot \vec{N}\, dS$$

(ii) furthermore, the flux of heat across the boundary is proportional to ∇u; $\quad \vec{F}(u) = -\gamma \nabla u$.

The constant $\gamma > 0$ and flux $\vec{F}$ has a negative sign, because heat flows from hotter regions to cooler ones.

Therefore, for all (smooth) subdomains $A \subseteq D$, we have a relationship

$$\int_A \sigma \partial_t u(\underset{\sim}{x},t)\,dx = \int_{\partial A} +\gamma \nabla u \cdot N\,dS = \int_A \gamma\,(\nabla\cdot(\nabla u))\,dx$$

using Stokes' formula. Since $\nabla\cdot\nabla u = \Delta u$, we have for each $A \subseteq D$

$$(2) \qquad \int_A \left(\partial_t u - \frac{\gamma}{\sigma}\Delta u\right) dx = 0,$$

where γ/σ = coefficient of heat diffusion. This is to hold for every subset $A \subseteq D$, thus as long as $\partial_t u$ and $\Delta u \in C(D)$ it implies that

$$\partial_t u = \varkappa \Delta u .$$

When $d=1$ this reduces to $\partial_t u = \varkappa \partial_x^2 u$ of our lecture on Monday.

Second derivation, via gradient flows. In general, for $u \in \mathbb{R}^N$, and $H(u)$ some C^2 function, the (downwards) gradient flow is given by

$$\dot{u} = -\nabla_u H(u) = -\mathrm{grad}_u H(u)$$

Proposition 1: Such flows decrease the functional $H(u)$.

proof:

$$\frac{d}{dt}H(u(t)) = \langle \mathrm{grad}_u H \cdot \dot{u} \rangle = \langle \nabla_u H \cdot (-\nabla_u H)\rangle = -|\nabla u|^2 \le 0$$

The same principle holds in a Hilbert space, for example. Define

$$H(u) = \int_D \tfrac{1}{2} |\nabla u(x)|^2 \, dx \qquad u \in H_0^1(D)$$

where the Hilbert space $H_0^1(D) \subseteq L^2(D)$ consists in all functions f in the closure of $C_0^\infty(D)$ with respect to the metric

$$\|f_1 - f_2\|_{H^1}^2 = \int |\nabla f_1 - \nabla f_2|^2 \, dx .$$

The functional derivative of $H(u)$: $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} H(u+\varepsilon v) :=$
$\langle \text{grad}_u H \cdot v \rangle = \int_D \text{grad}_u H \; v \, dx$.

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} H(u+\varepsilon v) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{1}{2} \int_D \nabla(u+\varepsilon v) \cdot \nabla(u+\varepsilon v) \, dx$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{1}{2} \int_D |\nabla u|^2 + 2\varepsilon \nabla u \cdot \nabla v + \varepsilon^2 |\nabla v|^2 \, dx$$

$$= \int_D \nabla u \cdot \nabla v \, dx = -\int_D \Delta u \; v \, dx + \underbrace{\cancel{\int_{\partial D} v \nabla u \cdot N \, dS}}_{=0}$$

Gradient flow is therefore, viewing $u = u(x)$ as a point of phase space:

$$\partial_t u = -\text{grad}_u H = -(-\Delta u) = \Delta u. \qquad \underline{\text{Done}}$$

The heat equation in two dimensions - in particular

$$\partial_t u = \kappa (\partial_{x_1}^2 + \partial_{x_2}^2) u$$

The steady state heat equation, describing time independent heat distributions.

(3) $\Delta u = (\partial^2_{x_1} + \partial^2_{x_2}) u = 0.$

Solutions are called harmonic functions. Consider a domain the disk $D := \{(x_1,x_2) : x_1^2 + x_2^2 < 1\}$.

The Dirichlet problem (for the Laplacian) on D is to find

(4) $\Delta u = 0 \qquad |x| < 1$
$u(x) = f(x) \qquad$ for $|x| = 1 \qquad$ boundary data on $\partial D = \{|x|=1\}$

This can be solved explicitly, using separation of variables in polar coordinates.

$(x_1, x_2) \to (r, \theta) \qquad r^2 = x_1^2 + x_2^2 \qquad x_1 = r\cos\theta$
$x_2 = r\sin\theta$

The Laplace operator in polar coordinates

$$\Delta u = \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u .$$

If u is harmonic, $\Delta u = 0$, then in polar coordinates

$$r^2 \Delta u = (r^2\partial_r^2 + r\partial_r u) + \partial_\theta^2 u = 0.$$

Assume the special form of a separation of variables solution
$u(r,\theta) = F(r)\, G(\theta)$,
then
$$(r^2 \partial_r^2 F + r\partial_r F)G + F\partial_\theta^2 G = 0.$$
Therefore this Ansatz leads to the two eqns w. a common constant
$$\partial_\theta^2 G - \lambda G = 0$$
$$r^2\partial_r^2 F + r\partial_r F + \lambda F = 0$$
Since $G(\theta + 2\pi) = G(\theta)$, it is of course 2π-periodic: $G_n(\theta) = c_n e^{i\theta n}$
$n = 0,1,2\ldots \qquad = a_n \sin(n\theta) + b_n\cos(n\theta).$

Therefore $\lambda_k = -k^2$.

Now solve for $F_k(r)$ solving
$$r^2\partial_r^2 F_k + r\partial_r F_k + \lambda_k F_k = 0.$$
use the power law Ansatz
$$F_k(r) = \alpha_k r^k + \beta_k r^{-k} \qquad k = 0, 1, 2 \ldots$$
and reject inverse powers as they are singular at $r = 0$.

$$u(x_1, x_2) = u(r, \theta) = \sum_{k=0}^{+\infty} [a_k \sin(k\theta) + b_k \cos(k\theta)] r^k$$
$$= \sum_{k\in\mathbb{Z}} c_k e^{ik\theta} r^k .$$

The boundary data for $u(x)$ on $r=1$ is given by

$$f(\theta) = \sum_{k\in\mathbb{Z}} c_k e^{ik\theta} .$$

This returns us to Fourier's question, in the context of the Dirichlet problem for harmonic functions on the disk D. Namely, which functions $f(\theta)$ can be written in the form

$$f(\theta) = \sum_{k\in\mathbb{Z}} c_k e^{ik\theta}$$

which give rise to harmonic functions

$$u(r,\theta) = \sum_{k\in\mathbb{Z}} c_k r^k e^{ik\theta}$$

The answer, again, is that $\hat{f}_k = \sqrt{2\pi}\, c_k = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(\theta) e^{-i\theta k} d\theta$ recovers "almost" all reasonable functions $f(\theta)$.

Up next: Dirichlet's theorem, that $f \in C^2(S^1)$ suffices.

(v) Dirichlet's approximation theorem

Return now to Fourier's original question, as to which functions $f(x)$ on $\mathbb{S}^1$ can be written in Fourier series

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx} = \sum_{n=-\infty}^{+\infty} \frac{\hat{f}_n}{\sqrt{2\pi}} e^{inx} .$$

Fourier suggested that all reasonable functions admit such a representation, but he did not produce a satisfactory mathematical answer in his 1822 monograph 'Théorie analytique de la chaleur'. However Dirichlet gave a correct proof in 1829, which we will discuss today. Since this time there have been numerous and increasingly technically sophisticated contributions, which continue today.

<u>Theorem 1</u> (Dirichlet) Suppose that $f \in C^2(\mathbb{S}^1)$, then the partial sums

$$S_n(f) = \sum_{-n \leq k \leq n} \hat{f}_k \frac{e^{ikx}}{\sqrt{2\pi}}$$

converge uniformly in $x \in \mathbb{S}^1$ to $f(x)$ as $n \to +\infty$.

Notations: $f \in C^1(\mathbb{S}^1)$ is the space of 2π-periodic functions of $x \in \mathbb{R}$ which are once continuously differentiable. Namely

- $f(x) = f(x+2\pi)$ $\forall x$ and is continuous
- $\partial_x f(x)$ is also continuous.

In fact smoothness of $f(x)$ is intimately related to the decay rate of Fourier coefficients $\hat{f}_h$, and the convergence rate of the partial sums $S_n(f)$. This is an important feature of Fourier analysis, and one of its constant themes.

<u>Theorem 2</u>: Suppose that $f \in C^r(S^1)$ for $r \geq 1$. Then

$$\sup_{x \in S^1} |S_n(f)(x) - f(x)| := \| S_n(f) - f \|_{L^\infty(S^1)} \leq \frac{C_r}{n^{r-1/2}}.$$

Note: Even for $r=1$ this is better than Dirichlet's theorem. However the proof will use Hilbert space methods, while Dirichlet's theorem will be given in historical context.

We are speaking about Fourier series of complex exponentials. But in fact we can relate it directly to the sine-series and cosine series through Euler's formula

$$e^{ihx} = \cos(hx) + i \sin(hx).$$

Giving $f(x)$ in terms of Fourier series, then

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \sum_{h=-\infty}^{\infty} \hat{f}_h e^{ihx} \\ &= \frac{1}{\sqrt{2\pi}} \sum_{h=-\infty}^{\infty} \hat{f}_h \left(\cos(hx) + i\sin(hx)\right) \\ &= \frac{1}{\sqrt{2\pi}} \Big[\hat{f}_0 + \sum_{h=1}^{+\infty} (\hat{f}_h + \hat{f}_{-h}) 2\cos(hx) \\ &\qquad + \sum_{h=1}^{+\infty} (\hat{f}_h - \hat{f}_{-h}) 2i\sin(hx) \Big] \end{aligned}$$

Thus the question of convergence of Fourier series for $f(x)$ and the convergence of the cosine series or the sine series are very closely related questions.

Proposition 3 . If $f(x) = f(-x)$ then $\hat{f}_k = \hat{f}_{-k}$ and the sine series vanishes.

- If $f(x) = -f(-x)$ then $\hat{f}_k = -\hat{f}_{-k}$ and the cosine series vanishes
- If $f(x)$ is real valued, then $\hat{f}_k = \overline{\hat{f}_{-k}}$.

proof: Clearly

$$\overline{\hat{f}_k} = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \overline{f(x)}\, e^{ikx}\, dx = \frac{1}{\sqrt{2\pi}} \overline{\int_0^{2\pi} f(x)\, e^{-ikx}\, dx}$$

When $\overline{f(x)} = f(x)$ then $\ast = \hat{f}_{-k}$.

If $f(x) = f(-x)$ is even, (and periodic)

$$\hat{f}_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} (\cos(kx) - i\sin(kx))\, f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \cos(kx)\, f(x)\, dx = \hat{f}_{-k}$$

If $f(x) = -f(-x)$ is odd (and periodic) then similarly

$$\hat{f}_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} (\cos(kx) - i\sin(kx))\, f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} -i\sin(kx)\, f(x)\, dx = -\hat{f}_{-k} . \qquad \square$$

Proof of Dirichlet's theorem (Thm 1):

The proof divides into two basic steps.

1) show that the partial sums $S_n(f)$ converge uniformly to something.
2) Show that they converge to $f(x)$.

The partial sums

$$S_n(f) = \sum_{-n\le k\le n} \hat{f}_k \frac{e^{ikx}}{\sqrt{2\pi}}$$

and using that $\hat{f}_k = \int_0^{2\pi} \frac{e^{-iky}}{\sqrt{2\pi}} f(y)\,dy$

$$= \sum_{-n\le k\le n} \frac{1}{2\pi} e^{ikx} \int_0^{2\pi} e^{-iky} f(y)\,dy$$

$$= \int_0^{2\pi} f(y) \left(\frac{1}{2\pi} \sum_{-n\le k\le n} e^{ik(x-y)} \right) dy$$

$$:= \int_0^{2\pi} f(y)\, D_n(x-y)\,dy$$

which has introduced the Dirichlet kernel $D_n(x)$:

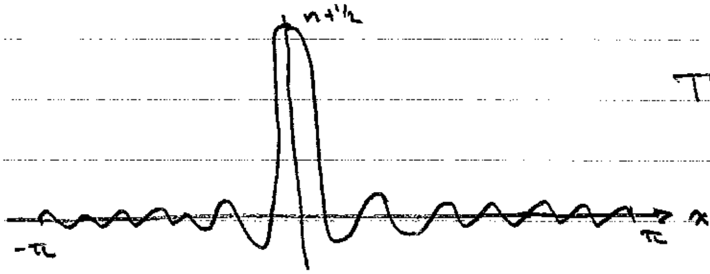
$$D_n(x) = \frac{1}{2\pi} \sum_{-n\le k\le n} e^{ikx}$$

$$= \frac{1}{2\pi} e^{-inx} \sum_{k=0}^{2n} e^{ikx}$$

$$= \frac{1}{2\pi} e^{-inx} \left(\frac{e^{i(2n+1)x} - 1}{e^{ix} - 1} \right)$$

$$= \frac{1}{2\pi} \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{1}{2}x)}$$

The kernel looks like this:



The formula tells us that $\int_0^{\pi} D_n(x)\,dx = 1$.

Thus $S_n(f)(x) = \int_{-\pi}^{\pi} D_n(x-y) f(y)\,dy$

Proposition 4 (differentiation and Fourier series)

$$\widehat{\frac{d}{dx}f}(k) = \int_0^{2\pi} e^{-ikx}\frac{df}{dx}\,dx = \left(f(x)\,e^{-ikx}\right)\Big|_{x=0}^{2\pi} - \int_0^{2\pi}(-ik)\,e^{-ikx}f(x)\,dx$$

$$= ik\,\hat{f}_k\,.$$

Therefore, if $f \in C^2(S^1)$ we have $\hat{f}_k = \frac{-1}{k^2}\left(\widehat{\frac{d^2f}{dx^2}}\right)_k$.
Furthermore, for any $g \in C^0(S^1)$

$$|\hat{g}_k| = \frac{1}{\sqrt{2\pi}}\left|\int_0^{2\pi} g(x)\,e^{-ikx}\,dx\right|$$

$$\le \frac{1}{\sqrt{2\pi}}\int_0^{2\pi}|g(x)|\,dx\,.$$

Therefore for $f \in C^2(S^1)$,

$$|(\widehat{f''})_k| \le C := \frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\left|\frac{d^2f}{dx^2}\right|dx\,.$$

We use this fact to show that the sequence of partial sums $\{S_n(f) : n = 1, 2, \cdots\}$ is a Cauchy sequence in $C_0(S^1)$.

Indeed

$$|S_n(f) - S_m(f)(x)| = \left|\sum_{n<|k|\le m}\hat{f}_k\,\frac{e^{ikx}}{\sqrt{2\pi}}\right|$$

$$\le \sum_{n<|k|\le m}\left|\frac{1}{k^2}\,(\widehat{f''})_k\,\frac{1}{\sqrt{2\pi}}\right|$$

$$\le \frac{C}{\sqrt{2\pi}}\sum_{n<|k|\le m}\frac{1}{k^2} \le \frac{C_1}{n} \qquad \text{since } \sum\frac{1}{n^2} < +\infty.$$

Therefore the sequence $S_n(f)(x)$ is (i) pointwise in $x \in S^1$ a Cauchy sequence, and therefore possesses a limit, say $h(x)$
(ii) Converges uniformly to this limit $h(x)$.
The issue is, what is the limit function $h(x)$? We would like it to be $f(x)$.

The difference between $S_n(f)(x)$ and $f(x)$ can be described as

$$\int_0^{2\pi} D_n(x-y)\, f(y)\, dy - f(x)$$

$$= \int_0^{2\pi} D_n(y)\, f(x-y)\, dy - \int_0^{2\pi} D_n(y)\, f(x)\, dy$$

recall that $\int_0^{2\pi} D_n(y)\, dy = 1$

$$= \int_0^{2\pi} D_n(y) \left[f(x-y) - f(x) \right] dx$$

Use the form of the Dirichlet kernel

$$= \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} \sin((n+\tfrac{1}{2})y) \left[\frac{f(x-y) - f(x)}{\sin(y/2)} \right] dy .$$

Because of l'Hopital's rule, the integrand as a function of y is $C^1(S^1)$, therefore

$$= \frac{1}{\sqrt{2\pi}} \left(- \frac{\cos(n+\frac{1}{2})y}{n+\frac{1}{2}}\, I(x,y) \right) \Big|_{y=0}^{2\pi}$$

$$+ \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \frac{\cos(n+\frac{1}{2})y}{n+\frac{1}{2}}\, \partial_y I(x,y)\, dy$$

The denominator is $(n+\frac{1}{2}) \to +\infty$, the other terms are bounded, hence these two terms converge to zero

$$| * | \le \frac{C_x}{n+1/2} . \qquad \blacksquare$$

The proof of theorem 2 will be put off until we have the tools of Hilbert spaces at hand.

(vi) heat kernels and the maximum principle

The point is to solve the heat equation with data $u(x,0) = f(x)$ on S^1,

$$\partial_t u = \frac{1}{2}\partial_x^2 u .$$

A formal solution can be given by a decomposition in Fourier series

$$u(x,t) = \sum_{k=-\infty}^{+\infty} c_k(t)\, e^{ikx}$$

$$\text{where } c_k(t) = \hat{u}_k(t) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(x,t)\, e^{-ikx}\, dx$$

as we did before.

The $c_k(t)$ satisfy a differential equation

$$\frac{dc_k}{dt} = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \partial_t u(x,t)\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \frac{1}{2}\partial_x^2 u(x,t)\, e^{-ikx}\, dx$$

$$= \frac{-k^2}{2\sqrt{2\pi}} \int_0^{2\pi} u(x,t)\, e^{-ikx}\, dx = -\frac{k^2}{2} c_k(t)$$

Solving this (as before), $c_k(t) = e^{-\frac{k^2 t}{2}} c_k(0)$, which allows us to rewrite

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} e^{-\frac{k^2 t}{2}} c_k(0)\, e^{ikx}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} e^{-\frac{k^2 t}{2}} \frac{1}{\sqrt{2\pi}} \left(\int_0^{2\pi} f(y)\, e^{-iky}\, dy \right) e^{ikx}$$

$$= \sum_{k=-\infty}^{\infty} e^{-\frac{k^2 t}{2}} \frac{1}{2\pi} \int_0^{2\pi} f(y)\, e^{ik(x-y)}\, dy$$

$$= \int_0^{2\pi} f(y) \left(\sum_{k=-\infty}^{\infty} \frac{e^{-\frac{k^2 t}{2}} e^{ik(x-y)}}{2\pi} \right) dy$$

$$= \int_0^{2\pi} h(x-y,t) f(y)\, dy \qquad \text{the heat kernel}$$

Theorem 1 The heat kernel $h(x,t) > 0$ whenever $t>0$. Therefore, whenever $f(y) \geq 0$, the solution $u(x,t) = \int_0^{2\pi} h(x-y,t) f(y)$ is strictly greater than zero. > 0

To prove this, we will prove an identity that

$$u(x,t) = \sum_n \hat{f}_k \, e^{-\frac{k^2 t}{2}} \frac{e^{ikx}}{\sqrt{2\pi}}$$

$$= \sum_k e^{-\frac{k^2 t}{2}} \int_0^{2\pi} \frac{e^{ik(x-y)}}{2\pi} f(y)\, dy$$

identity (proof to follow)

$$= \sum_n \frac{1}{\sqrt{2\pi t}} \int_0^{2\pi} e^{-(x-y-2\pi n)^2/2t} f(y)\, dy \; .$$

Using this expression, and thinking of $f(y)$ as a function on $\mathbb{R}$ which is periodic

$$u(x,t) = \frac{1}{\sqrt{2\pi t}} \sum_n \int_0^{2\pi} e^{-(x-y-2\pi n)^2/2t} f(y)\, dy$$

$$= \frac{1}{\sqrt{2\pi t}} \sum_n \int_{2\pi n}^{2\pi(n+1)} e^{-(x-y)^2/2t} f(y)\, dy$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/2t} f(y)\, dy$$

which is a much more transparent expression for the heat kernel on $\mathbb{R}$. Clearly $\frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t} > 0$ for $t > 0$, and the periodic heat kernel is a sum of translations of this, therefore it follows that

$$h(x-y,t) > 0 \; .$$

proof of the identity – Jacobi identity for the Θ-function.

<u>Definition 2</u>: $$\Theta(t) = \sum_{k=-\infty}^{+\infty} e^{-\pi k^2 t}$$

<u>Theorem 3</u> (Jacobi's identity) $$\Theta(t) = \frac{1}{\sqrt{t}}\,\Theta(1/t)$$

This is a functional equation that is the analog of many others in analytic number theory.

proof: Define a 2π-periodic function

$$g_t(x) = \sum_{n=-\infty}^{+\infty} e^{-(x-2\pi n)^2/2t},$$

a series which converges uniformly for each $t>0$.

Compute its Fourier series.

$$\hat{g}_k = \int_0^{2\pi} g(x)\,\frac{e^{-ikx}}{\sqrt{2\pi}}\,dx = \sum_{n=-\infty}^{+\infty} \int_0^{2\pi} \frac{e^{-ikx}}{\sqrt{2\pi}}\, e^{-(x-2\pi n)^2/2t}\,dx$$

$$= \sum_{n=-\infty}^{\infty} \int_{2\pi n}^{2\pi(n+1)} e^{-x^2/2t}\,\frac{e^{-ikx}}{\sqrt{2\pi}}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2/2t}\,\frac{e^{-ikx}}{\sqrt{2\pi}}\,dx = \sqrt{2\pi t}\, e^{-k^2 t/2}$$

a calculation to verify below.

For $t>0$ these Fourier coefficients converge rapidly to zero, so the formal Fourier series for $g(x)$ actually converges very well (certainly uniformly)

$$g_t(x) = \sum_{k=-\infty}^{+\infty} \frac{e^{ikx}}{\sqrt{2\pi}}\,\sqrt{2\pi t}\, e^{-k^2 t/2}$$

$$= \sum_{n=-\infty}^{+\infty} e^{-(x-2\pi n)^2/2t}.$$

We have proved that

$$\sum_{k=-\infty}^{\infty} e^{ikx} e^{-k^2t/2} = \frac{\sqrt{2\pi}}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-(x-2\pi n)^2/2t} .$$

Setting $x = 0$, we find $t = 2\pi\tau$

$$\sum_{k=-\infty}^{\infty} e^{-\pi k^2 \tau} = \frac{1 \sqrt{2\pi}}{\sqrt{\tau}_{2\pi}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/\tau} .$$

Verify the above calculation, that

$$\int_{-\infty}^{\infty} e^{-x^2/2t} \frac{e^{-ikx}}{\sqrt{2\pi}} dx = \sqrt{t}\, e^{-k^2t/2} .$$

Denote $f(k) = \int_{-\infty}^{\infty} e^{-x^2/2t} \frac{e^{-ikx}}{\sqrt{2\pi}} dx \qquad k \in \mathbb{R}$.

$$\frac{d}{dk} f(k) = -i \int_{-\infty}^{\infty} x e^{-x^2/2t} \frac{e^{-ikx}}{\sqrt{2\pi}} dx$$

$$= i t \int_{-\infty}^{\infty} \frac{d}{dx}\left(e^{-x^2/2t} \right) \frac{e^{-ikx}}{\sqrt{2\pi}} dx$$

$$= -i t \int_{-\infty}^{\infty} e^{-x^2/2t} \frac{d}{dx}\left(\frac{e^{-ikx}}{\sqrt{2\pi}} \right) dx$$

$$= -k t \int_{-\infty}^{\infty} e^{-x^2/2t} \frac{e^{-ikx}}{\sqrt{2\pi}} dx$$

$$= -k t f(k) .$$

Solve for $f(k) = f(0)\, e^{-k^2t/2}$.

Evaluate

$$f(0) = \int_{-\infty}^{\infty} e^{-x^2/2t} \frac{1}{\sqrt{2\pi}} dx = \sqrt{t} .$$

This goes as follows:

$$f(0)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2/2t} e^{-y^2/2t} \frac{1}{2\pi} dx\,dy$$

$$= \iint e^{-(x^2+y^2)/2t} \frac{1}{2\pi} dx\,dy = \int_0^{2\pi} d\theta \int_0^{+\infty} e^{-r^2/2t} \frac{1}{2\pi} r\,dr = t .$$