(iv) Fourier series on $L^2(\mathbb{T}^1)$

Math 4FT
W. Craig
March 7, 2013

The principal task today is to give coordinates to the Hilbert space

$$L^2(\mathbb{T}^1) = \{ f(x) : \int_0^{2\pi} |f(x)|^2 dx < +\infty \}$$

Recall that this space has a norm (a distance function to zero) given by a scalar product.

$$(f,g) = \int_0^{2\pi} f(x)\, \overline{g(x)}\, dx \quad , \quad \|f\| = \sqrt{\int_0^{2\pi} |f(x)|^2 dx} = \sqrt{(f,f)} .$$

We defined angles in terms of the scalar product

$$\mathrm{re}\,(f,g) = \|f\|\,\|g\| \cos(\theta) .$$

and we say that f and g are orthogonal when $(f,g)=0$.

Def'n 1: A family of functions $\{f_j(x)\}_{j=1}^{\infty}$ is an orthogonal family when any two are orthogonal

$$(f_j, f_k) = 0 \qquad j \neq k .$$

and orthonormal when

$$(f_j, f_j) = \|f_j\|^2 = 1 .$$

Note: As long as no $f_j \equiv 0$, we may always normalize an orthogonal family to be an orthonormal one:

$$f_n(x) \longrightarrow g_n(x) = \frac{1}{\|f_n\|} f_n(x) , \qquad \|g_n\| = 1 .$$

The set of functions $\{ g(x) \in L^2 : \|g\| = 1 \} = S^{\infty}$, the unit sphere of $L^2(\mathbb{T}^1)$. It is not compact.

Note: The L^2-distance between any two orthonormal functions is $\sqrt{2}$: indeed

$$\|g_1 - g_2\|^2 = \left(\int_0^{2\pi} |g_1(x) - g_2(x)|^2 dx \right)$$
$$= (g_1, g_1) - 2\,\mathrm{re}\,(g_1, g_2) + (g_2, g_2) = 2 .$$

Theorem 2: The complex exponentials $\frac{1}{\sqrt{2\pi}} e^{ikx} := e_k(x)$ form an orthonormal family.

proof: integrate. $(e_j, e_k) = \int_0^{2\pi} \frac{1}{2\pi} e^{i(j-k)x}\, dx = \begin{cases} 1 & j=k \\ \left.\dfrac{e^{i(j-k)x}}{2\pi i(j-k)}\right|_{x=0}^{x=2\pi} = 0 & \text{when } j \neq k. \end{cases}$

A family $\{f_j(x)\}_{j=1,2,\dots}$ spans $L^2(\mathbb{T}^1)$ if the finite linear combinations

$$\left\{ \sum_{j=1}^{N} c_j f_j(x) \ : \ c_j \in \mathbb{C} \right\} \quad \text{are dense in } L^2;$$

that is, $\forall f \in L^2\ \forall \varepsilon > 0\ \exists \{c_j\} : \| f - \sum_{j=1}^{N} c_j f_j \| < \varepsilon$.

Definition 3: A basis for L^2 is an orthonormal family which spans $L^2(\mathbb{T}^1)$.

The goal of this next portion is to show that our family given by the Fourier ~~series~~ exponentials $\{\frac{1}{\sqrt{2\pi}} e^{ikx}\}_{k\in\mathbb{Z}} = \{e_k(x)\}_{k\in\mathbb{Z}}$ forms a basis for $L^2(\mathbb{T}^1)$.

Theorem 4 (Least squares approximation) Let $\{g_n\}_{n=1,2,\dots}$ be any orthonormal family in $L^2(\mathbb{T}^1)$. For any $f \in L^2$ and any $N \geq 1$, and any complex vector $c \in \mathbb{C}^N$,

$$\| f - \sum_{j=1}^{N} (f, g_j) g_j \| \leq \| f - \sum_{j=1}^{N} c_j g_j \|.$$

The lower bound given by the LHS is achieved only with $c_j = (f, g_j)$, $\forall 1 \leq j \leq N$.

The coefficients (f, g_j) are called the "generalized Fourier coefficients" of $f(x)$. When $g_j = \frac{1}{\sqrt{2\pi}} e^{ijx}$, then $(f, g_j) = \hat{f}_j$ are the actual Fourier coefficients of $f(x)$.

proof (of Thm 4) Write the RHS as follows

$$\| f - \sum_{j=1}^{N} c_j g_j \|^2 = \| (f - \sum_{j=1}^{N} (f, g_j) g_j) + \sum_{j=1}^{N} ((f,g_j) - c_j) g_j \|^2$$

$$= \| (f - \sum_{j=1}^{N} (f, g_j) g_j) \|^2 + 2\,\mathrm{re}\, (f - \sum_{j=1}^{N} (f,g_j) g_j , \sum_{j=1}^{N} ((f,g_j) - c_j) g_j)$$

$$+ \| \sum_{j=1}^{N} ((f,g_j) - c_j) g_j \|^2 \qquad (*)$$

The middle term is computed as

$$2\,\mathrm{re} \sum_{j=1}^{N} \overline{((f,g_j) - c_j)} \, (f - \sum_{\ell=1}^{N} (f, g_\ell) g_\ell , g_j)$$

$$= 2\,\mathrm{re} \sum_{j=1}^{N} \overline{((f,g_j) - c_j)} \underbrace{\left[(f,g_j) - \sum_{\ell=1}^{N} (f, g_\ell)(g_\ell, g_j) \right]}_{=0}) \qquad = \delta_{\ell j}$$

We are left with

$$\| f - \sum_{j=1}^{N} c_j g_j \|^2 = \| (f - \sum_{j=1}^{N} (f, g_j) g_j) \|^2$$

$$+ \| \sum_{j=1}^{N} ((f,g_j) - c_j) g_j \|^2$$

The last term can be expressed using the Pythagorean theorem

$$0 \le \sum_{j=1}^{N} | (f,g_j) - c_j |^2 .$$

Hence the inequality

$$\| f - \sum_{j=1}^{N} c_j g_j \|^2 \ge \| (f - \sum_{j=1}^{N} (f, g_j) g_j) \|^2 ,$$

with equality only if all $c_j = (f, g_j)$.

Coordinates – Claims: Choosing $\{g_j\}_{j=1}^{\infty}$ an orthonormal family as above, and take any sequence $c = (c_j)_{j=1}^{\infty} \in \ell^2(\mathbb{Z}^+)$, the sequence of partial sums

$$f_N(x) = \sum_{j=1}^{N} c_j g_j(x)$$

is (1) a Cauchy sequence in $L^2(\mathbb{T}^1)$. Hence it converges to some $f \in L^2$,

where $f = \sum_{j=1}^{\infty} c_j g_j$ (2) Furthermore $\| f \|^2 = \sum_{j=1}^{\infty} |c_j|^2 = \| c \|^2_{\ell^2(\mathbb{Z}^+)}$

proof of these statements:

(1) Cauchy: $\| f_N(x) - f_n(x) \|^2 = \| \sum_{j=n+1}^{N} c_j g_j \|^2 = \sum_{j=n+1}^{N} |c_j|^2 \to 0$ with $n, N \to \infty$

(2) The limit $f_n(x) - f(x) = \lim_{N \to \infty} (f_n(x) - f_N(x))$

$$= \lim_{N \to \infty} \sum_{n+1 \le j \le N} c_j g_j(x) \underset{N \to \infty}{\longrightarrow} 0 \text{ in } L^2.$$

(3) Furthermore $\| f \|^2 = \lim_{N \to \infty} \| f_N \|^2 = \lim_{N \to \infty} \sum_{j=1}^{N} |c_j|^2 = \sum_{j=1}^{\infty} |c_j|^2$
$= \| c \|^2_{\ell^2}$.

<u>Theorem 5</u>: Given any <u>orthonormal family</u> $\{ g_j(x) \}_{j=1}^{\infty}$, then

$$0 \le \| f - \sum_{j=1}^{N} (f, g_j) g_j \|^2 = \| f \|^2 - \sum_{j=1}^{N} |(f, g_j)|^2 .$$

This is one way to state <u>Bessel's inequality</u>, that

$$\sum_{j=1}^{N} |(f, g_j)|^2 \le \| f \|^2 \qquad \forall N, \text{ and it holds in the limit as well.}$$

If $\{ g_j(x) \}_{j=1}^{\infty}$ is an <u>orthonormal</u> ~~family~~ <u>basis</u> as well then

$$\sum_{j=1}^{\infty} |(f, g_j)|^2 = \| f \|^2 .$$

<u>proof</u>: We have already done the calculation $(*) \cong$ <u>Bessel's inequality</u>, which shows that LHS = RHS = Thm 5. This holds for any N, and therefore holds for the limit $N \to \infty$.

If $\{ g_j \}_{j=1}^{\infty}$ is an orthonormal <u>basis</u>, then $\forall \varepsilon > 0$ there exist $\{ c_n \}_{n=1}^{N}$ such that

$$\| f - \sum_{j=1}^{N} (f, g_j) g_j \|^2 \le \| f - \sum_{j=1}^{N} c_j g_j \|^2 < \varepsilon ,$$

and we have shown that the LHS $= \| f \|^2 - \sum_{j=1}^{N} |(f, g_j)|^2$.
Therefore as $N \to \infty$ and $\{ c_j \}_{j=1}^{\infty}$ are improved, the two terms of the LHS converge

& so on.

<u>Theorem 6</u> The orthonormal family $\{ e_j(x) = \frac{1}{\sqrt{2\pi}} e^{ijx} \}$ is a basis.

<u>proof</u>: We will argue using the Dirichlet kernel $D_n(x) = \frac{1}{2\pi} \frac{\sin((n+\frac{1}{2})x)}{\sin(x/2)}$, using the theorem of Dirichlet

- For any $f \in C^2(\mathbb{T}^1)$ the partial sums $S_n(f)(x) = \sum_{|k| \le n} \hat{f}_k \frac{e^{ikx}}{\sqrt{2\pi}}$ converge uniformly to $f(x)$ as $n \to \infty$.

The proof is now in two steps. First, we prove that $C^2(\mathbb{T}^1) \subseteq L^2(\mathbb{T}^1)$ and it is dense. Secondly, we show that any function $f \in C^2$ can be approximated (in the L^2 sense) by finite sums of complex exponentials $e_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}$, i.e. an orthonormal family.

Proof of the first fact:

<u>Lemma 7</u> Given $\varepsilon > 0$, for each $f \in L^2$ there is an approximation $g \in C^2$ such that

$$\| f - g \| < \varepsilon / 2 .$$

The proof of this is deferred until we study convolution, in the next section.

Proof of the second fact: Let $f \in C^2$ and let $N \geq 1$ be sufficiently large so that

$$\sup_x | S_n(f)(x) - f(x) | < \frac{\varepsilon}{2\sqrt{2\pi}} \qquad \forall n \geq N .$$

This is possible by Dirichlet's theorem. Then we check the L^2 norm of the difference

$$\| S_n(f) - f \| = \left(\int_0^{2\pi} | S_n(f) - f |^2 dx \right)^{1/2} \le \left(\int_0^{2\pi} \frac{\varepsilon^2}{4(2\pi)} dx \right)^{1/2} = \frac{\varepsilon}{2} .$$

Given these two elements we can complete the proof of Thm 6, in which we need to show that finite linear combinations of the (orthonormal) family $\{e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}\}_{n\in\mathbb{Z}}$ are dense.

Let $\varepsilon > 0$; given any $f \in L^2$ choose $g \in C^2$ such that
$\|f - g\| < \varepsilon/2$,
by Lemma 7. Now approximate $|S_n(g) - g| < \frac{\varepsilon}{2\sqrt{2\pi}}$ by Dirichlet's theorem. Thus

$$\|f - S_n(g)\| \leq \|f - g\| + \|g - S_n(g)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \blacksquare$$

(v) convolution operators

Math 4FF
W. Craig
March 11, 2012

A motivation for the material in this section is to prove Lemma 7 of the last section.

<u>Definition 1</u>: Given periodic functions $f, g \in L^1(\mathbb{T}^1)$ we may define the <u>convolution product</u>

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x-y)\, g(y)\, dy$$

This can be thought of as a "weighted average" of the values of $f(x)$ by the "distribution" $g(x)$.

<u>Proposition 2</u>: (Elementary properties of the convolution product)

(i) $f * g(x) = g * f(x)$ commutivity

(ii) $f \mapsto g * f$ is a linear map in f

(iii) $(f * g) * h = f * (g * h)$ associativity

(iv) Fourier transform $(\widehat{f * g})_k = \hat{f}_k \hat{g}_k$

(v) When both $f, g \in L^1(\mathbb{T}^1)$ then $f * g(x) \in C(\mathbb{T}^1)$ (well, maybe this is not so elementary)

<u>proofs</u>: (i) $(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x-y)\, g(y)\, dy$

setting $y = x - y'$ in a change of variables

$$= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(y')\, g(x-y')\, dy' = (g * f)(x) \qquad \blacksquare$$

(iv) $\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-ikx} \left(\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x-y)\, g(y)\, dy \right) dx$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \left(\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-ik(x-y)} f(x-y)\, dx \right) e^{-iky} g(y)\, dy$$

$$= \hat{f}_k \hat{g}_k \qquad \blacksquare$$

An additional elementary property:

(vi) If $f \in L^1$ and $g \in C^p$ $0 \le p$, then $f * g \in C^p$.

C1

proof of (vi) $\partial_x^p (f * g) = \partial_x^p \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} g(x-y) f(y) dy$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} (\partial_x^p g)(x-y) f(y)\, dy,$$

therefore the case $p \geq 1$ reduces to the case $p = 0$, that is, where $g \in C^0(\mathbb{T}^1)$.

Take such $g \in C^0$; since $\mathbb{T}^1$ is compact, uniform continuity is automatic for continuous functions. That is, given $\varepsilon > 0$, there is $\delta > 0$ such that for any $|x_1 - x_2| < \delta$ then

$$|g(x_1) - g(x_2)| < \varepsilon.$$

Consider

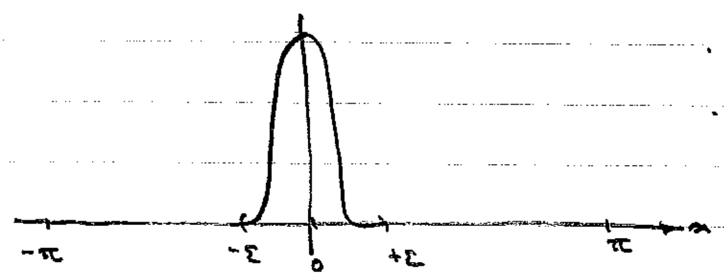
$$|(f * g)(x_1) - (f * g)(x_2)|$$
$$= \left| \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} (g(x_1 - y) - g(x_2 - y)) f(y)\, dy \right|$$
$$\leq \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} |g(x_1-y) - g(x_2-y)|\,|f(y)|\, dy$$

by Cauchy-Schwarz
$$\leq \varepsilon \frac{1}{\sqrt{2\pi}} \left(\int_0^{2\pi} dy \right)^{1/2} \left(\int_0^{2\pi} |f(y)|^2 dy \right)$$
$$\leq \varepsilon \|f\|.$$

Therefore $(f * g)(x)$ is continuous in x. ∎

Approximation of the identity.

Take a special family (or class of families) of C^2 functions; $g_\varepsilon \in C^2(\mathbb{T}^1)$ looking like this



- $g_\varepsilon > 0$
- $\mathrm{supp}(g_\varepsilon) \subset [-\varepsilon, \varepsilon]$
- $\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} g_\varepsilon(x)\, dx = 1$

Note that $\lim_{\varepsilon \to 0} (\hat{g}_\varepsilon)_k = 1$ for all k, indeed

$$\left| \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} g_\varepsilon(x)\, e^{-ikx}\, dx - 1 \right| = \left| \frac{1}{\sqrt{2\pi}} \int_{-\varepsilon}^{\varepsilon} g_\varepsilon(x)\, (e^{-ikx} - 1)\, dx \right|$$

$$\leq \sup_{-\varepsilon < x < \varepsilon} \left| e^{-ikx} - 1 \right|$$ and this is close to zero when $\varepsilon \ll 1/k$.

Handy ⟹ These families are approximations to Dirac-δ-functions.

__Lemma 3__ (translation is continuous in L^2) For any $f \in L^2(\mathbb{T}^1)$

$$\lim_{y \to 0} \| f(\cdot - y) - f(\cdot) \| = \lim_{y \to 0} \left(\int_0^{2\pi} |f(x-y) - f(x)|^2\, dx \right)^{1/2} = 0 .$$

Assume this for a moment, we will finish the proof of last section, that for each $f \in L^2$, $\varepsilon > 0$ there is a C^2 approximation h
$\| h - f \| < \varepsilon$.

We want to show that, using a family g_ε of approximate identities, that

$$\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} g_\varepsilon(x-y) f(y)\, dy \xrightarrow{L^2} f(x) \quad \text{as } \varepsilon \to 0.$$

Take a test function $h(x) \in L^2$, we will show that

$$\int_0^{2\pi} h(x) \left[\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} g(x-y) f(y)\, dy - f(x) \right] dx \to 0 \quad \text{as } \varepsilon \to 0$$

Indeed,

$$\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} h(x) \left[\int_0^{2\pi} g(y) (f(x-y) - f(x))\, dy \right] dx$$

exchanging integration

$$= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} g(y) \left[\int_0^{2\pi} h(x) (f(x-y) - f(x))\, dx \right] dy$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\varepsilon}^{+\varepsilon} g(y)\, \|h\|\, \| f(\cdot - y) - f(\cdot) \|\, dy$$

$$= \|h\|\, \| f(\cdot - y) - f(\cdot) \| \longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

This is good practice, but not quite enough for the result we want.

Now take on

$$\| g_\varepsilon * f - f \|^2 = \int_0^{2\pi} \left| \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} g(x-y)\,(f(y) - f(x))\,dy \right|^2 dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^{2\pi} g(x-y)\,(f(y) - f(x))\,dy \right) \overline{\left(\int_0^{2\pi} g(x-z)\,(f(z) - f(x))\,dz \right)} dx$$

Interchanging integration, and changing variables as above,

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} g(y)\,\overline{g(z)} \left[\int_0^{2\pi} (f(x-y) - f(x))\,\overline{(f(x-z) - f(x))}\,dx \right] dy\,dz$$

$$\le \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} g(y)\,\overline{g(z)}\ \| f(\cdot - y) - f(\cdot) \|\ \| f(\cdot - z) - f(\cdot) \|$$

and this $\to 0$ as $\varepsilon \to 0$ by Lemma 3.

Hence Lemma 3 is used to finish our proof of density of $C^2 \subseteq L^2$ in Lemma 7. of section 3 (iv).

We have to go back to prove Lemma 3 on the continuity of translation. This takes us back to Lebesgue theory.

Approximation theorems for $f \in L^2$ or L^1, by
(1) step functions
(2) continuous functions.

In fact these also hold for any $f \in L^p$ $1 \le p < +\infty$, where

$$L^p = \{ f \text{ measurable} : \underbrace{\left(\int_0^{2\pi} |f(x)|^p dx \right)^{1/p}}_{\| \cdot \|_p} < +\infty \}.$$

Step function: Given $f \in L^p$ $1 \le p < +\infty$ and given $\varepsilon > 0$ there is f_n bounded such that

$$\| f - f_n \|_p < \varepsilon.$$

proof: (we'll do the case $p=2$) Set $f_n(x) = \begin{cases} n & \text{if } f(x) \geq n \\ f(x) & \text{if } -n \leq f(x) \leq n \\ -n & \text{if } f(x) < -n \end{cases}$

The Chebyshev inequality states that
$$m(\{x : |f(x)| \geq n\}) \leq \frac{1}{n^2}\|f\|^2,$$
so $|f(x)|$ is big only on small sets.

The pointwise limit $\lim_{n\to\infty} f_n(x) = f(x)$ a.e.

Since $|f - f_n|^2 \leq |f|^2 := g(x)$ integrable,
by the dominated convergence theorem
$$\lim_{n\to\infty} \int_0^{2\pi} |f - f_n|^2 \, dx = 0.$$

This is to say that $f_n \xrightarrow{L^2} f$, as $n \to \infty$.

Now choose a step function φ such that $|f_n(x) - \varphi(x)| < \varepsilon$, $|\varphi| < n$
except on a set A of measure $m(A) < \delta$;
(This is a task for a measure theory class).
Then
$$\|f_n - \varphi\|^2 = \int_0^{2\pi} |f_n(x) - \varphi(x)|^2 \, dx$$
$$= \int_A |f_n(x) - \varphi(x)|^2 dx + \int_{[0,2\pi)\setminus A} |f_n(x) - \varphi(x)|^2 \, dx$$
$$\leq 2\varepsilon^2 n^2 + 2\delta n^2,$$
which can be made arbitrarily small.

Now we can compare translates.
$$\|f(\cdot - y) - f(\cdot)\| \leq \|f(\cdot - y) - f_n(\cdot - y)\| + \|f_n(\cdot - y) - f_n(\cdot)\| + \|f_n(\cdot) - f(\cdot)\|$$
$$\|f_n(x-y) - f_n(x)\| \leq \|f_n(x-y) - \varphi(x-y)\| + \|\varphi(x-y) - \varphi(x)\| + \|\varphi(x) - f_n(x)\|$$

c6,

Finally we touch upon the crux of the matter, which is to show that

$$\| \varphi(x-y) - \varphi(x) \| < \varepsilon \quad \text{for } |y| < \delta.$$

But $\varphi(x)$ is a step function, a finite linear combination of constant functions over intervals.

$$\varphi_j(x) = c_j \chi_{[a_j, b_j]}$$

so that

$$\int | \varphi_j(x-y) - \varphi_j(x) |^2 dx = |c_j|^2 \int | \chi_{[a_j,b_j]}(x-y) - \chi_{[a_j,b_j]}(x) |^2 dx$$

$$= |c| \int (\chi^2_{[a_j-y, a_j]} + \chi^2_{[b_j-y, b_j]}) dx \leq 2|c||y|. \blacksquare$$

We still have one item to discuss in this section, which is statement (ii) of Proposition 2; namely, if $f, g \in L^1$ then $f * g(x) \in C(\mathbb{T}^1)$.

Firstly, we can quite easily show that whenever $f, g \in L^1$ then $f * g \in L^1$. Indeed

$$\| f*g \|_{L^1} = \int_0^{2\pi} | f*g(x) | dx = \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} \left| \int_0^{2\pi} f(x-y) g(y) dy \right| dx$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \int_0^{2\pi} | f(x-y) | | g(y) | dx\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \left(\int_0^{2\pi} | f(x-y) | dx \right) | g(y) | dy$$

$$= \frac{1}{\sqrt{2\pi}} \| f \|_{L^1} \| g \|_{L^1} .$$

We could say that L^1 is an algebra of functions, with addition and convolution product as the two operations.

The statement of Proposition 2a is that more is known, that in fact $f * g$ is continuous, not simply integrable. Here is a sketch of a proof.

$f(x)$ and $g(x)$ $\in L^1$
therefore there are bounded
$f_n(x)$ and $g_n(x)$ with $\int |f - f_n| \, dx$ and $\int |g - g_n| \, dx < \varepsilon/3$

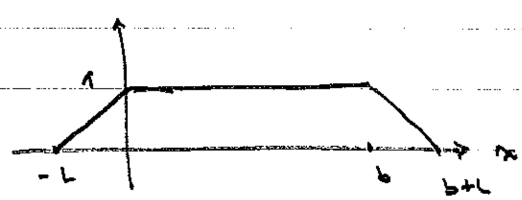
Next, there are step functions $\varphi(x)$ and $\psi(x)$ such that
$\int_0^{2\pi} |f_n(x) - \varphi(x)| \, dx < \varepsilon/3$ and $\int_0^{2\pi} |g_n(x) - \psi(x)| \, dx < \varepsilon/3$

Each of $\varphi(x)$ and $\psi(x)$ is a linear combination of ~~step~~ indicator functions over intervals $\varphi_j(x) = c_j \chi_{[a_j, b_j]}$, $\psi_j(x) = d_j \chi_{[e_j, f_j]}$.

The crux is therefore, how do convolutions of indicator functions of intervals behave.

$$\int_0^{2\pi} \chi_{[a,b]}(x-y) \, \chi_{[e,f]}(y) \, dy$$

Q: how does this behave? Take $[a,b] = [0,b]$, and $L = |[e,f]| < b$ (WLOG), then here it is explicitly:



, (students should check).

(3)(vi) differential operators

Math 4 FT
W. Craig
Th. March 14, 2013

Both differentiation and convolution are operations which respect translation, and therefore are nicely compatible with the Fourier transform.

Basic properties (review

Proposition 1

(i) $\widehat{(\frac{d}{dx}f)}_k = ik\hat{f}_k$

(ii) $\widehat{(e^{i\ell x}f)}_k = \hat{f}_{k-\ell}$

(iii) $\widehat{(f*g)}_k = \hat{f}_k\,\hat{g}_k$

(iv) $\widehat{(fg)}_k = \frac{1}{\sqrt{2\pi}}\sum_{\ell\in\mathbb{Z}} \hat{f}_{k-\ell}\,\hat{g}_\ell \;:= (\hat{f}*\hat{g})_k$

We have seen (i) (iii) and (iv) previously. Here is the proof of (ii)

(ii) $\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}(f(x)\,e^{i\ell x})e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} f(x)\,e^{-i(k-\ell)x}\,dx = \hat{f}_{k-\ell}.$

These properties allow us to give explicit solutions to a number of partial differential equations.

Let $P(\partial_x)$ be a differential operator

$$P(\partial_x)u = \sum_{j=0}^{m} p_j\,\partial_x^j u = f$$

Under Fourier transform

$$\widehat{(P(\partial_x)u)}_k = \sum_{j=0}^{m} p_j\,(ik)^j\,\hat{u}_k = \hat{f}_k$$

Suppose that P is *elliptic*, meaning that

$$\left| \sum_{j=0}^{m} p_j (ik)^j \right| \geq \Lambda (|k|^m + 1)$$

{ for convenience, this is stronger than the usual definition.

Then $P(\partial_x)u = f$ can be solved by the Fourier transform and algebra.

$$\sum_{j=0}^{m} p_j (ik)^j \hat{u}_k = \widehat{(P(\partial_x)u)}_k = \hat{f}_k \qquad \hat{u}_k = \frac{\hat{f}_k}{\sum_{j=0}^{m} p_j (ik)^j}$$

Examples:

1) Schrödinger operator $\left(-\frac{d^2}{dx^2} + Q\right)u = f$
take $Q > 0$ a constant

2) $\left(-\frac{d^2}{dx^2} + Q\right)u = \lambda u$ spectral problem w. periodic boundary conditions
under Fourier transform

$$\{-(ik)^2 + Q\}\hat{u}_k = \lambda \hat{u}_k$$

whose solutions are $\lambda_k = (k^2 + Q)$ eigenvalues

$$u_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}}, \quad \frac{e^{-ikx}}{\sqrt{2\pi}}$$

eigenfunctions

The spectrum consists of

$$\lambda_k = (k^2 + Q) \qquad k = 0, 1, 2 \ldots$$

with λ_0 multiplicity 1
λ_k multiplicity 2.

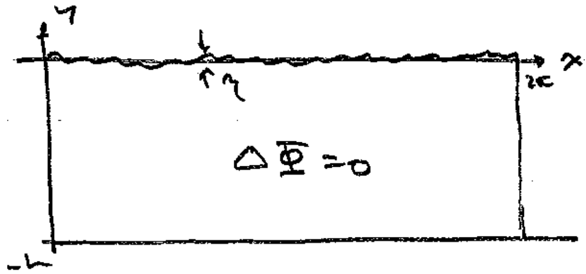
With different boundary conditions — $u(0) = 0 = u(\pi)$, the spectrum is different

$$u_j(x) = \frac{\sin(\frac{j}{2}x)}{\sqrt{\pi/2}} \qquad j = 1, 2, 3$$

$\lambda_j = 1, 2, 3 \ldots$ with multiplicity 1.

This procedure extends to construction of the Green's function.

A more unusual case would be to ocean waves.



velocity potential Φ,
$\vec{u}(x,y) = \nabla \Phi$

- bottom boundary condition
$\partial_y \Phi(x,-h) = 0.$
- lateral side boundary condition
$\Phi(x+2\pi, y) = \Phi(x,y)$

Top $y=0$ $\quad 0 \le x < 2\pi$
boundary condition
$\partial_y \Phi = \omega \eta$
$g \eta = -\omega \Phi$

ω = temporal frequency of surface waves

Reduce in Fourier series in x.

$$\Phi(x,0) = \sum_{k \in \mathbb{Z}} \xi_k \frac{e^{ikx}}{\sqrt{2\pi}} \qquad \eta(x) = \sum_{k \in \mathbb{Z}} \eta_k \frac{e^{ikx}}{\sqrt{2\pi}}$$

The harmonic extension of $\Phi(x,0)$ to $\Phi(x,y)$ is

$$\Phi(x,y) = \sum_{k \in \mathbb{Z}} \xi_k \frac{e^{ikx}}{\sqrt{2\pi}} \frac{\cosh(k(y+h))}{\cosh(kh)}$$

$$\partial_y \Phi(x, y=0) = \sum_{k \in \mathbb{Z}} \xi_k \frac{e^{ikx}}{\sqrt{2\pi}} \frac{k \sinh(kh)}{\cosh(kh)} = \sum_{k \in \mathbb{Z}} \xi_k \, k \tanh(kh) \, e^{ikx}$$

We find the differential operator expressed as follows

$$\begin{pmatrix} \partial_y \Phi \\ g \eta \end{pmatrix} = \begin{pmatrix} \omega \eta \\ -\omega \Phi \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} k \tanh(hk) \xi_k &= \omega \eta_k \\ g \eta_k &= -\omega \xi_k \end{aligned}$$

and solving for η_k =

$$g k \tanh(hk) \, \eta_k = -\omega_k^2 \eta_k .$$

That is, the normal mode frequencies are the eigenvalues, given by the dispersion relation $\omega^2 = g k \tanh(hk)$.

(vii) Fourier series on the torus $\mathbb{T}^d$.

We have been working on the circle $\mathbb{T}^1$, for which we identified the interval $[0, 2\pi)$ with periodic boundary conditions, and considered periodic functions

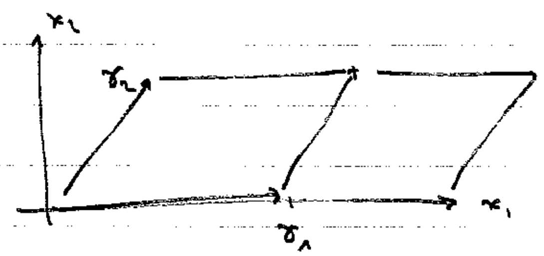
$$f(x+2\pi j) = f(x) \qquad \text{for all } j \in \mathbb{Z}, \ x \in \mathbb{R}^1.$$

One can generalise this to the d-dimensional torus as follows. Let $x \in \mathbb{R}^d$, and consider multiply-periodic functions

$$f(x+\gamma) = f(x) \qquad \text{for all } \gamma \in \Gamma,$$

where Γ is a lattice in $\mathbb{R}^d$.

Lattice



generators $\gamma_1, \gamma_2 \dots \gamma_d$ (a basis, such that $\det(\gamma_1, \dots \gamma_d) \neq 0$)
(these are not uniquely defined)

Then $\Gamma = \{ \gamma = j_1 \gamma_1 + \dots + j_d \gamma_d \ ; \ j \in \mathbb{Z}^d \}$

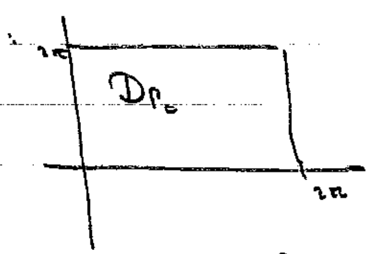
The standard lattice $\Gamma_0 = \{ 2\pi (j_0, j_1, \dots j_d) \ ; \ \text{each } j_i \in \mathbb{Z} \}$

A <u>fundamental domain</u> for this lattice is given by

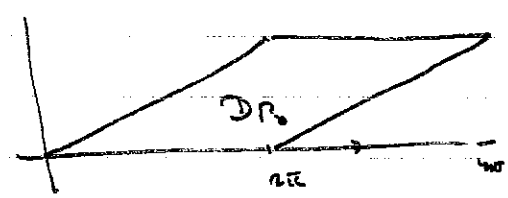
$$D_\Gamma = \{ x \in \mathbb{R}^d : \quad x = \theta_1 \gamma_1 + \theta_2 \gamma_2 + \dots \theta_d \gamma_d \ ; \ \theta_j \in [0,1) \}$$

volume $|\Gamma|$ of the domain. For example $|\Gamma_0| = (2\pi)^d$.

examples:



but also



so not unique.

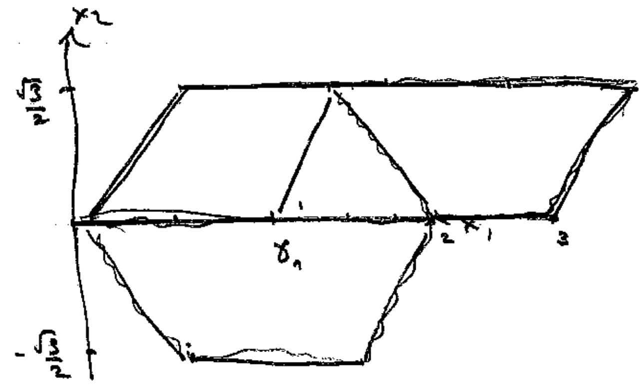
It suffices to know $f(x)$ on D_Γ in order to know it for all $x \in \mathbb{R}^d$

51

It is true that every periodic pattern can be described by such a lattice and fundamental domain.

example: hexagonal lattice

$$\gamma_1 = (1, 0)$$
$$\gamma_2 = (1/2, \sqrt{3}/2)$$



$1 = \frac{1}{4} + \frac{3}{4}$

Periodic functions are those such that
$$f(x+\gamma) = f(x) \qquad \text{for all } \gamma \in \Gamma.$$
For such functions, knowing their values on D_Γ suffices to know them for all x, hence we identify D_Γ with $\mathbb{T}^d$.

Fourier series representation. Consider the complex exponentials
$$e^{ih\cdot x} \qquad h = (h_1, \cdots, h_d) \in \mathbb{R}^d \qquad x = (x_1, \cdots x_d) \in \mathbb{R}^d$$
If this is to be periodic on $D_\Gamma = \mathbb{T}^d$, we need that
$$e^{ih\cdot(x+\gamma)} = e^{ih\cdot x} \qquad \text{for all } x, \text{ and for all } \gamma \in \Gamma.$$
Therefore h must satisfy
$$h\cdot\gamma = 2\pi n \qquad \text{for some } n \in \mathbb{Z}, \text{ for all } \gamma \in \Gamma.$$
The set of such h forms another lattice, $\Gamma' :=$ the dual lattice.

The following functions form a basis for $L^2(\mathbb{T}^d)$:
$$\left\{ \frac{e^{ih\cdot x}}{|\Gamma|^{1/2}} : h \in \Gamma' \right\}.$$

For example, the dual basis for the standard lattice

$$\Gamma_0 = \{ (2\pi j_1, 2\pi j_2, \dots 2\pi j_d) \quad (j_1, \dots j_d) \in \mathbb{Z}^d \}$$

The dual lattice is $\Gamma_0' = \{ (h_1, \dots h_d) : h_\ell \in \mathbb{Z} \}$.
Its volume $|\Gamma_0'| = (2\pi)^d$.

Fourier transform $\quad \hat{f}_k = \frac{1}{|\Gamma|^{1/2}} \int_{\mathbb{T}^d} f(x)\, e^{-ik\cdot x}\, dx \quad$ each $k \in \Gamma'$

Theorem (Fourier inversion formula)

$$f(x) = \sum_{k \in \Gamma'} \hat{f}_k \frac{e^{ik\cdot x}}{|\Gamma|^{1/2}}$$

where $\hat{f}_k = \frac{1}{|\Gamma|^{1/2}} \int_{\mathbb{T}^d} f(x)\, e^{-ik\cdot x}\, dx$.

Plancherel identity $\quad \int_{\mathbb{T}^d} |f(x)|^2 dx = \sum_{k \in \Gamma'} |\hat{f}_k|^2$.