Math 4 FT
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Section 4 Applications of Fourier series.

(i) Wirtinger's inequality.

Theorem 1: Let $I = [a,b]$, and consider $f \in C^1([a,b])$, with $f(a) = 0 = f(b)$. Then

$$\int_a^b |f(x)|^2 \, dx \le \left(\frac{b-a}{\pi}\right)^2 \int_a^b \left|\frac{\partial f}{\partial x}\right|^2 dx ,$$

and the constant is sharp.

proof: Translate and dilate the interval to be $[0,\pi]$, then the equivalent inequality reads that

$$\int_0^\pi |f(x)|^2 \, dx \le \int_a^b \left|\frac{\partial f}{\partial x}\right|^2 dx .$$

Extend $f(x) = -f(-x)$ to be odd and periodic on $[-\pi,\pi]$. In particular

$$\hat f_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) \, dx = 0 .$$

By the Plancherel theorem

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{k \in \mathbb{Z}} |\hat f_k|^2$$

$$\int_{-\pi}^{\pi} |\partial_x f(x)|^2 \, dx = \sum_{k \in \mathbb{Z} \setminus \{0\}} |k \hat f_k|^2 ,$$

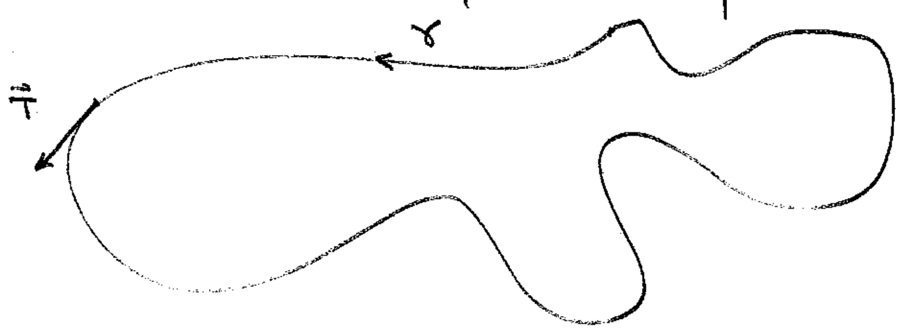
and now the inequality of Wirtinger becomes obvious, namely

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{k \in \mathbb{Z} - \{0\}} |\hat f_k|^2 \le \sum_{k \in \mathbb{Z} \setminus \{0\}} |k \hat f_k|^2 = \int_{-\pi}^{\pi} |\partial_x f(x)|^2 \, dx$$

The inequality is sharp when $\hat f_k = 0$ for all $k \neq 1$; namely for $a \sin(x)$.

(ii) Isoperimetric inequality

Consider a simple closed plane curve γ



Theorem 2 (Isoperimetric inequality) Suppose that γ is a simple closed plane curve of length $= L$. Then the curve that encloses the largest area is the circle.

proof (Hurwitz): Parametrise the plane curve by $(x(t), y(t))$, $0 < t < 2\pi$, such that

- $x(0) = x(2\pi)$ and $y(0) = y(2\pi)$ (closed)
- $x(t)$ and $y(t) \in C^1$ piecewise smooth
- $(x(t_1), y(t_1)) \neq (x(t_2), y(t_2))$ for $t_1 \neq t_2$ (simple) unless $t_1 = 0$ $t_2 = 2\pi$ or vice versa.
- Unit arc length
$$L = \int_0^{2\pi} \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\, dt = 2\pi$$

The theorem states that if $L = 2\pi$ (length of circle $= 2\pi R$, so $R = 1$) then the enclosed area A satisfies
$$\pi = \pi R^2 \geq A,$$
with equality only if γ is a circle.

Modify the parametrization of the circle so that it is arc length; that is $\dot{x}(t)^2 + \dot{y}(t)^2 = 1$ $\forall t$,
$$t = \int_0^t \sqrt{\dot{x}^2 + \dot{y}^2}\, ds.$$

There is an integral formula for area, which is that

$$A = \frac{1}{2}\int_0^{2\pi} (x(t)\,\dot{y}(t) - \dot{x}(t)\,y(t))\,dt$$

proof of this fact: Consider the vector field $V = (-y, x)$ in $\mathbb{R}^2$. Stokes theorem states that

$$\iint_{\Omega} \nabla \times V \,dx\,dy = \int_{\gamma} V\cdot T \,ds_t$$

The curl of $V = (-y, x)$ is $\partial_x V_2 - \partial_y V_1 = 2$.
The tangent vector to the curve is $T = (\dot{x}(t), \dot{y}(t))$.

Hence

$$2A = \iint_{\Omega} 2\,dx\,dy = \iint_{\Omega} \nabla\times V\,dx\,dy$$
$$= \int_{\gamma} V\cdot T\,ds_t = \int_0^{2\pi}(x\dot{y} - y\dot{x})\,dt$$

Now expand $x(t)$, $y(t)$ in Fourier series

$$x(t) = \sum_{n\in\mathbb{Z}} \hat{x}_n \frac{e^{int}}{\sqrt{2\pi}} \qquad \dot{x}(t) = \sum_{n\in\mathbb{Z}} in\,\hat{x}_n \frac{e^{int}}{\sqrt{2\pi}}$$

$$y(t) = \sum_{n\in\mathbb{Z}} \hat{y}_n \frac{e^{int}}{\sqrt{2\pi}} \qquad \dot{y}(t) = \sum_{n\in\mathbb{Z}} ik\,\hat{y}_k \frac{e^{int}}{\sqrt{2\pi}}$$

Using this to express the length

$$L = 2\pi = \sum_{n\in\mathbb{Z}} \int_0^{2\pi}(\dot{x}^2 + \dot{y}^2)\,dt = \sum_{n\in\mathbb{Z}} n^2(|\hat{x}_n|^2 + |\hat{y}_n|^2)$$

$$A = \frac{1}{2}\sum_{n\in\mathbb{Z}} \int_0^{2\pi}(x(t)\,\dot{y}(t) - \dot{x}(t)\,y(t))\,dt$$
$$= \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\hat{x}_n\overline{(in\hat{y}_n)} - in\,\hat{x}_n\overline{\hat{y}_n}\right) = -\sum_{n\in\mathbb{Z}} in\,\hat{x}_n\overline{\hat{y}_n}$$

Writing $(\frac{1}{2}L - A) = \sum_{n\in\mathbb{Z}} \frac{n^2}{2}(|\hat{x}_n|^2 + |\hat{y}_n|^2) + in\,\hat{x}_n\overline{\hat{y}_n}$.

Claim: this expression is nonnegative.

To prove the claim, write $\hat{x}_n = a_n + ib_n$
$\hat{y}_n = c_n + id_n$,
where a_n, b_n, c_n and d_n are real.

Since $x(t)$ and $y(t)$ are real functions, this means that
$a_n = a_{-n}$, $b_n = -b_{-n}$, $\hat{x}_{-n} = \overline{\hat{x}_n}$
$c_n = c_{-n}$ $d_n = -d_{-n}$ $\hat{y}_{-n} = \overline{\hat{y}_n}$.

Writing out $(\frac{1}{2}L - A) = \sum_{n\in\mathbb{Z}} \frac{n^2}{2}(a_n^2 + b_n^2 + c_n^2 + d_n^2)$

$+ n\left(-(a_n d_n + b_n c_n) + i(\cancel{a_n c_n - b_n d_n})\right)$

by the reality condition, the term $n(a_n c_n - b_n d_n)$ is odd in n.

Rewrite the sum as
$$(\tfrac{1}{2}L - A) = \frac{1}{2}\sum_{n\in\mathbb{Z}} (n a_n - d_n)^2 + (n b_n - c_n)^2 + (n^2-1)(d_n^2 + c_n^2).$$

This sum has only non negative terms. Indeed
$k=0$: $d_0^2 + c_0^2 - (d_0^2 + c_0^2) = 0$
$|k| \geq 1$: sum of positive quantities.

Hence always
$\frac{1}{2}L \geq A$

When does the case of equality occur?
$k=0$: always vanishes.
$|n| = 1$: zero when $a_1 = d_1$ and $b_1 = c_1$
$|n| \geq 2$ $a_n = b_n = c_n = d_n = 0$.

Therefore $x(t) = a_0 + a_1\cos(t) - b_1\sin(t)$ with $c_1 = b_1$,
$y(t) = c_0 + c_1\sin(t) + d_1\cos(t)$ and $a_1^2 + d_1^2 = 1$ $a_1 = d_1$, ∎

(iii) equidistribution of irrational rotations.

Consider a simple dynamical system, consisting of the circle $\mathbb{T}^1$ and a mapping.

$$x_0 \in \mathbb{T}^1 \to x_0 + \gamma := x_1 \pmod{2\pi}$$

where $\gamma \in [0, 2\pi)$ is the rotational 'frequency'. The orbit of x_0 is the sequence

$$x_0, \quad x_1 = x_0 + \gamma \pmod{2\pi}, \quad x_2 = x_0 + 2\gamma \pmod{2\pi} \text{ etc.}$$

An observable is a L^1 function $f(x)$ on $\mathbb{T}^1$ (we will take $f \in C^2$ for technical reasons).

The "time average" of the observable $f(x)$ on an orbit is the following.

$$\lim_{n\to\infty} \left(\frac{1}{n} \sum_{j=0}^{n-1} f(x_j) \right)$$

The "phase space average" of $f(x)$ is simply

$$\frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx \, .$$

Theorem 3 (H. Weyl) If $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ (is irrational) then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(x_j) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx$$

for almost all x_0.

This theorem exhibits the ~~truth~~ validity for this system of one of the central postulates of statistical mechanics, due to Boltzmann and Gibbs, namely the ergodic principle.

Namely – the time average of $\overset{\text{observables of a}}{\text{a}}$ "disordered" system should be equal to their phase space averages (or ensemble averages).

<u>proof</u> (of Theorem 3): First try out the result for the Fourier characters

$$e_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}} .$$

case $k=0$: $\quad e_0(x) = \frac{1}{\sqrt{2\pi}}$

$$\cdot \ \frac{1}{n}\sum_{j=0}^{n-1} e_0(x_j) = \frac{1}{n}\sum_{j=0}^{n-1}\frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}\left(\cancel{\frac{n}{n}}\right)$$

$$\cdot \ \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\sqrt{2\pi}}\,dx = \frac{1}{\sqrt{2\pi}} .$$

case $k \neq 0$: $\quad e_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}}$, so $\frac{e^{ik\gamma}}{\sqrt{2\pi}} \neq 1$.

$$\cdot \ \frac{1}{n}\sum_{j=0}^{n-1}\frac{e^{ikx_j}}{\sqrt{2\pi}} = \frac{1}{n}\sum_{j=0}^{n-1}\frac{e^{ik(x_0+j\gamma)}}{\sqrt{2\pi}}$$

$$= \frac{e^{ikx_0}}{\sqrt{2\pi}}\frac{1}{n}\left(\sum_{j=0}^{n-1} e^{ik(j\gamma)}\right)$$

$$= \underbrace{\frac{e^{ikx_0}}{\sqrt{2\pi}\,n}}\left(\frac{e^{ik(n\gamma)}-1}{e^{ik\gamma}-1}\right) \longrightarrow 0$$

as $n \to \infty$, and uniformly in x_0.

$$\cdot \ \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{ikx}}{\sqrt{2\pi}}\,dx = 0 .$$

The examples work.

Now consider a general $f \in C^2(\mathbb{T}^1)$, and approximate it uniformly by trigonometric polynomials; $N = N(\varepsilon)$,

$$f_N(x) = \sum_{|k|\le N}\hat{f}_k\frac{e^{ikx}}{\sqrt{2\pi}}, \quad |f_N(x) - f(x)| < \varepsilon/3$$

Then $\lim_{n\to\infty} \sup_{x_0} \left| \frac{1}{n}\sum_{j=0}^{n-1} f(x_j) - \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \right|$

$$\le \lim_{n\to\infty} \sup \left| \frac{1}{n}\left(\sum_{j=0}^{n-1} f(x_j) - f_N(x_j) \right) \right|$$

$$+ \lim_{n\to\infty} \sup \left| \frac{1}{n}\sum f_N(x_j) - \frac{1}{2\pi}\int_0^{2\pi} f_N(x)\,dx \right|$$

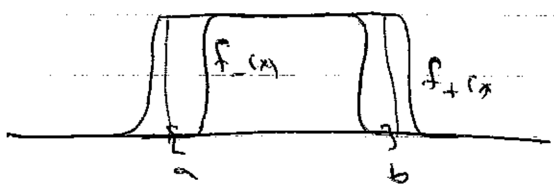
$$+ \lim_{n\to\infty} \sup \left| \frac{1}{2\pi}\int_0^{2\pi} f_N(x) - f(x)\,dx \right|$$

$$\le \frac{\varepsilon}{3} + \text{something small by our examples} + \frac{\varepsilon}{3} \quad \square$$

Corollary 4. Consider an interval $[a,b] \subseteq \mathbb{T}^1$. As $n\to\infty$,

$$\lim_{n\to\infty} \frac{1}{n} \# \{ j < n : a \le x_j \le b \} = \frac{b-a}{2\pi}$$

proof. Approximate $\chi_{[a,b]}(x)$ by C^1 functions f_N, and ~~then~~ before the limit, use Theorem 3.



$$\frac{1}{2\pi}\int_0^{2\pi} f_-(x)\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f^-(x_j) \le \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \chi_{[a,b]}(x_j)$$

$$\le \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \chi_{[a,b]}(x_j)$$

$$\le \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f_+(x_j) = \frac{1}{2\pi}\int_0^{2\pi} f_+(x)\,dx$$

Taking $f_-(x) \le \chi_{[a,b]}(x) \le f_+$ and a limit $\| f_+ - \chi_{[a,b]} \|_{L^1} \to 0$ $\| \chi_{ab} - f_- \|_{L^1} \to 0$, we are done $\square$

W1 recurrence of random walks

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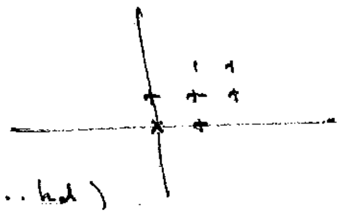
This topic is a beautiful application of Fourier series to the question of recurrence of random walks. The argument was discovered by Polya (1921).

- A random walk: a particle moves by steps of length 1 in the lattice $\mathbb{Z}^d$, one step per unit of time.

$\mathbb{Z}^d$ = ($d=1$ = $\mathbb{Z}$) ($d=2$ =



)

$d=3$

$h=(h_1, h_2, \dots h_d)$
$h_j \in \mathbb{Z}$ for arbitrary d.

Let us take basis vectors (or generators) for $\mathbb{Z}^d$

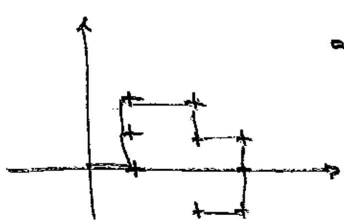
$$e_1 = (1,0,\cdots 0), \quad e_2 = (0,1,0\cdots), \quad e_d = (0,0\cdots,1)$$

Position $q(t)$ of our random particle at time $t=1$ is $q(t) = \pm e_j = a_1$, where $j = 1$ or $2 \dots d$.

At time $t=n$ ($t \in \mathbb{N}$ in this problem)

$$q(t) = a_1 + a_2 + \cdots + a_n$$

where each time step; a_j = one of $\{\pm e_1, \pm e_2 \cdots \pm e_d\}$ and each step is taken with equal probability $\frac{1}{2d}$, and the steps are chosen independently.



$t=9$

That is, $P(a_1 = e_{j_1}$ and $a_2 = e_{j_2}$ and $\dots a_n = e_{j_n})$

$$= P(a_1 = e_{j_1}) \times P(a_2 = e_{j_2}) \times \cdots \times P(a_n = e_{j_n}) = \frac{1}{(2d)^n}.$$

The position of the particle $a_j(t)$ at time t is $P_n^t = a_1 + a_2 + \cdots a_n$.

The question we are asked about is to compute $P(p_n = k)$, for some lattice site k, and to consider this for $n \to \infty$ large.

Polya's idea, take the probabilities $P(p_n = k)$ as Fourier coefficients $\hat{f}_k$ of a function $f(x) \in L^2(\mathbb{T}^d)$; using multidimensional Fourier series.

$$f(x) = \sum_{k \in \mathbb{Z}^d} \frac{1}{\sqrt{2\pi}^d} P(p_n = k) \; e^{ik\cdot x} \qquad x \in \mathbb{T}^d$$

$$:= \frac{1}{\sqrt{2\pi}^d} E(e^{ip_n \cdot x}) \qquad \text{the expected value of } e^{ik\cdot x}.$$

This expected value can be computed by independence

$$\sqrt{2\pi}^d f(x) = E(e^{ip_n\cdot x}) = E\left(\prod_{j=1}^{n} e^{ia_j\cdot x} \right)$$

since $p_n = a_1 + \cdots + a_n$

$$= \prod_{j=1}^{n} E(e^{ia_j\cdot x})$$

$$= \prod_{j=1}^{n} \left(\sum_{e=1}^{d} \left(\frac{1}{2d} e^{ie_e\cdot x} + \frac{1}{2d} e^{-ie_e\cdot x} \right) \right)$$

$$= \left(\frac{1}{d} \sum_{e=1}^{d} \left(\frac{e^{ix_e} + e^{-ix_e}}{2} \right) \right)^n$$

$$= \Big[\underbrace{\frac{1}{d} (\cos(x_1) + \cos(x_2) + \cdots + \cos(x_d))}_{F_d(x)} \Big]$$

$$= (F_d(x))^n \qquad \text{defining } F_d(x) \text{ as being this sum of cosines.}$$

Note that $|F_d(x)| \le 1$.

Taking the Fourier transform,

$$P(p_n = k) = \hat{f}_k = \frac{1}{\sqrt{2\pi}^d} (\widehat{F_d^n})_k = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} (F_d(x))^n e^{-ik\cdot x} dx$$

Our object is to calculate this quantity, which is explicit. The case $h=0$ is particularly interesting, having to do with the probability of returning to $q(T) = 0$ for some finite time $T > 0$.

$$P(p_n = 0) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} (F_d(x))^n \, dx,$$

this is the probability of the particle $q(T) = 0$ when $T = n$.

The expected number of times this particle visits the origin in the future is going to be

$$\sum_{n=0}^{+\infty} P(p_n = 0) = \lim_{\varepsilon \nearrow 1} \sum_{n=0}^{+\infty} \varepsilon^n P(p_n = 0)$$

$$= \lim_{\varepsilon \nearrow 1} \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \left(\sum_{n=0}^{\infty} \varepsilon^n F_d^n(x) \right) dx$$

$$= \lim_{\varepsilon \nearrow 1} \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{1}{1 - \varepsilon F_d(x)} \, dx$$

Using the monotone convergence theorem,

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{1}{1 - F_d(x)} \, dx.$$

The key point is to study $1 - F_d(x) = 1 - \frac{1}{d}[\cos(x_1) + \cos(x_2) + \cdots + \cos(x_d)]$ near $x = 0$.

$$(1 - F_d(x)) = 1 - \frac{1}{d}\left[\left(1 - \frac{x_1^2}{2}\right) + \left(1 - \frac{x_2^2}{2}\right) + \cdots \left(1 - \frac{x_d^2}{2}\right) + o(|x|^2)\right]$$

$$= \frac{1}{2d}(x_1^2 + x_2^2 + \cdots + x_d^2) + o(|x|^2)$$

Therefore the integral

$$\int_{\mathbb{T}^d} \frac{1}{(1 - F_d(x))} \, dx \sim \int_{\mathbb{T}^d} \frac{1}{\frac{1}{2d}|x|^2} \, dx.$$ How does this behave?

$$\int_{\mathbb{T}^d} \frac{2d}{|x|^2}(1 + o(1))\, dx \simeq 2d \iint \frac{1}{r^2} r^{d-1}\, dr\, dS_\theta$$

$$= +\infty \quad \text{when } d = 1, 2$$

$$< +\infty \quad \text{when } d = 3, 4, 5 \text{ etc.}$$

Conclusion: When $d = 1$ or 2, then $\sum_{n=0}^{\infty} P(p_n = 0) = +\infty$,

that is, the number of visits of $z(t)$ to $x = 0$ is expected to be infinite. The random path described by $z(t)$ is recurrent.

When $d = 3, 4, 5 \cdots$, $\sum_{n=0}^{\infty} P(p_n = 0) < +\infty$,

that is, the number of visits of a random path $z(t)$ to $k = 0$ is finite, (and expected in fact to be zero), this is non recurrence.

Restating this: $P(p_n = 0 \text{ i.o.}) = 1$ for $d = 1, 2$
$P(p_n = 0 \text{ i.o.}) = 0$ for $d \geq 3$.

Since $k = 0$ is not a special point, we could make the same asymptotic conclusion for any point $k \in \mathbb{Z}^d$. Hence we conclude that typical random paths leave any compact set in large time.