(ii) L^1 Fourier series

Math 4FT
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Mon Apr 8

Fourier series and its convergence properties in L^2 turned out to be essentially linear algebra, although in a space of infinite dimension. We turn now to study those periodic functions which are simply integrable – $L^1(\mathbb{T}^1)$.

$$f \in L^1 := \left\{ f(x+2\pi) = f(x) : \int_0^{2\pi} |f(x)|\,dx < +\infty \right\}$$

As we have discussed, L^1 is also a Banach space, a linear space with a norm

$$\|f\|_1 = \int_0^{2\pi} |f(x)|\,dx$$

where $\|\alpha f + \beta g\|_1 \leq |\alpha| \int |f|\,dx + |\beta| \int |g|\,dx = |\alpha|\,\|f\|_1 + |\beta|\,\|g\|_1$

Furthermore it is a complete space:

If $f_n(x)$ is a L^1-Cauchy sequence, namely $\forall \varepsilon > 0$ $\exists N = N(\varepsilon)$ such that for any $m, n \geq N$,

$$\|f_m - f_n\|_1 = \int_0^{2\pi} | f_m(x) - f_n(x) |\,dx < \varepsilon,$$

then there exist a limiting function $f \in L^1$ such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^1} = \lim_{n\to\infty} \int_0^{2\pi} |f_n(x) - f(x)|\,dx = 0$$

and $\|f\|_1 = \int_0^{2\pi} |f(x)|\,dx < +\infty$.

In the problem set we prove that L^1 is not a Hilbert space, that is, its norm is not given by a scalar product.

By the way, we have shown that

$$C^\infty \subseteq C \subseteq L^2 \subseteq L^1,$$

and each inclusion is dense (in the respective topology).

The Fourier coefficients of an L^1 function f are well defined, indeed

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)\, e^{-inx}\, dx$$

$$|\hat{f}_n| \le \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} |f(x)\, e^{-inx}|\, dx = \frac{1}{\sqrt{2\pi}} \|f\|_1 .$$

The situation with respect to the recovery of an arbitrary function $f \in L^1$ by resumming its Fourier series is however not so good as that of L^2.

$$S_n(f)(x) \overset{\text{a?}}{\longrightarrow} f(x) \quad ;$$

There are functions $f \in L^1$ such that the partial sums diverge at every point $x \in \mathbb{T}^1$.

Turning to Cesaro summability, there is a clearer picture:

Theorem 1 The arithmetic mean of the partial sums of every $f \in L^1$ converge to f in L^1 norm.

$$\lim_{n\to\infty} \left\| \frac{1}{n}(S_0(f) + \cdots + S_{n-1}(f)) - f \right\|_1 = 0.$$

proof: This uses the Fejer kernel, as one would imagine.

$$\frac{1}{n} \sum_{j=0}^{n-1} S_j(f) = \int_0^{2\pi} F_n(x-y)\, f(y)\, dy$$

Therefore

$$\left(\frac{1}{n} \sum_{j=0}^{n-1} S_j(f)(x) - f(x)\right) = \int_0^{2\pi} F_n(x-y)(f(y) - f(x))\, dy$$

$$= \int_0^{2\pi} F_n(y)(f(x-y) - f(x))\, dy$$

The $L^1(\mathbb{T}^1)$ norm of the difference can be estimated

$$\| \tfrac{1}{n} \sum_{j=0}^{n-1} S_j(f) - f \|_{L^1} \leq \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_n(y) \, |f(x-y) - f(x)| \, dy \, dx$$
$$= \int_{-\pi}^{\pi} F_n(y) \left(\int_{-\pi}^{\pi} |f(x-y) - f(x)| \, dx \right) dy .$$

Turn to the fact that translation is continuous in L^1, (we had this before in L^2). For $|y| < \delta$ sufficiently small then

$$\| f(\cdot - y) - f(\cdot) \|_1 < \varepsilon/2 .$$

And as before, split the above integral into two parts:

$$\int_{-\pi}^{\pi} F_n(y) \left(\int_{-\pi}^{\pi} |f(x-y) - f(x)| \, dx \right) dy$$
$$= \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} F_n(y) \, \| f(\cdot - y) - f(\cdot) \|_1 \, dy \; + \int_{-\delta}^{\delta} F_n(y) \, \| f(\cdot - y) - f(\cdot) \|_1$$

The second integral is less than $\varepsilon/2$ because $\int_{-\delta}^{\delta} F_n \leq \int_{-\pi}^{\pi} F_n = 1$ and because of the continuity of translation.

The first integral is bounded by the fact that for n sufficiently large, and for $|y| \geq \delta$, we have

$$0 \leq F_n(y) \leq \frac{4\pi \| f \|_1}{n \sin(\delta/2)} < \varepsilon/2 \qquad \boxtimes$$

When $f \in L^2$, we know that $|\hat{f}_k| \to 0$ sufficiently fast that $\sum_k |\hat{f}_k|^2 < +\infty$.

For $f \in L^1$ there is not a neat analog. We know that

$$|\hat{f}_k| \leq \frac{1}{\sqrt{2\pi}} \| f \|_1 ,$$

but not too much more in general. The other fact is as follows.

Theorem 2 (Riemann-Lebesgue Lemma) If $f \in L^1(\mathbb{T}^1)$ then the Fourier coefficients $\hat{f}_k$ satisfy

$$\lim_{|k| \to \infty} |\hat{f}_k| = 0.$$

proof: We can express the k^{th} Fourier coefficient

$$\hat{f}_k = \int_0^{2\pi} f(x) \frac{e^{-ikx}}{\sqrt{2\pi}} dx = -\int_0^{2\pi} f(x) \frac{e^{-ik(x-\frac{\pi}{k})}}{\sqrt{2\pi}} dx$$

$$= -\int_0^{2\pi} f(x'+\tfrac{\pi}{k}) \frac{e^{-ikx'}}{\sqrt{2\pi}} dx'$$

Therefore we may write

$$\hat{f}_k = \frac{1}{2} \int_0^{2\pi} (f(x) - f(x+\tfrac{\pi}{k})) \frac{e^{-ikx}}{\sqrt{2\pi}} dx$$

$$|\hat{f}_k| \le \frac{1}{2\sqrt{2\pi}} \int_0^{2\pi} |f(x) - f(x+\tfrac{\pi}{k})| dx$$

$$= \frac{1}{2\sqrt{2\pi}} \| f(\cdot) - f(\cdot + \tfrac{\pi}{k}) \|_1 .$$

We know that translations are continuous in L^1, hence the latter term converges to zero as $|k| \to +\infty$. ∎

No rate of convergence is given by this theorem, and indeed there is most likely no rate available for general $f \in L^1$ ∎

iii. Gibbs' phenomenon

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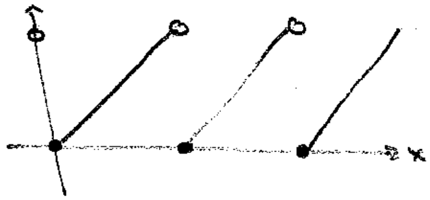
All of our efforts for now have been focused on how well a Fourier series converges.
The phenomenon discussed by Gibbs (1899) has to do with how badly it does, at least in the vicinity of a discontinuity. This is relevant to numerical methods that rely on the FFT to examine functions with discontinuities.

Gibbs' paper discussed simply the Fourier series partial sums for the sawtooth function

$$f(x) = \begin{cases} x & 0 \le x < 2\pi \\ & \text{repeated periodically} \end{cases}$$

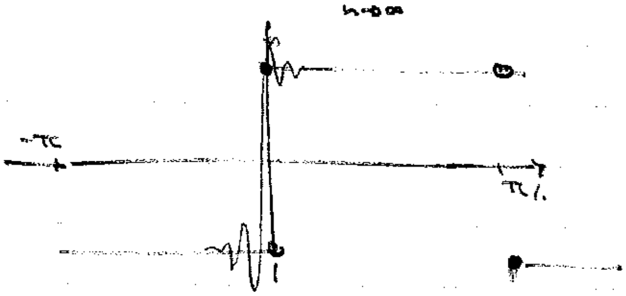


It is more straightforward for us to study

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ +1 & 0 \le x < \pi . \end{cases}$$

Fact: Each $S_n(f)$ overshoots the values of ± 1 in a nbhd of zero by about $.089490\ldots$

$$\lim_{n\to\infty} \| S_n \|_\infty = \lim_{n\to\infty} \sup_{-\pi < x < \pi} | S_n(x) | = 1.089\ldots$$



The region of the overshoot becomes smaller as $n \to \infty$, but the absolute size of the overshoot does not diminish to zero.

proof of Gibbs phenomenon:

Start with the Dirichlet kernel to express the partial sum

$$S_n(x) = \int_{-\pi}^{\pi} D_n(x-y) f(y)\, dy$$

$$= -\int_{-\pi}^{0} D_n(x-y)\, dy + \int_0^{\pi} D_n(x-y)\, dy$$

use symmetry of $D_n(y)$

$$= -\int_{-\pi}^{0} D_n(y-x)\, dy + \int_0^{\pi} D_n(y-x)\, dy$$

$$= -\int_{-x-\pi}^{-x} D_n(y)\, dy + \int_{-x}^{-x+\pi} D_n(y)\, dy$$

$$= -\int_{x}^{x+\pi} D_n + \int_{x}^{-x+\pi} D_n(y) + \int_{-x}^{x} D_n$$

$$= \int_{-x}^{x} D_n - \int_{-x+\pi}^{x+\pi} D_n$$

Suppose that $|x| < \pi/2$, then the second integral is bounded by Const $\frac{1}{n}$;

$$\int_{-x+\pi}^{x+\pi} D_n = \int_{-x+\pi}^{x+\pi} \frac{1}{2\pi} \frac{\sin((n+\frac{1}{2})y)}{\sin(y/2)}\, dy$$

$$= \frac{1}{2\pi(n+\frac{1}{2})} \int_{-x+\pi}^{x+\pi} \frac{d}{dy} \cos((n+\tfrac{1}{2})y) \frac{1}{\sin(y/2)}\, dy$$

$$= \frac{1}{2\pi} \frac{1}{n+\frac{1}{2}} \left[\frac{\cos((n+\frac{1}{2})y)}{\sin(y/2)} \right]_{-x+\pi}^{x+\pi} - \frac{1}{2\pi(n+\frac{1}{2})} \int_{-x+\pi}^{x+\pi} \frac{1}{2} \frac{\cos((n+\frac{1}{2})y)\cos(\frac{y}{2})}{\sin^2(y/2)}$$

Therefore

$$\left| S_n - \int_{-x}^{x} D_n(y)\, dy \right| \le \frac{\text{Const}}{n} .$$

We can therefore study something quite concrete, such as $\int_{-x}^{x} D_n(y)\, dy$ in order to quantify the overshoot in Gibbs' phenomenon.

p3

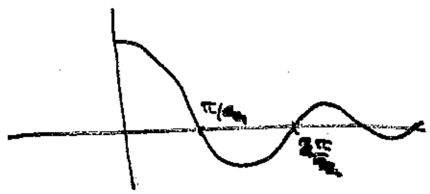
Now
$$\int_{-x}^{x} D_n(y)\,dy = \frac{1}{2\pi}\int_{-x}^{x} \frac{\sin((n+\frac{1}{2})y)}{\sin(y/2)}\,dy$$

Simplify again by writing
$$\frac{1}{\sin(y/2)} - \frac{2}{y} = \frac{y}{12} + \frac{7y^3}{1440} + \cdots$$

therefore
$$\int_{-x}^{x} D_n(y)\,dy - \int_{-x}^{x} \frac{\sin((n+\frac{1}{2})y)}{y/2}\,dy$$
$$= \int_{-x}^{x} \left[\frac{1}{\sin(y/2)} - \frac{2}{y}\right] \sin((n+\tfrac{1}{2})y)\,dy$$
$$= \int_{-x}^{x} \frac{y}{12} \sin((n+\tfrac{1}{2})y)\,dy + \text{even smaller terms in } x.$$

Finally, we are left with the bare-bones approximation:
$$\int_{-x}^{x} D_n(y)\,dy = 2\int_{-x}^{x} \frac{\sin((n+\frac{1}{2})y)}{y}\,dy$$
$$= 4\int_{0}^{x(n+\frac{1}{2})} \frac{\sin(y)}{y}\,dy$$

The function $\frac{\sin(y)}{y}$ looks like this



The integral is maximum at the first zero:
$$4\int_{0}^{\pi} \frac{\sin(y)}{y}\,dy = 1.089490\ldots$$

The interval in which oscillations are significant has width $\left(\frac{1}{n+1/2}\right)$. This happens not only when there is a jump (of amplitude 2) as in the square wave case, but in the vicinity of any jump in $f(x)$.

(iv) Lacunary series

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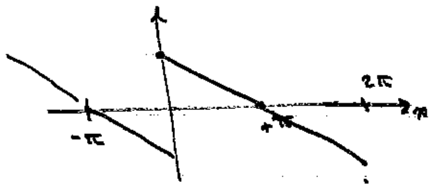
This is a section on the hazards of convergence of Fourier series. It is one of the standard tricks of harmonic analysis to make counterexamples of functions which oscillate at different scales

$$f(x) = \sum_{\ell=1}^{\infty} f_\ell(x/2^\ell) .$$

We will use this idea to construct a continuous function whose Fourier series $S_N(x)$ diverges at a dense set of points.

Start with the sawtooth function



$$f(x) = \begin{cases} i(\pi - x) & 0 < x < \pi \\ \text{odd in } x \end{cases}$$

This has Fourier series coefficients

$$\hat{f}_k = c \sum_{k \neq 0} \frac{e^{ikx}}{k} .$$

Partial sums, as we have defined them, are symmetric about $k=0$

$$S_n(f)(x) = \sum_{-n \le k \le n} \hat{f}_k \frac{e^{ikx}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \sum_{\substack{-n \le k \le n \\ k \neq 0}} \frac{1}{k} e^{ikx} .$$

Suppose however we summed only half at a time

$$S_n^-(f)(x) = \frac{1}{\sqrt{2\pi}} \sum_{-n \le k \le -1} \frac{1}{k} e^{ikx} .$$

Proposition 1 As n increases, $S_n^-(f)(0)$ diverges logarithmically.

proof: $S_n^-(f)(0) = \frac{1}{\sqrt{2\pi}} \sum_{-n \le k \le -1} \frac{1}{k} \simeq \mathrm{Log}(k) \nearrow +\infty$.

In fact, the series $\frac{1}{\sqrt{2\pi}} \frac{1}{k} \quad -\infty < k \le -1$ is not the Fourier series of a L^∞ function, it can be shown.

To make the counterexample do converge, we will make two targets of resummation from the sawtooth function $f(x)$:

$$\tilde{f}_N(x) = \sum_{-N \le k \le -1} \frac{e^{ikx}}{k} = S_N^-(f) \text{ and } f_N(x) = \sum_{\substack{-N \le k \le -1 \\ \text{and} \\ 1 \le k \le N}} \frac{e^{ikx}}{k} = S_N^{\leftrightarrow}(f)$$

Proposition 2 $f_N(x)$ is uniformly bounded in $x \in \mathbb{T}^1$, and uniformly so in N.

proof: We will use a sense of Abel summability to prove this.

The Abel means of a series $\sum_{k=1}^{\infty} c_k$ are given by

$$A_r(c) = \sum_{k=1}^{\infty} r^k c_k .$$

Lemma 3 If the Abel means $A_r(c)$ are bounded as $r \nearrow 1$ and if $|c_k| < \frac{C}{|k|}$, then the partial sums

$$S_N(c) := \sum_{k=1}^{N} c_k \qquad \text{are bounded} \quad |S_N(c)| \le C_0.$$

proof: Let $r = 1 - \frac{1}{N}$ and choose M so large that $k|c_k| \le M$. Estimate the difference

$$S_N - A_r = \sum_{k=1}^{N} c_k - r^k c_k \quad - \sum_{k=N+1}^{+\infty} r^k c_k$$

therefore

$$|S_N - A_r| \le \sum_{k=1}^{N} |c_k| (1 - r^k) + \sum_{k=N+1}^{\infty} r^k |c_k|$$

$$\le M \sum_{k=1}^{N} (1-r) + \frac{M}{N} \sum_{k=N+1}^{+\infty} r^k$$

where we used that

$$1 - r^k = (1-r)(1 + r + \dots + r^{k-1}) \le (1-r)k .$$

$$\sum_{k=N+1}^{\infty} r^k = \frac{1 - r^N}{1-r} \le \frac{1}{1-r} .$$

$$\le M N(1-r) + \frac{M}{N}\frac{1}{1-r} \le 2M .$$

Therefore, if $|A_r| \le M'$ and $M \le M'$, then $|S_N| \le 3M' =: C_0$. ☐

Apply this lemma to the sawtooth Fourier series partial sums.

$$\sum_{k\neq 0} \frac{e^{ikx}}{k} = \sum_{k\ge 1} \frac{e^{ikx}-e^{-ikx}}{k} =: \sum_{k\ge 1} c_k(x)$$

Thus $|c_k| \le \frac{C_1}{|k|}$. The Abel means of this series are ~~the~~ given by the Poisson kernel.

$$A_r(f)(x) = \int P_r(x-y) f(y)\, dy$$

$$|A_r(f)(x)| \le \|f\|_\infty \int P_r(y)\, dy = \|f\|_\infty, \text{ bounded.}$$

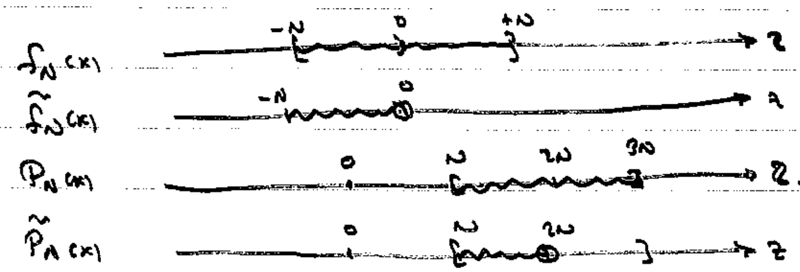
Using the lemma, $S_N(f)(x)$ are uniformly bounded in x and N, proving proposition 2.

Using the two facts in Proposition 1 and Proposition 2, we will construct a new function $g(x)$ whose Fourier series is <u>asymmetric</u> in a certain way, for which the partial sums <u>diverge</u> at $x=0$.

Let $P_N(x) = e^{i2Nx} f_N(x)$

$\tilde{P}_N(x) = e^{i2Nx} \tilde{f}_N(x)$, trigonometric polynomials.

Fourier Support of the above functions:



Finally, put this construction together using a lacunary series:

$$\{N_n\}_{n=1}^{\infty} \quad \{\alpha_n\}_{n=1}^{\infty}$$

such that

- $3N_n < N_{n+1}$
- $\sum_{n=1}^{\infty} \alpha_n$ converges, $\alpha_n > 0$.
- $\alpha_n \log(N_n) \to \infty$ with $n \to \infty$.

For example $\alpha_n = \frac{1}{n^2}$, $N_n = 3^{2^n}$.

—— a lacunary sequence based on this choice of scales.

Define $g(x) = \sum_{n=1}^{\infty} \alpha_n P_{N_n}(x)$.

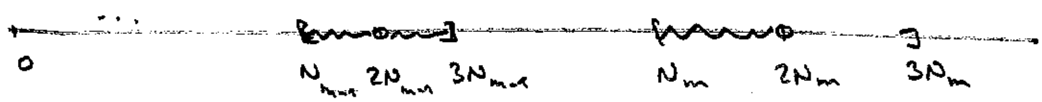
Since $|P_{N_n}(x)| = |f_{N_n}(x)| \leq C_0$, and $\sum \alpha_n < +\infty$, then this sum converges uniformly

$$\sum_{n=1}^{L} \alpha_n |P_{N_n}(x)| \to g(x) \quad \text{as } L \to \infty,$$

and $g(x) \in C(\mathbb{T}^1)$ is continuous and periodic.

However, if we break up the summands $P_{N_n}(x)$, by taking partial Fourier series sums $S_{2N_n}(g)(x)$, we find

$$|S_{2N_m}(g)(0)| \geq c\, \alpha_m \log|N_m| + O(C_0) \to \infty \text{ as } m \to \infty.$$



This proves that the partial sums for $g(x)$ do not converge for $x = 0$.

Finally, take some convergent series of translates of $g(x)$:

$$G(x) = \sum_{k=1}^{\infty} \beta_k\, g(x - \gamma_k) \qquad \gamma_k \text{ dense}, \ \sum \beta_k < +\infty.$$