Branching Processes and the Effect of Parlaying Bets on Lottery Odds

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Abstract. We introduce and analyse the effect of parlaying bets in lotteries. This occurs when a ticket purchased for one lottery game wins either a ticket for a succeeding game or a small dollar prize that is then used to purchase tickets for future games. It is shown that this behaviour can be modeled as a branching process, and a result of Dwass on the total progeny is used to show that the probability of winning some large prize can be increased by as much as 40% as a result of parlaying in Lotto Super 7 and by as much as 100% in the instant scratch game Crossword.

1 Introduction

It is commonly accepted that a simple combinatorial argument based on the hypergeometric distribution will yield the probability that a single ticket will win a prize in the various flavours of Lotto-type games. These probabilities are commonly quoted in the media, especially when there is a particularly large jackpot. What has been overlooked, however, is that there are lesser prizes, including free tickets, that may then be played in future drawings. For instance, in the Canadian Lotto Super 7 the fifth and sixth prizes each pay $10 and the seventh prize is a free ticket. Each ticket costs $2 and allows three plays of seven numbers. Presumably, regular players who win $10 will purchase five additional tickets giving them 15 more chances for the next game. As a result, a single $2 ticket may lead to a win in succeeding games beyond the game for which it was originally purchased. This is called *parlaying*[2] and its effect is to increase the overall probability that a single ticket will win a prize in some future lottery, in fact by as much as 100% in some lotteries.

Recently the Canadian Broadcasting Corporation claimed on its program *The Fifth Estate* (CBC 2006) that an unusually large number of lottery-ticket

* Supported by an NSERC Discovery Grant.

[2] From the *American Heritage College Dictionary*, third edition: "parlay. *tr. v.* 1. *Games.* To bet (an original wager and its winnings) on a subsequent event. —*n.* 1. *Games.* A bet comprising a sum of a prior wager plus its winnings or a series of such bets."

retailers had won major prizes in Ontario lotteries. This resulted in an announcement the following day of an investigation by the provincial ombudsman (Ontario Ombudsmans Office 2006). The CBC's story centered on an elderly man, his $250,000 lottery ticket, and the clerk at a convenience store where he had gone to check whether his ticket had won. It was noted that the ticket the man submitted had resulted from a free ticket won two weeks previously. In other words, the winning ticket arose from a parlay.[3]

The purpose of this paper is then threefold: to introduce parlaying and point out the need to take it into account in computing the probabilities of winning different prizes; to discuss a model, called a Bienaymé–Galton–Watson branching process, that can be used to carry out the analysis; and to calculate the probabilities of relevant events and the distributions of interesting random variables.

2 Structure as a single-type branching process

Branching processes are used as models to describe the growth of populations of individuals over time (Harris 1963, Athreya and Ney 1972) and have found numerous applications in physics, biology, and genetics since their original introduction (Watson 1873) to describe the loss of surnames in England. The branching processes considered here arise in discrete time and are known as Bienaymé–Galton–Watson processes. They may be described as follows.

Consider a population of individuals whose size $\{X_n,\ n = 0, 1, \ldots\}$ changes at each unit of discrete time. Each individual lives one unit of time after which he disappears and is replaced with a random number of offspring ξ according to a probability distribution $\mathbb{P}[\xi = i] = p_i$ for $i = 0, 1, \ldots$. All individuals reproduce independently of each other and the random variable X_n counts the population size at the end of generation n.

For concreteness we present the model and results as applied to the Lotto Super 7, although a similar analysis will apply to other lotteries (as discussed in Section 6). A ticket costs $2 and allows a player to select seven numbers from 1 to 47 or to have these selected by a ticket machine (called a "quick-pick"). The machine then selects two additional sets each of seven numbers. Thus, a $2 ticket gives the player three sets of seven numbers each. Every Friday, lottery officials randomly draw seven regular numbers and one bonus number, for a total of eight numbers drawn from $\{1, 2, \ldots, 47\}$. A win occurs whenever at least three of the numbers in any of the sets of the player's ticket match some of the numbers drawn by the lottery, although the major prizes require either six or seven numbers matched. An initial $2.5 million jackpot grows until it is won. The prize structure is given in Table 1. For instance, the second prize requires matching of six of the seven regular numbers plus the bonus number.

[3] From CBC (2006): "July 27, 2001: Bob Edmonds goes to the Coby Milk & Variety to check the free Super 7 ticket with Encore he bought on July 13."

Table 1. Prizes and winning probabilities of a single set of seven numbers in Lotto Super 7.

match	prize	probability
7 numbers	share 73% of pool	$p_1 = \binom{7}{7}\binom{40}{0}/\binom{47}{7}$
6 numbers + bonus number	share 5% of pool	$p_2 = \binom{7}{6}\binom{39}{0}\binom{1}{1}/\binom{47}{7}$
6 numbers	share 5% of pool	$p_3 = \binom{7}{6}\binom{39}{1}\binom{1}{0}/\binom{47}{7}$
5 numbers	share 17% of pool	$p_4 = \binom{7}{5}\binom{40}{2}/\binom{47}{7}$
4 numbers	\$10	$p_5 = \binom{7}{4}\binom{40}{3}/\binom{47}{7}$
3 numbers + bonus number	\$10	$p_6 = \binom{7}{3}\binom{39}{3}\binom{1}{1}/\binom{47}{7}$
3 numbers	free play (\$2 value)	$p_7 = \binom{7}{3}\binom{39}{4}\binom{1}{0}/\binom{47}{7}$

If a player parlays winnings by purchasing additional tickets, then the number of tickets produced by parlaying may be viewed as a branching process $\{X_n\}$ with $X_0 = 3$ corresponding to one initial ticket with three selections. For instance, consider the three lesser prizes, having fixed values. A selection of seven numbers that wins the free play, which is a \$2 ticket, may be said to give birth to three more individuals by spawning a single ticket with three selections. A \$10 win can be used to buy five more tickets (15 new individuals) and we will assume that players who win \$10 reinvest all their winnings into new tickets. This assumption is appropriate for players who play regularly for whom small prizes are not their ultimate goal. (For infrequent players, probability calculations are meaningless.)

The remaining prizes have variable payouts depending on the size of the pool and the number of individuals who share the pool. For instance a fourth-place prize typically pays on the order of \$100. Presumably, someone who wins one of the top four prizes may also parlay some of his winnings. For the purposes of explaining the methodology and presenting a numerical example, we will in fact make this assumption that such a person will parlay his winnings into five more tickets (\$10), that is 15 selections of seven numbers. However, the methodology does not depend on which prizes the player parlays and the results vary little if the top four prizes are parlayed or not (see Section 6).

With this assumption, $\mathbb{P}[\xi = 3] = p_7$, $\mathbb{P}[\xi = 15] = p_1+p_2+p_3+p_4+p_5+p_6$, and $\mathbb{P}[\xi = 0] = 1 - (p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7)$. The distribution of the random variable ξ is therefore given by Table 2.[4] This results in a subcritical branching process with mean $m \equiv \mathbb{E}[X_1 \mid X_0 = 1] = 0.30006 < 1$. The probability generating function (p.g.f.) of ξ is

$$F(x) \equiv \mathbb{E}[x^{\xi}] = 0.94338 + 0.045774x^3 + 0.010849x^{15}, \quad 0 \le x \le 1. \qquad (1)$$

[4] The decimal expansions of the actual probabilities have been rounded to five significant digits.

Table 2. Lotto Super 7 offspring distribution.

i	$f_i = \mathbb{P}[\xi = i]$
0	0.94338
3	0.045774
15	0.010849

Classical branching processes require that the individuals behave independently. If the selections are quick-picked then the offspring numbers will also be independent, as the following argument shows. Suppose that E and F are two sets of seven numbers with offspring numbers ξ and η. Let A be the lottery's selection of seven numbers. The values of ξ and η depend only on E, F, and A. Then, conditioning on A, $\mathbb{P}[\xi = i,\ \eta = j] = \mathbb{E}[\mathbb{P}[\xi = i,\ \eta = j \mid A]]$. But $\mathbb{P}[\xi = i,\ \eta = j \mid A] = \mathbb{P}[\xi = i]\mathbb{P}[\eta = j]$ because the quick-pick selections are independent and identically distributed, and independent of the lottery's picks. Therefore ξ and η are independent[5] and this argument extends to independence of the numbers of offspring of all individuals in a given generation based on quick-picks.

Independence persists if one of the selections is not a quick-pick but is chosen by the player (a self-pick) by a nonrandom mechanism. Thus, let E and F be two quick-picks and G a self-pick, corresponding to a \$3 ticket where the players selects his own numbers for the first selection of seven numbers. G is therefore deterministic, although, since A is random, the offspring numbers are still random. Let ξ, η, ζ denote the corresponding offspring numbers. As before $\mathbb{P}[\xi = i,\ \eta = j,\ \zeta = k] = \mathbb{E}[\mathbb{P}[\xi = i,\ \eta = j,\ \zeta = k \mid A]]$. Since A is fixed in the conditional expectation, and ζ is completely determined by the overlap between G (deterministic) and A (fixed), it must be that $\zeta = k$ if and only if G lies in a collection of selections $S(A, G, k)$ and the conditional distribution of $\zeta = k$ given A is either 1 or 0 depending on whether A lies in this set or not. Moreover, the conditional distribution of ζ is also independent of the conditional distribution of ξ and η because the latter are obtained by a mechanism independent of A and the choice of G. Hence

$$\begin{aligned}\mathbb{P}[\xi = i,\ \eta = j,\ \zeta = k \mid A] &= 1_{S(A,G,k)}\mathbb{P}[\xi = i,\ \eta = j \mid A] \\ &= 1_{S(A,G,k)}\mathbb{P}[\xi = i]\mathbb{P}[\eta = j]\end{aligned}$$

where $1_{S(A,G,k)}$ is the indicator of the event $S(A, G, k)$. Therefore

$$\begin{aligned}\mathbb{P}[\xi = i,\ \eta = j,\ \zeta = k] &= \mathbb{E}[\mathbb{P}[\xi = i,\ \eta = j,\ \zeta = k \mid A]] \\ &= \mathbb{E}[1_{S(A,G,k)}\mathbb{P}[\xi = i]\mathbb{P}[\eta = j]] \\ &= \mathbb{E}[1_{S(A,G,k)}]\mathbb{P}[\xi = i]\mathbb{P}[\eta = j] \\ &= \mathbb{P}[\zeta = k]\mathbb{P}[\xi = i]\mathbb{P}[\eta = j]\end{aligned}$$

[5] Independence of the choices E and F is inherited by the offspring numbers ξ and η as long as shared pools always pay out at least \$10 to each winner, a virtual certainty.

since $\mathbb{E}[1_{S(A,G,k)}] = \mathbb{P}[\zeta = k]$ because A is randomly generated by the lottery and so the chance of any specified match is the same for either a self-pick or quick-pick. This proves independence when there is one self-pick.

What if a player makes two or more self-picks, say G and H? In this case, there will be a correlation between the respective winnings ζ and τ which depends on the size of the intersection $G \cap H$.[6] Since the number of possible selections is enormous in comparison to the number of tickets that might be accumulated by parlaying, the correlation will be close to zero. While even zero correlation is not the same as independence, the branching model should still give very accurate numerical results even for self-picks.

Branching processes where the offspring numbers are not independent and identically distributed have been considered, for instance in Quine (1994), but our usage is simple enough not to need reliance on general theorems, and we calculate probabilities assuming independence.

3 Maximum duration of a parlay

Subcritical branching processes have finite lifetime L, which is the first generation n such that $X_n = 0$, and die out with probability 1. The distribution of L is obtained from the nth iterate or functional composition of F with itself, that is,

$$F_n(x) = \underbrace{F(F(\cdots F}_{n \text{ times}}(x)\cdots)),$$

where we define $F_0(x) = x$. Observe that

$$\mathbb{P}[L \leq n \mid X_0 = 1] = \mathbb{P}[X_n = 0 \mid X_0 = 1] = F_n(0)$$

and therefore, if $X_0 = 1$,

$$\begin{aligned} \mathbb{P}[L = n] &= \mathbb{P}[L \leq n] - \mathbb{P}[L \leq n-1] \\ &= \mathbb{P}[X_n = 0] - \mathbb{P}[X_{n-1} = 0] = F_n(0) - F_{n-1}(0). \end{aligned} \tag{2}$$

This gives the distribution of the duration of the process starting at $X_0 = 1$. We call L the length of the parlay. Thus if $L = 3$ then an initial ticket parlays two generations forward and allows play in three lotteries, including the initial game for which it was bought. A Super 7 ticket has three sets of numbers, which means that $X_0 = 3$ and hence

$$\begin{aligned} \mathbb{P}[L = n \mid X_0 = 3] &= \mathbb{P}[X_n = 0 \mid X_0 = 3] - \mathbb{P}[X_{n-1} = 0 \mid X_0 = 3] \\ &= F_n^3(0) - F_{n-1}^3(0). \end{aligned} \tag{3}$$

The values of $\mathbb{P}[L = n \mid X_0 = 3]$ for $1 \leq n \leq 6$, shown in Table 3, were computed using *Maple*. The expected value of L is $\mathbb{E}[L] = 1.2154$.

[6] This relates to lottery wheels which are sets of tickets with the property that purchase of the entire set will guarantee that at least one ticket will match at least a specified number of the lottery's selections.

Table 3. Duration of the parlay (distribution of L).

n	1	2	3	4	5	6
$\mathbb{P}[L = n \mid X_0 = 3]$	0.83956	0.11999	0.02886	0.00815	0.00240	0.00072

4 Total progeny

In view of the distribution of L it is necessary to determine the distribution of the total number of individuals resulting from a single \$2 ticket, including the initial three for the first game and those for subsequent games that are generated by parlaying. This corresponds to what is known as the total progeny in a branching process (Dwass 1969) and is defined by

$$Y = \sum_{n=0}^{L-1} X_n = \sum_{n=0}^{\infty} X_n. \tag{4}$$

The p.g.f. of Y, starting with $X_0 = 1$, denoted by $g(x)$, is given by the solution to

$$g(x) = xF(g(x)). \tag{5}$$

As long as F is analytic and nonvanishing at $x = 0$, which is the case here, the implicit function theorem guarantees a unique solution in a neighborhood of the origin. The appropriate tool for determining the distribution of Y is a result of Dwass (1969).

Table 4. Distribution of total number of parlayed tickets Y starting with $X_0 = 3$.

n	$\mathbb{P}[Y = n]$	n	$\mathbb{P}[Y = n]$	n	$\mathbb{P}[Y = n]$
3	0.8395677097	21	0.0098741179	39	0.0012456088
6	0.1026041086	24	0.0050797802	42	0.0008024231
9	0.0167190852	27	0.0021253383	45	0.0004360456
12	0.0031216321	30	0.0007906917	48	0.0003104600
15	0.0006312035	33	0.0012004956	51	0.0003148655
18	0.0122173770	36	0.0014946270		

Theorem 1.

$$\mathbb{P}[Y = n \mid X_0 = k] = \frac{k}{n}\mathbb{P}[X_1 = n - k \mid X_0 = n], \quad k \leq n. \tag{6}$$

A single ticket has three random selections so the associated branching process starts at $X_0 = 3$. By independence, the p.g.f. of the total progeny starting with three individuals is $G(x) = (g(x))^3$. Since one initial ticket is equivalent to $k = 3$ in (6) we have

$$\mathbb{P}[Y = n \mid X_0 = 3] = \frac{3}{n}\mathbb{P}[X_1 = n - 3 \mid X_0 = n] \tag{7}$$

and the term on the right side of (7) can be obtained from the expansion of $(F(x))^n$ which can be obtained with *Maple*. The distribution is not monotone in the number of tickets, as might be expected. For instance, the probability of accumulating 18 tickets is approximately four times as large as the probability of accumulating 12 tickets, which seems counter-intuitive. Table 4 gives the probability distribution of Y, shown more graphically in the serial plot of $\mathbb{P}[Y = n]$ in Figure 1. In view of the scale, the point $(3, \mathbb{P}[Y = 3])$ is not plotted.

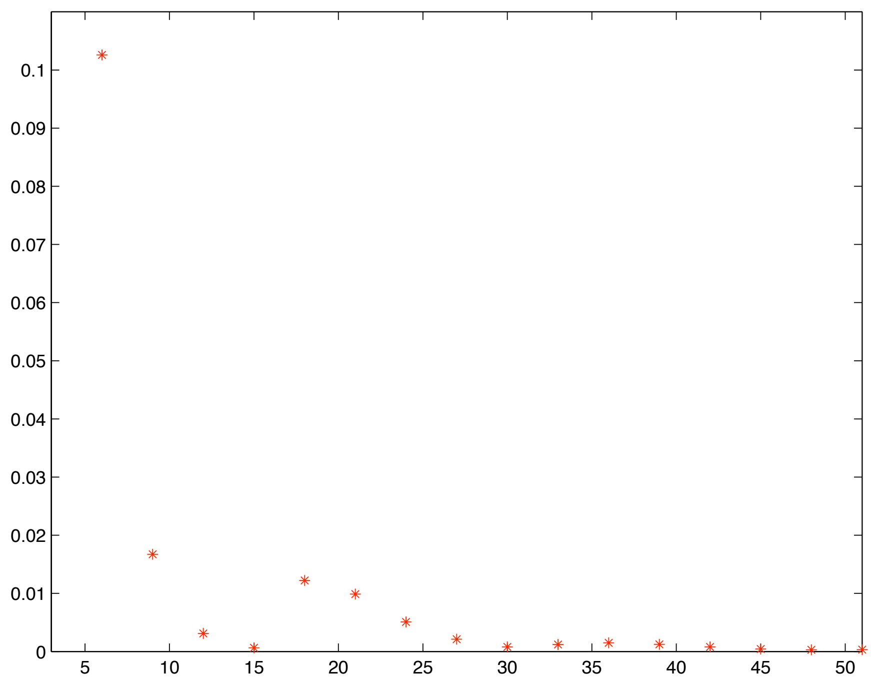

Fig. 1. Probability distribution of Y.

A referee has suggested that it might be of interest to explain the multimodality of the distribution of Y. In accordance, we explain using sample paths why $\mathbb{P}[Y = 18]$ is so much larger than $\mathbb{P}[Y = 12]$. Table 5 shows, in the first column, the possible sequences of generation sizes $\{X_1, X_2, \ldots\}$ for which

$Y = 12$. There are four sequences that result in $Y = 12$. The column headed "# of paths" counts how many such sample paths there are to obtain each sequence. These are obtained by enumerating which offspring contribute to successive generations. The column headed "$\mathbb{P}$[1 path]" gives the probability of each such sample path. It turns out that the individual sample path probabilities are all the same, regardless of the sequence of generation sizes. By multiplying these probabilities by the number of paths and then summing, we arrive at $\mathbb{P}[Y = 12] = 0.003121632$. Comparison with Table 4 shows that this is the same value obtained by the result of Dwass by which the distribution of Y is arrived at more simply without recourse to counting sample paths. Table 6 shows the corresponding computations for $\mathbb{P}[Y = 18]$. There are 14 distinct sequences $\{X_1, X_2, \ldots\}$ that yield $Y = 18$. The major contributor is the first sample path which dominates all others and whose probability is different from the other sample paths.

Table 5. All sample paths that achieve $Y = 12$.

X_0	X_1	X_2	X_3	X_4	# of paths	$\mathbb{P}$[1 path]	$\mathbb{P}[X_n$ sequence]
3	9	0	0	0	$\binom{3}{3} = 1$	$f_3^3 f_0^9$	0.000056757
3	6	3	0	0	$\binom{3}{2}\binom{6}{1} = 18$	$f_3^3 f_0^9$	0.001021625
3	3	6	0	0	$\binom{3}{1}\binom{3}{2} = 9$	$f_3^3 f_0^9$	0.000510812
3	3	3	3	0	$\binom{3}{1}\binom{3}{1}\binom{3}{1} = 27$	$f_3^3 f_0^9$	0.001532437
totals					55		0.003121632

5 The parlay factor

Let p denote the probability that a set of seven randomly selected numbers wins the jackpot for a specific draw. The probability that a ticket with three selections of seven numbers wins a jackpot for a specific draw is $1-(1-p)^3 = 3p - 3p^2 + p^3 \approx 3p$ for $0 < p << 1$. Let J represent the event that a single initial ticket wins at least one future jackpot taking into account parlaying.

$$
\begin{aligned}
\mathbb{P}[J] &= \sum_i \mathbb{P}[J \mid Y = i]\mathbb{P}[Y = i \mid X_0 = 3] \\
&= \sum_i (1 - \mathbb{P}[\overline{J} \mid Y = i])\mathbb{P}[Y = i \mid X_0 = 3] \\
&= \sum_i (1 - (1-p)^i)\mathbb{P}[Y = i \mid X_0 = 3] \\
&= 1 - G(1-p).
\end{aligned} \tag{8}
$$

Table 6. All sample paths that achieve $Y = 18$.

X_0	X_1	X_2	X_3	X_4	X_5	X_6	# of paths	$\mathbb{P}$[1 path]	$\mathbb{P}[X_n$ sequence]
3	15	0	0	0	0	0	$\binom{3}{1} = 3$	$f_{15} f_0^{17}$	0.01208287684
3	9	6	0	0	0	0	$\binom{3}{3}\binom{9}{2} = 36$	$f_3^5 f_0^{13}$	0.00000339076
3	9	3	3	0	0	0	$\binom{3}{3}\binom{9}{1}\binom{3}{1} = 27$	$f_3^5 f_0^{13}$	0.00000254307
3	6	9	0	0	0	0	$\binom{3}{2}\binom{6}{3} = 60$	$f_3^5 f_0^{13}$	0.00000565127
3	6	6	3	0	0	0	$\binom{3}{2}\binom{6}{2}\binom{6}{1} = 270$	$f_3^5 f_0^{13}$	0.00002543070
3	6	3	6	0	0	0	$\binom{3}{2}\binom{6}{1}\binom{3}{2} = 54$	$f_3^5 f_0^{13}$	0.00000508614
3	6	3	3	3	0	0	$\binom{3}{2}\binom{6}{1}\binom{3}{1}\binom{3}{1} = 162$	$f_3^5 f_0^{13}$	0.00001525842
3	3	9	3	0	0	0	$\binom{3}{1}\binom{3}{3}\binom{9}{1} = 27$	$f_3^5 f_0^{13}$	0.00000254307
3	3	6	6	0	0	0	$\binom{3}{1}\binom{3}{2}\binom{6}{2} = 135$	$f_3^5 f_0^{13}$	0.00001271535
3	3	6	3	3	0	0	$\binom{3}{1}\binom{3}{2}\binom{6}{1}\binom{3}{1} = 162$	$f_3^5 f_0^{13}$	0.00001525842
3	3	3	9	0	0	0	$\binom{3}{1}\binom{3}{1}\binom{3}{3} = 9$	$f_3^5 f_0^{13}$	0.00000084769
3	3	3	6	3	0	0	$\binom{3}{1}\binom{3}{1}\binom{3}{2}\binom{6}{1} = 162$	$f_3^5 f_0^{13}$	0.00001525842
3	3	3	3	6	0	0	$\binom{3}{1}\binom{3}{1}\binom{3}{1}\binom{3}{2} = 81$	$f_3^5 f_0^{13}$	0.00000762921
3	3	3	3	3	3	0	$\binom{3}{1}\binom{3}{1}\binom{3}{1}\binom{3}{1}\binom{3}{1} = 243$	$f_3^5 f_0^{13}$	0.00002288763
			totals				1431		0.01221737700

We call the ratio

$$\Lambda = \frac{1 - G(1-p)}{1 - (1-p)^3}$$

the *parlay factor*. For p in a neighbourhood of 0, since $G(1) = 1$ we may write

$$\begin{aligned} \frac{1 - G(1-p)}{1 - (1-p)^3} &\approx \frac{1 - G(1-p)}{3p} \approx \frac{G'(1)}{3} \\ &\equiv \frac{\mathbb{E}[Y \mid X_0 = 3]}{3} = \mathbb{E}[Y \mid X_0 = 1] = g'(1). \end{aligned} \tag{9}$$

Differentiate (5) to obtain $g'(x) = F(g(x)) + xF'(g(x))g'(x)$, set $x = 1$, and solve $g'(1) = F(g(1)) + F'(g(1))g'(1) = 1 + mg'(1)$ to obtain

$$g'(1) = \frac{1}{1-m} = 1.4287.$$

This gives a nice interpretation of Λ as [7]

$$\Lambda = \frac{1}{1-m},$$

[7] More precisely $\Lambda \approx 1/(1-m)$, as in (9), but because the approximation is so good, we have replaced the approximate equal sign with a true equal sign, here and elsewhere below where Λ is given.

and as a result

$$\mathbb{E}[Y \mid X_0 = 3] = \frac{3}{1-m} = 4.286.$$

For Super 7, therefore, $\Lambda = 1.4287$ and the effect of parlaying increases the probability that a single ticket will ultimately win a jackpot by slightly over 40% and

$$\mathbb{P}[J] = 1 - G(1-p) = 3p\Lambda = \frac{3p}{1-m} = 4.286p.$$

We see that the effect of parlaying can be obtained very simply, without computing the distribution of L, using only the mean of X_1, not the entire distribution of L. This makes it very easy to examine the effect of parlaying on other games. By examining the distribution of Y one also finds that the contribution to $\mathbb{E}[Y]$ with the first three terms is only 3.285 which indicates that many events with low probabilities contribute to the mean.

Although this analysis has concentrated on the parlay effect on the jackpot, it clearly applies to any of the top four prizes, in particular to the first or second prizes combined, whose probability is also very small, and which are considered the major prizes, being the only ones typically in excess of $50,000. For instance, in the March 2, 2007 draw, the jackpot was $12,000,000 and the second prize was $207,317.80. There were no winners. However, the third prize was $1,470.30, paid to each of 141 winners. A reasonable strategy for parlaying would be to save all parlayed tickets to be played when the jackpot is very high.

6 Other games

A similar analysis can be carried out for other lotteries. We illustrate with Lotto 6/49 and Instant Crossword, and also very briefly with multistate lotteries such as Powerball and Hot Lotto. In view of the simplicity of the parlay factor, it is not necessary to compute the coefficients of generating functions. All that is needed is the mean of the associated offspring p.g.f.

6.1 Lotto 6/49

Lotto 6/49 is also an online game with drawings held twice per week on Wednesdays and Saturdays. A ticket costs $2 and players choose six numbers from 1 to 49. The prize structure and probabilities are given in Table 5.

As before, we assume that the lowest prize $5 is used to purchase two additional tickets (with $1 left over to combine with future winnings), the second lowest prize $10 is used to purchase five additional tickets, and the top four prizes are also used to purchase five additional tickets. Thus the associated branching process has an offspring distribution concentrated on $\{0, 2, 5\}$ with mean $m = 0.1178$ and parlay factor $\Lambda = 1.1336$, which is substantially less than for Lotto Super 7. If we assume that only the lowest

Table 7. Prizes and winning probabilities in Lotto 6/49.

match	prize	probability
6 numbers	share 80.50% of pool	$p_1 = \binom{6}{6}\binom{43}{0}/\binom{49}{6}$
5 numbers + bonus number	share 5.75% of pool	$p_2 = \binom{6}{5}\binom{42}{0}\binom{1}{1}/\binom{49}{6}$
5 numbers	share 4.75% of pool	$p_3 = \binom{6}{5}\binom{42}{1}\binom{1}{0}/\binom{49}{6}$
4 numbers	share 9% of pool	$p_4 = \binom{6}{4}\binom{43}{2}/\binom{49}{6}$
3 numbers	\$10	$p_5 = \binom{6}{3}\binom{43}{3}/\binom{49}{6}$
2 numbers + bonus number	\$5	$p_6 = \binom{6}{2}\binom{42}{3}\binom{1}{1}/\binom{49}{6}$

two prizes are parlayed, then the mean $m = 0.1129$ and $\Lambda = 1.1272$, showing that the results change little, as stated in Section 2. Actually, the parlay factor is somewhat higher, taking into account the $1 left over from a $5 prize. This is briefly discussed after Instant Crossword.

6.2 Instant Crossword

Another popular game in Ontario is Instant Crossword, which costs $3 for a card that the player scratches to reveal letters. The probabilities vary slightly from game to game because of either the ticket run (number of tickets produced) or if there are bonus games, but the corresponding prize structure remains constant. We assume in the branching structure that only the $3, $5, and $10 prizes are each used to purchase additional 1, 1, and 3 tickets, respectively, while the larger prizes are also parlayed into 3 additional tickets. Table 6 shows the probabilities for Crossword games #1250, #1252, #1259, and #1264 between November 2006 and February 2007 and the corresponding means and parlay factors.

In modeling Lotto 6/49 and Instant Crossword as branching processes we have not taken into consideration that the probabilities change slightly from game to game. However, the mean remains fairly constant so the parlay factor based on a strict branching assumption still gives a reasonable measure of the gain in probability obtained by parlaying. We have also ignored parlaying any winnings that do not make up the full cost of a ticket. In 6/49, $1 remains after the lowest prize of $5 is used to purchase two additional $2 tickets, while in Crossword $2 remains after one $3 ticket is bought with a $5 prize and $1 remains after three tickets are purchased with the proceeds of a $10 prize. These amounts will accumulate over time and could also be used to purchase additional tickets, if this were the strategy. To carry out a rigourous analysis which would account for such accumulation it would be necessary to generalize the theory of branching processes to allow fractional offspring that reproduce when they accumulate to an integer. Generalizations of branching processes do exist for nonpositive integer offspring such as continuous state branching

Table 8. Prizes and probabilities for Instant Crossword games.

prize	game #1250	bonus game #1252	game #1259	Quest for Gold game #1264
$50,000	$p_1 = 1/1{,}000{,}000$	1/1,000,000	1/1,000,000	1/1,666,666.67
$25,000	$p_2 = 1/1{,}000{,}000$	1/1,000,000	1/1,000,000	1/1,666,666.67
$10,000	$p_3 = 1/500{,}000$	1/1,000,000	1/1,000,000	1/1,666,666.67
$5,000	$p_4 = 1/500{,}000$	1/1,000,000	1/1,000,000	1/1,666,666.67
$100	$p_5 = 1/2{,}000$	1/10,000	1/2,000	1/1,754.39
$50	$p_6 = 1/1{,}000$	1/4,000	1/1,000	1/1,612.90
$25	$p_7 = 1/500$	1/500	1/250	1/172.12
$10	$p_8 = 1/14.47$	1/18.6	1/15.85	1/22.22
$5	$p_9 = 1/25$	1/10	1/17.54	1/11.11
$3	$p_{10} = 1/5.32$	1/5.26	1/5.32	1/8.33
m	0.4458	0.4585	0.4508	0.3661
Λ	1.8044	1.8466	1.8207	1.5775
$m^\star$	0.5355	0.6431	0.5668	0.5311
$\Lambda^\star$	2.1529	2.8015	2.3085	2.1326

processes (see references in Athreya and Ney 1972) and recently branching processes with negative offspring distribution (Dumitriu, Spencer, and Yan 2003), but the problem of accumulation has not been addressed.

To understand the effect that accumulation might have, as a first approximation the means can be adjusted fractionally and proportionally. They are denoted as $m^\star$ with corresponding parlay factors $\Lambda^\star$ in Table 6 for Instant Crossword. For Lotto 6/49 the mean increases to $m^\star = 0.1240$ and the parlay factor to $\Lambda^\star = 1.1415$.

6.3 Multistate Lotteries

For comparison we include the means and parlay factors for two multistate lotteries. For consistency with the bettor's parlay policy in the previous three numerical illustrations, we assume in Powerball that the ninth (lowest) prize of $3 is parlayed into three $1 tickets, the eighth prize of $4 is parlayed into four tickets, the seventh and sixth prizes of $7 each are parlayed into seven tickets, while the remaining prizes whose values are $100 or more are parlayed into ten tickets. For Hot Lotto the ninth (lowest) prize of $2 is parlayed into two $1 tickets, the eighth prize of $3 is parlayed into three tickets, the seventh and sixth prizes of $4 each are parlayed into seven tickets, while the remaining prizes whose values are $50 or more are parlayed into ten tickets.

- Powerball: $m = 0.1096, \Lambda = 1.1231$.
- Hot Lotto: $m = 0.1650, \Lambda = 1.1977$.

Acknowledgment

Thanks to a referee for some excellent comments that improved the presentation.

References

1. Athreya, K. B. and Ney, P. E. (1972) *Branching Processes*. Springer-Verlag, New York.
2. Canadian Broadcasting Corporation (2006) `http://www.cbc.ca/fifth/luckofthedraw/index.html`.
3. Dumitriu, I., Spencer, J., and Yan, C. (2003) Branching processes with negative offspring distributions. *Ann. Combin.* **7** 35–47.
4. Dwass, M. (1969) The total progeny in a branching process and a related random walk. *Jour. Appl. Probab.* **6** 682–686.
5. Harris, T. E. (1963) *The Theory of Branching Processes*. Springer-Verlag, New York.
6. Ontario Lottery and Gaming Corporation (2007) `http://www.olg.ca/lotteries/games/howtoplay.do?game=super7`.
7. Ontario Ombudsmans Office (2006) `http://www.ombudsman.on.ca/smuploads/OwnmotionrelOLGC.pdf`.
8. Quine, M. P. and Szczotka, W. (1994) Generalisations of the Bienaymé–Galton–Watson branching process via its representation as an embedded random walk. *Ann. Appl. Probab.* **4** 1206–1222.
9. Watson, H. W. (1873) Solution to problem 4001. *Educational Times* (August 1) 115–116.