LAST (family) NAME: ________________________

FIRST (given) NAME: ________________________

ID # : ____________________

MATHEMATICS 2P04

McMaster University Final Examination
Duration of Examination: 3 hours

SAMPLE FINAL EXAM B: SOLUTIONS

THIS EXAMINATION PAPER INCLUDES 24 PAGES AND 15 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

Instructions:

- Indicate your answers clearly in the spaces provided. In the full answer questions in Part I (questions 1 - 5) show all your work to receive full credit. For the multiple choice questions in Part II (questions 6 - 15) be sure to circle the correct letters on page 12 to receive full credit.
- No books, notes, or "cheat sheets" allowed. The only calculator permitted is the McMaster Standard Calculator, the Casio fx 991.
- The total number of points is 100.
- There is a formula sheet included with the exam on page 24.
- Pages 23 of the test is for scratch work or overflow. If you continue the solution to a question on this page, you must indicate this clearly on the page containing the original question. **GOOD LUCK!**

#	Mark	#	Mark
1.		**4.**	
2.		**5.**	
3.		**6-15.**	
		TOTAL	

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Some formulas you may use:

$$y_p(x) = -\left[\int \frac{y_2(x)\,f(x)}{W(y_1,y_2)}\,dx\right]\,y_1(x) + \left[\int \frac{y_1(x)\,f(x)}{W(y_1,y_2)}\,dx\right]\,y_2(x)$$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\,e^{-st}\,dt.$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{e^{a\,t}\} = \frac{1}{s-a},$$

$$\mathcal{L}\{\sin(k\,t)\} = \frac{k}{s^2+k^2}, \quad \mathcal{L}\{\cos(k\,t)\} = \frac{s}{s^2+k^2},$$

$$\mathcal{L}\{f'(t)\} = s\,F(s) - f(0), \quad \mathcal{L}\{f''(t)\} = s^2\,F(s) - s\,f(0) - f'(0),$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\,F(s) - s^{n-1}\,f(0) - s^{n-2}\,f'(0) - \cdots - s\,f^{(n-2)}(0) - f^{(n-1)}(0),$$

$$\mathcal{L}\{e^{a\,t}\,f(t)\} = F(s-a), \quad \mathcal{L}\{\mathcal{U}(t-a)\,f(t-a)\} = e^{-s\,a}\,F(s),$$

$$\mathcal{L}\{t^n\,f(t)\} = (-1)^n\,\frac{d^n}{ds^n}F(s),$$

$$\mathcal{L}\{f*g\} = \mathcal{L}\left\{\int_0^t f(t-\tau)\,g(\tau)\,d\tau\right\} = F(s)\,G(s),$$

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s},$$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-s\,t_0}.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\,\cos\left(\frac{n\pi x}{p}\right) + b_n\,\sin\left(\frac{n\pi x}{p}\right)\right\}, \quad \text{where}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\,\cos\left(\frac{n\pi x}{p}\right)dx,\ n \ge 0, \quad b_n = \frac{1}{p}\int_{-p}^{p} f(x)\,\sin\left(\frac{n\pi x}{p}\right)dx,\ n \ge 1.$$

$$2\,\sin(A)\,\sin(B) = \cos(A-B) - \cos(A+B),$$

$$2\,\sin(A)\,\cos(B) = \sin(A+B) + \sin(A-B),$$

$$2\,\cos(A)\,\cos(B) = \cos(A+B) + \cos(A-B).$$

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Part I: Provide all details and fully justify your answer in order to receive credit.

1. (a) *(4 pts.)* If the method of undetermined coefficients is used to determine a particular solution $y_p(x)$ of the linear differential equation

$$\frac{d^4y}{dx^4} - 4\,\frac{d^3y}{dx^3} + 24\,\frac{d^2y}{dx^2} - 40\,\frac{dy}{dx} + 100\,y = (x-1)\,e^x\,\cos(3x) + x\,\sin(3x),$$

what is the correct form to use to find $y_p(x)$? (**Do not solve for the coefficients in $y_p(x)$!**)
(**Hint:** $m^4 - 4\,m^3 + 24\,m^2 - 40\,m + 100 = \left(m^2 - 2\,m + 10\right)^2$)

Solution: The auxiliary equation is

$$m^4 - 4\,m^3 + 24\,m^2 - 40\,m + 100 = 0,$$

or

$$(m^2 - 2\,m + 10)^2 = \left((m-1)^2 + 3^2\right)^2 = (m - 1 - 3\,i)^2\,(m - 1 + 3\,i)^2 = 0.$$

Hence, $m = 1 \pm 3\,i$ are both complex roots of order 2. The correct form for $y_p(x)$ is thus

$$\begin{aligned} y_p(x) &= (A\,x + B)\,x^2\,e^x\,\cos(3x) + (C\,x + D)\,x^2\,e^x\,\sin(3x) \\ &\quad + (E\,x + F)\,\cos(3x) + (G\,x + H)\,\sin(3x). \end{aligned}$$

(b) *(4 pts.)* Use the method of **undetermined coefficients** to find the general solution of the differential equation

$$y'' - y' = \sin x + 1.$$

Solution: The auxiliary equation is

$$m^2 - m = m\,(m-1) = 0,$$

so the solution of the associated homogeneous equation is

$$y_c(x) = C_1 + C_2\,e^x.$$

A particular solution has the form

$$y_p(x) = A\,\cos(x) + B\,\sin(x) + C\,x.$$

Since

$$y_p'(x) = -A\,\sin(x) + B\,\cos(x) + C$$

and

$$y_p''(x) = -A\,\cos(x) - B\,\sin(x),$$

we have

$$y_p''(x) - y_p'(x) = -(A+B)\,\cos(x) + (A-B)\,\sin(x) - C.$$

Hence,

$$A + B = 0, \quad A - B = 1, \quad \text{and} \quad -C = 1,$$

from which we deduce that

$$A = \frac{1}{2}, \quad B = -,\frac{1}{2} \quad \text{and} \quad C = -1.$$

So,

$$y_p(x) = \frac{1}{2}\,\cos(x) - \frac{1}{2}\,\sin(x) - x,$$

and the general solution

$$y = y_p(x) + y_c(x) = \frac{1}{2}\,\cos(x) - \frac{1}{2}\,\sin(x) - x + C_1 + C_2\,e^x.$$

Continued...

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2. *(8 pts.)* Use Laplace transforms to solve the first order linear system

$$\begin{cases} y_1'(t) = -6\,y_1(t) - 2\,y_2(t) \\ y_2'(t) = 5\,y_1(t) - 4\,y_2(t) \end{cases}$$

with initial conditions $y_1(0) = -2$ and $y_2(0) = 1$.

Solution: Let $Y_1(s) = \mathcal{L}\{y_1(t)\}$ and $Y_2(s) = \mathcal{L}\{y_2(t)\}$. Taking Laplace transform, we obtain

$$\begin{cases} s\,Y_1(s) + 2 = -6\,Y_1(s) - 2\,Y_2(s) \\ s\,Y_2(s) - 1 = 5\,Y_1(s) - 4\,Y_2(s) \end{cases}$$

or,

$$\begin{cases} (s+6)\,Y_1(s) + 2\,Y_2(s) = -2 \\ -5\,Y_1(s) + (s+4)\,Y_2(s) = 1 \end{cases}$$

or, in matrix form,

$$\begin{bmatrix} (s+6) & 2 \\ -5 & (s+4) \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Using Cramer's rule, we get

$$Y_1(s) = \frac{\begin{vmatrix} -2 & 2 \\ 1 & (s+4) \end{vmatrix}}{\begin{vmatrix} (s+6) & 2 \\ -5 & (s+4) \end{vmatrix}} = \frac{-2\,s - 10}{s^2 + 10\,s + 34} = \frac{-2\,(s+5)}{(s+5)^2 + 3^2},$$

from which we obtain, taking inverse Laplace transform, that

$$y_1(t) = -2\,e^{-5t}\,\cos(3t).$$

Similarly,

$$\begin{aligned} Y_2(s) &= \frac{\begin{vmatrix} s+6 & -2 \\ -5 & 1 \end{vmatrix}}{\begin{vmatrix} (s+6) & 2 \\ -5 & (s+4) \end{vmatrix}} = \frac{s-4}{s^2 + 10\,s + 34} = \frac{(s+5) - 9}{(s+5)^2 + 3^2} \\ &= \frac{(s+5)}{(s+5)^2 + 3^2} - 3\,\frac{3}{(s+5)^2 + 3^2}, \end{aligned}$$

and

$$y_2(t) = e^{-5t}\,\cos(3t) - 3\,e^{-5t}\,\sin(3t).$$

Continued...

3. (a) *(6 pts.)* Compute the Fourier series expansion of the function

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$$

on the interval $-\pi < x < \pi$.

Solution: Since $f(x)$ is odd on $(-\pi, \pi)$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, \cos(n\,x)\, dx = 0, \quad n = 0, 1, 2, \ldots$$

Furthermore,

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, \sin(n\,x)\, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x)\, \sin(n\,x)\, dx = \frac{2}{\pi} \int_{0}^{\pi} \sin(n\,x)\, dx \\ &= \frac{2}{\pi} \left[-\frac{\cos}{(} n\,x) n \right]_0^{\pi} = \frac{2}{\pi} \left[\frac{1 - \cos(n\,\pi)}{n} \right] = \frac{2}{\pi} \left[\frac{1 - (-1)^n}{n} \right], \quad n = 1, 2, 3, \ldots. \end{aligned}$$

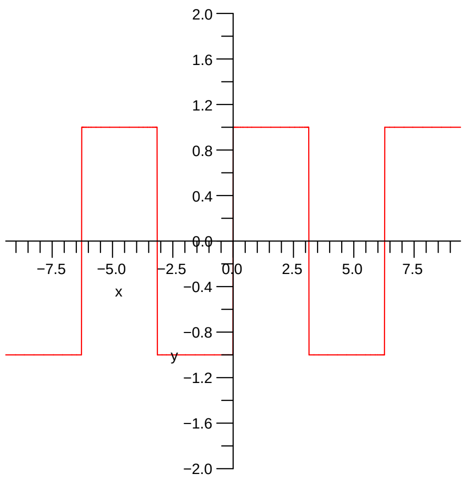
The Fourier series expansion of $f(x)$ on $-\pi < x < \pi$ is thus

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{(1 - (-1)^n)}{n} \sin(n\,x) = \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{4}{n\,\pi} \sin(n\,x) \\ &= \sum_{k=0}^{\infty} \frac{4}{(1 + 2\,k)\,\pi} \sin((1 + 2\,k)\,x) \\ &= \frac{4}{\pi} \sin(x) + \frac{4}{3\,\pi} \sin(3\,x) + \frac{4}{5\,\pi} \sin(5\,x) + \frac{4}{7\,\pi} \sin(7\,x) + \ldots \end{aligned}$$

Continued...

NAME: ______________________ **ID #:**______________________

(b) *(2 pts.)* Consider the Fourier series expansion obtained in part (a) as a function defined on the whole real line (i.e. as a function defined for all values of x).
Sketch the graph of that function for $-3\,\pi < x < 3\pi$.



Continued...

NAME: ______________________ **ID #:**______________________

4. Consider the eigenvalue problem

$$y'' - 4\,y' + \lambda\,y = 0,$$

with boundary conditions $y(0) = y(\pi) = 0$.

(a) *(4 pts.)* Show that there are no eigenvalues λ satisying $\lambda \leq 4$.
(Consider the case $\lambda = 4$ and the case $\lambda < 4$ separately.)

Solution: The auxiliary equation is

$$m^2 - 4\,m + \lambda = 0,$$

and it has roots

$$m = 2 \pm \sqrt{4 - \lambda}.$$

If $4 - \lambda > 0$, iėif $\lambda < 4$, let $\omega = \sqrt{4 - \lambda} > 0$. So the roots of the auxiliary equations are $m_1 = 2 + \omega$ and $m_2 = 2 - \omega$ and the general solution of the corresponding differential equation is

$$y(x) = C_1\,e^{(2+\omega)\,x} + C_2\,e^{(2+\omega)\,x}.$$

The boundary condition $y(0) = 0$ implies $C_2 = -C_1$, so

$$y(x) = C_2\,e^{2\,x}\,(e^{\omega\,x} - e^{-\omega\,x}).$$

The boundary condition $y(\pi) = 0$ implies that

$$C_2\,e^{2\,\pi}\,(e^{\omega\,\pi} - e^{-\omega\,\pi}) = 0.$$

Since $\omega\,\pi > 0$, $e^{\omega\,\pi} \neq e^{-\omega\,\pi}$ and we conclude that $C_2 = 0$, so there is no non-trivial solution. When $\lambda = 4$, $m = 2$ is a double root of the auxiliary equation and the general solution of the D.E. has the form

$$y(x) = C_1\,e^{2\,x} + C_2\,x\,e^{2\,x}$$

. The condition $y(0) = 0$ implies that $C_1 = 0$ and the condition $y(\pi) = 0$ implies then that $C_2\,\pi\,e^{2\pi} = 0$, which shows again that $C_2 = 0$ and there is no non-trivial solution in this case either.

Continued...

NAME: ______________________ **ID #:**______________________

(b) *(6 pts.)* Find all eigenvalues $\lambda > 4$ and all corresponding eigenfunctions.

Solution: When $\lambda > 4$, the solutions of the auxiliary equation are

$$m = 2 \pm i\,\sqrt{\lambda - 4}$$

. Letting $\omega = \sqrt{\lambda - 4} > 0$, we have two roots $m_1 = 2 + i\,\omega$ and $m_2 = 2 - i\,\omega$ and the general solution of the D.E. has the form

$$y(x) = C_1\, e^{2\,x}\, \cos(\omega\, x) + C_2\, e^{2\,x}\, \sin(\omega\, x).$$

The condition $y(0) = 0$ implies that $C_1 = 0$ so that

$$y(x) = C_2\, e^{2\,x}\, \sin(\omega\, x).$$

The condition $y(\pi) = 0$ implies that

$$C_2\, e^{2\,\pi}\, \sin(\omega\, \pi) = 0,$$

If $C_2 \neq 0$, we must have $\sin(\omega\, \pi) = 0$, i. e. $\omega\, \pi = k\, \pi$ or $\omega = k$, for some integer $k \geq 1$. The eigenfunctions are thus of the form

$$\phi_k(x) = C_k\, \sin(k\, x), \quad k = 1, 2, 3, \ldots$$

and the corresponding eigenvalues are

$$\lambda_k = 4 + k^2, \quad k = 1, 2, 3, \ldots$$

Continued...

NAME: ______________________ **ID #:**______________________

5. *(8 pts.)* Use Laplace transforms to compute the solution $x(t)$ of the initial value problem

$$\begin{cases} x''(t) + 2\,x'(t) = g(t) \\ x(0) = 0, \quad x'(0) = 2, \end{cases}$$

where

$$g(t) = \begin{cases} 0, & 0 < t < 1, \\ 1, & 1 < t < 2, \\ 0, & 2 < t. \end{cases}$$

Solution: We have $g(t) = \mathcal{U}(t-1) - \mathcal{U}(t-2)$. Letting $X(s) = \mathcal{L}\{x(t)\}$ and taking Laplace transforms, we obtain

$$(s^2\,X(s) - 2) + 2\,(s\,X(s)) = \frac{e^{-s}}{s} - \frac{e^{-2\,s}}{s},$$

or,

$$(s^2 + 2\,s)\,X(s) = [s\,(s+2)]\,X(s) = 2 + \frac{e^{-s}}{s} - \frac{e^{-2\,s}}{s},$$

so,

$$X(s) = \frac{2}{s(s+2)} + \frac{e^{-s}}{s^2\,(s+2)} - \frac{e^{-2\,s}}{s^2\,(s+2)}.$$

Using partial fractions,

$$\frac{2}{s(s+2)} = \frac{a}{s} + \frac{b}{s+2} = \frac{s(a+b) + 2\,a}{s\,(s+2)},$$

so $a = 1$, $b = -2$ and $\frac{2}{s(s+2)} = \frac{1}{s} - \frac{1}{s+2}$. Also,

$$\begin{aligned} \frac{1}{s^2\,(s+2)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C\,s + D}{s+2} = \frac{A\,s(s+2) + B\,(s+2) + (C\,s + D)\,s^2}{s^2\,(s+2)} \\ &= \frac{C\,s^3 + (A+D)\,s^2 + (2\,A + B)\,s + 2\,B}{s^2\,(s+2)}, \end{aligned}$$

so, $B = 1/2$, $A = -1/4$, $D = 1/4$ and $C = 0$. Thus, $\frac{1}{s^2\,(s+2)} = \frac{1}{4}\left(\frac{-1}{s} + \frac{2}{s^2} + \frac{1}{s+2}\right)$. Therefore,

$$X(s) = \frac{1}{s} - \frac{1}{s+2} + \frac{e^{-s}}{4}\left(\frac{-1}{s} + \frac{2}{s^2} + \frac{1}{s+2}\right) - \frac{e^{-2\,s}}{4}\left(\frac{-1}{s} + \frac{2}{s^2} + \frac{1}{s+2}\right),$$

and, taking inverse Laplace transforms,

$$x(t) = 1 - e^{-2\,t} + \left[-1 + 2\,(t-1) + e^{-2(t-1)}\right]\frac{\mathcal{U}(t-1)}{4} - \left[-1 + 2\,(t-2) + e^{-2(t-2)}\right]\frac{\mathcal{U}(t-2)}{4}.$$

Continued...

NAME: ______________________ **ID #:**______________________

6. *(8 pts.)* Solve the Cauchy-Euler initial value problem

$$\begin{cases} 2\,x^2\,y'' - x\,y' + y = x, \quad x > 0, \\ y(1) = 0,\ y'(1) = 0. \end{cases}$$

Solution: The auxiliary equation for the associated Cauchy-Euler homogeneous equation is

$$2\,m\,(m-1) - m + 1 = 0 \quad \text{or} \quad 2\,m^2 - 3\,m + 1 = 0$$

which has solutions $m = \frac{3\pm\sqrt{9-8}}{4}$, i. e. $m = 1$ and $m = 1/2$. The complementary solution is thus

$$y_c(x) = C_1\,x + C_2\,x^{1/2}.$$

Using the method of variation of parameters with $y_1(x) = x$ and $y_2(x) = x^{1/2}$, we compute

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^{1/2} \\ 1 & \frac{1}{2}\,x^{-1/2} \end{vmatrix} = -\frac{1}{2}\,x^{1/2}$$

Since the right-hand side of the differential equation written in standard form is $f(x) = \dfrac{1}{2\,x}$, there exist a particular solution $y_p(x)$ given by

$$\begin{aligned} y_p(x) &= -\left(\int \frac{f(x)\,y_2(x)}{W(y_1,y_2)(x)}\,dx\right)\,y_1(x) + \left(\int \frac{f(x)\,y_1(x)}{W(y_1,y_2)(x)}\,dx\right)\,y_2(x) \\ &= -\left(\int \frac{\left(\frac{1}{2\,x}\right)\,x^{1/2}}{-\frac{1}{2}\,x^{1/2}}\,dx\right)\,x + \left(\int \frac{\left(\frac{1}{2\,x}\right)\,x}{-\frac{1}{2}\,x^{1/2}}\,dx\right)\,x^{1/2} \\ &= \left(\int \frac{1}{x}\,dx\right)\,x - \left(\int x^{-1/2}\,dx\right)\,x^{1/2} \\ &= (\ln x)\,x - \left(2\,x^{1/2}\right)\,x^{1/2} = (\ln x)\,x - 2\,x \end{aligned}$$

The general solution is thus

$$y(x) = C_1\,x + C_2\,x^{1/2} + (\ln x)\,x - 2\,x, \quad C_1, C_2 \text{ arbitrary.}$$

or, more simply,

$$y(x) = C_1\,x + C_2\,x^{1/2} + (\ln x)\,x, \quad C_1, C_2 \text{ arbitrary.}$$

The condition $y(1) = 0$ yields $C_1 + C_2 = 0$. Since

$$y'(x) = C_1 + \frac{C_2}{2}\,x^{-1/2} + 1 + \ln x,$$

the condition $y'(1) = 0$ yields $C_1 + \frac{C_2}{2} = -1$. Hence $C_1 = -2$ and $C_2 = 2$. The solution is thus

$$\boxed{y(x) = -2\,x + 2\,x^{1/2} + (\ln x)\,x.}$$

Continued...

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PART II: Multiple choice part. Circle the letter corresponding to the correct answer for each of the questions 7 to 16 in the box below. Indicate your choice very clearly. Ambiguous answers will be marked as wrong. There is only one correct choice for each question and an incorrect answer scores 0 marks.

QUESTION #	ANSWER:				
7.	**A**	**B**	**C**	**D**	**E**
8.	**A**	**B**	**C**	**D**	**E**
9.	**A**	**B**	**C**	**D**	**E**
10.	**A**	**B**	**C**	**D**	**E**
11.	**A**	**B**	**C**	**D**	**E**
12.	**A**	**B**	**C**	**D**	**E**
13.	**A**	**B**	**C**	**D**	**E**
14.	**A**	**B**	**C**	**D**	**E**
15.	**A**	**B**	**C**	**D**	**E**
16.	**A**	**B**	**C**	**D**	**E**

Continued...

7. *(5 pts.)* Let $y(t)$ be the solution of the initial value problem

$$\begin{cases} y' - 2\,t\,y = 2\,t, \\ y(0) = 1. \end{cases}$$

Then, $y(1)$ is

(A) e^{-1}.

(B) $e^2 - 2$.

$\rightarrow$ **(C)** $e - 1$.

(D) $2\,e - 1$.

(E) $e^3 + \frac{1}{2}$.

Solution. This is a 1st order linear DE. The integrating factor is

$$u(x) = e^{\int (-2\,t)\,dt} = e^{-t^2}.$$

We have thus

$$e^{-t^2}\,y' - 2\,t\,e^{-t^2}\,y = \left(e^{-t^2}\,y\right)' = 2\,t\,e^{-t^2}.$$

and, by integration,

$$e^{-t^2}\,y = -e^{-t^2} + C$$

The condition $y(0) = 1$ yields $C = 1$. Hence,

$$y(t) = -1 + e^{t^2}$$

and

$$y(1) = e - 1$$

Continued...

8. *(5 pts.)* Let $f(t)$ be the inverse Laplace transform of

$$F(s) = \frac{(1 + e^{-\pi s})}{s\,(s^2 + 9)}.$$

Then, for $t > \pi$, $f(t)$ is equal to:

(A) $\frac{1}{9}\,(1 - \cos(3\,t))$.

(B) $e^{3\,t} + t$.

$\rightarrow$ **(C)** $\frac{2}{9}$.

(D) $\frac{2}{9} + 2\,\cos(3\,t)$.

(E) $\frac{1}{9}\,(\sin(3\,t))$.

Solution. We have

$$\frac{1}{s\,(s^2+9)} = \frac{A}{s} + \frac{B\,s + C}{s^2+9} = \frac{(A+B)\,s^2 + C\,s + 9\,A}{s\,(s^2+9)}$$

which yields $A = \frac{1}{9}$, $B = -\frac{1}{9}$ and $C = 0$. Thus,

$$\frac{1}{s\,(s^2+9)} = \frac{1}{9}\,\left\{\frac{1}{s} - \frac{s}{s^2+9}\right\}$$

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+9}\right\} = 1 - \cos(3\,t),$$

we have

$$\begin{aligned} f(t) &= \frac{1}{9}\,\{[1 - \cos(3\,t)] + [1 - \cos(3\,(t-\pi))]\,\mathcal{U}(t-\pi)\} \\ &= \frac{1}{9}\,\{[1 - \cos(3\,t)] + [1 + \cos(3\,t)]\,\mathcal{U}(t-\pi)\} \end{aligned}$$

Therefore,

$$f(t) = \frac{2}{9} \quad \text{for } t > \pi.$$

Continued...

NAME: ______________________ **ID #:**______________________

9. *(5 pts.)* Let $F(s)$ be the Laplace transform of

$$f(t) = \int_0^t \tau^2 \sin(\tau)\, d\tau.$$

Then, $F(2)$ is equal to:

(A) 0.

(B) $\frac{5}{43}$.

(C) $-\frac{4}{173}$.

$\rightarrow$ **(D)** $\frac{11}{125}$.

(E) $-\frac{7}{56}$.

Solution. Since $\mathcal{L}\left\{\sin(t)\right\} = \frac{1}{s^2+1}$, we have

$$\begin{aligned}
\mathcal{L}\left\{t^2 \sin(t)\right\} &= \frac{d}{ds^2}\left\{\frac{1}{s^2+1}\right\} = \frac{d}{ds}\left\{-\frac{2\,s}{(s^2+1)^2}\right\} \\
&= -\frac{2\,(s^2+1)^2 - (2\,s)\,2\,(s^2+1)(2\,s)}{(s^2+1)^4} \\
&= \frac{-2\,(s^2+1) + 8\,s^2}{(s^2+1)^3} \\
&= \frac{2\,(3\,s^2-1)}{(s^2+1)^3}
\end{aligned}$$

and thus

$$F(s) = \mathcal{L}\left\{\int_0^t \tau^2 \sin(\tau)\, d\tau\right\} = \frac{2\,(3\,s^2-1)}{s\,(s^2+1)^3}.$$

Therefore,

$$F(2) = \frac{2\,((3)\,(4)-1)}{2\,(4+1)^3} = \frac{22}{2\,(5^3)} = \boxed{\frac{11}{125}.}$$

Continued...

NAME: ______________________ **ID #:**______________________

10. *(5 pts)* The **best** one can say, without actually solving the differential equation

$$(x+2)\,(x^2+9)\,y'' + 3\,(x+5)\,y' + (x^2+1)\,y = 0,$$

is that the radius of convergence of a power series solution about $x_0 = 0$ is at least:

(A) ∞.

(B) 1.

$\rightarrow$ **(C)** 2.

(D) 3.

(E) 5.

Solution. In standard form, the DE is written as

$$y'' + \frac{3\,(x+5)}{(x+2)\,(x^2+9)}\,y' + \frac{(x^2+1)}{(x+2)\,(x^2+9)}\,y = 0.$$

The singular points are $x = -2$ and $x = \pm 3\,i$. The closest singular point to 0 is thus $x = -2$ which is at a distance of 2 from 0. The radius of convergence of a power series solution about $x_0 = 0$ is thus at least 2.

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11. *(5 pts)* Consider the differential equation

$$(1+x)\,y'' - 2\,x\,y' - 4\,y = 0.$$

The recurrence formula for the coefficients c_n, of the power series solution $y(x) = \sum_{n=0}^{\infty} c_n\,x^n$ about the ordinary point $x_0 = 0$ is given by:

$\rightarrow$ **(A)** $c_{n+2} = -\frac{n}{n+2}\,c_{n+1} + \frac{2}{n+1}\,c_n,\ n = 0, 1, 2, \ldots$

(B) $c_{n+2} = \frac{n+4}{n+2}\,c_n,\ n = 0, 1, 2, \ldots$

(C) $c_{n+2} = -\frac{n+1}{n+2}\,c_{n+1} + \frac{2\,n}{n+1}\,c_n,\ n = 0, 1, 2, \ldots$

(D) $c_{n+2} = -\frac{n+2}{n}\,c_{n+1} + \frac{1}{n+1}\,c_n,\ n = 0, 1, 2, \ldots$

(E) $c_{n+2} = -\frac{n+3}{n+2}\,c_{n+1} + \frac{n}{n+3}\,c_n,\ n = 0, 1, 2, \ldots$

Solution. Let $y(x) = \sum_{n=0}^{\infty} c_n\,x^n$. Then, we have

$$\begin{aligned}
0 &= (1+x)\,y'' - 2\,x\,y' - 4\,y \\
&= (1+x)\left\{\sum_{n=2}^{\infty} c_n\,n\,(n-1)\,x^{n-2}\right\} - 2\,x\left\{\sum_{n=0}^{\infty} c_n\,n\,x^{n-1}\right\} - 4\left\{\sum_{n=0}^{\infty} c_n\,x^n\right\} \\
&= \left\{\sum_{n=2}^{\infty} c_n\,n\,(n-1)\,x^{n-2}\right\} + \left\{\sum_{n=2}^{\infty} c_n\,n\,(n-1)\,x^{n-1}\right\} \\
&+ \left\{\sum_{n=0}^{\infty}(-2\,c_n)\,n\,x^n\right\} + \left\{\sum_{n=0}^{\infty}(-4\,)c_n\,x^n\right\} \\
&= \left\{\sum_{n=0}^{\infty} c_{n+2}\,(n+2)\,(n+1)\,x^n\right\} + \left\{\sum_{n=0}^{\infty} c_{n+1}\,(n+1)\,n\,x^n\right\} \\
&+ \left\{\sum_{n=0}^{\infty}(-2\,c_n)\,n\,x^n\right\} + \left\{\sum_{n=0}^{\infty}(-4\,)c_n\,x^n\right\} \\
&= \sum_{n=0}^{\infty}\left\{c_{n+2}\,(n+2)\,(n+1)\ + c_{n+1}\,(n+1)\,n - (2\,n+4)\ c_n\right\}\,x^n
\end{aligned}$$

This shows that

$$c_{n+2}\,(n+2)\,(n+1)\ + c_{n+1}\,(n+1)\,n - (2\,n+4)\ c_n = 0, \quad n \geq 0,$$

or, equivalently, that

$$\boxed{c_{n+2} = -\frac{n}{n+2}\,c_{n+1} + \frac{2}{n+1}\,c_n, \quad n \geq 0.}$$

Continued...

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12. *(5 pts)* Consider the differential equation

$$y'' + P(x)\,y' + Q(x)\,y = 0,$$

where $P(x)$ and $Q(x)$ are polynomials. The recurrence relation for the coefficients c_n of the power series solution $y(x) = \sum_{n=0}^{\infty} c_n\,x^n$ about the ordinary point $x_0 = 0$ is

$$c_{n+2} = \frac{4-n}{(2\,n+1)\,(n+2)}\,c_n, \quad n = 0, 1, 2, \ldots$$

A solution of the differential equation is given by the polynomial:

(A) $y(x) = x + 2\,x^3 + \frac{5}{3}\,x^5$.

(B) $y(x) = 1 + x^2 + \frac{1}{10}\,x^4$.

(C) $y(x) = 3\,x + \frac{3}{35}\,x^3 - \frac{3}{77}\,x^5$.

→ **(D)** $y(x) = 1 + 2\,x^2 + \frac{1}{5}\,x^4$.

(E) $y(x) = x^2 + \frac{1}{20}\,x^4$.

Solution. We have, using the recurrence relation,

$$c_2 = \frac{4}{(1)\,(2)}\,c_0 = 2\,c_0$$

$$c_4 = \frac{2}{(5)\,(4)}\,c_2 = \frac{1}{5}\,c_0$$

and $c_{2\,n} = 0$ for $n \geq 3$. Also,

$$c_3 = \frac{3}{(3)\,(3)}\,c_1 = \frac{1}{3}\,c_1,$$

$$c_5 = \frac{1}{(7)\,(5)}\,c_3 = \frac{1}{105}\,c_1.$$

The general solution is thus

$$y(x) = c_0\,\left\{1 + 2\,x^2 + \frac{1}{5}\,x^4\right\} + c_1\,\left\{x + \frac{1}{3}\,x^3 + \frac{1}{105}\,x^5 + \ldots\right\}.$$

Choosing $c_0 = 1$ and $c_1 = 0$, we obtain the solution

$$y(x) = 1 + 2\,x^2 + \frac{1}{5}\,x^4.$$

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13. *(5 pts)* A complete list of the **regular singular points** for the differential equation

$$x\,(x-1)^2\,(x+2)^2\,(x-2)^4\,y'' - (x-1)\,(x-2)^3\,y' + x\,(x-2)\,y = 0$$

is given by:

(A) $1, -2.$

(B) $0, 1, 2.$

(C) $-1, 0, 1.$

(D) $0, 2.$

→ **(E)** $0, 1.$

Solution. In standard form, the DE is written as

$$y'' - \left\{\frac{1}{x\,(x-1)\,(x+2)^2\,(x-2)}\right\} y' + \left\{\frac{1}{(x-1)^2\,(x+2)^2\,(x-2)^3}\right\} y = 0.$$

The singular points are thus -2, 0, 1, 2. We have

$$P(x) = \frac{1}{x\,(x-1)\,(x+2)^2\,(x-2)} \quad \text{and} \quad Q(x) = \frac{1}{(x-1)^2\,(x+2)^2\,(x-2)^3}$$

Since

$$(x+2)\,P(x) = \frac{1}{x\,(x-1)\,(x+2)\,(x-2)}$$

is not analytic at $x = -2$, -2 is an irregular singular point. Since

$$(x-2)^2\,Q(x) = \frac{1}{(x-1)^2\,(x+2)^2\,(x-2)}$$

is not analytic at $x = 2$, 2 is an irregular singular point. Since

$$x\,P(x) = \frac{1}{(x-1)\,(x+2)^2\,(x-2)} \quad \text{and} \quad x^2\,Q(x) = \frac{x^2}{(x-1)^2\,(x+2)^2\,(x-2)^3}$$

are both analytic at $x = 0$, 0 is a regular singular point. Since

$$(x-1)\,P(x) = \frac{1}{x\,(x+2)^2\,(x-2)} \quad \text{and} \quad (x-1)^2\,Q(x) = \frac{1}{(x+2)^2\,(x-2)^3}$$

are both analytic at $x = 1$, 1 is a regular singular point.
The regular singular points are thus 0 and 1.

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14. *(5 pts)* Consider the differential equation

$$2\,(x-1)^2\,y'' - (x-1)\,y' - 2\,y = 0.$$

The roots of the indicial equation for the series solution about the regular singular point $x_0 = 1$ are:

(A) $-\frac{1}{3}, 3.$

(B) $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}.$

$\rightarrow$ **(C)** $-\frac{1}{2}, 2.$

(D) $\frac{1+\sqrt{17}}{4}, \frac{1-\sqrt{17}}{4}.$

(E) $-\frac{1}{3}, 1.$

Solution. In standard form, the DE is written as

$$y'' - \frac{1}{2\,(x-1)}\,y' - \frac{1}{(x-1)^2}\,y = 0.$$

We have thus

$$P(x) = -\frac{1}{2\,(x-1)} \quad \text{and} \quad Q(x) = -\frac{1}{(x-1)^2}.$$

Hence,

$$(x-1)\,P(x) = -\frac{1}{2} \quad \text{and} \quad (x-1)^2\,Q(x) = -1.$$

Hence, $a_0 = -\frac{1}{2}$ and $b_0 = -1$. The indicial equation is thus

$$r\,(r-1) - \frac{1}{2}\,r - 1 = 0 \quad \text{or} \quad r^2 - \frac{3}{2}\,r - 1 = 0$$

and its roots are

$$r = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4}+4}}{2} = \frac{\frac{3}{2} \pm \sqrt{\frac{25}{4}}}{2} = \frac{\frac{3}{2} \pm \frac{5}{2}}{2} = \boxed{2 \text{ or } -\frac{1}{2}.}$$

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15. *(5 pts)* Let $f(x)$ be a 1-periodic function (i. e. $f(x+1) = f(x)$ for all x) and suppose that

$$f(x) = \begin{cases} x^2, & 0 \le x < \frac{1}{2} \\ \frac{1}{4}, & \frac{1}{2} \le x < 1. \end{cases}$$

Consider the expansion of $f(x)$ as a 1-periodic Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(2n\pi x) + b_n \sin(2n\pi x)\}.$$

To which value does the Fourier series converge at $x = 2$?

(A) $\frac{1}{2}$.

(B) $-\frac{1}{4}$.

(C) $+\infty$.

$\rightarrow$ **(D)** $\frac{1}{8}$.

(E) 0.

Solution. Since $f(2^+) = f(0^+) = 0$ and $f(2^-) = f(1^-) = \frac{1}{4}$, the Fourier series evaluated at $x = 2$ converges to

$$\frac{f(2^+) + f(2^-)}{2} = \boxed{\frac{1}{8}}.$$

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16. *(5 pts)* Suppose the function $f(x)$ defined for $0 \le x \le \pi$ satisfies

$$\frac{2}{\pi}\int_0^{\pi} f(x)\,\sin(nx)\,dx = \frac{1}{2^n}.$$

Then, the solution $u(x,t)$ to the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = 2\,\dfrac{\partial^2 u}{\partial x^2}, & 0 < x < \pi, \qquad t > 0, \\ u(0,t) = 0, & u(\pi,t) = 0, \qquad t > 0, \\ u(x,0) = f(x), & 0 < x < \pi, \end{cases}$$

is given by the series:

(A) $\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{2^n}\, e^{-2nt}\, \cos(nx)$.

(B) $\sum_{n=1}^{\infty} \frac{1}{2^n}\, e^{-2n^2\pi^2 t}\, \sin(n\pi x)$.

→ **(C)** $\sum_{n=1}^{\infty} \frac{1}{2^n}\, e^{-2n^2 t}\, \sin(nx)$.

(D) $\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{2^n}\, e^{-n^2t^2}\, \sin(nx)$.

(E) $\sum_{n=1}^{\infty} \frac{1}{2^n}\, e^{-n\pi t}\, \sin(n^2 x)$.

Solution. The solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n\, e^{-2n^2 t}\, \sin(nx)$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,\sin(nx)\,dx = \frac{1}{2^n}, \quad n \ge 1.$$

Thus,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1}{2^n}\, e^{-2n^2 t}\, \sin(nx)$$

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