LAST (family) NAME: ______________________

FIRST (given) NAME: ______________________

ID # : ____________________

MATHEMATICS 2P04

McMaster University Final Examination
Duration of Examination: 3 hours

SAMPLE FINAL EXAM B

THIS EXAMINATION PAPER INCLUDES 24 PAGES AND 15 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

Instructions:

- Indicate your answers clearly in the spaces provided. In the full answer questions in Part I (questions 1 - 5) show all your work to receive full credit. For the multiple choice questions in Part II (questions 6 - 15) be sure to circle the correct letters on page 12 to receive full credit.
- No books, notes, or "cheat sheets" allowed. The only calculator permitted is the McMaster Standard Calculator, the Casio fx 991.
- The total number of points is 100.
- There is a formula sheet included with the exam on page 24.
- Pages 23 of the test is for scratch work or overflow. If you continue the solution to a question on this page, you must indicate this clearly on the page containing the original question. **GOOD LUCK!**

#	Mark	#	Mark
1.		**4.**	
2.		**5.**	
3.		**6-15.**	
		TOTAL	

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Some formulas you may use:

$$y_p(x) = -\left[\int \frac{y_2(x)\,f(x)}{W(y_1,y_2)}\,dx\right]\,y_1(x) + \left[\int \frac{y_1(x)\,f(x)}{W(y_1,y_2)}\,dx\right]\,y_2(x)$$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\,e^{-st}\,dt.$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{e^{a\,t}\} = \frac{1}{s-a},$$

$$\mathcal{L}\{\sin(k\,t)\} = \frac{k}{s^2+k^2}, \quad \mathcal{L}\{\cos(k\,t)\} = \frac{s}{s^2+k^2},$$

$$\mathcal{L}\{f'(t)\} = s\,F(s) - f(0), \quad \mathcal{L}\{f''(t)\} = s^2\,F(s) - s\,f(0) - f'(0),$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\,F(s) - s^{n-1}\,f(0) - s^{n-2}\,f'(0) - \cdots - s\,f^{(n-2)}(0) - f^{(n-1)}(0),$$

$$\mathcal{L}\{e^{a\,t}\,f(t)\} = F(s-a), \quad \mathcal{L}\{\mathcal{U}(t-a)\,f(t-a)\} = e^{-s\,a}\,F(s),$$

$$\mathcal{L}\{t^n\,f(t)\} = (-1)^n\,\frac{d^n}{ds^n}F(s),$$

$$\mathcal{L}\{f*g\} = \mathcal{L}\left\{\int_0^t f(t-\tau)\,g(\tau)\,d\tau\right\} = F(s)\,G(s),$$

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s},$$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-s\,t_0}.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\,\cos\left(\frac{n\pi x}{p}\right) + b_n\,\sin\left(\frac{n\pi x}{p}\right)\right\}, \quad \text{where}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\,\cos\left(\frac{n\pi x}{p}\right)dx,\ n \geq 0, \quad b_n = \frac{1}{p}\int_{-p}^{p} f(x)\,\sin\left(\frac{n\pi x}{p}\right)dx,\ n \geq 1.$$

$$2\,\sin(A)\,\sin(B) = \cos(A-B) - \cos(A+B),$$

$$2\,\sin(A)\,\cos(B) = \sin(A+B) + \sin(A-B),$$

$$2\,\cos(A)\,\cos(B) = \cos(A+B) + \cos(A-B).$$

Continued...

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Part I: Provide all details and fully justify your answer in order to receive credit.

1. (a) *(4 pts.)* If the method of undetermined coefficients is used to determine a particular solution $y_p(x)$ of the linear differential equation

$$\frac{d^4y}{dx^4} - 4\,\frac{d^3y}{dx^3} + 24\,\frac{d^2y}{dx^2} - 40\,\frac{dy}{dx} + 100\,y = (x-1)\,e^x\,\cos(3x) + x\,\sin(3x),$$

what is the correct form to use to find $y_p(x)$? (**Do not solve for the coefficients in** $y_p(x)$**!**)
(**Hint:** $m^4 - 4\,m^3 + 24\,m^2 - 40\,m + 100 = \left(m^2 - 2\,m + 10\right)^2$)

Continued...

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(b) *(4 pts.)* Use the method of **undetermined coefficients** to find the general solution of the differential equation

$$y'' - y' = \sin x + 1.$$

Continued...

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2. *(8 pts.)* Use Laplace transforms to solve the first order linear system

$$\begin{cases} y_1'(t) = -6\,y_1(t) - 2\,y_2(t) \\ y_2'(t) = 5\,y_1(t) - 4\,y_2(t) \end{cases}$$

with initial conditions $y_1(0) = -2$ and $y_2(0) = 1$.

Continued...

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3. (a) *(6 pts.)* Compute the Fourier series expansion of the function

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ \quad 1, & 0 \leq x < \pi \end{cases}$$

on the interval $-\pi < x < \pi$.

Continued...

(b) *(2 pts.)* Consider the Fourier series expansion obtained in part (a) as a function defined on the whole real line (i.e. as a function defined for all values of x).
Sketch the graph of that function for $-3\,\pi < x < 3\pi$.

Continued...

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4. Consider the eigenvalue problem

$$y'' - 4\,y' + \lambda\,y = 0,$$

with boundary conditions $y(0) = y(\pi) = 0$.

(a) *(4 pts.)* Show that there are no eigenvalues λ satisying $\lambda \leq 4$.
(Consider the case $\lambda = 4$ and the case $\lambda < 4$ separately.)

Continued...

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(b) *(6 pts.)* Find all eigenvalues $\lambda > 4$ and all corresponding eigenfunctions.

Continued...

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5. *(8 pts.)* Use Laplace transforms to compute the solution $x(t)$ of the initial value problem

$$\begin{cases} x''(t) + 2\,x'(t) = g(t) \\ x(0) = 0, \quad x'(0) = 2, \end{cases}$$

where

$$g(t) = \begin{cases} 0, & 0 < t < 1, \\ 1, & 1 < t < 2, \\ 0, & 2 < t. \end{cases}$$

Continued...

NAME: ______________________ **ID #:**______________________

6. *(8 pts.)* Solve the Cauchy-Euler initial value problem

$$\begin{cases} 2\,x^2\,y'' - x\,y' + y = x, \quad x > 0, \\ y(1) = 0,\ y'(1) = 0. \end{cases}$$

Continued...

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PART II: Multiple choice part. Circle the letter corresponding to the correct answer for each of the questions 7 to 16 in the box below. Indicate your choice very clearly. Ambiguous answers will be marked as wrong. There is only one correct choice for each question and an incorrect answer scores 0 marks.

QUESTION #	ANSWER:				
7.	A	B	C	D	E
8.	A	B	C	D	E
9.	A	B	C	D	E
10.	A	B	C	D	E
11.	A	B	C	D	E
12.	A	B	C	D	E
13.	A	B	C	D	E
14.	A	B	C	D	E
15.	A	B	C	D	E
16.	A	B	C	D	E

Continued...

NAME: ________________________ **ID #:**________________________

7. *(5 pts.)* Let $y(t)$ be the solution of the initial value problem

$$\begin{cases} y' - 2\,t\,y = 2\,t, \\ y(0) = 1. \end{cases}$$

Then, $y(1)$ is

(A) e^{-1}.

(B) $e^2 - 2$.

(C) $e - 1$.

(D) $2\,e - 1$.

(E) $e^3 + \frac{1}{2}$.

Continued...

NAME: ______________________ **ID #:**______________________

8. *(5 pts.)* Let $f(t)$ be the inverse Laplace transform of

$$F(s) = \frac{(1 + e^{-\pi s})}{s\,(s^2 + 9)}.$$

Then, for $t > \pi$, $f(t)$ is equal to:

(A) $\frac{1}{9}\,(1 - \cos(3\,t))$.

(B) $e^{3\,t} + t$.

(C) $\frac{2}{9}$.

(D) $\frac{2}{9} + 2\,\cos(3\,t)$.

(E) $\frac{1}{9}\,(\sin(3\,t))$.

Continued. . .

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9. *(5 pts.)* Let $F(s)$ be the Laplace transform of

$$f(t) = \int_0^t \tau^2 \sin(\tau)\, d\tau.$$

Then, $F(2)$ is equal to:

(A) 0.

(B) $\frac{5}{43}$.

(C) $-\frac{4}{173}$.

(D) $\frac{11}{125}$.

(E) $-\frac{7}{56}$.

Continued...

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10. *(5 pts)* The **best** one can say, without actually solving the differential equation

$$(x+2)\,(x^2+9)\,y'' + 3\,(x+5)\,y' + (x^2+1)\,y = 0,$$

is that the radius of convergence of a power series solution about $x_0 = 0$ is at least:

(A) ∞.

(B) 1.

(C) 2.

(D) 3.

(E) 5.

Continued...

NAME: ______________________ **ID #:**______________________

11. *(5 pts)* Consider the differential equation

$$(1+x)\,y'' - 2\,x\,y' - 4\,y = 0.$$

The recurrence formula for the coefficients c_n, of the power series solution $y(x) = \sum_{n=0}^{\infty} c_n\, x^n$ about the ordinary point $x_0 = 0$ is given by:

(A) $c_{n+2} = -\frac{n}{n+2}\, c_{n+1} + \frac{2}{n+1}\, c_n,\ n = 0, 1, 2, \ldots$

(B) $c_{n+2} = \frac{n+4}{n+2}\, c_n,\ n = 0, 1, 2, \ldots$

(C) $c_{n+2} = -\frac{n+1}{n+2}\, c_{n+1} + \frac{2\,n}{n+1}\, c_n,\ n = 0, 1, 2, \ldots$

(D) $c_{n+2} = -\frac{n+2}{n}\, c_{n+1} + \frac{1}{n+1}\, c_n,\ n = 0, 1, 2, \ldots$

(E) $c_{n+2} = -\frac{n+3}{n+2}\, c_{n+1} + \frac{n}{n+3}\, c_n,\ n = 0, 1, 2, \ldots$

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12. *(5 pts)* Consider the differential equation

$$y'' + P(x)\,y' + Q(x)\,y = 0,$$

where $P(x)$ and $Q(x)$ are polynomials. The recurrence relation for the coefficients c_n of the power series solution $y(x) = \sum_{n=0}^{\infty} c_n\,x^n$ about the ordinary point $x_0 = 0$ is

$$c_{n+2} = \frac{4-n}{(2\,n+1)\,(n+2)}\,c_n, \quad n = 0, 1, 2, \ldots$$

A solution of the differential equation is given by the polynomial:

(A) $y(x) = x + 2\,x^3 + \frac{5}{3}\,x^5$.

(B) $y(x) = 1 + x^2 + \frac{1}{10}\,x^4$.

(C) $y(x) = 3\,x + \frac{3}{35}\,x^3 - \frac{3}{77}\,x^5$.

(D) $y(x) = 1 + 2\,x^2 + \frac{1}{5}\,x^4$.

(E) $y(x) = x^2 + \frac{1}{20}\,x^4$.

Continued...

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13. *(5 pts)* A complete list of the **regular singular points** for the differential equation

$$x\,(x-1)^2\,(x+2)^2\,(x-2)^4\,y'' - (x-1)\,(x-2)^3\,y' + x\,(x-2)\,y = 0$$

is given by:

(A) $1, -2.$

(B) $0, 1, 2.$

(C) $-1, 0, 1.$

(D) $0, 2.$

(E) $0, 1.$

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14. *(5 pts)* Consider the differential equation

$$2\,(x-1)^2\,y'' - (x-1)\,y' - 2\,y = 0.$$

The roots of the indicial equation for the series solution about the regular singular point $x_0 = 1$ are:

(A) $-\frac{1}{3}, 3.$

(B) $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}.$

(C) $-\frac{1}{2}, 2.$

(D) $\frac{1+\sqrt{17}}{4}, \frac{1-\sqrt{17}}{4}.$

(E) $-\frac{1}{3}, 1.$

Continued...

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15. *(5 pts)* Let $f(x)$ be a 1-periodic function (i. e. $f(x+1) = f(x)$ for all x) and suppose that

$$f(x) = \begin{cases} x^2, & 0 \le x < \frac{1}{2} \\ \frac{1}{4}, & \frac{1}{2} \le x < 1. \end{cases}$$

Consider the expansion of $f(x)$ as a 1-periodic Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(2n\pi x) + b_n \sin(2n\pi x) \right\}.$$

To which value does the Fourier series converge at $x = 2$?

(A) $\frac{1}{2}$.

(B) $-\frac{1}{4}$.

(C) $+\infty$.

(D) $\frac{1}{8}$.

(E) 0.

Continued...

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16. *(5 pts)* Suppose the function $f(x)$ defined for $0 \le x \le \pi$ satisfies

$$\frac{2}{\pi}\int_0^{\pi} f(x)\, \sin(nx)\, dx = \frac{1}{2^n}.$$

Then, the solution $u(x,t)$ to the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = 2\,\dfrac{\partial^2 u}{\partial x^2}, & 0 < x < \pi, \qquad t > 0, \\ u(0,t) = 0, & u(\pi,t) = 0, \qquad t > 0, \\ u(x,0) = f(x), & 0 < x < \pi, \end{cases}$$

is given by the series:

(A) $\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{2^n}\, e^{-2nt}\, \cos(nx)$.

(B) $\sum_{n=1}^{\infty} \frac{1}{2^n}\, e^{-2n^2\pi^2 t}\, \sin(n\pi x)$.

(C) $\sum_{n=1}^{\infty} \frac{1}{2^n}\, e^{-2n^2 t}\, \sin(nx)$.

(D) $\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{2^n}\, e^{-n^2 t^2}\, \sin(nx)$.

(E) $\sum_{n=1}^{\infty} \frac{1}{2^n}\, e^{-n\pi t}\, \sin(n^2 x)$.

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SCRATCH

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SCRATCH

*** **END** ***