

LAST (Family) NAME: _____

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MATHEMATICS 2P04

Sample Final Exam

Duration of Examination: 3 hours

THIS EXAMINATION PAPER INCLUDES 26 PAGES AND 16 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

Instructions:

- Indicate your answers clearly in the spaces provided. In the full answer questions in Part I (questions 1 - 6) show all your work to receive full credit. For the multiple choice questions in Part II (questions 7-16) be sure to circle the correct letters on page 14 to receive full credit.
- No books, notes, or “cheat sheets” allowed.
- The only calculator permitted is the McMaster Standard Calculator, the Casio fx 991.
- The total number of points is 100.
- There is a formula sheet included with the exam on page 24.
- Page 23 of the test is for scratch work or overflow. If you continue the solution to a question on one this page, you must indicate this clearly on the page containing the original question. **GOOD LUCK!**

#	Mark	#	Mark
1.		6.	
2.			
3.			
4.			
5.		7-16.	
		TOTAL	

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Part I: Provide all details and fully justify your answer in order to receive credit.

1. (a) (4 pts.) If the method of undetermined coefficients is used to determine a particular solution $y_p(x)$ of the linear differential equation

$$y^{(4)} - 4y' + 3y = x^2 e^x + x \cos(2x),$$

what is the correct form to use to find $y_p(x)$? (**Do not solve for the coefficients in $y_p(x)$!**)

Hint: $m^4 - 4m + 3 = (m - 1)^2 (m^2 + 2m + 3)$

Solution. Since the complementary solution is

$$y_c(x) = C_1 e^x + C_2 x e^x + C_3 e^{-x} \cos(\sqrt{2}x) + C_4 e^{-x} \sin(\sqrt{2}x),$$

a particular solution has the form

$$\begin{aligned} y_p(x) &= x^2 [Ax^2 + Bx + C] e^x + Dx \cos(2x) + Ex \sin(2x) + G \cos(2x) + H \sin(2x) \\ &= [Ax^4 + Bx^3 + Cx^2] e^x + Dx \cos(2x) + Ex \sin(2x) + G \cos(2x) + H \sin(2x). \end{aligned}$$

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(b) (4 pts.) Use the method of **undetermined coefficients** to find the general solution of the differential equation

$$y'' + y' = \sin x + x.$$

Solution. The auxiliary equation is $m^2 + m = 0$ or $m(m + 1) = 0$ and $m = 0, -1$ are simple roots. The complementary solution is thus

$$y_c(x) = C_1 + C_2 e^{-x}.$$

A particular solution has the form

$$y_p(x) = A \cos(x) + B \sin(x) + C x^2 + D x.$$

We have

$$y'_p(x) = B \cos(x) - A \sin(x) + 2 C x + D,$$

and

$$y'_p(x) = -A \cos(x) - B \sin(x) + 2 C.$$

Hence,

$$y''_p + y'_p = (B - A) \cos(x) - (A + B) \sin(x) + 2 C x + (D + 2 C) = \sin x + x$$

from which we deduce that $A = B = -\frac{1}{2}$, $C = \frac{1}{2}$ and $D = -1$. The general solution is thus

$$y(x) = y_c(x) + y_p(x) = C_1 + C_2 e^{-x} - \frac{1}{2} \cos(x) - \frac{1}{2} \sin(x) + \frac{1}{2} x^2 - x.$$

Continued ...

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2. (8 pts.) Use Laplace transforms to solve the first order linear system

$$\begin{cases} y_1'(t) = -y_1(t) - y_2(t) + 4 \\ y_2'(t) = y_1(t) - 3y_2(t) \end{cases}$$

with initial conditions $y_1(0) = 0$ and $y_2(0) = 1$.

Solution. Letting $Y_1(s) = \mathcal{L}\{y_1(t)\}$ and $Y_2(s) = \mathcal{L}\{y_2(t)\}$, we obtain by taking Laplace transforms

$$\begin{cases} sY_1(s) = -Y_1(s) - Y_2(s) + \frac{4}{s} \\ sY_2(s) - 1 = Y_1(s) - 3Y_2(s), \end{cases}$$

or,

$$\begin{cases} (s+1)Y_1(s) + Y_2(s) = \frac{4}{s} \\ -Y_1(s) + (s+3)Y_2(s) = 1, \end{cases}$$

which can be written in matrix form as

$$\begin{bmatrix} s+1 & 1 \\ -1 & s+3 \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{4}{s} \\ 1 \end{bmatrix}.$$

Using Cramer's rule (or any other appropriate method), we obtain

$$Y_1(s) = \frac{\begin{vmatrix} \frac{4}{s} & 1 \\ 1 & s+3 \end{vmatrix}}{\begin{vmatrix} s+1 & 1 \\ -1 & s+3 \end{vmatrix}} = \frac{3s+12}{s(s^2+4s+4)} = \frac{3s+12}{s(s+2)^2}$$

Using partial fractions,

$$\frac{3s+12}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2} = \frac{A(s^2+4s+4) + B(s^2+2s) + Cs}{s(s+2)^2}$$

and solving for the constants yield $A = 3$, $B = -3$ and $C = -3$. Hence,

$$Y_1(s) = \frac{3}{s} - \frac{3}{s+2} - \frac{3}{(s+2)^2}$$

and

$$y_1(t) = 3 - 3e^{-2t} - 3te^{-2t}.$$

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Similarly,

$$Y_2(s) = \frac{\begin{vmatrix} s+1 & \frac{4}{s} \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} s+1 & 1 \\ -1 & s+3 \end{vmatrix}} = \frac{3s+12}{s(s^2+4s+4)} = \frac{s^2+s+4}{s(s+2)^2}$$

Using partial fractions,

$$\frac{s^2+s+4}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2} = \frac{A(s^2+4s+4) + B(s^2+2s) + Cs}{s(s+2)^2}$$

and solving for the constants yield $A = 1$, $B = 0$ and $C = -3$. Hence,

$$Y_2(s) = \frac{1}{s} - \frac{3}{(s+2)^2}$$

and

$$y_2(t) = 1 - 3te^{-2t}.$$

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3. (a) (6 pts.) Compute the Fourier series expansion of the function

$$f(x) = \begin{cases} 0, & -\pi < x < -\frac{\pi}{2} \\ 1, & -\frac{\pi}{2} \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi \end{cases}$$

on the interval $-\pi < x < \pi$.

Solution. We have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 1 dx = \frac{\pi}{\pi} = 1,$$

and, for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(nx) dx \\ &= \frac{1}{\pi} \left[\frac{\sin(nx)}{n} \right]_{-\pi/2}^{\pi/2} = \frac{1}{\pi n} \left(\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{-n\pi}{2}\right) \right) \\ &= \frac{2}{\pi n} \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n \text{ even} \\ (-1)^k \frac{2}{\pi n}, & n = 1 + 2k \end{cases} \end{aligned}$$

Since $f(x)$ is even on $-\pi < x < \pi$, we have $b_n = 0$, for all $n \geq 1$. Hence,

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(nx) + b_n \sin(nx)\} = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{1+2k} \cos((1+2k)x) \\ &= \frac{1}{2} + \frac{2}{\pi} \left\{ \cos(x) - \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} - \frac{\cos(7x)}{7} + \dots \right\} \end{aligned}$$

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(b) (2 pts.) Consider the Fourier series expansion obtained in part (a) as a function defined on the whole real line (i.e. as a function defined for all values of x).

Sketch the graph of that function for $-3\pi < x < 3\pi$.

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4. Consider the linear differential equation

$$9x^2 y'' - 6xy' + (9x^2 + 4)y = 0.$$

(a) (4 pts.) Show that $x = 0$ is a regular singular point for the differential equation above. Find the indicial equation and compute its roots.

Solution. In standard form, the DE is written as

$$y'' - \frac{2}{3x} y' + \left(1 + \frac{4}{9x^2}\right) y = 0.$$

We have $P(x) = -\frac{2}{3x}$ and $Q(x) = 1 + \frac{4}{9x^2}$, and since these functions are undefined at 0, $x = 0$ is a singular point.

Since $xP(x) = -\frac{2}{3}$ and $x^2 Q(x) = x^2 + \frac{4}{9}$ are both analytic at $x = 0$ (they are both polynomials), it follows that $x = 0$ is a regular singular point. We have $a_0 = -\frac{2}{3}$ and $b_0 = \frac{4}{9}$. The indicial equation is $r(r-1) + a_0 r + b_0 = 0$ or

$$r(r-1) - \frac{2}{3}r + \frac{4}{9} = r^2 - \frac{5}{3}r + \frac{4}{9} = \left(r - \frac{1}{3}\right)\left(r - \frac{4}{3}\right) = 0$$

The roots are thus $r = \frac{4}{3}$ and $r = \frac{1}{3}$

(b) (6 pts.) Compute the first three non-zero terms of the series solution corresponding to the **largest** root of the indicial equation found in part (a).

Solution. Let $y = \sum_{n=0}^{\infty} c_n x^{n+r}$. Then, $y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$. Therefore,

$$\begin{aligned} 0 &= 9x^2 y'' - 6xy' + (9x^2 + 4)y \\ &= 9x^2 \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} - 6x \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} + (9x^2 + 4) \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= \sum_{n=0}^{\infty} 9c_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} (-6)c_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} 9c_n x^{n+r+2} \\ &\quad + \sum_{n=0}^{\infty} 4c_n x^{n+r} \\ &= \sum_{n=0}^{\infty} c_n \{9(n+r)(n+r-1) - 6(n+r) + 4\} x^{n+r} + \sum_{n=2}^{\infty} 9c_{n-2} x^{n+r} \end{aligned}$$

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Thus,

$$0 = c_0 \{9r(r-1) - 6r + 4\} x^r + c_1 \{9(1+r)r - 6(1+r) + 4\} x^{1+r} \\ + \sum_{n=2}^{\infty} [c_n \{9(n+r)(n+r-1) - 6(n+r) + 4\} + 9c_{n-2}] x^{n+r}$$

If $r = \frac{4}{3}$, the largest root of the indicial equation, we obtain

$$0 = 0 c_0 x^{\frac{4}{3}} + c_1 18 x^{\frac{7}{3}} + \sum_{n=2}^{\infty} \left[c_n \left\{ 9 \left(n + \frac{4}{3} \right) \left(n + \frac{1}{3} \right) - 6 \left(n + \frac{4}{3} \right) + 4 \right\} + 9 c_{n-2} \right] x^{n+\frac{4}{3}} \\ = 18 c_1 x^{\frac{7}{3}} + \sum_{n=2}^{\infty} [c_n \{9n^2 + 9n\} + 9c_{n-2}] x^{n+\frac{4}{3}}.$$

This yields $c_1 = 0$ and the recurrence formula

$$c_n = -\frac{c_{n-2}}{n(n+1)}, \quad n \geq 2.$$

Since $c_1 = 0$, the recurrence formula implies that $c_n = 0$ if n is odd. We have also

$$c_2 = -\frac{c_0}{6}, \quad c_4 = -\frac{c_2}{20} = \frac{c_0}{120}.$$

The corresponding solution is thus

$$y = c_0 x^{4/3} \left\{ 1 - \frac{x^2}{6} + \frac{x^4}{120} + \dots \right\}$$

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5. (8 pts.) Use Laplace transforms to compute the solution $x(t)$ of the initial value problem

$$\begin{cases} x''(t) + 2x'(t) + 5x(t) = g(t) \\ x(0) = 1, \quad x'(0) = 0, \end{cases}$$

where

$$g(t) = \begin{cases} 2 + 5t, & 0 < t < 1, \\ -1 + 10t, & 1 < t < \infty. \end{cases}$$

Solution. We can write $g(t) = 2 + 5t + [2 + 5(t - 1)]\mathcal{U}(t - 1)$. Let $Y(s) = \mathcal{L}\{y(t)\}$. Taking the Laplace transform of both sides of the DE yields

$$s^2 Y(s) - s + 2[Y(s) - 1] + 5Y(s) = \frac{2}{s} + \frac{5}{s^2} + \left[\frac{2}{s} + \frac{5}{s^2}\right]e^{-s},$$

or

$$(s^2 + 2s + 5)Y(s) = s + 2 + \frac{2s + 5}{s^2} + \frac{2s + 5}{s^2}e^{-s}.$$

Thus,

$$Y(s) = \frac{s + 2}{s^2 + 2s + 5} + \frac{2s + 5}{s^2(s^2 + 2s + 5)} + \frac{2s + 5}{s^2(s^2 + 2s + 5)}e^{-s}.$$

Using partial fractions,

$$\begin{aligned} \frac{2s + 5}{s^2(s^2 + 2s + 5)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 2s + 5} \\ &= \frac{A(s^3 + 2s^2 + 5s) + B(s^2 + 2s + 5) + Cs^3 + Ds^2}{s^2(s^2 + 2s + 5)} \\ &= \frac{(A + C)s^3 + (2A + B + D)s^2 + (5A + 2B)s + 5B}{s^2(s^2 + 2s + 5)} \end{aligned}$$

and solving for the coefficients yield $A = 0$, $B = 1$, $C = 0$ and $D = -1$. Thus,

$$\begin{aligned} Y(s) &= \frac{s + 2}{s^2 + 2s + 5} + \frac{1}{s^2} - \frac{1}{s^2 + 2s + 5} + \left(\frac{1}{s^2} - \frac{1}{s^2 + 2s + 5}\right)e^{-s} \\ &= \frac{s + 1}{(s + 1)^2 + 2^2} + \frac{1}{s^2} + \left(\frac{1}{s^2} - \frac{1}{(s + 1)^2 + 2^2}\right)e^{-s} \end{aligned}$$

and, taking inverse Laplace transform, we obtain

$$y(t) = e^{-t} \cos(2t) + t + \left[(t - 1) - \frac{1}{2}e^{-(t-1)} \sin(2(t-1))\right] \mathcal{U}(t - 1).$$

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6. (8 pts.) Use the method of variation of parameters to solve the non-homogeneous Cauchy-Euler equation

$$4x^2 y'' - 4xy' + 3y = 4x^{5/2} e^x, \quad x > 0.$$

Solution. The Cauchy-Euler auxiliary equation is

$$4r(r-1) - 4r + 3 = 0 \quad \text{or} \quad 4r^2 - 8r + 3 = 0.$$

Its roots are $r = \frac{8 \pm \sqrt{64-48}}{8} = \frac{8 \pm \sqrt{16}}{8} = 1 \pm \frac{1}{2}$, i. e. $r = \frac{1}{2}$ and $r = \frac{3}{2}$. The complementary solution is thus

$$y_c(x) = C_1 x^{1/2} + C_2 x^{3/2}.$$

Letting $y_1(x) = x^{1/2}$ and $y_2(x) = x^{3/2}$, we have

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^{1/2} & x^{3/2} \\ \frac{1}{2}x^{-1/2} & \frac{3}{2}x^{1/2} \end{vmatrix} = x$$

and, when the DE is put in standard form, the right-hand side of the DE is $f(x) = x^{1/2} e^x$. A particular solution is given by

$$\begin{aligned} y_p(x) &= - \left[\int \frac{y_2(x) f(x)}{W(y_1, y_2)} dx \right] y_1(x) + \left[\int \frac{y_1(x) f(x)}{W(y_1, y_2)} dx \right] y_2(x) \\ &= - \left[\int \frac{x^{3/2} (x^{1/2} e^x)}{x} dx \right] x^{1/2} + \left[\int \frac{x^{1/2} (x^{1/2} e^x)}{x} dx \right] x^{3/2} \\ &= - \left[\int x e^x dx \right] x^{1/2} + \left[\int e^x dx \right] x^{3/2} \\ &= -(x-1)e^x x^{1/2} + e^x x^{3/2} = x^{1/2} e^x. \end{aligned}$$

The general solution is

$$y(x) = x^{1/2} e^x + C_1 x^{1/2} + C_2 x^{3/2}, \quad C_1, C_2 \text{ arbitrary.}$$

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PART II: Multiple choice part. Circle the letter corresponding to the correct answer for each of the questions 7 to 16 in the box below. Indicate your choice very clearly. Ambiguous answers will be marked as wrong. There is only one correct choice for each question and an incorrect answer scores 0 marks.

QUESTION #	ANSWER:				
7.	A	B	C	D	E
8.	A	B	C	D	E
9.	A	B	C	D	E
10.	A	B	C	D	E
11.	A	B	C	D	E
12.	A	B	C	D	E
13.	A	B	C	D	E
14.	A	B	C	D	E
15.	A	B	C	D	E
16.	A	B	C	D	E

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7. (5 pts.) Let $y(t)$ be the solution of the initial value problem

$$\begin{cases} y' + 4ty = e^{-2t^2}, \\ y(0) = -1. \end{cases}$$

Then, $y(1)$ is

→ (A) 0

(B) -2

(C) e^{-2}

(D) $2e^{-2}$

(E) $4e^{-4}$

Solution. The integrating factor is $u(t) = e^{\int 4t dt} = e^{2t^2}$. Thus

$$y' e^{2t^2} + 4t e^{2t^2} y = (y e^{2t^2})' = e^{2t^2} e^{-2t^2} = 1$$

and

$$y e^{2t^2} = t + C$$

and

$$y = (t + C) e^{-2t^2}.$$

The condition $y(0) = -1$ implies $C = -1$. Thus, $y = (t - 1) e^{-2t^2}$ and $y(1) = 0$.

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8. (5 pts.) Let $f(t)$ be the inverse Laplace transform of

$$F(s) = \frac{1 - e^{-\pi s}}{s^2 (s^2 + 4)}.$$

Then, for $t > \pi$, $f(t)$ is equal to:

(A) $\frac{1}{4} (t - \frac{1}{2} \sin(2t))$

(B) 0

→ (C) $\frac{\pi}{4}$

(D) $\frac{1}{2} (t - \pi + \sin(2t))$

(E) $\frac{1}{4} (t - \cos(2t))$

Solution. We have

$$\frac{1}{s^2 (s^2 + 4)} = \frac{1}{4} \left\{ \frac{1}{s^2} - \frac{1}{(s^2 + 4)} \right\}$$

Therefore,

$$\begin{aligned} f(t) &= \frac{t}{4} - \frac{1}{8} \sin(2t) - \left[\frac{t - \pi}{4} - \frac{1}{8} \sin(2(t - \pi)) \right] \mathcal{U}(t - \pi) \\ &= \frac{t}{4} - \frac{1}{8} \sin(2t) - \left[\frac{t - \pi}{4} - \frac{1}{8} \sin(2t) \right] \mathcal{U}(t - \pi) \end{aligned}$$

and, for $t > \pi$,

$$f(t) = \frac{\pi}{4}.$$

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9. (5 pts.) Let $F(s)$ be the Laplace transform of

$$f(t) = \int_0^t e^{-\tau} \cos(\tau) d\tau.$$

Then, $F(2)$ is equal to:

(A) $\frac{1}{10}$

→ (B) $\frac{3}{20}$

(C) $\frac{1}{2}$

(D) $-\frac{1}{e}$

(E) 0

Solution. We have

$$F(s) = \frac{1}{s} \frac{s+1}{(s+1)^2 + 1}$$

and

$$F(2) = \frac{1}{2} \frac{3}{(3)^2 + 1} = \frac{3}{20}.$$

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10. (5 pts) The differential equation

$$(x - 3)(x - 4)y'' + xy' - (x^2 + 1)y = 0,$$

has a power series solution $y(x)$ defined around $x_0 = 0$. The **best** one can say, without actually solving the differential equation, is that the radius of convergence of that power series solution about $x_0 = 0$ is at least:

(A) 4

(B) 1

→ (C) 3

(D) ∞

(E) 3.5

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11. (5 pts) Consider the differential equation

$$y'' + x y' - y = 0$$

solved by a power series $y(x) = \sum_{n=0}^{\infty} c_n x^n$ about the ordinary point $x_0 = 0$. The recurrence formula for the coefficients c_n , $n \geq 0$, is given by:

(A) $c_{n+2} = -\frac{c_n}{n+2}$

(B) $c_{n+2} = \frac{n c_{n+1} - c_n}{(n+1)(n+2)}$

(C) $c_{n+2} = \frac{c_n}{(n+1)(n+2)}$

→ (D) $c_{n+2} = -\frac{(n-1)c_n}{(n+1)(n+2)}$

(E) $c_{n+2} = -\frac{c_{n+1}}{n+1} - \frac{c_n}{n}$

Solution. We have

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + x \sum_{n=0}^{\infty} c_n n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=0}^{\infty} c_{n+2} (n+2)(n+1)x^n + \sum_{n=0}^{\infty} c_n n x^n - \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=0}^{\infty} \{c_{n+2} (n+2)(n+1) + c_n (n-1)\} x^n \end{aligned}$$

which yields the recurrence relation

$$c_{n+2} = -\frac{(n-1)c_n}{(n+1)(n+2)}, \quad n \geq 0.$$

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12. (5 pts) The general solution of the differential equation

$$(1 + x^2)y'' - 2xy' + 2y = 0$$

has the form:

(A) $y(x) = C_0(1 - x + x^2) + C_1(2 - x + x^4)$

→ (B) $y(x) = C_0(1 - x^2) + C_1x$

(C) $y(x) = C_0(1 - x^2 + x^4 + \dots) + C_1(x - x^3 + x^5 + \dots)$

(D) $y(x) = C_0(1 + \frac{x^2}{2} + \frac{x^4}{4} + \dots) + C_1(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots)$

(E) $y(x) = C_0(1 - 2x^2 + 4x^4 + \dots) + C_1(x - 3x^3 + 5x^5 + \dots)$

Solution. Letting $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have

$$\begin{aligned} 0 &= (1 + x^2) \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - 2x \sum_{n=0}^{\infty} c_n n x^{n-1} + 2 \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=0}^{\infty} c_{n+2} (n+2)(n+1)x^n + \sum_{n=2}^{\infty} c_n n(n-1)x^n + \sum_{n=0}^{\infty} (-2n)c_n x^n + \sum_{n=0}^{\infty} 2c_n x^n \\ &= [2c_2 + 2c_0] + 6c_3x + \sum_{n=2}^{\infty} \{c_{n+2}(n+2)(n+1) + c_n(n^2 - 3n + 2)\} x^n \end{aligned}$$

We deduce that $c_2 = -c_0$, $c_3 = 0$ and

$$c_{n+2} = -\frac{c_n(n^2 - 3n + 2)}{(n+2)(n+1)} = -\frac{c_n(n-1)(n-2)}{(n+2)(n+1)}, \quad n \geq 2$$

which implies that $c_n = 0$ for $n \geq 3$. Thus, $y(x) = c_0(1 - x^2) + c_1x$.

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13. (5 pts) A complete list of the **regular singular points** for the differential equation

$$x^2(x-1)^2(x+2)^2y'' - x^2(x+3)(x+2)y' - y = 0$$

is given by:

(A) 0, 1, -2.

(B) 0, 1, -3.

(C) 1, -2.

(D) 1, -2, -3.

→ (E) 0, -2.

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14. (5 pts) Consider the eigenvalue problem

$$\begin{cases} y''(x) + \lambda y(x) = 0 \\ y(0) = 0, y'(\frac{\pi}{2}) = 0. \end{cases}$$

The eigenvalues are given by:

(A) $\lambda = 1 + 2n, \quad n = 0, 1, 2, 3, \dots$

(B) $\lambda = n^2, \quad n = 1, 2, 3, \dots$

(C) $\lambda = \frac{(2n)^2}{\pi}, \quad n = 0, 1, 2, 3, \dots$

\rightarrow (D) $\lambda = (1 + 2n)^2, \quad n = 0, 1, 2, 3, \dots$

(E) $\lambda = 1 + n^2, \quad n = 0, 1, 2, 3, \dots$

Solution. The auxiliary equation is $m^2 + \lambda = 0$. If $\lambda < 0$, $\lambda = -\omega^2$ where $\omega > 0$. The general solution is then

$$y(x) = C_1 e^{\omega x} + C_2 e^{-\omega x}.$$

The condition $y(0) = 0$ implies $C_2 = -C_1$, so

$$y(x) = C_1 \{e^{\omega x} - e^{-\omega x}\} \quad \text{and} \quad y'(x) = C_1 \omega \{e^{\omega x} + e^{-\omega x}\}$$

Clearly, $y'(\frac{\pi}{2}) \neq 0$ unless $C_1 = 0$. Thus, no non-trivial solutions exist in this case. If $\lambda = 0$, the general solution is $y(x) = C_1 + C_2 x$. The condition $y(0) = 0$ implies $C_1 = 0$ and $y'(\frac{\pi}{2}) = 0$ implies $C_2 = 0$, so no non-trivial solutions exist in this case either. If $\lambda > 0$, $\lambda = \omega^2$ where $\omega > 0$. The general solution is then

$$y(x) = C_1 \cos(\omega x) + C_2 \sin(\omega x).$$

The condition $y(0) = 0$ implies $C_1 = 0$, so

$$y(x) = C_2 \sin(\omega x) \quad \text{and} \quad y'(x) = C_2 \omega \cos(\omega x)$$

The condition $y'(\frac{\pi}{2}) = 0$ implies $C_2 \cos(\omega \frac{\pi}{2}) = 0$, so non-trivial solution exist when $\omega \frac{\pi}{2} = \frac{\pi}{2} + n\pi$, $n \geq 0$, n integer or when $\omega = 1 + 2n$ for such n . The corresponding eigenvalues are

$$\lambda = \omega^2 = (1 + 2n)^2, \quad n \geq 0.$$

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15. (5 pts) Let $f(x)$ be a 2-periodic function (i. e. $f(x+2) = f(x)$ for all x) and suppose that

$$f(x) = \begin{cases} x+1, & -1 \leq x < 0, \\ -2, & 0 < x < 1. \end{cases}$$

Consider the expansion of $f(x)$ as a 2-periodic Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(n\pi x) + b_n \sin(n\pi x)\}.$$

To which value does the Fourier series converge at $x = 3$?

(A) $\frac{1}{2}$

(B) $-\frac{1}{4}$

(C) $-\frac{1}{2}$

→ (D) -1

(E) 0

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16. (5 pts) Let $u(x, t)$ be the temperature on a rod placed on the x -axis between $x = 0$ and $x = \pi$ and suppose that $u(x, t)$ is solution of the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, & 0 < x < \pi, & t > 0, \\ u(0, t) = 0, & u(\pi, t) = 0, & t > 0, \\ u(x, 0) = \sin(3x), & 0 < x < \pi, & \end{cases}$$

Then, at time $t > 0$, the average temperature on the rod for $0 < x < \pi$ is given by

(A) $\frac{1}{\pi} e^{-2t}$

→ (B) $\frac{2}{3\pi} e^{-9t}$

(C) $\frac{1}{4\pi} e^{-4t}$

(D) $\frac{3}{\pi} e^{-16t}$

(E) $\frac{2}{5\pi} e^{-25t}$

Note that the average temperature on the rod at time t is given by $\frac{1}{\pi} \int_0^\pi u(x, t) dx$.

Solution. The solution is $u(x, t) = \sin(3x) e^{-3^2 t} = \sin(3x) e^{-9t}$. The average temperature on the rod at time t is thus

$$\frac{1}{\pi} \int_0^\pi \sin(3x) e^{-9t} dx = \frac{1}{\pi} e^{-9t} \left[\frac{-\cos(3x)}{3} \right]_0^\pi = \frac{2}{3\pi} e^{-9t}.$$

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Some formulas you may use:

$$y_p(x) = - \left[\int \frac{y_2(x) f(x)}{W(y_1, y_2)} dx \right] y_1(x) + \left[\int \frac{y_1(x) f(x)}{W(y_1, y_2)} dx \right] y_2(x)$$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t) e^{-st} dt.$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a},$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2 + k^2}, \quad \mathcal{L}\{\cos(kt)\} = \frac{s}{s^2 + k^2},$$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0), \quad \mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0),$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0),$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a), \quad \mathcal{L}\{\mathcal{U}(t-a) f(t-a)\} = e^{-sa} F(s),$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s),$$

$$\mathcal{L}\{f * g\} = \mathcal{L}\left\{ \int_0^t f(t-\tau) g(\tau) d\tau \right\} = F(s) G(s),$$

$$\mathcal{L}\left\{ \int_0^t f(\tau) d\tau \right\} = \frac{F(s)}{s},$$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right\}, \quad \text{where}$$

$$a_n = \frac{1}{p} \int_{-p}^p f(x) \cos\left(\frac{n\pi x}{p}\right) dx, \quad n \geq 0, \quad b_n = \frac{1}{p} \int_{-p}^p f(x) \sin\left(\frac{n\pi x}{p}\right) dx, \quad n \geq 1.$$

$$2 \sin(A) \sin(B) = \cos(A-B) - \cos(A+B),$$

$$2 \sin(A) \cos(B) = \sin(A+B) + \sin(A-B),$$

$$2 \cos(A) \cos(B) = \cos(A+B) + \cos(A-B).$$

*** END ***