LAST (Family) NAME: ______________________________

INITIALS: _____________

ID # : ______________________

Tutorial # : ______________________

MATHEMATICS 2P04

Sample Final Exam
Day Class
Duration of Examination: 3 hours

THIS EXAMINATION PAPER INCLUDES 26 PAGES AND 16 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

Instructions:

- Indicate your answers clearly in the spaces provided. In the full answer questions in Part I (questions 1 - 6) show all your work to receive full credit. For the multiple choice questions in Part II (questions 7-16) be sure to circle the correct letters on page 14 to receive full credit.
- No books, notes, or "cheat sheets" allowed.
- The only calculator permitted is the McMaster Standard Calculator, the Casio fx 991.
- The total number of points is 100.
- There is a formula sheet included with the exam on page 24.
- Page 23 of the test is for scratch work or overflow. If you continue the solution to a question on one this page, you must indicate this clearly on the page containing the original question. **GOOD LUCK!**

#	Mark	#	Mark
1.		**6.**	
2.			
3.			
4.			
5.		**7-16.**	
		TOTAL	

Continued ...

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Part I: Provide all details and fully justify your answer in order to receive credit.

1. (a) *(4 pts.)* If the method of undetermined coefficients is used to determine a particular solution $y_p(x)$ of the linear differential equation

$$y^{(4)} - 4\,y' + 3\,y = x^2\,e^x + x\,\cos(2\,x),$$

what is the correct form to use to find $y_p(x)$? (**Do not solve for the coefficients in** $y_p(x)$**!**)
Hint: $m^4 - 4\,m + 3 = (m-1)^2\,(m^2 + 2\,m + 3)$

Continued ...

(b) *(4 pts.)* Use the method of **undetermined coefficients** to find the general solution of the differential equation

$$y'' + y' = \sin x + x.$$

Continued . . .

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2. *(8 pts.)* Use Laplace transforms to solve the first order linear system

$$\begin{cases} y_1'(t) = -y_1(t) - y_2(t) + 4 \\ y_2'(t) = y_1(t) - 3\, y_2(t) \end{cases}$$

with initial conditions $y_1(0) = 0$ and $y_2(0) = 1$.

Continued ...

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3. (a) *(6 pts.)* Compute the Fourier series expansion of the function

$$f(x) = \begin{cases} 0, & -\pi < x < -\frac{\pi}{2} \\ 1, & -\frac{\pi}{2} \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi \end{cases}$$

on the interval $-\pi < x < \pi$.

Continued . . .

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(b) *(2 pts.)* Consider the Fourier series expansion obtained in part (a) as a function defined on the whole real line (i.e. as a function defined for all values of x).
Sketch the graph of that function for $-3\,\pi < x < 3\pi$.

Continued ...

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4. Consider the linear differential equation

$$9\,x^2\,y'' - 6\,x\,y' + (9\,x^2 + 4)\,y = 0.$$

(a) *(4 pts.)* Show that $x = 0$ is a regular singular point for the differential equation above. Find the indicial equation and compute its roots.

(b) *(6 pts.)* Compute the first three non-zero terms of the series solution corresponding to the **largest** root of the indicial equation found in part (a).

Continued . . .

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5. *(8 pts.)* Use Laplace transforms to compute the solution $x(t)$ of the initial value problem

$$\begin{cases} x''(t) + 2\,x'(t) + 5\,x(t) = g(t) \\ x(0) = 1, \quad x'(0) = 0, \end{cases}$$

where

$$g(t) = \begin{cases} 2 + 5\,t, & 0 < t < 1, \\ -1 + 10\,t, & 1 < t < \infty. \end{cases}$$

Continued ...

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6. *(8 pts.)* Use the method of variation of parameters to solve the non-homogeneous Cauchy-Euler equation

$$4\,x^2\,y'' - 4\,x\,y' + 3\,y = 4\,x^{5/2}\,e^x, \quad x > 0.$$

Continued ...

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PART II: Multiple choice part. <u>Circle</u> the letter corresponding to the correct answer for each of the questions 7 to 16 in the box below. Indicate your choice very clearly. Ambiguous answers will be marked as wrong. There is only one correct choice for each question and an incorrect answer scores 0 marks.

QUESTION #	ANSWER:				
7.	A	B	C	D	E
8.	A	B	C	D	E
9.	A	B	C	D	E
10.	A	B	C	D	E
11.	A	B	C	D	E
12.	A	B	C	D	E
13.	A	B	C	D	E
14.	A	B	C	D	E
15.	A	B	C	D	E
16.	A	B	C	D	E

Continued ...

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7. *(5 pts.)* Let $y(t)$ be the solution of the initial value problem

$$\begin{cases} y' + 4\,t\,y = e^{-2\,t^2}, \\ y(0) = -1. \end{cases}$$

Then, $y(1)$ is

(A) 0

(B) -2

(C) e^{-2}

(D) $2\,e^{-2}$

(E) $4\,e^{-4}$

Continued ...

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8. *(5 pts.)* Let $f(t)$ be the inverse Laplace transform of

$$F(s) = \frac{1 - e^{-\pi s}}{s^2\,(s^2+4)}.$$

Then, for $t > \pi$, $f(t)$ is equal to:

(A) $\frac{1}{4}\left(t - \frac{1}{2}\,\sin(2\,t)\right)$

(B) 0

(C) $\frac{\pi}{4}$

(D) $\frac{1}{2}\left(t - \pi + \sin(2\,t)\right)$

(E) $\frac{1}{4}\,\left(t - \cos(2\,t)\right)$

Continued ...

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9. *(5 pts.)* Let $F(s)$ be the Laplace transform of

$$f(t) = \int_0^t e^{-\tau} \cos(\tau)\, d\tau.$$

Then, $F(2)$ is equal to:

(A) $\frac{1}{10}$

(B) $\frac{3}{20}$

(C) $\frac{1}{2}$

(D) $-\frac{1}{e}$

(E) 0

Continued . . .

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10. *(5 pts)* The differential equation

$$(x-3)\,(x-4)\,y'' + x\,y' - (x^2+1)\,y = 0,$$

has a power series solution $y(x)$ defined around $x_0 = 0$. The **best** one can say, without actually solving the differential equation, is that the radius of convergence of that power series solution about $x_0 = 0$ is at least:

(A) 4

(B) 1

(C) 3

(D) ∞

(E) 3.5

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11. *(5 pts)* Consider the differential equation

$$y'' + x\,y' - y = 0$$

solved by a power series $y(x) = \sum_{n=0}^{\infty} c_n\, x^n$ about the ordinary point $x_0 = 0$. The recurrence formula for the coefficients c_n, $n \geq 0$, is given by:

(A) $c_{n+2} = -\frac{c_n}{n+2}$

(B) $c_{n+2} = \frac{n\, c_{n+1} - c_n}{(n+1)\,(n+2)}$

(C) $c_{n+2} = \frac{c_n}{(n+1)\,(n+2)}$

(D) $c_{n+2} = -\frac{(n-1)\, c_n}{(n+1)\,(n+2)}$

(E) $c_{n+2} = -\frac{c_{n+1}}{n+1} - \frac{c_n}{n}$

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12. *(5 pts)* The general solution of the differential equation

$$(1 + x^2)\, y'' - 2\, x\, y' + 2\, y = 0$$

has the form:

(A) $y(x) = C_0\,(1 - x + x^2) + C_1\,(2 - x + x^4)$

(B) $y(x) = C_0\,(1 - x^2) + C_1\,x$

(C) $y(x) = C_0\,(1 - x^2 + x^4 + \dots) + C_1\,(x - x^3 + x^5 + \dots)$

(D) $y(x) = C_0\,(1 + \frac{x^2}{2} + \frac{x^4}{4} + \dots) + C_1\,(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots)$

(E) $y(x) = C_0\,(1 - 2\,x^2 + 4\,x^4 + \dots) + C_1\,(x - 3\,x^3 + 5\,x^5 + \dots)$

Continued ...

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13. *(5 pts)* A complete list of the **regular singular points** for the differential equation

$$x^2\,(x-1)^2\,(x+2)^2\,y'' - x^2\,(x+3)\,(x+2)\,y' - y = 0$$

is given by:

(A) $0, 1, -2.$

(B) $0, 1, -3.$

(C) $1, -2.$

(D) $1, -2, -3.$

(E) $0, -2.$

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14. *(5 pts)* Consider the eigenvalue problem

$$\begin{cases} y''(x) + \lambda\, y(x) = 0 \\ y(0) = 0,\; y'(\frac{\pi}{2}) = 0. \end{cases}$$

The eigenvalues are given by:

(A) $\lambda = 1 + 2\,n, \quad n = 0, 1, 2, 3, \ldots$

(B) $\lambda = n^2, \quad n = 1, 2, 3, \ldots$

(C) $\lambda = \frac{(2\,n)^2}{\pi}, \quad n = 0, 1, 2, 3, \ldots$

(D) $\lambda = (1 + 2\,n)^2, \quad n = 0, 1, 2, 3, \ldots$

(E) $\lambda = 1 + n^2, \quad n = 0, 1, 2, 3, \ldots$

Continued ...

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15. *(5 pts)* Let $f(x)$ be a 2-periodic function (i. e. $f(x+2) = f(x)$ for all x) and suppose that

$$f(x) = \begin{cases} x+1, & -1 \le x < 0, \\ -2, & 0 < x < 1. \end{cases}$$

Consider the expansion of $f(x)$ as a 2-periodic Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(n\pi x) + b_n \sin(n\pi x) \right\}.$$

To which value does the Fourier series converge at $x = 3$?

(A) $\frac{1}{2}$

(B) $-\frac{1}{4}$

(C) $-\frac{1}{2}$

(D) -1

(E) 0

Continued ...

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16. *(5 pts)* Let $u(x,t)$ be the temperature on a rod placed on the x-axis between $x = 0$ and $x = \pi$ and suppose that $u(x,t)$ is solution of the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & 0 < x < \pi, \qquad t > 0, \\ u(0,t) = 0, & u(\pi,t) = 0, \qquad t > 0, \\ u(x,0) = \sin(3\,x), & 0 < x < \pi, \end{cases}$$

Then, at time $t > 0$, the average temperature on the rod for $0 < x < \pi$ is given by

(A) $\frac{1}{\pi}\, e^{-2\,t}$

(B) $\frac{2}{3\,\pi}\, e^{-9\,t}$

(C) $\frac{1}{4\,\pi}\, e^{-4\,t}$

(D) $\frac{3}{\pi}\, e^{-16\,t}$

(E) $\frac{2}{5\pi}\, e^{-25\,t}$

Note that the average temperature on the rod at time t is given by $\frac{1}{\pi}\int_0^\pi u(x,t)\,dx$.

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SCRATCH

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Some formulas you may use:

$$y_p(x) = -\left[\int \frac{y_2(x)\,f(x)}{W(y_1,y_2)}\,dx\right] y_1(x) + \left[\int \frac{y_1(x)\,f(x)}{W(y_1,y_2)}\,dx\right] y_2(x)$$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\,e^{-st}\,dt.$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{e^{a\,t}\} = \frac{1}{s-a},$$

$$\mathcal{L}\{\sin(k\,t)\} = \frac{k}{s^2+k^2}, \quad \mathcal{L}\{\cos(k\,t)\} = \frac{s}{s^2+k^2},$$

$$\mathcal{L}\{f'(t)\} = s\,F(s) - f(0), \quad \mathcal{L}\{f''(t)\} = s^2\,F(s) - s\,f(0) - f'(0),$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\,F(s) - s^{n-1}\,f(0) - s^{n-2}\,f'(0) - \cdots - s\,f^{(n-2)}(0) - f^{(n-1)}(0),$$

$$\mathcal{L}\{e^{a\,t}\,f(t)\} = F(s-a), \quad \mathcal{L}\{\mathcal{U}(t-a)\,f(t-a)\} = e^{-s\,a}\,F(s),$$

$$\mathcal{L}\{t^n\,f(t)\} = (-1)^n\,\frac{d^n}{ds^n}F(s),$$

$$\mathcal{L}\{f*g\} = \mathcal{L}\left\{\int_0^t f(t-\tau)\,g(\tau)\,d\tau\right\} = F(s)\,G(s),$$

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s},$$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-s\,t_0}.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\,\cos(\frac{n\pi x}{p}) + b_n\,\sin(\frac{n\pi x}{p})\right\}, \quad \text{where}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\,\cos(\frac{n\pi x}{p})\,dx,\ n \ge 0, \quad b_n = \frac{1}{p}\int_{-p}^{p} f(x)\,\sin(\frac{n\pi x}{p})\,dx,\ n \ge 1.$$

$$2\,\sin(A)\,\sin(B) = \cos(A-B) - \cos(A+B),$$

$$2\,\sin(A)\,\cos(B) = \sin(A+B) + \sin(A-B),$$

$$2\,\cos(A)\,\cos(B) = \cos(A+B) + \cos(A-B).$$

*** **END** ***