Math 4A03: Practice problems on Multivariable Calculus

Problem 1. *Consider the mapping* $\mathbf{f} = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by*

$$\mathbf{f}(x, y) = (e^y + x, e^x - y) \quad (x, y) \in \mathbb{R}^2.$$

(a) *Is it possible to express* (x, y) *as a differentiable function of* (u, v) *near the origin* $(x_0, y_0) = (0, 0)$*?*

(b) *Compute* $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial u}$, $\frac{\partial y}{\partial v}$ *at* $(x, y) = (0, 0)$*, i.e. when* $(u, v) = (1, 1)$.

Problem 2. *Consider the mapping* $\mathbf{f} = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by*

$$\mathbf{f}(x, y) = (x^2 - y^2, 2\,x\,y), \quad (x, y) \in \mathbb{R}^2.$$

(a) *Show that the range of* $\mathbf{f}$ *is* $\mathbb{R}^2$ *and that if* $(u_0, v_0) \neq (0, 0)$*, there are exactly two points in* $\mathbb{R}^2$ *that are mapped to* (u_0, v_0) *by* $\mathbf{f}$.

(b) *Show that the mapping* $\mathbf{f}$ *is locally invertible at the point* $(x_0, y_0) = (1, 1)$*. Find an explicit formula for its local inverse* $\mathbf{g}(u, v)$ *defined in a neighborhood of* $\mathbf{f}(1, 1) = (0, 2)$.

Problem 3. *Consider the system of equations*

$$\begin{cases} w\,x\,y\,z = 0 \\ w^4 + x^4 + y^4 + z^4 = 18 \end{cases}$$

(a) *Is it possible to express* (x, y) *as a differentiable function of* (w, z) *near the solution* $(w, x, y, z) = (-1, 0, 1, 2)$*? (Use the implicit function theorem to answer this question.)*

(b) *If so, what are* $\frac{\partial x}{\partial w}(-1, 2)$ *and* $\frac{\partial x}{\partial z}(-1, 2)$*? (Use the implicit function theorem to answer this question.)*

(c) *Compute explicitely the differentiable function of* (w, z) *in part (a) and verify your answers in part (b) by a direct computation (i.e. without using the implicit function theorem).*

Problem 4. *Determine if the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$f(x, y) = \begin{cases} x + y, & x \neq y, \\ x^2 + x, & x = y \end{cases}$$

is differentiable at $(0, 0)$ *using the definition of differentiablity.*

Solution. We first compute the first-order partial derivatives at $(0,0)$.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

and

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

We know that if $f'(0,0)$ exists then

$$f'(0,0) = \left[\frac{\partial f}{\partial x}(0,0) \quad \frac{\partial f}{\partial y}(0,0)\right] = [1\ 1]\,.$$

To see if f is differentiable at $(0,0)$, we need to check if

$$\frac{|f(h_1,h_2) - f(0,0) - \frac{\partial f}{\partial x}(0,0)\,h_1 - \frac{\partial f}{\partial y}(0,0)\,h_2|}{\sqrt{h_1^2+h_2^2}}$$
$$= \frac{|f(h_1,h_2) - f(0,0) - h_1 - h_2|}{\sqrt{h_1^2+h_2^2}} \to 0,$$

as $(h_1,h_2)\to(0,0)$. If $h_1 \neq h_2$, we have $f(h_1,h_2) = h_1 + h_2$ and since $f(0,0) = 0$, $f(h_1,h_2) - f(0,0) - h_1 - h_2 = 0$ so, clearly, the previous quotient goes to 0 if $(h_1,h_2)\to(0,0)$ with $h_1 \neq h_2$. On the other if $h_1 = h_2$, we have

$$\frac{|f(h_1,h_2) - f(0,0) - h_1 - h_2|}{\sqrt{h_1^2+h_2^2}} = \frac{|h_1^2 + h_1 - 2\,h_1|}{\sqrt{2\,h_1^2}} = \frac{|h_1 - 1|}{\sqrt{2}}$$

and this large expression does not converge to 0 as $h_1 \to 0$. Hence, f is not differentiable at $(0,0)$.

Problem 5. *Let $E \subset \mathbb{R}^n$ be open and let $f : E \to \mathbb{R}$ be a function having partial derivatives $\frac{\partial f}{\partial x_j}$, $j = 1,\dots,n$, bounded on E. Prove that f is continuous on E.*

Hint: *Proceed as in the proof done in class that, if these partial derivatives are continuous on E, then $f \in \mathcal{C}^1(E)$.*

Solution. Let $\mathbf{x} \in E$ and choose $r > 0$ small enough so that

$$\{\mathbf{y} \in \mathbb{R}^n,\ ||\mathbf{x} - \mathbf{y}|| < r\} \subset E.$$

where $||.||$ denotes the usual euclidean norm on $\mathbb{R}^n$. let $\mathbf{h} = \sum_{j=1}^n h_j\,\mathbf{e_j}$ with $||\mathbf{h}|| < r$ (where $\mathbf{e_1},\dots,\mathbf{e_n}$ is the standard orthonormal basis in $\mathbb{R}^n$). Define $\mathbf{v_0} = \mathbf{0}$ and $\mathbf{v_k} = \sum_{j=1}^k h_j\,\mathbf{e_j}$ for $k = 1,\dots n$. We can thus write

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = \sum_{j=1}^n \left[f(\mathbf{x}+\mathbf{v_j}) - f(\mathbf{x}+\mathbf{v_{j-1}})\right].$$

Define

$$g_j(t) = f(\mathbf{x} + \mathbf{v_{j-1}} + t\,(\mathbf{v_j} - \mathbf{v_{j-1}})) = f(\mathbf{x} + \mathbf{v_{j-1}} + t\,h_j\,\mathbf{e_j}), \quad 0 \le t \le 1,$$

for $j = 1, \dots, n$. Then,

$$g_j'(t) = \frac{\partial f}{\partial x_j}(\mathbf{x} + \mathbf{v_{j-1}} + t\, h_j\, \mathbf{e_j})\, h_j.$$

By the mean value theorem, there exists θ_j with $0 < \theta_j < 1$ such that

$$g_j(1) - g_j(0) = g_j'(\theta_j),$$

or, equivalently,

$$f(\mathbf{x} + \mathbf{v_j}) - f(\mathbf{x} + \mathbf{v_{j-1}}) = \frac{\partial f}{\partial x_j}(\mathbf{x} + \mathbf{v_{j-1}} + \theta_j\, h_j\, \mathbf{e_j})\, h_j.$$

Since all first-order partial derivatives are bounded by M on E, it follows that

$$|f(\mathbf{x} + \mathbf{v_j}) - f(\mathbf{x} + \mathbf{v_{j-1}})| = |\frac{\partial f}{\partial x_j}(\mathbf{x} + \mathbf{v_{j-1}} + \theta_j\, h_j\, \mathbf{e_j})|\, |h_j| \leq M\, |h_j|.$$

This leads to

$$|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})| \leq \sum_{j=1}^{n} |f(\mathbf{x} + \mathbf{v_j}) - f(\mathbf{x} + \mathbf{v_{j-1}})| \leq M \sum_{j=1}^{n} |h_j| \to 0,$$

as $||\mathbf{h}|| \to 0$. This shows that

$$\lim_{||\mathbf{h}|| \to 0} f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}),$$

which means that f is continuous at $\mathbf{x}$.

Problem 6. *Let $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^2$ be the mapping defined by*

$$\mathbf{f}(x, y) = (e^x\, \cos y, e^x\, \sin y), \quad (x, y) \in \mathbf{R}^2.$$

(a) What is the range of $\mathbf{f}$.

(b) Show that $\mathbf{f}$ is locally one-to-one on $\mathbf{R}^2$ (i.e. one-to-one on a neighborhood of every point $(x_0, y_0) \in \mathbf{R}^2$), but not globally one-to-one (i.e. not one-to-one on $\mathbf{R}^2$).

(c) Let $\mathbf{g}$ be the continuous inverse of $\mathbf{f}$ defined in a neighborhood of $(\frac{1}{2}, \frac{\sqrt{3}}{2}) = \mathbf{f}(0, \frac{\pi}{3})$. Find an explicit formula for $\mathbf{g}$. Compute $\mathbf{f}'\left(0, \frac{\pi}{3}\right)$ and $\mathbf{g}'\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and verify that

$$\mathbf{g}'\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \left(\mathbf{f}'\left(0, \frac{\pi}{3}\right)\right)^{-1}.$$

(d) What are the images under $\mathbf{f}$ of lines parallel to the coodinate axes?

Solution.

(a) Since any point $(u,v)\in\mathbb{R}^2$ can be written in polar coordinate as $(u,v) = (r\,\cos(y), r\,\sin(y))$ where $r\ge 0$ and $r=0$ if and only $(u,v)=(0,0)$ while $r=e^x$, for some real x if and only if $r>0$ or $(u,v)\neq(0,0)$, it follows that the range of f is the set $\mathbb{R}^2\setminus\{(0,0)\}$.

(b) If $(x_0,y_0)\in\mathbb{R}^2$, we have

$$\mathbf{f}'(x_0,y_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(x_0,y_0) & \frac{\partial f_1}{\partial y}(x_0,y_0) \\ \frac{\partial f_2}{\partial x}(x_0,y_0) & \frac{\partial f_2}{\partial y}(x_0,y_0)\end{bmatrix} = \begin{bmatrix} e^{x_0}\,\cos(y_0) & -e^{x_0}\,\sin(y_0) \\ e^{x_0}\,\sin(y_0) & e^{x_0}\,\cos(y_0)\end{bmatrix}$$

Since $\det\left(\mathbf{f}'(x_0,y_0)\right) = e^{2\,x_0}\left(\cos^2(x_0)+\sin^2(x_0)\right) = e^{2\,x_0}\neq 0$, it follows that $\mathbf{f}'(x_0,y_0)$ is invertible and the inverse function theorem shows that $\mathbf{f}$ is one-to-one on a neighborhood of (x_0,y_0). Nevertheless, $\mathbf{f}$ is not one-to-one on $\mathbb{R}^2$, since

$$\begin{aligned}\mathbf{f}(x,y+2\pi) &= (e^x\,\cos(y+2\pi), e^x\,\sin(y+2\pi)) \\ &= (e^x\,\cos(y), e^x\,\sin(y)) \\ &= \mathbf{f}(x,y).\end{aligned}$$

(c) Since $\mathbf{f}(0,\frac{\pi}{3}) = (\frac{1}{2},\frac{\sqrt{3}}{2})$, we need to solve the system

$$\begin{cases} e^x\,\cos(y) = u \\ e^x\,\sin(y) = v\end{cases}$$

for (x,y) in term of (u,v), for (u,v) close to $(\frac{1}{2},\frac{\sqrt{3}}{2})$. Since $u^2+v^2 = e^{2\,x}$, it follows that $x=\ln(\sqrt{u^2+v^2})$. Since $\tan(y)=\frac{v}{u}$, we have $y=\tan^{-1}(\frac{v}{u})$ (Note that $\tan^{-1}(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$). The inverse mapping $\mathbf{g}$ is thus given by

$$\mathbf{g}(u,v) = \left(\ln(\sqrt{u^2+v^2}),\,\tan^{-1}(\frac{v}{u})\right).$$

Letting $\mathbf{g}=(g_1,g_2)$, we have

$$\mathbf{g}'(u_0,v_0) = \begin{bmatrix} \frac{\partial g_1}{\partial u}(u_0,v_0) & \frac{\partial g_1}{\partial v}(u_0,v_0) \\ \frac{\partial g_2}{\partial u}(u_0,v_0) & \frac{\partial g_2}{\partial v}(u_0,v_0)\end{bmatrix} = \begin{bmatrix} \frac{u}{u^2+v^2} & \frac{v}{u^2+v^2} \\ -\frac{v}{u^2+v^2} & \frac{u}{u^2+v^2}\end{bmatrix}$$

We have

$$\mathbf{f}'\left(0,\frac{\pi}{3}\right) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix},\quad \mathbf{g}'\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$$

and

$$\mathbf{f}'\left(0,\frac{\pi}{3}\right)\,\mathbf{g}'\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$$

which shows that

$$\mathbf{g}'\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right) = \left(\mathbf{f}'\left(0,\frac{\pi}{3}\right)\right)^{-1}.$$

(d) The line $x = x_0$ has for image the circle of radius e^{x_0} centered at $(0,0)$ while the line $y = y_0$ has for image a ray starting at the origin (but not including it) that makes an angle y_0 with the positive x-axis.

Problem 7.

Solution.

Problem 8. *Consider the mapping* $\mathbf{f} = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by*

$$\mathbf{f}(x, y) = (x + y, 2\,x\,y), \quad (x, y) \in \mathbb{R}^2.$$

(a) *Show that the range of* $\mathbf{f}$ *is the set*

$$L = \{(u, v) \in \mathbb{R}^2,\ u^2 - 2\,v \geq 0\}$$

Hint: *Show first that* $\mathbf{f}$ *maps* $\mathbb{R}^2$ *into* L *and then that every point in* L *is the image under* $\mathbf{f}$ *of some point in* $\mathbb{R}^2$*.*

(b) *(9 pts.) Show that the mapping* $\mathbf{f}$ *is locally invertible at the point* $(x_0, y_0) = (2, -1)$*. Find an explicit formula for its local inverse* $\mathbf{g}(u, v)$ *defined in a neighborhood of* $\mathbf{f}(2, -1) = (1, -4)$*.*

(c) *(5 pts.) Compute* $\mathbf{g}'(1, -4)$*.*

Solution. (a) Let $u = x + y$ and $v = 2\,x\,y$, then

$$u^2 - 2\,v = (x+y)^2 - 4\,x\,y = x^2 + 2\,x\,y + y^2 - 4\,x\,y = x^2 - 2\,x\,y + y^2 = (x-y)^2 \geq 0.$$

This shows that the range of $\mathbf{f}$ is contained in L.

Now let (u, v) belong to L, so $u^2 - 2\,v \geq 0$.

If $v = 0$, we have $\mathbf{f}(0, u) = (u, 0)$ (or $\mathbf{f}(u, 0) = (u, 0)$) showing that $(u, 0)$ belongs to the range of $\mathbf{f}$ for any value of u.

If $v \neq 0$ we have $y = \frac{v}{2\,x}$ and $u = x + \frac{v}{2\,x}$ or $x^2 - u\,x + \frac{v}{2} = 0$, so $x = \frac{u \pm \sqrt{u^2 - 2\,v}}{2}$. There are thus two points mapped to (u, v) by $\mathbf{f}$, namely

$$(x, y) = \left(\frac{u + \sqrt{u^2 - 2\,v}}{2}, \frac{v}{u + \sqrt{u^2 - 2\,v}} \right)$$

and

$$(x, y) = \left(\frac{u - \sqrt{u^2 - 2\,v}}{2}, \frac{v}{u - \sqrt{u^2 - 2\,v}} \right).$$

Thus (u, v) also belongs to the range of $\mathbf{f}$ when $(u, v) \in L$ and $v \neq 0$.

(b) We have

$$\mathbf{f}'(x, y) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2\,y & 2\,x \end{bmatrix}.$$

Since the first-order partial derivatives of u and v are continuous, $\mathbf{f}$ belongs to $\mathcal{C}^1(\mathbb{R}^2)$. Since

$$\mathbf{f}'(2,-1) = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

and

$$\begin{vmatrix} 1 & 1 \\ -2 & 4 \end{vmatrix} = 6 \neq 0,$$

the inverse mapping theorem shows that $\mathbf{f}$ is locally invertible at the point $(2,-1)$. Using part (a), the local inverse is given by

$$\begin{aligned} \mathbf{g}(u,v) &= \left(\frac{u + \sqrt{u^2 - 2\,v}}{2}, \frac{v}{u + \sqrt{u^2 - 2\,v}} \right) \\ &= \left(\frac{u + \sqrt{u^2 - 2\,v}}{2}, \frac{u - \sqrt{u^2 - 2\,v}}{2} \right). \end{aligned}$$

(c)

$$\mathbf{g}'(1,-4) = (\mathbf{f}'(2,-1))^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

Equivalently, we can use the explicit expression of $\mathbf{g}$ to compute $\mathbf{g}'$. We have

$$\mathbf{g}'(u,v) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{u}{2\sqrt{u^2-2\,v}} & -\frac{1}{2\sqrt{u^2-2\,v}} \\ \frac{1}{2} - \frac{u}{2\sqrt{u^2-2\,v}} & \frac{1}{2\sqrt{u^2-2\,v}} \end{bmatrix}$$

and

$$\mathbf{g}'(1,-4) = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} \end{bmatrix}.$$

Problem 9. (a) *Show that the system*

$$\begin{cases} x^2 + y^2 - u\,v - 3 = 0 \\ x\,u + y\,v - 2 = 0 \end{cases}$$

can be solved for x, y in terms of u, v near the point $(x,y,u,v) = (1,0,2,-1)$.

(b) *Compute the partial derivative*

$$\frac{\partial x}{\partial u}(2,-1).$$

Solution. Let $\mathbf{f}(x,y,u,v) = (f_1,f_2) = (x^2 + y^2 - u\,v - 3, x\,u + y\,v - 2)$. Note that $\mathbf{f}$ belongs to $\mathcal{C}^1(\mathbb{R}^2)$ and

$$\mathbf{f}'(x,y,u,v) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{bmatrix} = \begin{bmatrix} 2\,x & 2\,y & -v & -u \\ u & v & x & y \end{bmatrix}$$

For the system to be solvable for x, y in terms of u, v, we need to verify, according to the implicit function theorem, that the matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

evaluated at the point $(x, y, u, v) = (1, 0, 2, -1)$ is invertible. This matrix is

$$\begin{bmatrix} 2\,x & 2\,y \\ u & v \end{bmatrix}_{(1,0,2,-1)} = \begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix}$$

which has determinant $-2 \neq 0$ and is thus invertible.

(b) Since

$$f_1(x(u,v), y(u,v), u, v) = 0,$$

we have

$$\frac{\partial f_1}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f_1}{\partial u} = 0.$$

Similarly,

$$\frac{\partial f_2}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f_2}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f_2}{\partial u} = 0.$$

This can be written is matrix form as

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{bmatrix} = - \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}$$

from which we deduce that

$$\begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{bmatrix} = - \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}$$

Evaluating the previous expression at the point $(x, y, u, v) = (1, 0, 2, -1)$, we obtain

$$\begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{bmatrix}_{(2,-1)} = - \begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$$

In particular,

$$\frac{\partial x}{\partial u}(2, -1) = -\frac{1}{2}.$$