

THE MÖBIUS STRIP AND STOKES' THEOREM

1. STOKES' THEOREM

Let us recall Stokes' Theorem:

Theorem 1.1. *Let M be an oriented surface in $\mathbb{R}^3$ with boundary given by the closed curve γ , with orientation induced from that of M (by the right hand rule.) Let $\mathbf{F}(x, y, z)$ be a vector field. Then*

$$(1.1) \quad \iint_M (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \oint_{\gamma} \mathbf{F} \cdot d\mathbf{r}$$

In words, the flux of the *curl* of the vector field $\mathbf{F}$ through the surface M in the direction of $\mathbf{n}$ is equal to the circulation of the field $\mathbf{F}$ around the boundary curve γ in the associated direction.

2. THE MÖBIUS STRIP

Let us now describe the *Möbius strip* and try to use Stokes' Theorem on it. The Möbius strip is obtained by taking a rectangular strip of paper, and gluing two sides together after performing a twist. Let's be more precise: First, imagine constructing a surface of revolution, by taking a curve in the y - z plane and rotating it about the z -axis. Our curve will be the straight line segment $x = R$, $-L \leq y \leq L$, for some positive constants R and L . In this case we just get a cylinder as in Figure 1:



FIGURE 1. Cylinder

However, suppose that as the straight line segment rotates about the z -axis, it also rotates counterclockwise about its centre at half the rate, so it returns to its starting point upside down. (Refer to Figure 2.) Note that we assume that $L < R$ to ensure that the resulting surface has no self-intersections.

In this case we can write down a parametrization using $x = r \cos(v)$, $y = r \sin(v)$ and $z = z$ and we obtain:

$$(2.1) \quad \begin{aligned} x(t, v) &= \left(R - t \sin\left(\frac{v}{2}\right) \right) \cos(v) \\ y(t, v) &= \left(R - t \sin\left(\frac{v}{2}\right) \right) \sin(v) \\ z(t, v) &= t \cos\left(\frac{v}{2}\right) \\ 0 \leq v \leq 2\pi &\quad -L \leq t \leq L \end{aligned}$$

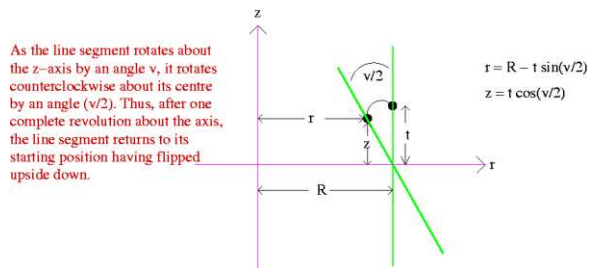


FIGURE 2. Parametrization of the Möbius Strip

The resulting surface is shown from two different viewpoints in Figure 3.

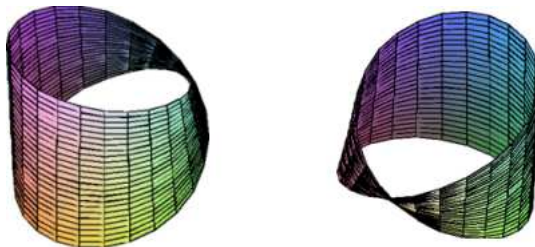


FIGURE 3. Front and back views of the Möbius Strip

We now identify the boundary curve γ to this surface. Note that since the line segment has turned upside down by the time it comes back to its starting point, if we follow the point at the top of the segment it has come to the bottom after rotating by an angle 2π , and then it has come *back to the top* after an angle of 4π . Hence the boundary curve can be parametrized by taking $t = L$ in the surface parametrization and $0 \leq v \leq 4\pi$.

$$\begin{aligned}
 (2.2) \quad x(v) &= \left(R - L \sin\left(\frac{v}{2}\right)\right) \cos(v) \\
 y(v) &= \left(R - L \sin\left(\frac{v}{2}\right)\right) \sin(v) \\
 z(v) &= L \cos\left(\frac{v}{2}\right) \\
 0 &\leq v \leq 4\pi
 \end{aligned}$$

You should check that this is equivalent (using trig identities) to taking both of the pieces where $t = L$ and $t = -L$ for $0 \leq v \leq 2\pi$.

3. APPLYING STOKES' THEOREM TO THE MÖBIUS STRIP

Let us try to apply Stokes' Theorem to this surface. We need a vector field $\mathbf{F}$. We will take

$$\mathbf{F} = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right\rangle$$

This vector field is well defined everywhere except on the z -axis, which is not part of our surface. It is easy to check that $\nabla \times \mathbf{F} = 0$. Hence the left hand side of Stokes' Theorem gives:

$$\iint_M (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = 0$$

Let us compute the line integral on the right hand side.

$$\oint_{\gamma} \mathbf{F} \cdot d\mathbf{r} = \int_0^{4\pi} \mathbf{F}(x(v), y(v), z(v)) \cdot \mathbf{r}'(v) dv$$

where $\mathbf{r}(v)$ is the parametrization. We have

$$(3.1) \quad \begin{aligned} x'(v) &= -\frac{L}{2} \cos\left(\frac{v}{2}\right) \cos(v) - \left(R - L \sin\left(\frac{v}{2}\right)\right) \sin(v) \\ y'(v) &= -\frac{L}{2} \cos\left(\frac{v}{2}\right) \sin(v) + \left(R - L \sin\left(\frac{v}{2}\right)\right) \cos(v) \\ z'(v) &= -\frac{L}{2} \sin\left(\frac{v}{2}\right) \end{aligned}$$

and also

$$\mathbf{F}(\mathbf{r}(v)) = \left\langle \frac{-\sin(v)}{\left(R - L \sin\left(\frac{v}{2}\right)\right)}, \frac{\cos(v)}{\left(R - L \sin\left(\frac{v}{2}\right)\right)}, 0 \right\rangle$$

From here it is easy to check that $\mathbf{F}(\mathbf{r}(v)) \cdot \mathbf{r}'(v) = 1$ and hence the line integral is

$$\oint_{\gamma} \mathbf{F} \cdot d\mathbf{r} = \int_0^{4\pi} dv = 4\pi \neq 0$$

It seems Stokes' Theorem is violated! The answer to this paradox is that Stokes' Theorem was *not applicable* to this surface, because in fact the Möbius strip is *not orientable*. Notice that the curl of $\mathbf{F}$ was zero so we never actually had to calculate the flux integral. To see what would have happened, let us compute the normal to the surface $\mathbf{n} = \frac{\partial \mathbf{X}}{\partial t} \times \frac{\partial \mathbf{X}}{\partial v}$ where $\mathbf{X}(t, v) = \langle x(t, v), y(t, v), z(t, v) \rangle$ is the parametrization. To simplify our calculations, and since it is all we will need anyway, we will consider only the part of the Möbius strip corresponding to the parameter $t = 0$. This is a closed curve on the surface. Note that it is the path followed by the centre of the line segment as it revolves around the z -axis and simultaneously rotates about its centre. We have

$$\frac{\partial \mathbf{X}}{\partial t} = \left\langle -\sin\left(\frac{v}{2}\right) \cos(v), -\sin\left(\frac{v}{2}\right) \sin(v), \cos\left(\frac{v}{2}\right) \right\rangle$$

and the calculations from equations (3.1) show that, for $t = 0$,

$$\frac{\partial \mathbf{X}}{\partial v} = \langle -R \sin(v), R \cos(v), 0 \rangle$$

and hence

$$\mathbf{n} = \frac{\partial \mathbf{X}}{\partial t} \times \frac{\partial \mathbf{X}}{\partial v} = \left\langle -R \cos(v) \cos\left(\frac{v}{2}\right), -R \sin(v) \cos\left(\frac{v}{2}\right), -R \sin\left(\frac{v}{2}\right) \right\rangle$$

We see that as v changes from 0 to 2π , $\mathbf{n}$ changes to $-\mathbf{n}$. But this corresponds to the *same point* on the surface. Hence we do not have a well-defined unit normal vector field, and the Möbius strip is *not orientable*. Another way to say it is that this surface has *only one side*. Orientable surfaces have two sides, and we can only do surface integrals over two-sided surfaces.