FREE FINITE GROUP ACTIONS ON RATIONAL HOMOLOGY 3–SPHERES

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ABSTRACT. We use methods from the cohomology of groups to describe the finite groups which can act freely and homologically trivially on closed 3–manifolds which are rational homology spheres.

1. Introduction

Cooper and Long [8] have shown that every finite group can act freely and smoothly on some closed, oriented 3-manifold M with the rational homology of the 3-sphere (for brevity we shall call such an object *a rational homology 3-sphere*). However, under the natural condition that the action must induce the identity on the integral homology of M, new group theoretic restrictions arise. In this note, we apply group cohomology to establish necessary conditions for such *homologically trivial* actions and use this information to construct some new examples.

Theorem A. *Let G denote a finite group acting freely and homologically trivially on a rational homology 3–sphere M. Let π denote the product of precisely those primes which divide both $|G|$ and $|H_1(M;\mathbb{Z})|$. Then there exists an extension*

$$1 \to H \to Q_\pi \to G \to 1$$

where $H \cong H_1(M;\mathbb{Z})_{(\pi)}$ is a central, cyclic subgroup and Q_π is a group with periodic cohomology of period two or four.

Theorem A and more generally the results in §2 and §3 hold for finite G–CW complexes with the cohomology ring of a rational homology sphere.

The corresponding existence statement would follow from a positive answer to the following question:

Question. Let G be a finite group with periodic cohomology of period four. Does G act freely and homologically trivially on some rational homology 3–sphere ?

A complete list of such groups is given in Milnor [24, §3], and those which can act freely and othogonally on S^3 were listed by Hopf [16]. Perelman [18] showed that the remaining groups in Milnor's list do not arise as the fundamental group of any closed, oriented 3-manifold. For some of these we have a non-existence result in our setting.

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Theorem B. *Let Q be a finite group of period four which is not the fundamental group of a closed, oriented 3-manifold. If G is a quotient of Q by a central cyclic subgroup, and the order of Q is divisible by 16, then G can not act freely and homologically trivially on a rational homology 3-sphere.*

Among the remaining groups in Milnor's list, the groups $Q(8n, k, l)$, for n odd, have been much studied, and it is known that some (but not all) can act freely on integral homology 3-spheres (see Madsen [22]). This work gives some new examples of existence in the setting of Theorem A via quotients by the action of central cyclic subgroups (see Proposition 4.5). The results of Pardon [26] provided free actions of period four groups on rational homology 3-spheres with some control on the torsion, but did not address the homological triviality requirement (see Proposition 4.1). More information about the actions of the groups $Q(8n, k, l)$ is given in Theorem 4.13.

This article is organized as follows: in §2 we apply methods from group cohomology to actions on rational homology spheres; in §3 we consider the restrictions arising in the homologically trivial case; in §4 we discuss the existence of homologically trivial actions, and finally §5 deals with how our approach applies quite generally to finite quotients of fundamental groups of closed 3–manifolds.

2. Application of cohomological methods

Let G denote a finite group acting freely, smoothly, and preserving orientation on a closed 3–manifold M that is a rational homology sphere. In dimension three, free actions of finite groups by homeomorphisms are equivalent to smooth actions, and the quotient manifolds are homotopy equivalent to finite CW complexes.

We denote by $\Omega^r(\mathbb{Z})$ the $\mathbb{Z}G$ module uniquely defined in the stable category (where $\mathbb{Z}G$-modules are identified up to stabilization by projectives) as the r–fold dimension–shift of the trivial module $\mathbb{Z}$. Note the isomorphism of $\mathbb{Z}G$–modules $H_1(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z})$; we may use either version depending on the context. We refer to [2] and [6] for background on group cohomology.

Proposition 2.1. *If a finite group G acts freely on a rational homology 3–sphere M, then there is a short exact sequence of $\mathbb{Z}G$–modules in the stable category of $\mathbb{Z}G$–modules of the form*

$$0 \to \Omega^{-2}(\mathbb{Z}) \to \Omega^2(\mathbb{Z}) \to H_1(M;\mathbb{Z}) \to 0 \tag{2.2}$$

Proof. We will assume that M is a G–CW complex with cellular chains $C_*(M)$. Then we have exact sequences of $\mathbb{Z}G$–modules

$$\begin{gathered} 0 \to \mathbb{Z} \to C_3(M) \to C_2(M) \to B_1 \to 0 \\ 0 \to Z_1 \to C_1(M) \to C_0(M) \to \mathbb{Z} \to 0 \\ 0 \to B_1 \to Z_1 \to H_1(M;\mathbb{Z}) \to 0 \end{gathered}$$

where B_1 denotes the module of boundaries and Z_1 the module of cycles respectively. The result follows from (stably) identifying Z_1 with $\Omega^2(\mathbb{Z})$ and B_1 with $\Omega^{-2}(\mathbb{Z})$ respectively. $\square$

Note that the stable map $\Omega^{-2}(\mathbb{Z}) \to \Omega^2(\mathbb{Z})$ defines an element

$$\sigma \in \underline{\mathrm{Hom}}_{\mathbb{Z}G}(\Omega^{-2}(\mathbb{Z}), \Omega^2(\mathbb{Z})) \cong \underline{\mathrm{Hom}}_{\mathbb{Z}G}(\mathbb{Z}, \Omega^4(\mathbb{Z})) \cong \widehat{H}^{-4}(G, \mathbb{Z})$$

This class appears when applying Tate cohomology to (2.2).

Corollary 2.3. *The short exact sequence* (2.2) *yields a long exact sequence in Tate cohomology*

$$\cdots \to \widehat{H}^{i+2}(G, \mathbb{Z}) \xrightarrow{\cup\sigma} \widehat{H}^{i-2}(G, \mathbb{Z}) \to \widehat{H}^{i}(G, H_1(M;\mathbb{Z})) \to \widehat{H}^{i+3}(G, \mathbb{Z}) \to \dots$$

determined by the class $\sigma \in \widehat{H}^{-4}(G, \mathbb{Z})$ which is the image of the generator $1 \in \widehat{H}^0(G, \mathbb{Z}) \cong \mathbb{Z}/|G|$.

Next we identify the class σ geometrically.

Proposition 2.4. *If $[M/G] \in H_3(M/G, \mathbb{Z})$ denotes the fundamental class of the quotient manifold, then σ is the image of $c_*[M/G] \in H_3(BG, \mathbb{Z})$, under the natural isomorphism $H_3(BG, \mathbb{Z}) \cong \widehat{H}^{-4}(G, \mathbb{Z})$, where $c\colon M/G \to BG$ is the classifying map of the covering.*

Proof. To see this, recall that the description due to MacLane [19, Chap. V.8] of

$$\mathrm{Tor}_3^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) \cong H_3(G; \mathbb{Z})$$

via chain complexes, shows that the image of the fundamental class

$$c_*[M/G] \in H_3(G; \mathbb{Z})$$

is represented by the chain complex $C_*(M)$ of finitely-generated free $\mathbb{Z}G$-modules. We can apply dimension-shifting in the "complete" Ext-theory to the formula:

$$\mathrm{Tor}_3^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) = \widehat{H}^{-4}(G; \mathbb{Z}) = \mathrm{Ext}_{\mathbb{Z}G}^{-4}(\mathbb{Z}; \mathbb{Z}) = \underline{\mathrm{Hom}}_{\mathbb{Z}G}(\mathbb{Z}, \Omega^4\mathbb{Z})$$

to identify $c_*[M/G]$ with the extension class of the sequence (2.2) (see Wall [30, §2] for more background). □

In a similar way the map $\Omega^2(\mathbb{Z}) \to H_1(M, \mathbb{Z})$ defines an extension class

$$\mathcal{E}_M \in H^2(G, H_1(M; \mathbb{Z}))$$

which appears in the long exact sequence from Corollary 2.3 as the image of the generator under the map

$$\widehat{H}^0(G, \mathbb{Z}) \to \widehat{H}^2(G, H_1(M; \mathbb{Z})). \tag{2.5}$$

This algebraic map arises geometrically as folllows. Let $X \subset M$ denote a connected one dimensional $G-CW$ sub–complex such that $\pi_1(X) \to \pi_1(M)$ is onto. If we denote $F = \pi_1(X)$, then we have a diagram of extensions

$$\begin{tikzcd} 1 \arrow[r] & R \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \pi_1(M) \arrow[r] & \pi_1(M/G) \arrow[r] & G \arrow[r] & 1 \end{tikzcd} \tag{2.6}$$

Abelianizing kernels gives rise to the diagram

$$
\begin{tikzcd}
1 \arrow[r] & R_{ab} \arrow[r] \arrow[d, two heads] & \Phi \arrow[r] \arrow[d, two heads] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & H_1(M;\mathbb{Z}) \arrow[r] & Q \arrow[r] & G \arrow[r] & 1
\end{tikzcd}
\tag{2.7}
$$

where Φ is the associated free abelianized extension. This extension realizes the universal class of highest exponent in $\widehat{H}^2(G, \Omega^2(\mathbb{Z}))$; note that R_{ab} is a free abelian group which as a $\mathbb{Z}G$–module is stably equivalent to $\Omega^2(\mathbb{Z})$. By construction, the bottom extension represents the class $\mathcal{E}_M \in H^2(G, H_1(M, \mathbb{Z}))$ (see [15] page 207).

It is known (see [5]) that free group actions can be fruitfully analyzed using exponents. For actions on rational homology 3–spheres the analysis can be done quite explicitly.

Corollary 2.8. *If G acts freely on a rational homology 3–sphere M, then*

$$|G| = \exp\{\sigma\} \cdot \exp\{\mathcal{E}_{\mathrm{M}}\}.$$

G has periodic cohomology with periodicity induced by σ if and only if the extension Q representing $\mathcal{E}_M$ is split. In particular, if $|G|$ is relatively prime to $|H_1(M;\mathbb{Z})|$, then σ is a periodicity class for the cohomology of G.

Proof. Look at the exact sequence in Corollary 2.3 at $i = 2$:

$$\cdots \to \widehat{H}^4(G;\mathbb{Z}) \xrightarrow{\cup\sigma} \widehat{H}^0(G;\mathbb{Z}) \to \widehat{H}^2(G; H_1(M;\mathbb{Z})) \to \dots$$

and let $\exp(\sigma)$, respectively $\exp(\mathcal{E}_M)$, denote the order of the image, respectively cokernel, of the map induced by $\cup\,\sigma$. Recall that by Tate duality (see Brown [6, Chap.VI.7]) there is an element $\sigma^* \in \widehat{H}^4(G, \mathbb{Z})$ such that

$$\sigma \cup \sigma^* = |G|/\exp\sigma \in \mathbb{Z}/|\mathrm{G}|,$$

and the exponent expression follows. A finite group has periodic cohomology if and only if it has an element of positive degree with exponent equal to $|G|$ (for example, combine Swan [29, Corollary 2.2 and Lemma 4.2]). This element is invertible in Tate cohomology. Hence if the class σ has this highest exponent, then it must be a periodicity class, inducing isomorphisms throughout the long exact sequence, and this occurs if and only if $\widehat{H}^2(G; H_1(M;\mathbb{Z})) = 0$ and the extension Q is split. $\square$

Remark 2.9. On the other extreme, if σ is trivial, then the extension class $\mathcal{E}_M$ has highest exponent equal to $|G|$. Using the stable isomorphism

$$\underline{\mathrm{Hom}}_{\mathbb{Z}G}(\Omega^2(\mathbb{Z}), H_1(M,\mathbb{Z})) \cong \underline{\mathrm{Hom}}_{\mathbb{Z}G}(\mathbb{Z}, \Omega^{-2}(H_1(M,\mathbb{Z})))$$

we can represent $\mathcal{E}_M$ by a rank one trivial submodule in $\Omega^{-2}(H_1(M,\mathbb{Z}))$ (note that any finitely generated $\mathbb{Z}G$–module is stably equivalent to a $\mathbb{Z}$–torsion free module via dimension–shifting). By [1, Theorem 1.1] and its proof, the short exact sequence

$$0 \to \mathbb{Z} \to \Omega^{-2}(H_1(M,\mathbb{Z})) \to \Omega^{-5}\mathbb{Z} \to 0$$

in the stable category is split exact. After shifting back, we obtain a stable decomposition $H_1(M,\mathbb{Z}) \cong \Omega^2(\mathbb{Z}) \oplus \Omega^{-3}(\mathbb{Z})$. This will occur for rational homology spheres with a free

G–action where $H^4(G,\mathbb{Z}) = 0$. An example of this phenomenon is given by the Mathieu group M_{23} (see [23]).

Remark 2.10. For $G = (\mathbb{Z}/p)^r$, we have $p \cdot \widehat{H}^k(G,\mathbb{Z}) = 0$ for $k \neq 0$, so we see that the exponent of $\mathcal{E}_M$ is at least p^{r-1}, and in particular the module $H_1(M;\mathbb{Z})$ must have p^{r-1}–torsion.

Remark 2.11. It is also an interesting problem to determine which groups can act homologically trivially on higher dimensional rational homology spheres. Using exponents it can be shown that if G acts freely and homologically trivially on a simply–connected rational homology n–sphere, then the rank of G can be at most $n-2$. We expect that further group theoretic restrictions will play a role.

3. Restrictions in the homologically trivial case

In this section we focus on the special case when the G–action on M is trivial in homology. This imposes some drastic restrictions.

Proposition 3.1. *If G acts freely and homologically trivially on a rational homology 3–sphere M, then every elementary abelian subgroup of G has rank at most two.*

Proof. From Corollary 2.8 and the fact that $\widehat{H}^r(G,L)$ has exponent p when L has trivial action and $r \neq 0$, we see that if $(\mathbb{Z}/p)^r$ acts freely and homologically trivially on M, then p^r divides p^2 and the result follows. □

Let us write the trivial $\mathbb{Z}G$–module $H_1(M;\mathbb{Z})$ as a direct sum of finitely generated, finite abelian p–groups $A_p = H_1(M;\mathbb{Z})_{(p)}$. Then we have

Lemma 3.2. *If p is a prime number dividing the order of G, then $H_1(M;\mathbb{Z})_{(p)}$ is either trivial or cyclic.*

Proof. Consider a cyclic $C \cong \mathbb{Z}/p$ in G. Since $H^7(C;\mathbb{Z}) = 0$, from the sequence 2.3 for C at $i = 4$, we see that $H^4(C, H_1(M;\mathbb{Z}))$ is a homomorphic image of $H^2(C,\mathbb{Z}) = \mathbb{Z}/p$. It follows that

$$H^4(C, H_1(M;\mathbb{Z})) \cong H^4(C, A_p) \cong A_p/pA_p$$

is either trivial or $\mathbb{Z}/p$, which proves the result. □

A well–known example of a free action on a rational homology sphere is given by the free action of $\mathbb{Z}/2 \times \mathbb{Z}/2$ on $\mathbb{R}\mathbf{P}^3$, which comes from the free action of the quaternions on the 3–sphere. Associated to it there is a central extension of the form

$$1 \to \mathbb{Z}/2 \to Q(8) \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1.$$

Next we will show that all groups acting freely and homologically trivially on rational homology spheres can be modeled in this way.

Theorem 3.3. *Let G denote a finite group acting freely and homologically trivially on a rational homology 3–sphere M. Let $\pi = p_1 \dots p_r$ denote the product of precisely those primes which divide both $|G|$ and $|H_1(M;\mathbb{Z})|$. Then there exists an extension*

$$1 \to H \to Q_\pi \to G \to 1$$

where $H \cong H_1(M;\mathbb{Z})_{(\pi)}$ is a central, cyclic subgroup and Q_π is a group with periodic cohomology.

Proof. Suppose that G acts freely on a rational homology 3-sphere M and consider the group extension

$$1 \to \pi_1(M) \to \pi_1(M/G) \to G \to 1.$$

Let L denote the kernel of the map $\pi_1(M) \to H_1(M;\mathbb{Z})_{(\pi)}$; then it is normal in both $\pi_1(M)$ and $\pi_1(M/G)$ and we can consider the associated central quotient extension:

$$0 \to H_1(M;\mathbb{Z})_{(\pi)} \to Q_\pi \to G \to 1.$$

Note that $H^2(G, H_1(M;\mathbb{Z})) \cong H^2(G, H_1(M;\mathbb{Z})_{(\pi)})$ and our construction is the obvious quotient of the extension Q representing the class $\mathcal{E}_M$ apprearing in Corollary 2.8.

Suppose p is a prime that divides $|G|$ but which is relatively prime to $|H_1(M;\mathbb{Z})|$. Then $H^2(\mathrm{Syl}_p(G), H_1(M;\mathbb{Z})) = 0$ and so $\mathrm{Syl}_p(G) = \mathrm{Syl}_p(Q_\pi)$ is periodic by Corollary 2.8. Now suppose that p is a prime which divides π, and let $C \subset G$ denote a cyclic subgroup of order p. By naturality we have a commutative diagram, where the rows are exact sequences:

$$\begin{tikzcd}
\widehat{H}^0(G,\mathbb{Z}) \arrow[r] \arrow[d] & \widehat{H}^2(G, H_1(M;\mathbb{Z})) \arrow[r] \arrow[d] & \widehat{H}^5(G,\mathbb{Z}) \arrow[d] \\
\widehat{H}^0(C,\mathbb{Z}) \arrow[r, "\approx"] & \widehat{H}^2(C, H_1(M;\mathbb{Z})) \arrow[r] & 0
\end{tikzcd}$$

The isomorphism in the lower row of this diagram comes from the rest of the sequence

$$0 \to \widehat{H}^1(C; H_1(M;\mathbb{Z})) \to \widehat{H}^4(C;\mathbb{Z}) \to \widehat{H}^0(C;\mathbb{Z}) \to \widehat{H}^2(C, H_1(M;\mathbb{Z})) \to 0$$

since

$$\widehat{H}^1(C; H_1(M;\mathbb{Z})) \cong \mathbb{Z}/p \cong \widehat{H}^4(C;\mathbb{Z}).$$

By Lemma 3.2 the p–component of $H_1(M;\mathbb{Z})$ is a finite cyclic p–group with a trivial C–action. Hence $\widehat{H}^i(C, H_1(M;\mathbb{Z})) \neq 0$ for all i. The map $\widehat{H}^0(G,\mathbb{Z}) \to \widehat{H}^0(C,\mathbb{Z})$ sends a generator to a generator so the extension class $\mathcal{E}_M \in H^2(G, H_1(M;\mathbb{Z}))$ restricts non–trivially on all such subgroups C, and hence the corresponding restricted extensions of the form

$$0 \to H_1(M;\mathbb{Z})_{(\pi)} \to Q_\pi|_C \to C \to 1$$

are all non–split.

If we take $H = H_1(M;\mathbb{Z})_{(\pi)}$, which we know to be cyclic by Lemma 3.2, then the extension expresses G as the quotient Q_π/H where H is a central, cyclic subgroup and every restricted group of the form $Q_\pi|C$ is non–split, where $C \cong \mathbb{Z}/p\mathbb{Z}$, and p divides π.

Let $u \in Q_\pi$ denote an element of order p; if the subgroup generated by H and u is not cyclic, then it must be split abelian, a contradiction. Therefore all elements of order p in Q_π lie in H, a cyclic subgroup, and so Q_π has no rank two p–elementary abelian subgroups. We have already established this for the primes which do not divide π, whence we infer that Q has periodic cohomology. □

Proposition 3.4. *The period of Q_π is two or four.*

Proof. Consider the central group extension

$$1 \to H \to Q \to G \to 1$$

where $H := H_1(M, \mathbb{Z})_{(\pi)}$ and $Q := Q_\pi$. By Swan [28], the period of Q is the least common multiple of the p-periods of Q taken over all primes p dividing $|Q|$. The p-periods are determined by group cohomology with p-local coefficients. By [28, Theorem 1], the 2-period of Q is 2 or 4. Moveover, by [28, Theorem 2], the p-period of Q for p odd is twice the order of $\Phi_p(Q) \cong N_Q(\mathrm{Syl}_p(Q))/C_Q(\mathrm{Syl}_p(Q))$, the group of automorphisms of $\mathrm{Syl}_p(Q)$ induced by inner automorphisms of Q. Note that as $\mathrm{Syl}_p(Q)$ is cyclic, its automorphism group is also cyclic and hence $\Phi_p(Q)$ is cyclic of order prime to p.

As explained in [28, Lemma 3], the action on $\widehat{H}^{2i}(\mathrm{Syl}_p(Q), \mathbb{Z}_{(p)})$ is given by multiplication by r^i, where r is an integer prime to p that is a multiplicative generator of $\Phi_p(Q)$. Hence this action has invariants only when i is a multiple of $|\Phi_p(Q)|$, and $\widehat{H}^*(Q, \mathbb{Z}_{(p)}) \neq 0$ only in degrees which are multiples of $2|\Phi_p(Q)|$.

If p doesn't divide $|G|$ but does divide $|Q|$, then the p–period of Q is equal to that of the central cyclic subgroup H and thus equal to two. If p divides $|G|$ then the projection $Q \to G$ induces an isomorphism $\Phi_p(Q) \cong \Phi_p(G)$, and hence the p-periods of Q and G are equal. Consider now the following portion of the p–local version of the long exact cohomology sequence from Corollary 2.3:

$$\widehat{H}^4(G; \mathbb{Z}_{(p)}) \to \widehat{H}^0(G; \mathbb{Z}_{(p)}) \to \widehat{H}^2(G; H_p) \tag{3.5}$$

where $H_p = \mathrm{Syl}_p(H) = H_1(M; \mathbb{Z}_{(p)})$. As $|G|$ is divisible by p, the middle term is non-zero. Now if $\widehat{H}^2(G; H_p) = 0$, then $\widehat{H}^4(G; \mathbb{Z}_{(p)}) \neq 0$ and we conclude that G has p-period dividing four (a p–local version of Corollary 2.8). However, by the universal coefficient theorem applied to the trivial G–module H_p, we see that $\widehat{H}^2(G, H_p) = 0$ if the p–period of G is four or higher. Hence we conclude that the p-periods of G and Q must both be either two or four. □

The structure of G is more explicit for p-groups.

Corollary 3.6. *A finite p–group G acts freely and homologically trivially on some rational homology 3–sphere M with non–trivial p–torsion in $H_1(M; \mathbb{Z})$ if and only if (1) G is cyclic or (2) $p = 2$, $H_1(M; \mathbb{Z})_{(2)} \cong \mathbb{Z}/2\mathbb{Z}$ and G is a dihedral group.*

Proof. The finite groups of the form Q/H where Q is a periodic p–group and H is a non–trivial central cyclic subgroup are precisely the cyclic groups and $Q_{2^n}/Z(Q_{2^n})$, where Q_{2^n} is a generalized quaternion group of order 2^n, $n \geq 3$, with centre isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and quotient a dihedral group of order 2^{n-1}. Conversely all the groups Q appearing above act freely on $\mathbb{S}^3$, hence all the quotients $G = Q/H$ act freely on a rational homology sphere. □

Corollary 3.7. *Let G act freely and homologically trivially on a rational homology 3–sphere M.*

(i) *If both $|G|$ and $|H_1(M; \mathbb{Z})|$ are even, then $\mathrm{Syl}_2(G)$ is either cyclic or dihedral.*
(ii) *If p is an odd prime dividing $|G|$, then $\mathrm{Syl}_p(G)$ is cyclic.*
(iii) *If $(|H_1(M; \mathbb{Z})|, p) = 1$ then $\mathrm{Syl}_p(G)$ is either cyclic or generalized quaternion.*

4. Existence of homologically trivial actions

As mentioned in the Introduction, any finite group can act freely on some rational homology 3-sphere if there is no homological triviality assumption. This was first proved by Pardon [26] using local surgery theory, extending a result of Browder and Hsiang [4, p. 267]. The direct 3-dimensional argument of Cooper and Long [8] avoids the surgery formalism, but does not give any control on the torsion in $H_1(M;\mathbb{Z})$.

Proposition 4.1. *Let G be a finite group, and p a prime such that $(p,|G|)=1$. Then G acts freely on some p-local homology 3-sphere.*

Proof. This statement is a special case of Pardon [26, Theorem B], together with the standard remark that high-dimensional surgery existence results in dimensions $4k+3 \geq 7$, imply existence results in dimension three up to homology. □

It appears to be much more difficult to solve the existence problem for a *given* rational homology 3-sphere M. For example, what if we consider only the space form groups but do not require homologically trivial actions ?

Question. If G acts freely on S^3, can it act freely on a given rational homology 3-sphere M with $(|G|, |H_1(M;\mathbb{Z})|) = 1$?

We will now use the information in Theorem A to make some remarks about the existence of *homologically trivial* actions on rational homology 3-spheres.

The finite groups which can act freely on S^3 are now known (by the work of Perelman [18]): they are precisely the periodic groups in Hopf's list [16]. For any of these groups we can obtain examples of homologically trival actions by quotients S^3/H, where H is a central cyclic subgroup. Of the remaining period four groups, we first consider those which do not satisfy Milnor's $2p$-condition.

Proposition 4.2. *Let G be a finite group with periodic cohomology of period four containing a non-cyclic subgroup of order $2p$, for some odd prime p. Then G is a the product of a dihedral group by a cyclic group of relatively prime order. Any quotient of G by a central cyclic subgroup can act freely and homologically trivially on some rational homology 3-sphere.*

Proof. This follows by checking the list of periodic groups, taking into account our period four assumption. A convenient reference is Wall [31, Theorem 4.5], which states that G is an extension of a normal subgroup G_0 of odd order by a group G_1 isomorphic to one of the form $C(2^k)$, $Q(2^k)$, T_v^*, O_v^*, $SL_2(p)$ or $TL_2(p)$. The periods of these groups are listed in [31, Corollary 5.6]. In our case, G_0 must be cyclic (the only odd order group with period ≤ 4), and $G_1 = C(2^k)$ since there is an unique element of order two in the other cases. The action of G_1 on G_0 must be faithful to violate the $2p$ condition, and G of period four implies the claimed structure for G.

Since any quotient of G by a central subgroup is again of the same form, the required actions arise by quotients of a free action of a binary dihedral group $Q(4n)$ on S^3. □

Milnor [24, §3] listed the period four groups which do satisfy the $2p$ conditions, and identified two families of such group which (by Perelman [18]) can not act freely on S^3. These are:

(i) $Q(8n, k, l)$, with $n > k > l \geq 1$, and $8n$, k, l pairwise relatively prime;
(ii) $O(48, k, l)$, with l odd, $3 \nmid l$ and 48, k, l pairwise relatively prime.

One can also take the product of any one of these groups with a cyclic group of relatively prime order. We will refer to these as *type A* if n is odd, *type B* if $n \geq 2$ is even, or *type C* for $O(48, k, l)$. The groups of type B or C have order divisible by 16.

Proposition 4.3. *Suppose that G is a period four group of type* A, B *or* C. *If G acts freely and homologically trivially on a rational homology* 3*-sphere M, then $H_1(M; \mathbb{Z}) = \mathbb{Z}/d$, where $(d, |G|) = 1$.*

Proof. From diagram (2.7), we have short exact sequence:

$$1 \to H_1(M; \mathbb{Z}) \to Q \to G \to 1$$

where Q is a period four group. Since both Q and G are period four groups, and G has type A, B or C, it follows that d must be odd. Since $H_1(M; \mathbb{Z})$ is central, it follows that Q has the same type as G. In particular, this implies that $H_1(M; \mathbb{Z}) = \mathbb{Z}/d$, with $(d, |G|) = 1$. □

This results allows us to rule out types B and C. Note the condition $(d, |G|) = 1$ implies that d is an *odd* integer, so M would have to be a $\mathbb{Z}_{(2)}$-homology sphere for such an action to exist. Theorem B follows from the following result.

Proposition 4.4. *Let G be the quotient of a type* B *or* C *period four group by a central cyclic subgroup. Then G can not act freely and homologically trivially on a rational homology* 3*-sphere.*

Proof. The period four groups of type B or C themselves can not act freely on any $\mathbb{Z}_{(2)}$-homology 3-sphere (see Ronnie Lee [17, Corollary 4.15, Corollary 4.17]), so they are ruled out by Proposition 4.3.

Now suppose that some non-periodic quotient G of a type B or C group acts freely and homologically trivially on a rational homology 3-sphere M. We then have a covering space

$$M \to M/G \to BG.$$

From (2.7), we have an exact sequence

$$1 \to H_1(M; \mathbb{Z}) \to Q \to G \to 1$$

where Q is a period four group and $H_1(M; \mathbb{Z}) = \mathbb{Z}/2d$ is a central cyclic subgroup of Q. It follows that Q must again be of type B or C, d must be odd, and $H_1(M; \mathbb{Z})$ must contain the unique central subgroup $T = \mathbb{Z}/2$ of order two in Q.

The group Q is constructed by a pushout from $\pi_1(M)$, and we can form a further pushout over the projection $H_1(M; \mathbb{Z}) \to T = \mathbb{Z}/2$ to obtain the group extension

$$1 \to T \to Q' \to G \to 1$$

in which Q' is again a period four group of type B or C.

The 2-fold covering $M' \to M$ given by the quotient $\pi_1(M) \to T$, followed by the G-covering $M \to M/G$, is just the Q'-covering $M' \to M/G$.

To obtain a contradiction, we will now show that M' is a $\mathbb{Z}_{(2)}$-homology sphere. From the structure of $M' \to M$ as a 2-fold covering, we have an exact sequence

$$0 \to H_0(\mathbb{Z}/2; H_1(M'; \mathbb{Z})) \to H_1(M; \mathbb{Z}) \to \mathbb{Z}/2 \to 0$$

and $H_1(M; \mathbb{Z}) = \mathbb{Z}/2d$, with d odd. It follows that the co-invariants

$$H_0(\mathbb{Z}/2; H_1(M'; \mathbb{Z})) = \mathbb{Z}/d$$

are of odd order, and hence $H_1(M'; \mathbb{Z})$ has no 2-torsion. We have an exact sequence of $\mathbb{Z}/2$-modules of the form:

$$0 \to H_1(M'; \mathbb{Z})_{odd} \to H_1(M'; \mathbb{Z}) \to \mathbb{Z}^r \to 0$$

and by applying group homology $H_*(\mathbb{Z}/2; -)$ to the sequence, we conclude that $r = 0$ and $H_1(M'; \mathbb{Z})$ is all odd torsion. In other words, M' is a $\mathbb{Z}_{(2)}$-homology 3-sphere and the free Q'-action can not exist. □

The remaining existence question concerns central quotients of the period four groups of type A. It is enough to consider the period four groups themselves.

Proposition 4.5. *Let G be the quotient of a type* A *period four group Q by a central cyclic subgroup $T \leq Q$. If Q acts freely and homologically trivially on a rational homology* 3*-sphere M, then G acts freely and homologically trivially on M/T, which is again a rational homology* 3*-sphere.*

Proof. Let G be the quotient of a type A period four group Q by a central cyclic subgroup $T \leq Q$. If Q acts freely and homologically trivially on a rational homology 3-sphere M, then G acts freely on the rational homology 3-sphere M/T.

It remains to show that and the G-action on M/T is homologically trivial. Since T is central, the covering space $M \to M/T$ is preserved by Q, and we have an exact sequence:

$$0 \to H_1(M; \mathbb{Z}) \to H_1(M/T; \mathbb{Z}) \to H_1(T; \mathbb{Z}) \to 0$$

since $H_2(T; \mathbb{Z}) = 0$ and T acts homologically trivially on M. By applying group cohomology to this sequence, we obtain:

$$0 \to H_1(M; \mathbb{Z})^Q \to H_1(M/T; \mathbb{Z})^Q \to H_1(T; \mathbb{Z})^Q \to H^1(Q; H_1(M; \mathbb{Z}) \ .$$

However, by Proposition 4.3 we have $H_1(M; \mathbb{Z}) = \mathbb{Z}/d$, with $(d, |Q|) = 1$, and hence $H^1(Q; H_1(M; \mathbb{Z}) = 0$, Therefore $H_1(M/T; \mathbb{Z})^Q = H_1(M/T; \mathbb{Z})$, and the G-action on M/T is homologically trivial. □

The period four groups $G = Q(8n, k, l)$ of type A can not act freely on S^3 (by Perelman), but some members of this family do act freely on integral homology 3-spheres. For the existence of such actions, there are two obstacles: a finiteness obstruction and a surgery obstruction. Swan [29] showed that for every period four group, there exists a finitely dominated Poincaré 3-complex X with $\pi_1(X) = G$ and universal covering $\widetilde{X} \simeq S^3$. Such a complex is called a *Swan complex* of type $(G, 3)$.

We recall that the homotopy types of $(G, 3)$-complexes are in bijection (via the first k-invariant) with the invertible elements in $\widehat{H}^4(G; \mathbb{Z}) \cong \mathbb{Z}/|G|$.

Lemma 4.6. *Let G be a period four group which acts freely and homologically trivially on a rational homology 3-sphere M. Then there exists a $(G, 3)$-complex X, unque up to homotopy, and a degree 1 map $f \colon M/G \to X$ compatible with the classifying maps of the G-fold coverings.*

Proof. The classifying map $c \colon M/G \to K(G, 1)$ of the covering $M \to M/G$ gives a class $c_*[M] \in H_3(G, \mathbb{Z})$. By Proposition 2.4, this class corresponds to a generator

$$\sigma^* \in \widehat{H}^4(G; \mathbb{Z}) \cong \widehat{H}^{-4}(G; \mathbb{Z}) \cong H_3(G; \mathbb{Z}).$$

Let X be the $(G, 3)$-complex defined (up to homotopy) by this k-invariant. Since the classifying map $c \colon M/G \to K(G, 1)$ is surjective on fundamental groups, it follows that c lifts to a map $f \colon M/G \to X$. Since the images of the fundamental classes of M/G and X agree in $H_3(G; \mathbb{Z})$, it follows that f has degree 1. □

Remark 4.7. Any degree 1 map $f \colon N \to X$ from a closed oriented 3-manifold to a $(G, 3)$-complex provides a degree 1 normal map by pulling-back a framing of the trivial bundle over X.

The Wall finiteness obstruction $\sigma(X) \in \widetilde{K}_0(\mathbb{Z}G)$ vanishes if and only if there exists a finite Swan complex of type $(G, 3)$. This is the first obstruction to existence. By varying the homotopy type of X, Swan defined defines an invariant

$$\sigma(G) \in \widetilde{K}_0(\mathbb{Z}G)/S(G),$$

depending only on G, where $S(G) \subseteq \widetilde{K}_0(\mathbb{Z}G)$ is the *Swan subgroup* generated by projective ideals of the form $\langle r, \Sigma \rangle \subset \mathbb{Z}G$, where $(r, |G|) = 1$ and Σ denotes the norm element. Then $\sigma(G) = 0$ if and only if $\sigma(X) \in S(G)$ for every Swan complex X of type $(G, 3)$.

Proposition 4.8. *Let $G = Q(8n, k, l)$, with n odd, be a period four group of type* A. *If G acts freely and homologically trivially on a rational homology 3-sphere, then $\sigma(G) = 0$.*

Proof. Under the given assumptions, G acts freely and homologically trivially on a rational homology 3-sphere M, such that $H_1(M; \mathbb{Z}) = \mathbb{Z}/d$, where $(d, |G|) = 1$. By Lemma 4.6. the classifying map $M/G \to K(G, 1)$ of the covering $M \to M/G$ lifts to a degree 1 map $f \colon M/G \to X$, to a uniquely defined $(G, 3)$-complex X. Since the map f induces a surjection on fundamental groups, the argument of Mislin [25, Theorem 3.3] shows that

$$\sigma(M/G) = \sigma(X) + \langle d, \Sigma \rangle \in \widetilde{K}_0(\mathbb{Z}G),$$

and hence $\sigma(X) \in S(G)$. Since varying the homotopy type of X changes $\sigma(X)$ only by an element of the Swan subgroup (see Swan [29, Lemma 7.3]), we conclude that $\sigma(G) = 0$. □

The secondary obstruction comes from surgery theory (and is defined only if the finiteness obstruction is zero). It can be computed in some cases to show existence (see Madsen [22]). For the type A groups, a $(G, 3)$-complex X has *almost linear* k-invariant

$e_0 \in H^4(G;\mathbb{Z})$ if the restriction of e_0 to each Sylow subgroup of G is the k-invariant of a standard free orthogonal action on S^3 (see [22, p. 195]).

Definition 4.9. We will say that a free homologically trivial action of a type A group G on a rational homology 3-sphere M has *almost linear k-invariant* if there exists a degree 1 map $f\colon M/G \to X$ to a *finite* $(G,3)$-complex with almost linear k-invariant $e_0 \in H^4(G;\mathbb{Z})$.

Remark 4.10. If G acts freely and smoothly on an integral homology 3-sphere Σ, then the quotient manifold $\Sigma/G = X$ is a finite $(G,3)$-complex with almost linear k-invariant (see [13, Corollary C] and the discussion of [22, Conjecture D]). By Proposition 4.5, any quotient of such a group G by a central cyclic subgroup would act freely and homologically trivially on a rational homology 3-sphere.

Conversely, we expect that the following existence statement holds:

Conjecture 4.11. *Let $G = Q(8n,k,l)$, with n odd, be a period four group of type* A*. Then G acts freely and homologically trivially on a rational homology* 3*-sphere with almost linear k-invariant if and only if G acts freely on an integral homology* 3*-sphere.*

In the remainder of this section, we prove this conjecture under some additional assumptions. If $f\colon N \to X$ denotes a degree 1 normal map to a finite $(G,3)$-complex, with $\pi_1(X) = G$, then there is *weakly simple* surgery obstruction $\lambda'(f) \in L'_3(\mathbb{Z}G)$. This is defined since every finite Poincaré 3-complex with finite fundamental group is weakly simple (meaning that its Poincaré torsion lies in $SK_1(\mathbb{Z}G)$). We let $\lambda^h(f) \in L^h_3(\mathbb{Z}G)$, the image of $\lambda'(f)$ under the natural map, denote the obstruction to surgery on f up to homotopy equivalence.

Let $H = Q(4ab)$ denote the index two subgroup of G, containing the subgroup $C(4) \leq Q(8)$ which acts by inversion on the normal subgroup of order ab.

Theorem 4.12 (Madsen [22])**.** *Suppose that $G = Q(8n,k,l)$, with n odd, is a period four group of type* A *such that $\sigma(G) = 0$. Let $f\colon N \to X$ be a degree 1 normal map to a finite $(G,3)$-complex with almost linear k-invariant. Then $\lambda'(f) = 0$ if and only if $\mathrm{Res}_H(\lambda'(f)) = 0$ for each subgroup $H \leq G$ of the form $H = Q(4ab)$. Furthermore, $\lambda^h(f) = 0$ if and only if $\mathrm{Res}_K(\lambda^h(f)) = 0$ for each subgroup $K \leq G$ of the form $K = Q(8a,b)$.*

Proof. This a a summary statement of the calculations in [22, §4-5]. See in particular [22, Theorems 4.19, 4.21 and Corollary 5.12]. $\square$

For the groups $G = Q(8a,b) = Q(8a,b,1)$, the *top component* $S(ab) \subseteq S(G) \subset \widetilde{K}_0(\mathbb{Z}G)$ of the Swan subgroup is defined as the the kernel of the restrictions to all odd index subgroups. For example, Bentzen and Madsen [3, Proposition 4.6] computed $S(Q(8p,q))$, for p,q odd primes, almost completely, and showed that $S(pq) = 0$ in a many cases (e.g $(p,q) \equiv (\pm 3, \pm 3) \mod 8$; or $(p,q) \equiv (1, \pm 3) \mod 8$, and 2 has odd order mod p.

Theorem 4.13. *Let $G = Q(8p,q)$, for odd primes $p > q$, and assume that $S(pq) = 0$. Then G acts freely and homologically trivially on a rational homology* 3*-sphere with almost linear k-invariant if and only if G acts freely on an integral homology* 3*-sphere.*

Proof. Remark 4.10 explains the sufficiency part. For the converse, suppose that $G = Q(8p, q)$ with $S(pq) = 0$ acts freely and homologically trivially on a rational homology 3-sphere M with almost linear k-invariant. Then there exists a finite $(G, 3)$-complex X with almost linear k-invariant, and a degree 1 normal map $f\colon M/G \to X = X(G)$. by [12, Theorem 3.1], we may assume that the covering space $X(H)$ is homotopy equivalent to an orthogonal spherical space form, for $H = Q(4pq) \leq G$, and that the normal invariant restricts to the normal invariant of an orthogonal spherical space form over the 2-Sylow covering $X(Q(8))$. In particular, since $H_1(M; \mathbb{Z}) = \mathbb{Z}/d$ for d odd, we must have

$$\sigma(X(Q(8)) = \langle d, N\rangle = 0 \in \widetilde{K}_0(\mathbb{Z}Q(8).$$

Hence $d \equiv \pm 1 \pmod 8$, and $\operatorname{Res}_{Q(8)}(\lambda(f)) = 0$ by [9, Theorem 5.1(ii)]). This information about the normal map $f\colon M/G \to X$ was extracted from the work of Madsen, Thomas and Wall (see [20, 21]).

Now we consider the restriction of the surgery obstruction $\operatorname{Res}_H(\lambda^h(f) \in L_3^h(\mathbb{Z}H)$. Since $X(H)$ is homotopy equivalent to an orthogonal space form, $\operatorname{Res}_H(\lambda^h(f))$ is the surgery obstruction of a normal map between closed manifolds. Therefore, $\operatorname{Res}_H(\lambda^h(f))$ is detected by further restriction to the 2-Sylow subgroup $C(4)$, and hence $\operatorname{Res}_H(\lambda^(f)) = 0$. It follows that $H_1(M; \mathbb{Z}) = \mathbb{Z}/d$ stably supports a hyperbolic linking form, and hence $d \equiv r^2 \mod (8ab)$ is a square. Since $S(pq) = 0$, it follows that $\langle r, N\rangle = 0 \in \widetilde{K}_0(\mathbb{Z}G)$. Now by [9, Theorem 5.1(ii)] applied to $\lambda^h(f) \in L_3^h(\mathbb{Z}G)$, we see that $\lambda^h(f) = 0$. Therefore G acts freely on an integral homology 3-sphere. □

Remark 4.14. By taking full advantage of Madsen's results as summarized in Theorem 4.12, we could give a statement for the groups $Q(8a, b)$, under the assumption that $S(a', b') = 0$ for all divisors $1 \neq a' \mid a$, and $1 \neq b' \mid b$.

Remark 4.15. We would like to remove the almost linear k-invariant assumption. However, the group $G = Q(8)$ acts freely and homologically trivially on $M = RP^3$ with $H_1(M; \mathbb{Z}) = \mathbb{Z}/3$, since $Q(8) \times \mathbb{Z}/3$ acts freely on S^3. This action has non-linear k-invariant in our sense. Indeed, by the proof of Proposition 4.8, there is a degree 1 map $f\colon M/G \to X$, where X is a Swan complex for $Q(8)$ with non-trivial finiteness obstruction.

5. Finite quotients of fundamental groups of 3-manifolds

In this section we consider closed 3–manifolds with finite coverings which are rational homology spheres. The associated finite covering groups act freely on rational homology 3-spheres, so they afford examples to which our methods will apply. Note that according to [8, Theorem 2.6] every finite group in fact acts freely on some *hyperbolic* (hence aspherical) closed rational homology sphere In such cases the fundamental group determines the topology, and we are really just considering finite index subgroups of Poincaré duality groups with vanishing first Betti number.

Recall that for any group Q and integer $n \geq 0$ we define the n–th term of its *derived series* as $Q^{(n)} = [Q^{(n)}, Q^{(n)}]$, where $Q^{(0)} = Q$. The derived series for a finite group stabilizes at a perfect normal subgroup, but may not terminate for an infinite group. In fact an interesting open question is whether or not the derived series for the fundamental group Γ of a closed orientable 3–manifold stabilizes if $\Gamma/\Gamma^{(n)}$ is finite for all n. If it does stabilize then $\Gamma/\Gamma^{(i)}$ is a solvable group with periodic cohomology (of period four), as it acts freely on an integral homology 3–sphere. Independently of the stability question, one can ask (as in [7]) about possible restrictions on the finite quotient groups $\Gamma/\Gamma^{(n)}$.

Let L denoted a closed 3–manifold such that for some $n > 0$ the quotient $\pi_1(L)/\pi_1(L)^{(n)}$ is finite. Let $\Gamma = \pi_1(L)$. From the extensions

$$1 \to \Gamma^{(i)}/\Gamma^{(i+1)} \to \Gamma/\Gamma^{(i+1)} \to \Gamma/\Gamma^{(i)} \to 1$$

for $1 \leq i \leq n-1$, we infer that all the groups $\Gamma/\Gamma^{(i)}$ and $\Gamma^{(i)}/\Gamma^{(i+1)}$ are finite in that range. Hence the corresponding covering spaces L_i are rational homology spheres. The finite groups $\Gamma/\Gamma^{(i)}$ act freely on them, with quotient L; note that $H^2(L_i, \mathbb{Z}) \cong \Gamma^{(i)}/\Gamma^{(i+1)}$. Applying 2.3 we obtain

Proposition 5.1. *Let M denote a closed 3–manifold with $\Gamma = \pi_1(L)$ such that $\Gamma/\Gamma^{(n)}$ is finite for some $n > 0$. Then for all $1 \leq i \leq n-1$ there are long exact sequences*

$$\cdots \to \widehat{H}^{i+2}(\Gamma/\Gamma^{(i)}, \mathbb{Z}) \xrightarrow{\cup \sigma_i} \widehat{H}^{i-2}(\Gamma/\Gamma^{(i)}, \mathbb{Z}) \to \widehat{H}^{i}(\Gamma/\Gamma^{(i)}, \Gamma^{(i)}/\Gamma^{(i+1)}) \to \widehat{H}^{i+3}(\Gamma/\Gamma^{(i)}, \mathbb{Z}) \to \dots$$

These sequences are determined by elements $\sigma_i \in \widehat{H}^{-4}(\Gamma/\Gamma^{(i)}, \mathbb{Z})$, which are images of the respective generators in $\widehat{H}^0(\Gamma/\Gamma^{(i)}, \mathbb{Z}) \cong \mathbb{Z}/|\Gamma/\Gamma^{(i)}|$.

Corollary 5.2. *If $\Gamma^{(i)}$ is perfect, then $\sigma_i \in \widehat{H}^{-4}(\Gamma/\Gamma^{(i)}, \mathbb{Z})$ is a periodicity generator for the cohomology of $\Gamma/\Gamma^{(i)}$.*

As we have seen, Proposition 5.1 can be used to obtain restrictions on the finite groups $\Gamma/\Gamma^{(i)}$. As an application we take the opportunity to quickly obtain some of the results in [7] and [27].

Proposition 5.3 (Cavendish [7])**.** *Let L denote a closed 3–manifold and $q : \Gamma \to G$, a surjective homomorphism from its fundamental group $\Gamma = \pi_1(L)$ onto a finite group G which induces an isomorphism $H_1(\Gamma, \mathbb{Z}) \cong H_1(G, \mathbb{Z})$. Then $\phi : \widehat{H}^2(G, \mathbb{Z}) \to \widehat{H}^{-2}(G, \mathbb{Z})$ given by $x \mapsto \sigma \cup x$ is an isomorphism and cup product defines a non–degenerate pairing*

$$\widehat{H}^2(G, \mathbb{Z}) \otimes \widehat{H}^2(G, \mathbb{Z}) \to \widehat{H}^4(G, \mathbb{Z}).$$

If in addition $ker(q) \subset \Gamma^{(2)}$, then this pairing factors through a cyclic subgroup of $\widehat{H}^4(G, \mathbb{Z})$.

Proof. There is a covering space $\tilde{L}$ of L corresponding to $ker(q)$, with a free action of G. Consider the fibration $\tilde{L} \to L \to BG$ and its associated 5-term exact sequence

$$0 \to H^2(BG, \mathbb{Z}) \to H^2(L, \mathbb{Z}) \to H^2(\tilde{L}, \mathbb{Z})^G \to H^3(BG, \mathbb{Z}) \to 0$$

where the last map is surjective because $H^3(L, \mathbb{Z})$ is torsion–free. On the other hand, from 2.3 we have an exact sequence

$$0 \to \operatorname{coker}(\phi) \to \widehat{H}^0(G, H^2(\tilde{L}, \mathbb{Z})) \to H^3(G, \mathbb{Z}) \to 0.$$

As $H^2(BG, \mathbb{Z}) \cong H^2(L, \mathbb{Z})$, then $H^2(\tilde{L}, \mathbb{Z})^G \cong H^3(G, \mathbb{Z})$, but it factors through Tate cohomology so we obtain that the norm map N is trivial on $H^2(\tilde{L}, \mathbb{Z})$,

$$\widehat{H}^0(G, H^2(\tilde{L}, \mathbb{Z})) \cong H^3(G, \mathbb{Z})$$

and that ϕ is surjective. As the domain and codomain of ϕ have the same number of elements this implies that it is an isomorphism. Now given $y \in \widehat{H}^2(G,\mathbb{Z})$ we can choose $z \in \widehat{H}^2(G,\mathbb{Z})$ such that the Tate dual $y^* = \sigma \cup z$. Then $0 \neq z \cup y$ because

$$0 \neq y^* \cup y = \sigma \cup z \cup y,$$

showing that the pairing is nondegenerate.

Let us now assume that $\ker(q) \subset \Gamma^{(2)}$ and let $J = [G,G]$, $R = G/[G,G]$ and $S = [\pi_1(L), \pi_1(L)] = \pi_1(\tilde{L}/J)$. Γ maps onto G, so S maps onto J and $H_1(S,\mathbb{Z})$ maps onto $H_1(J,\mathbb{Z})$. The condition $\ker(q) \subset \Gamma^{(2)} = S^{(1)}$ means that the kernel of $q|_S \colon S \to J$ is contained in $S^{(1)}$, and so the abelianized map is an isomorphism $H_1(S,\mathbb{Z}) \cong H_1(J,\mathbb{Z})$. Given that $\tilde{L}/K$ is a rational homology sphere,

$$H^2(\tilde{L}/J,\mathbb{Z}) \cong H^2(J,\mathbb{Z}) \cong H_1(J,\mathbb{Z}).$$

Using this identification and applying 2.3 to the R–action on $\tilde{L}/J$, we obtain the exact sequence

$$0 \to \widehat{H}^1(R, H^2(J,\mathbb{Z})) \xrightarrow{d} \widehat{H}^4(R,\mathbb{Z}) \to \widehat{H}^0(R,\mathbb{Z}).$$

As before we can identify d with the differential $d_3 \colon E_3^{1,2} \to E_3^{4,0}$ for the LHS spectral sequence for the group extension

$$1 \to J \to G \to R \to 1.$$

Therefore the image of d goes to zero under the inflation map $H^4(R,\mathbb{Z}) \to H^4(G,\mathbb{Z})$, which therefore factors through coker $d \subset \widehat{H}^0(R,\mathbb{Z}) \cong \mathbb{Z}/|R|$, a cyclic group. Using the isomorphism $\widehat{H}^2(R,\mathbb{Z}) \cong \widehat{H}^2(G,\mathbb{Z})$ and naturality of the cup product completes the proof. □

We apply this to obtain a quick proof of a result due to Reznikov [27], following the approach in [7].

Corollary 5.4 (Reznikov [27])**.** *Let L denote a closed three–manifold such that $G = \pi_1(L)/\pi_1(L)^{(n)}$ is a finite 2–group, and $H_1(L,\mathbb{Z}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Then G is a generalized quaternion group.*

Proof. As G is a quotient of $\pi_1(L)$ mapping onto its abelianization, then

$$G/[G,G] \cong \mathbb{Z}/2 \times \mathbb{Z}/2$$

and so G is a 2-group of maximal class and thus must be either (generalized) quaternion, dihedral or semi-dihedral (see [11], Section 5.4). However the condition that the cup product pairing be non–singular eliminates the semi–dihedral groups (see [10]), and the fact that the image has rank one eliminates the dihedral groups (see [14]). Thus we conclude that G is a generalized quaternion group. □

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