CYCLIC GROUP ACTIONS ON CONTRACTIBLE 4-MANIFOLDS

NIMA ANVARI AND IAN HAMBLETON

Abstract. There are known infinite families of Brieskorn homology 3-spheres which can be realized as boundaries of smooth contractible 4-manifolds. In this paper we show that free periodic actions on these Brieskorn spheres do not extend smoothly over a contractible 4-manifold. We give a new infinite family of examples in which the actions extend locally linearly but not smoothly.

1. Introduction

The Brieskorn homology spheres $\Sigma(a, b, c)$ provide important examples of Seifert fibered 3-manifolds [26], and have been extensively studied as test cases for questions about smooth 4-manifolds and gauge theory invariants (see Anvari [1], Lawson [22], Fintushel and Stern [14, 15], Saveliev [27]). In this paper we answer a well-known question (asked by Allan Edmonds at Oberwolfach in 1988) about extending free cyclic group actions on $\Sigma(a, b, c)$ to certain smooth 4-manifolds which they bound.

Kwasik and Lawson [20] found an infinite family of Brieskorn homology 3-spheres which admit free $\mathbb{Z}/p$ actions and bound smooth contractible 4-manifolds W, such that the actions extend locally linearly with one fixed point in W, but no such extended action exists smoothly. Their examples come from the list of Casson and Harer [5] of Brieskorn homology 3-spheres which bound smooth contractible 4-manifolds:

$$\begin{aligned} &\Sigma(r, rs-1, rs+1) \quad \text{r even, s odd} \\ &\Sigma(r, rs \pm 1, rs \pm 2) \quad \text{r odd, s arbitrary.} \end{aligned}$$

Necessary and sufficient conditions for a locally linear extension of a free action on an integral homology three sphere to its bounding contractible 4-manifold are contained in the work of Edmonds [8]. To show non-smoothability, Kawsik and Lawson apply the gauge theoretic results of Fintushel and Stern [13] in the orbifold setting.

In this paper we demonstrate a new technique to detect non-smoothability of these actions and apply it to obtain a complete answer:

Theorem A. *The free $\pi = \mathbb{Z}/p$ actions on a Brieskorn homology 3-sphere $\Sigma(a, b, c)$ do not extend to smooth actions on any contractible smooth 4-manifold W that it bounds.*

Remark 1.1. By P. A. Smith theory, any smooth or locally linear extension of a free π-action on $\Sigma(a, b, c)$ to a contractible manifold W must have exactly one fixed point.

Recall that the Brieskorn homology spheres for a, b, c pairwise relatively prime can be realized as the link of a complex surface singularity:

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$$\Sigma(a,b,c) = \{(x,y,z) \in \mathbb{C}^3 \,|\, x^a + y^b + z^c = 0\} \cap S^5$$

with its induced orientation. As a Seifert fibered homology sphere it admits a smooth fixed-point free circle action with three orbits of finite isotropy (see [26]). For p relatively prime to a, b, c the action of $\mathbb{Z}/p$ is the unique free action contained in the circle action giving the Seifert fibered structure on $\Sigma(a,b,c)$ and is referred to as the *standard action* (see [23, Prop. 4.3]).

We give new infinite family of examples admitting a locally linear extension to W. These are from the second of the infinite families found by Stern [28]:

$$\begin{aligned}
&\Sigma(r, rs \pm 1, 2r(rs \pm 1) + rs - (\pm 1)) \quad \text{r even, s odd}\\
&\Sigma(r, rs \pm 1, 2r(rs \pm 1) + rs \pm 2) \quad \text{r odd, s arbitrary}\\
&\Sigma(r, rs \pm 2, 2r(rs \pm 2) + rs \pm 1) \quad \text{r odd, s arbitrary.}
\end{aligned}$$

where we take $s = kp$ for any positive integer k.

Theorem B. *Let r and p be odd primes such that $p > r$. Then for each positive integer k, the standard free $\pi = \mathbb{Z}/p$ action on $\Sigma(r, rkp \pm 1, 2r(rkp \pm 1) + rkp \pm 2)$ extends to a locally linear action on a smooth contractible 4-manifold W with one fixed point.*

We point out the following application to equivariant embeddings of $\Sigma(a,b,c)$ in smooth homotopy 4-spheres with semifree π-actions.

Corollary C. *There are infinite families of Brieskorn homology spheres $\Sigma(a,b,c)$ with free $\pi = \mathbb{Z}/p$ action, for $p \nmid abc$, which embed equivariantly in homotopy 4-spheres with locally linear π-actions. No such smooth equivariant embeddings of $\Sigma(a,b,c)$ exist into any smooth π-action on a homotopy 4-sphere.*

Here is brief outline of the paper. The links of complex surface singularities that are integral homology three spheres are *plumbed* homology spheres; that is, they can be realized as the boundaries of smooth 4-manifolds obtained by plumbing disk bundles over 2-spheres with an intersection matrix that is negative definite. Among these is the canonical negative definite resolution in that it admits no (-1)-blowdowns. To prove non-smoothability of locally linear extensions, we extend the free action on a Brieskorn homology sphere $\Sigma = \Sigma(a,b,c)$ to its canonical negative definite resolution $M(\Gamma)$ by equivariant plumbing on the resolution graph. From this we form the closed, simply connected 4-manifold

$$X = M(\Gamma) \cup_{\Sigma} (-W)$$

which by Donaldson's Theorem A [6] has intersection matrix that is diagonalizable over the integers. If the action on W is smoothable, then X admits a smooth $\mathbb{Z}/p$ action which equivariantly splits along a free action on $\Sigma(a,b,c)$. The idea is that the global orientation of the moduli space prevents the configuration of invariant and fixed 2-spheres in $M(\Gamma)$ obtained from plumbing to embed equivariantly and smoothly in a connected sum of linear actions on complex projective spaces. We use equivariant Yang-Mills moduli spaces as developed in Hambleton-Lee [17, 18].

In the next section we collect results from equivariant gauge theory that we will need for the proof of Theorem A, and in Section 5 we prove that locally linear extensions exist for the infinite family in Theorem B. We work out explicit examples for the infinite family $\Sigma(3, 3s+1, 6(3s+1)+3s+2)$.

2. Equivariant Moduli Spaces

Let (Σ, π) denote a Brieskorn integral homology 3-sphere $\Sigma = \Sigma(a, b, c)$, together with a free $\pi = \mathbb{Z}/p$ action contained in the natural circle action of the Seifert fibration, and suppose this action extends smoothly to a contractible 4-manifold W. Now consider $(M(\Gamma), \pi)$ to be the negative definite resolution of $\Sigma(a, b, c)$ together with the free π-action extending via equivariant plumbing on the graph Γ. Then $X = M(\Gamma) \cup_\Sigma (-W)$ denotes a simply connected, smooth negative definite 4-manifold together with a homologically trivial $\mathbb{Z}/p$-action. As mentioned in the introduction, our strategy will be to study the equivariant instanton moduli spaces to obtain a contradiction to the action of π extending smoothly to W. We begin this section by collecting results about equivariant Yang-Mills moduli spaces that we will need to prove non-smoothability of the extension (see [7] and [17]).

Let $P \to X$ denote a principal $SU(2)$-bundle over a closed, smooth and simply connected 4-manifold X whose intersection form is odd and negative definite. By results of Donaldson and Wall, it follows that X is homotopy equivalent to a connected sum of copies of $\overline{\mathbb{C}P}^2$ (see [7]). Suppose that $\pi = \mathbb{Z}/p$ with p an odd prime acts smoothly on X inducing identity on homology. We fix a real analytic structure on X compatible with the group action and a real analytic π-invariant metric so the action is given by real analytic isometries. Let $\mathcal{A}$ denote the space of $SU(2)$ connections and $\mathcal{B} = \mathcal{A}/\mathcal{G}$ the space of connections modulo the gauge group $\mathcal{G}$. Since $SU(2)$-bundles are classified by the second Chern class $c_2(P) \in H^4(X; \mathbb{Z})$ and since the π-action on X preserves the orientation, there are lifts of the isometries $g\colon X \to X$ that are generalized bundle maps $\hat{g}\colon P \to P$. Let $\mathcal{G}(\pi)$ denote the group of all lifts, then there is an action of $\mathcal{G}(\pi)$ on the space of connections $\mathcal{A}$ which is well-defined modulo gauge and there is a short exact sequence

$$1 \to \mathcal{G} \to \mathcal{G}(\pi) \to \pi \to 1 \tag{2.1}$$

we then get a well-defined π-action on $\mathcal{B}$. The metric induces a decomposition of 2-forms $\Omega^2(\operatorname{ad} P) = \Omega^2_+(\operatorname{ad} P) \oplus \Omega^2_-(\operatorname{ad} P)$. We are interested in the "charge one" bundle, with $c_2(P) = 1$, and the Yang-Mills moduli space defined by connections modulo gauge with anti-self-dual (ASD) curvature:

$$\mathcal{M}_1(X) = \{[A] \in \mathcal{B}(P) \mid F_A^+ = 0\} \tag{2.2}$$

Since the curvature is gauge invariant there is a natural π-action on $\mathcal{M}_1(X)$. The stabilizer $\mathcal{G}_A(\pi)$ has compact isotropy subgroups; when A is irreducible $\Gamma_A = \{\pm 1\}$ and when $[A]$ is reducible, $\Gamma_A = U(1)$ and the associated complex vector bundle $E \to X$ splits as $L \oplus L^{-1}$ for a complex line bundle L over X. The stabilizer $\mathcal{G}_A(\pi)$ is an extension in the short exact sequence

$$1 \to \Gamma_A \to \mathcal{G}_A(\pi) \to \pi \to 1 \tag{2.3}$$

and there exists a lift of the π-action on X to a π-action on the principal bundle P if and only if this sequence splits. The local finite-dimensional model of the moduli space is given by a $\mathcal{G}_A(\pi)$-equivariant Kuranishi map

$$\phi_A\colon H^1_A \to H^2_A \tag{2.4}$$

where H^1_A and H^2_A are the cohomology group of the $\mathcal{G}_A(\pi)$-equivariant fundamental elliptic complex

$$0 \to \Omega^0(X; \operatorname{ad} P) \xrightarrow{d_A} \Omega^1(X; \operatorname{ad} P) \xrightarrow{d_A^+} \Omega^2_+(X; \operatorname{ad} P) \to 0 \tag{2.5}$$

where $D^+_A = d^*_A + d^+_A$ is the linearization of the ASD equation and the tangent space $T_{[A]}\mathcal{M}$ is identified with H^1_A. When $H^2_A = 0$, the origin is a regular value for ϕ_A and the infinitesimal deformations in H^1_A can be integrated so that a neighborhood of an ASD connection $[A]$ in the equivariant moduli space $(\mathcal{M}_1(X), \pi)$ is locally isomorphic to $(\phi^{-1}(0)/\Gamma_A, \pi)$ and gives manifold charts on the moduli space.

However, in this equivariant setting it is known that there are obstructions to equivariant transversaliy: for example, the virtual representation $[H^1_A] - [H^2_A] \in RO(\pi)$ must be an actual representation. Moreover it may not be possible to make an equivariant perturbation of the ASD equations to get $H^2_A = 0$. Hambleton and Lee in [17] used the notion of equivariant general position as developed by Bierstone [3] and applied it to the setting of Yang-Mills moduli spaces. The idea is to make generic equivariant perturbations chart by chart giving the moduli space the structure of a equivariant stratified space. Here we list the main properties of the instanton moduli space in our setting when X is negative definite.

(i) The equivariant moduli space $(\mathcal{M}_1(X), \pi)$ is a Whitney stratified space which inherits an effective π-action and has open manifold strata parametrized by isotropy subgroups $\mathcal{M}^*_{(\pi')} = \{x \in \mathcal{M}^* \mid x \text{ has isotropy subgroup } \pi_x = \pi'\}$.

(ii) The fixed set in the moduli space $\operatorname{Fix}(\mathcal{M}_1(X), \pi)$ correspond to equivariant lift of the π-action to π-bundle structure on P which admits π-invariant anti-self-dual connections. The connected components of the fixed set in the irreducible component of the moduli space correspond to distinct equivalence classes of lifts [4]. Moreover, the dimension of the fixed set can be computed from the π-fixed set of the fundamental elliptic complex:

$$0 \to \Omega^0(X; \operatorname{ad} P)^\pi \xrightarrow{d_A} \Omega^1(X; \operatorname{ad} P)^\pi \xrightarrow{d_A^+} \Omega^2_+(X; \operatorname{ad} P)^\pi \to 0$$

for a connection $[A]$ in $\operatorname{Fix}(\mathcal{M}^*_1(X), \pi)$. A fixed stratum is non-empty if its formal dimension is positive. In particular, the free stratum $\mathcal{M}^*_{(e)}$ is a 5-dimensional, smooth, noncompact manifold consisting of irreducible ASD connections.

(iii) The strata have topologically locally trivial equivariant cone bundle neighborhoods.

(iv) There is an ideal boundary in the moduli space leading to π-equivariant Uhlenbeck-Taubes compactification $(\overline{\mathcal{M}_1(X)}, \pi)$ consisting of highly-concentrated ASD connections parametrized by a copy of X:

$$\overline{\mathcal{M}_1(X)} = \mathcal{M}_1(X) \cup X$$

where $\mathcal{M}_1(X)$ has a π-equivariant collar neighborhood diffeomorphic to $X \times (0, \lambda)$ for small λ with the product action being trivial on $(0, \lambda)$.

(v) There are equivariantly transverse charts at each reducible connection; that is $H^2_A = 0$ for each reducible connection $[A]$ and there exists a π-invariant neighborhood which is equivariantly homemorphic to a cone over some linear action on complex projective space $\overline{\mathbb{C}P}^2$.

By equivariant general position, the closures of singular strata of dimension ≥ 5 are disjoint from the closure of the free stratum. Moreover, there is a connected component of the free stratum containing the collar and the set of reducible connections. The fixed sets that occur in $\mathcal{M}_{(e)}(X)$ have even codimension.

The Yang-Mills moduli spaces inherit a canonical orientation from that of X. The top exterior power of the tangent space $T_{[A]}\mathcal{M} = \operatorname{Ker} D^+_A$ can be identified with the determinant line bundle of the elliptic complex

$$\det D^+_A = \Lambda^{\max}(\operatorname{Ker} D^+_A) \otimes_{\mathbb{R}} \Lambda^{\max}(\operatorname{Coker} D^+_A) \tag{2.6}$$

when $H^2_A = 0$ and in [7] it is shown that the determinant line bundle $\Lambda(P)$ is independent of deformations of A and extends to $\mathcal{B}^*$. Moreover, $\Lambda(P)$ admits a canonical trivialization giving an orientation on the free stratum $\mathcal{M}_{(e)}$ and inducing the given orientation on $\tau(X)$ times the inward normal, where τ denotes the equivariant Taubes embedding of (X, π) in $(\mathcal{M}_1(X), \pi)$. In [18] it is shown that the $\pi = \mathbb{Z}/p$, for p an odd prime, induces a preferred orientation on the fixed set. The idea is that for any π-fixed ASD connection $[A]$ there is a splitting of the fundamental elliptic complex $\Omega^* = (\Omega^*)^\pi \oplus (\Omega^*)^\perp$ and

$$\Lambda(P) = \Lambda((\Omega^*)^\pi) \otimes \Lambda((\Omega^*)^\perp). \tag{2.7}$$

Since the fixed set has even codimension and p is odd, the action induces a complex structure on $\Lambda((\Omega^*)^\perp)$ and hence a preferred orientation. Together with the canonical orientation of the moduli space, this induces a preferred orientation on the fixed set of any connected component containing $[A]$.

A final ingredient in the Yang-Mills setting is the map

$$\mu\colon H_2(X;\mathbb{Z}) \to H^2(\mathcal{B}^*;\mathbb{Z})$$

defined in [7, §5.2]. If $\tau\colon X \to \overline{\mathcal{M}_1(X)}$ denotes the inclusion of the Taubes boundary, then $\tau^*(\mu(\alpha)) = PD(\alpha)$, for any class $\alpha \in H_2(X;\mathbb{Z})$. Furthermore, for the restriction of $\mu(\alpha)$ to the copy of $\mathbb{C}P^\infty$ which links the reducible connection A, we have

$$\mu(\alpha)\,|_{\,\mathrm{lk}[A]} = -\langle c_1(L), \alpha\rangle h$$

where $h \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$ is the positive generator, and $L \to X$ is the complex line bundle given by the reduction $E = L \oplus L^{-1}$ induced by A (see [7, 5.1.21]).

Lemma 2.8. *For any $\alpha \in H_2(X;\mathbb{Z})$, the class $\mu(\alpha) \in H^2(\mathcal{B}^*;\mathbb{Z})$ corresponds to a π-equivariant line bundle $\mathcal{L}_\alpha \to \mathcal{B}^*$. Moreover, there exists an equivariant section s of $\mathcal{L}_\alpha$ restricted to $\mathcal{M}^*_1(X)$, so that the zero set $V_\alpha = s^{-1}(0)$ is in equivariant general position in the moduli space.*

Proof. For π a finite cyclic group, equivariant line bundles L over a space (Y, π) are classified by a cohomology class $[L] \in H^2(Y \times_\pi E\pi;\mathbb{Z})$. The natural map $H^2(Y \times_\pi E\pi;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ sends $[L] \mapsto c_1(L)$, and a spectral sequence calculation shows that this map is surjective. This shows that there exists a π-equivariant line bundle $\mathcal{L}_\alpha \to \mathcal{B}^*$ with $c_1(\mathcal{L}_\alpha) = \mu(\alpha)$. We may now restrict this line bundle to $\mathcal{M}^*_1(X)$ and perturb the zero section into equivariant general position by the method of [17]. The perturbation may be chosen so that V_α also intersects the links of the reducible connections in equivariant general position. □

For a class $\alpha \in H_2(X;\mathbb{Z})$ represented by an invariant 2-sphere $F \subset X$, the zero section V_α of $\mathcal{L}_\alpha$ is a stratified codimension two cobordism whose intersection with the Taubes collar may be chosen to be $F = \tau(X) \cap V_\alpha$. The other boundary components provide surfaces in the links of the reducible connections, that are π-invariant under the linear actions on complex projective spaces.

3. Smooth actions on negative definite 4-manifolds

The equivariant moduli space provides an equivariant stratfied cobordism that relates a smooth π-action on a negative definite 4-manifold to an equivariant connected sum of linear actions on complex projective spaces.

Example 3.1 (Linear Models)**.** The complex projective plane $\mathbb{C}P^2$ admits linear actions of any finite cyclic group $\pi = \mathbb{Z}/m$, given in homogeneous coordinates by the formula

$$t \cdot [z_0 : z_1 : z_2] = [z_0 : \zeta^a z_1 : \zeta^b z_2], \tag{3.2}$$

where $t \in \pi$ is a generator, $\zeta = e^{2\pi i/m}$ is a primitive root of unity, and a and b are integers such that the greatest common divisor $(a, b, m) = 1$. For these actions, π induces the identity on homology, and the singular set always contains the three fixed points $x_1 = [1 : 0 : 0]$, $x_2 = [0 : 1 : 0]$, and $x_3 = [0 : 0 : 1]$. In addition, the three projective lines through the points x_i and x_j, for $i \neq j$, are smoothly embedded π-invariant or π-fixed 2-spheres with various isotropy subgroups depending on the values of a and b (see [19, §1]).

If we take the standard complex orientation on $\mathbb{C}P^2$, then for m odd the π-action provides a standard orientation at a fixed point $x_0 \in F$ for the normal bundle to an embedded surface $F \subset \mathbb{C}P^2$. In this way, any connected π-invariant surface containing a fixed point inherits a standard orientation. For the projective lines $\mathbb{C}P^1 \subset \mathbb{C}P^2$, each representing a primitive generator of $H_2(\mathbb{C}P^2) = \mathbb{Z}$, the induced orientation is the standard complex orientation.

In this paper, we will work with negative definite 4-manifolds. The formula (3.2) gives a smooth, orientation-preserving π-action on $\overline{\mathbb{C}P}^2$, and for m odd the induced orientation on the projective lines is opposite to the complex orientation. We will use the notation $\overline{\mathbb{C}P}^1 \subset \overline{\mathbb{C}P}^2$ for this standard embedding.

Examples of smooth homologically trivial π-actions on a connected sum $X = \#_1^{\mathrm{n}} \overline{\mathbb{C}P}^2$ are constructed by a *tree* of equivariant connected sums, where we connect linear actions on $\overline{\mathbb{C}P}^2$ at fixed points. In order to preserve orientation, the tangential rotation numbers at the attaching points must be of the form (c, d) and $(c, -d)$. In this way, we obtain many model actions.

Definition 3.3. Let (X, π) be a smooth cyclic group action on $X \simeq \#_1^{\mathrm{n}} \overline{\mathbb{C}P}^2$. A diagonal basis $\{e_1, e_2, \dots, e_n\}$ for $H_2(X;\mathbb{Z})$, with $e_i \cdot e_j = -\delta_{ij}$, is called a *standard* basis if

$$\mu(e_i) \,|\,_{\mathrm{lk}[A_i]} = -\langle c_1(L_i), e_i\rangle h_i = h_i \in H^2(\mathcal{B}^*;\mathbb{Z}),$$

for $1 \leq i \leq n$, where $[A_i]$ denotes the reducible connection in $\overline{\mathcal{M}_1(X)}$ determined by $\{\pm e_i\}$. A standard basis is unique up to re-ordering of the basis elements.

The equivariant moduli space shows that every smooth π-action on an odd negative definite 4-manifold strongly resembles an equivariant connected sum of linear actions.

Theorem 3.4 ([18, Theorem C]). *Let (X, π) be a smooth cyclic group action on $X \simeq \#_1^n \overline{\mathbb{C}P}^2$ inducing the identity on homology. Then there exists an equivariant connected sum of linear actions on $\overline{\mathbb{C}P}^2$ with the same fixed point data and tangential isotropy representations.*

Let F denote a fixed 2-sphere for the π-action on an equivariant connected sum of linear actions on $\#_1^n \overline{\mathbb{C}P}^2$. We give F the standard orientation, and then it is clear that the homology class $[F]$ can be written as $\sum_i a_i e_i$ for $a_i \in \{0, 1\}$ in the diagonal basis e_i represented as $\overline{\mathbb{C}P^1} \subset \overline{\mathbb{C}P^2}$. The same statement holds for smooth π-actions on $X \simeq \#_1^n \overline{\mathbb{C}P}^2$. If (X, π) is a homologically trivial action, then the fixed set consists of a disjoint union of isolated points and smoothly embedded 2-spheres (see [9, Proposition 2.4]).

Theorem 3.5 ([18, Thm. 16]). *Let $\pi = \mathbb{Z}/p$, for p an odd prime, act smoothly and homologically trivally on a smooth, oriented 4-manifold $X \simeq \#_1^n \overline{\mathbb{C}P}^2$. Then the integral homology class for each standardly oriented fixed 2-sphere $F \subset X$ is given by an expression:*

$$[F] = \sum_i a_i e_i$$

where $\{e_i\}$ is a standard diagonal basis and $a_i \in \{0, 1\}$.

Proof. Since $X \simeq \#_1^n \overline{\mathbb{C}P}^2$, we have a standard diagonal basis $\{e_1, \ldots, e_n\}$ for the intersection form on $H_2(X; \mathbb{Z})$. We can express $[F] = \sum_i a_i e_i$, for some integers a_i. Let $\hat{e}_i = PD(e_i)$ be the Poincaré dual to e_i, so that $\langle \hat{e}_i, e_j \rangle = \langle \hat{e}_i \cup \hat{e}_j, [X] \rangle = -\delta_{ij}$. Let L_i denote the corresponding line bundle over X, with $c_1(L_i) = \hat{e}_i$, which provides the reduction $E = L \oplus L^{-1}$ and a reducible ASD connection $[A_i]$ on L_i.

In the compactified equivariant moduli space $\overline{\mathcal{M}_1(X)}$, the fixed set of the π-action is path connected (see [18, Theorem C] or [19, Theorem 3.11]). It follows that the links of the reducible connections all inherit the same standard orientation as $\overline{\mathbb{C}P}^2$.

If V denotes the 3-dimensional π-fixed stratum which is the zero set in Bierstone general position for $\mu([F]) = \sum a_i \mu(e_i)$, then V inherits a preferred orientation from the free stratum, and the induced orientation on each component $\partial V_i = V \cap \mathrm{lk}[A_i]$ depends only on its homology class.

Since the fixed stratum in the links arise from a linear π-action on complex projective space, we see that $\partial V = F \cup \bigcup \partial V_i$, where each non-empty component ∂V_i in the link $\mathrm{lk}[A_i]$ is a fixed 2-sphere representing the homology class of $\mathbb{C}P^1 \subset \mathbb{C}P^2$. We now evaluate

$$0 = \langle \mu(e_k), [\partial V] \rangle = \langle \mu(e_k), \tau_*[F] \rangle + \sum \langle \mu(e_k), [\partial V_i] \rangle \tag{3.6}$$

But $\langle \mu(e_k), \tau_*[F] \rangle = \langle PD(e_k), [F] \rangle = -a_k$, and $\langle \mu(e_k), [\partial V_i] \rangle = -\langle c_1(L_k), e_k \rangle \langle h_k, [\partial V_i] \rangle = \delta_{ik}$, since h_k is the positive generator. Since $\langle h_k, [\partial V_i] \rangle = 1$, it follows that the coefficients in $[F] = \sum a_i e_i$ all have values in $\{0, 1\}$. □

We will now generalize the statement of Theorem 3.5 to handle smoothly embedded π-invariant 2-spheres.

Theorem 3.7. *Let $\pi = \mathbb{Z}/p$, for p an odd prime, act smoothly and homologically trivally on a smooth, oriented 4-manifold $X \simeq \#_1^n \overline{\mathbb{C}P}^2$. Let $F \subset X$ be a smoothly embedded π-invariant 2-sphere with the standard orientation. Then the homology class $[F] \in H_2(X;\mathbb{Z})$ is given by the formula*

$$[F] = \sum_i a_i e_i$$

where $\{e_i\}$ is a standard diagonal basis and each $a_i \geq 0$.

Remark 3.8. If F does not have the standard orientation, then each $a_i \leq 0$.

Proof. Let F be a smoothly embedded π-invariant 2-sphere in the action (X, π). We assume that F is not π-fixed, hence it contains exactly two isolated fixed points $x_0, x_1 \in F$. Let $\alpha = [F] \in H_2(X;\mathbb{Z})$ and let $V \subset \overline{\mathcal{M}_1(X)}^*$ be the zero set of an equivariant section (in Bierstone general position) of the line bundle $\mathcal{L}_\alpha$ given by $\mu(\alpha) \in H^2(\mathcal{B}^*;\mathbb{Z})$. We may assume that $V \cap X = F$ at the Taubes boundary, and that $\partial V_i := V \cap \operatorname{lk}[A_i]$ is a π-invariant surface in a linear action $(\overline{\mathbb{C}P}^2, \pi)$ for each reducible connection $[A_i]$.

Note that without additional information, we can only conclude that the π-invariant surfaces ∂V_i are smoothly embedded in $\overline{\mathbb{C}P}^2$ except possibly in small neighbourhoods around the fixed points, where the surfaces might contain cones over π-invariant knots in (S^3, π). At these points the embeddings are only topological (and not locally flat).

However, we observe that the compactification $\overline{V}$ contains two 1-dimensional π-fixed strata of $\overline{\mathcal{M}_1(X)}^*$ joining each of the isolated fixed points on F to reducible connections, and passing through isolated fixed points on two components, say on ∂V_0 and ∂V_t. By Bierstone general position, the intersections $V \cap \operatorname{lk}[A_i]$ are equivariantly transverse at these points (for $i = 0, t$). Moreover, the fixed set

$$Z := \operatorname{Fix}(\overline{\mathcal{M}_1(X)}, \pi) \cap \overline{V}$$

is a tree by [19, Theorem 3.11]. Since each link $(\overline{\mathbb{C}P}^2, \pi)$ has at most three isolated fixed points, and there is a unique path in Z from x_0 to x_1 (up to homotopy), it follows that $\operatorname{Fix}(\partial V_i, \pi)$ contains exactly two fixed points for each non-empty π-invariant surface ∂V_i. At the "initial" component ∂V_0, that contains a fixed point connected to $x_0 \in F$, we also see that the standard orientation on ∂V_0 agrees with the standard "positive" orientation on $\mathbb{C}P^1 \subset \mathbb{C}P^2$. Since Z is connected, each non-empty component ∂V_i also inherits the positive orientation. It follows that $\langle h_k, [\partial V_i] \rangle \geq 0$, and the same calculation given in (3.6) completes the proof. □

Remark 3.9. Note that if $F \subset X$ is π-invariant but not π-fixed, and there exists a π-fixed 2-sphere S, standardly oriented and with $[S]^2 = -1$, then F has the standard orientation if $[F] \cdot [S] = -1$.

4. Proof of Theorem A

The minimal negative definite resolution for a Brieskorn homology sphere is obtained from the dual resolution graph of the singularity whose link is the Brieskorn homology 3-sphere $\Sigma(a_1, a_2, a_3)$ (see Saveliev [27, Ex. 1.17]). For these singularities, the graph is a tree with weight δ on the central node, and weights on the branches given by a continued fraction decomposition $a_i/b_i = [t_{i1}, t_{i2}, ..., t_{im_i}]$ of the Seifert invariants. These weights are uniquely determined by the condition $t_{ij} \leq -2$ $-a_i < b_i < 0$ and

$$a_1a_2a_3b_i/a_i \equiv 1 \pmod{a_i}. \tag{4.1}$$

where δ satisfies

$$\delta = \frac{-1}{a_1a_2a_3} + \sum_{i=1}^{3} \frac{b_i}{a_i} \leq 1. \tag{4.2}$$

Fintushel and Stern defined the R-invariant for Brieskorn homology spheres, which is an odd integer $R(a_1, a_2, a_3) \geq -1$. Moveover, if $\Sigma(a_1, a_2, a_3)$ bounds a smooth contractible manifold, then $R(a_1, a_2, a_3) = -1$ (see [13, Theorem 1.1]). Neumann and Zagier [25] gave the calculation

$$R(a_1, a_2, a_3) = -2\delta - 3.$$

This implies that the central node in the resolution graph of $\Sigma(a_1, a_2, a_3)$ has weight $\delta = -1$.

Equivariant plumbing on the defining graph Γ gives the minimal negative definite resolution $M(\Gamma)$, where each node in the graph is represented by an embedded 2-sphere with self-intersection number given by its weight. The circle action on $\Sigma(a, b, c)$ which arises from its Seifert fibering structure extends over the plumbing (see [26], [24]). By construction, the central node sphere is fixed under the circle action.

By restricting this circle action to $\pi = \mathbb{Z}/p$, for $p \nmid abc$, we obtain a simply connected, smooth 4-manifold $M(\Gamma)$ with a smooth, homologically trivial π-action, whose boundary is $\Sigma = \Sigma(a, b, c)$ with the standard free $\mathbb{Z}/p$-action.

Suppose that the standard free action on $\Sigma(a, b, c)$ also extends smoothly over another compact smooth 4-manifold W with $\partial W = \Sigma$. Then we obtain a smooth, closed 4-manifold

$$X = M(\Gamma) \cup_\Sigma (-W) \tag{4.3}$$

together with smooth, homologically trivial π-action. If W is *acyclic*, meaning that W has the integral homology of a point, and $\pi_1(W)$ is the normal closure of the image of $\pi_1(\Sigma)$, then X will be closed, *simply connected*, smooth 4-manifold with *odd* negative definite intersection form. In other words, $X \simeq \#_1^n \overline{\mathbb{C}P}^2$ where $n = b_2(M(\Gamma))$.

Theorem 4.4. *Suppose $\Sigma(a, b, c)$ bounds a smooth acyclic 4-manifold W, such that $\pi_1(W)$ is the normal closure of the image of $\pi_1(\Sigma(a, b, c))$. If p is an odd prime with $p \nmid abc$, then the free $\pi = \mathbb{Z}/p$ action on $\Sigma(a, b, c)$ does not extend to a smooth action on W.*

Proof. We form the manifold $X = M(\Gamma) \cup_\Sigma (-W)$ from the given acyclic manifold W and the plumbed manifold $M(\Gamma)$ as described in (4.3). We have $X \simeq \#_1^n \overline{\mathbb{C}P}^2$ where $n = b_2(M(\Gamma))$. There is a basis for $H_2(X; \mathbb{Z})$ represented by the nodal 2-spheres in the plumbing construction. Since the plumbing is done equivariantly, we obtain a configuration of smoothly embedded π-fixed 2-spheres and π-invariant 2-spheres in X, with at least one π-fixed 2-sphere F_1 of self-intersection -1 (namely the central node in the graph Γ). We fix an ordering on the other nodes so that F_2 and F_3 are adjacent to F_1.

We give each of these 2-spheres the complex orientation induced by the plumbing construction and the negative definite orientation on X. Let

$$Q_X\colon H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}$$

denote the intersection form of X, expressed as a matrix with respect to the basis

$$\mathcal{F} = \{[F_1], [F_2], \dots, [F_n]\}.$$

In other words, Q_X is the plumbing matrix defined by the graph Γ, in which $[F_i]\cdot[F_j] = 1$, for $i \neq j$, whenever this intersection is non-zero.

Let $\mathcal{E} = \{e_1, e_2, \dots, e_n\}$ denote a standard diagonal basis given by an (orientation-preserving) homotopy equivalence $X \simeq \#_1^n \overline{\mathbb{C}P}^2$, and the orientation convention given in Definition 3.3. Let C denote the change of basis matrix (with respect to $\mathcal{E}$ and $\mathcal{F}$), so that $C^t Q_X C = -I$ is in diagonal form with respect to the basis $\mathcal{E}$. Then the columns of C give the components of each e_i in terms of the basis $\mathcal{F}$, and similarly the columns of C^{-1} give the expressions for each F_i in terms of the standard diagonal basis $\mathcal{E}$.

Since F_1 is a fixed 2-sphere with $[F_1]\cdot[F_1] = -1$, we may assume that $e_1 = \pm[F_1]$ in the diagonal basis $\mathcal{E}$. Suppose first that $e_1 = [F_1]$. The inverse C^{-1} then has the form

$$C^{-1} = \begin{pmatrix} 1 & -1 & -1 & * & \cdots & * \\ 0 & a_2 & b_2 & * & \cdots & * \\ 0 & a_3 & b_3 & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_n & b_n & * & \cdots & * \end{pmatrix} \tag{4.5}$$

where we have labelled the base node F_1 and two adjacent nodes F_2 and F_3, such that $[F_2]\cdot[F_1] = [F_2]\cdot[F_1] = 1$, but $[F_2]\cdot[F_3] = 0$. By construction, F_2 and F_3 are π-invariant (but not fixed) embedded 2-spheres. This configuration always occurs in the plumbing graph for $M(\Gamma)$. By Remark 3.9, the plumbing orientation for F_2 and F_3 is opposite to the standard orientation.

It follows that $[F_2] = -e_1 + a_2e_2\dots$, and similarly that $[F_3] = -e_1 + b_2e_2\dots$. By Theorem 3.7 we can conclude that all the non-zero entries in the second and third column are actually *negative*. On the other hand, since

$$0 = [F_2]\cdot[F_3] = -1 - \sum_{i=2}^{n} a_ib_i$$

and each term $a_ib_i \geq 0$, we have a contradiction. If $e_1 = -[F_1]$, then F_2 and F_3 have the standard orientation and all the non-zero entries in the second and third columns of C^{-1} must be positive (by Theorem 3.7). We obtain a contradiction as before. □

5. Locally Linear Extensions

In this section we briefly survey some results of Edmonds [8] and Kwasik-Lawson [20]. First it should be noted, by the work of Freedman [16], that every integral homology 3-sphere Σ bounds a topological contractible 4-manifold Δ. That every free action on Σ can be extended to a topological action on a topological contractible 4-manifold was first noted by Ruberman and Kwasik-Vogel [21]. The question of extending a free $\pi = \mathbb{Z}/p$ action on Σ to a locally linear action on a contractible 4-manifold was studied by Edmonds

[8] for p a given prime, including the case of an involution $p = 2$; we will only consider the case of odd primes.

The result for locally linear actions will involve additional spectral and torsion invariants. The equivariant eta invariant is the g-signature defect term for manifolds with boundary. Let $\partial W = \Sigma$ and $Q = \Sigma/\pi$, then the relation between the rho invariants $\rho(Q,\gamma)$ of the orbit space and the equivariant eta invariant is given by

$$\eta_t(\Sigma) = \sum_{\gamma} \rho(Q,\gamma)\overline{\chi}_\gamma(t) \tag{5.1}$$

where the sum is over the irreducible representations of $\pi = \mathbb{Z}/p$. There is also a Fourier-type transform [2] expressing rho invariants in terms of the equivariant eta invariant η_t:

$$\rho(Q,\gamma) = \frac{1}{p}\sum_{t\neq 1} \eta_t(\Sigma)(\chi_\gamma(t) - \dim(\gamma)). \tag{5.2}$$

As an example that we will use later, the rho invariants of classical lens spaces are given by

$$\rho(L(p;r,s),\gamma(\ell)) = \frac{4}{p}\sum_{k=1}^{p-1} \cot(\frac{\pi k r}{p})\cot(\frac{\pi k s}{p})\sin^2(\frac{\pi k \ell}{p}) \tag{5.3}$$

which can be easily computed from the above formula using the equivariant eta invariant $\eta_t(S^3)$ of the 3-sphere with the action extending to a disk with rotation number (r,s).

$$\eta_t(S^3) = \frac{(t^r+1)(t^s+1)}{(t^r-1)(t^s-1)} \tag{5.4}$$

where $t = e^{2\pi i/p}$ is a generator of $\mathbb{Z}/p$. We will also need the notion of Reidemeister torsions before we state the main result about locally linear extensions. This torsion invariant arises from an acyclic chain complex as follows. Give Q a cell structure and let Σ be given the induced cell structure from the regular covering. Then $C_*(\Sigma)$ is a chain complex of free $\mathbb{Z}[\pi]$ modules. Using the natural homomorphisms

$$\mathbb{Z}[\pi] \to \mathbb{Z}[\zeta] \to \mathbb{Q}[\zeta]$$

where $\zeta = e^{2\pi i/p}$, we see that the twisted homology of $C_*(\Sigma)\otimes\mathbb{Q}[\zeta]$ is acyclic with torsion $\Delta(Q)$ in $\mathbb{Q}[\zeta]^\times$. The Reidemeister torsion of the lens space $L(p;r,s)$ is $\Delta(L(p;r,s)) \sim (\zeta^r - 1)(\zeta^s - 1)$.

Theorem 5.5 (Edmonds [8], Kwasik-Lawson [20, p. 32])**.**

(i) *A free action of $\pi = \mathbb{Z}/p$ on an integral homology 3-sphere Σ extends to a locally linear action on a contractible 4-manifold W with one fixed point if and only if the quotient rational homology sphere $Q = \Sigma/\pi$ is $\mathbb{Z}[\pi]$ h-cobordant to a classical lens space L.*

(ii) *A rational homology sphere $Q = \Sigma/\pi$ is $\mathbb{Z}[\pi]$ h-cobordant to classical lens space L if and only if there is a $\mathbb{Z}[\pi]$-homology equivalence $f\colon Q \to L$ under which their rho invariants are equal and the Reidemeister torsions satisfy $\Delta(Q) \sim u^2\Delta(L)$ where u is the image of a unit in $\mathbb{Z}[\pi]$.*

Recall that a $\mathbb{Z}[\pi]$ h-cobordism V between Q and L is one where $H_*(V,Q;\mathbb{Z}[\pi]) = 0$ with local coefficients; equivalently, the $\mathbb{Z}/p$-cover is an integral h-cobordism. To find a locally linear extension, one needs to find a lens space $L(p,q)$ for some integer $q \pmod p$ and a $\mathbb{Z}[\pi]$-homology equivalence $f\colon Q \to L(p,q)$ satisfying the conditions above. To do this, start with the classifying map of the cover $Q = \Sigma/\pi$, so a map $f\colon Q \to B\pi$. By general position arguments we can take the image to be a 3-dimensional lens space $L(p;r,s)$ and arrange so that the map is of degree one [8], thus giving a $\mathbb{Z}[\pi]$-homology equivalence $f\colon Q \to L(p;r,s)$. When Σ is a Seifert fibered space the following theorem gives the constraint on the lens space:

Theorem 5.6 (Kwasik-Lawson [20, p. 35])**.** *Let Q be a Seifert fibered space with Seifert invariants $\{(a_i,b_i)\}$ with $\alpha\sum b_i/a_i = p$ where α is the product of the a_i. Then there is a degree one map $f\colon Q \to L(p;r,s)$ which is a $\mathbb{Z}[\pi]$-homology equivalence if and only if $\alpha \equiv rs \pmod p$.*

In the case when we have a simple homology equivalence the Reidemeister torsion condition is fulfilled.

Theorem 5.7 (Kwasik-Lawson [20, p. 37])**.** *There is a simple $\mathbb{Z}[\pi]$-homology equivalence between the rational homology sphere $Q = \Sigma(a,b,c)/\pi$ by the standard π-action and a lens space $L(p;r,s)$, respecting the orientation and the preferred generators of $H_1(Q)$ and $H_1(L)$, if and only if $\{a,b,c\}$ are congruent to $\{r,s,1\} \pmod p$ up to sign and the product $abc \equiv rs \pmod p$.*

We now use the above results to show an infinite family in the list of Stern [28] admits locally linear pseudo free extensions to a contractible 4-manifold. First we will need the following

Lemma 5.8. *For each positive integer k, each of the Brieskorn homology 3-spheres $\Sigma(r, rs\pm 1, 2r(rs\pm 1)+rs\pm 2)$ bounds an indefinite smooth 4-manifold X_0 with signature equal to -2.*

Proof. We can realize $\Sigma(r, rs \pm 1, 2r(rs \pm 1) + rs \pm 2)$ as the boundary of the following plumbed indefinite 4-manifold X_0 (see Fickle [11]): The signature is determined via an

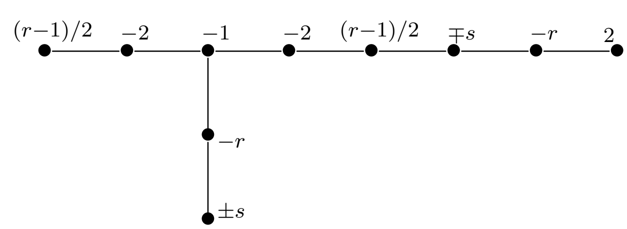


Figure 1. The boundary of this plumbed 4-manifold is the homology 3-sphere $\Sigma(r, rs \pm 1, 2r(rs \pm 1) + rs \pm 2)$.

algorithm which amounts to a graph version of the Gaussian diagonalization process over the rationals (see [10, p. 153]). ☐

Theorem 5.9. *Let r and p be an odd primes such that $p > r$. Then for each positive integer k the standard free $\pi = \mathbb{Z}/p$ action on $\Sigma(r, rkp\pm 1, 2r(rkp\pm 1)+rkp\pm 2)$ extends* *to locally linear action on a smooth contractible* 4*-manifold* W *with a single fixed point of rotation number* $(r, 2r+2)$.

Proof. There is a simple $\mathbb{Z}[\pi]$-homology equivalence from $Q = \Sigma(r, rkp \pm 1, 2r(rkp \pm 1) + rkp \pm 2)/\pi$ to the classical lens space $L(p; r, 2r+2)$. We need to show that these have equivalent rho invariants and we do this by showing that their equivariant eta invariants are equal. Equivariant plumbing (see Fintushel [12, §4]) on the graph in Figure 1 simplifies the computation: we will see that it produces cancelling pairs of rotation numbers .

For each integer a, let $D^2(a)$ denote the unit disk in $\mathbb{C}$ with S^1-action given by $z \mapsto z^a$. Given relatively prime integers a and b, we have a circle action on $D^2(a) \times D^2(b)$ given by the formula

$$z \cdot (re^{i\theta}, se^{i\tau}) = (re^{i(\theta+at)}, se^{i(\tau+bt)}), \tag{5.10}$$

where $z = e^{it} \in S^1$. Write $S^2 = D^2_+ \cup D^2_-$ as the upper and lower hemispheres and consider the trivial D^2-bundle over each hemisphere. The formula in (5.10) defines an S^1-action on the trivial bundle $D^2_+ \times D^2$, and similarly for the lower hemisphere with a and b replaced with c and d. We glue these trivial equivariant bundles together using the map

$$F \colon \partial D^2_+ \times D^2 \to \partial D^2_- \times D^2$$

defined by $F(e^{i\theta}, se^{i\tau}) = (e^{-i\theta}, se^{i(-k\theta+\tau)})$. We obtain an S^1-equivariant D^2-bundle E_k over $S^2 = D^2_+(a) \cup D^2_-(-a)$ with Euler number k, provided that

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{5.11}$$

To equivariantly plumb with another such D^2 bundle we identify over a trivialized hemisphere by interchanging base and fibre coordinates.

The extended $\mathbb{Z}/p$-action is part of the circle action and is therefore isotopic to the identity (hence homologically trivial). We can thus identify the equivariant signature of the manifold X_0 with its usual signature: $\operatorname{sign}(X_0) = -2$ (see Lemma 5.8). The rotation numbers arising from equivariant plumbing on the graph in Figure 1 are

$$(2, r), (2, r), (-1, 2), (-1, 2), (r, -2), (r, -2), (-1, r), (1, r), (r, 2r+2)$$

and one fixed 2-sphere with self-intersection -1 with rotation number 1 acting on the normal fiber. The Euler characteristic of the fixed set $\chi(\mathrm{Fix}(X_0)) = 11$ and signature equal to -2. After removing the cancelling pairs the G-signature theorem for manifolds with boundary simplifies to

$$\eta_t(\Sigma) = -2\Big(\frac{t+1}{t-1}\Big)\Big(\frac{t^2+1}{t^2-1}\Big) + \frac{4t}{(t-1)^2} - \operatorname{Sign}(X_0) + \Big(\frac{t^r+1}{t^r-1}\Big)\Big(\frac{t^{2r+2}+1}{t^{2r+2}-1}\Big). \tag{5.12}$$

It is easy to check that the first three terms above cancel leaving the equivarient eta invariant of the classical lens space $L(p; r, 2r+2)$ as was to be shown. $\square$

6. An Infinite Family of Examples

In this section we give an infinite family of examples of non-smoothable locally linear extensions.

Example 6.1. The Brieskorn homology 3-sphere $\Sigma = \Sigma(3, 16, 113)$ bounds a smooth contractible 4-manifold W, and admits free $\pi = \mathbb{Z}/5$ action. It is part of the infinite family of the form $\Sigma(r, rs+1, 2r(rs+1)+rs+2)$ given by Stern's examples with $r = 3$ and $s = 5$. It follows from Theorem B that the standard $\mathbb{Z}/5$ action on $\Sigma(3, 16, 113)$ extends to a locally linear action on W with one fixed point whose rotation data is $(3, 3)$. However, Theorem A shows that there is no such smooth action. It follows that $\Sigma(3, 16, 113)$ admits a $\mathbb{Z}/5$-equivariant embedding into a homotopy 4-sphere with a locally linear $\mathbb{Z}/5$-action.

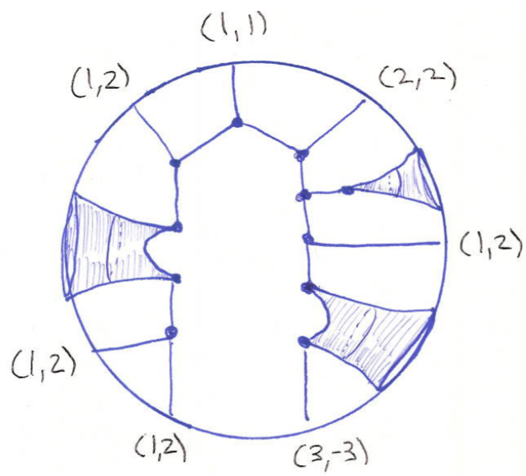


Figure 2. The fixed set pattern in the moduli space $(\mathcal{M}_1(X), \pi)$ for $\Sigma(3, 16, 113)$. Each vertex in the interior is a reducible connection whose link is a complex projective space with a linear π-action. The isotropy representations then resemble that of an equivariant connected sum of linear actions on complex projective spaces.

The associated negative definite smooth 4-manifold $M(\Gamma)$ has signature -11. Equivariant

$\Gamma =$

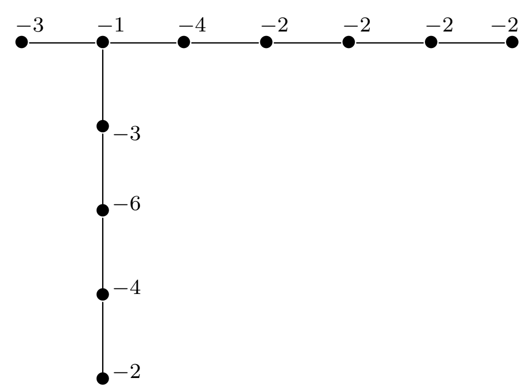


Figure 3. The canonical negative definite plumbing diagram for $\Sigma(3, 16, 113)$.

plumbing beginning with the central vertex produces 6 fixed points with rotation data $\{(1,1), (1,2), (1,2), (1,2), (1,2), (2,2)\}$ and 3 fixed 2-spheres F_1, F_2, F_3, two of which represent homology classes of self-intersection -2 with normal rotation number $c_F = 3$ and one representing a homology class (center vertex) of self-intersection -1 with normal rotation $c_F = 1$. For the locally linear action on $X = M(\Gamma) \cup_{\Sigma(3,16,113)} -W$, we have one additional fixed point with rotation data $(3,-3)$ coming from $-W$.

The intersection form Q_X is given by

$$Q_X = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix} \tag{6.2}$$

and by Donaldson's Theorem A there exists an invertible integer matrix C such that $C^tQC = -I$, then the change of basis matrix C^{-1} taking the basis in the plumbing diagram to a diagonal basis $\{e_i\}$ can be computed to be

$$C^{-1} = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \tag{6.3}$$

The fixed 2-spheres are given in terms of a diagonal basis as the first, sixth and tenth columns:

$$\begin{aligned} F_1 &= e_1 \\ F_6 &= -e_6 - e_7 \\ F_{10} &= -e_5 - e_{11}. \end{aligned}$$

The rest of the columns give expressions for the invariant 2-spheres in terms of the diagonal basis. Now any other such matrix is obtained from C by either permutations of the standard basis $\{e_i\}$ or a change of sign ($e_i \mapsto -e_i$) since these are the automorphisms of the standard diagonal form. As we have seen, this information contradicts the existence of the general position equivariant moduli space $\overline{\mathcal{M}_1(X)}$. Note that Figure 2 shows that this action is not ruled out just by the rotation numbers and the singular set in the moduli space.

Example 6.4. Before finding the general argument presented above, we worked out a particular infinite family of examples. Here is a way to simultaneously diagonalize all their intersection forms. Let (M_k, π) denote the canonical negative definite resolution of

$$\Sigma_k = \Sigma(3, 3kp+1, 6(3kp+1)+3kp+2)$$

together with a $\pi = \mathbb{Z}/p$ action extending the standard free π-action on Σ_k via equivariant plumbing. If the action also extends to a smooth action on a contractible 4-manifold W

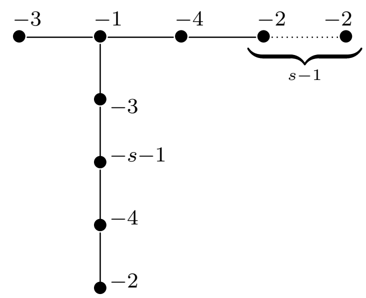
$\Gamma_k =$ 

Figure 4. The canonical negative definite resolution plumbing diagram for $\Sigma(3, 3s+1, 6(3s+1)+3s+2)$, where $k = sp$.

then $X_k = M_k \cup -W$ is a simply connected, negative definite 4-manifold with a smooth, homologically trivial π-action. The intersection form of X_k is given by the symmetric matrix indexed by k (of size depending on $s = kp$):

$$Q_{X_k} = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & -s-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 1 & -4 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We now claim that the matrix C^{-1} in $C^t Q_{X_k} C = -I$ is given in terms of a diagonal basis by the following expressions. Those which do not depend on the parameter are given by:

$$F_1 = e_1, \quad F_2 = -e_1 - e_2 + e_3, \quad F_3 = -e_1 + e_2 + e_4, \quad F_5 = -e_2 - e_3 + e_4 + e_6,$$
$$F_6 = -e_6 - e_7, \quad F_7 = -e_1 - e_3 - e_4 + e_8, \quad F_8 = -e_8 + e_9$$

and the rest of the basis is obtained inductively by:

$$F_4 = -e_4 + e_5 - e_8 - e_9 - e_{10} \cdots - e_n, \quad F_n = -e_5 - e_n \quad F_{n-1} = -e_{n-1} + e_n$$

where $n = 6 + s$, with $s \geq 3$. Once again, the point is that there is no consistent choice of sign in the expression of all the $[F_i]$, and moreover one cannot achieve such consistency by an automorphism of the standard form. Thus the actions in Theorem B for $r = 3$ and $s = 5$ do not extend smoothly.

The proof of Corollary C. If $\Sigma = \Sigma(a, b, c)$ is a Brieskorn homology 3-sphere which bounds a smooth contractible 4-manifold W, then the manifold $N = W \cup_\Sigma (-W)$ is a smooth homotopy 4-sphere in which $\Sigma(a, b, c)$ is a smoothly embedded submanifold. Now the examples of Theorem B provide a locally linear extension of the free $\pi = \mathbb{Z}/p$ actions to N with two isolated fixed points. Conversely, suppose that (N, π) is a smooth π-action on a homotopy 4-sphere. Then if (Σ, π) embeds smoothly and equivariantly into N, it follows that the action on N has two isolated fixed points, and that $N = W \cup_\Sigma W'$ is a smooth equivariant decomposition of N as the union of compact 4-manifolds W and W' with boundary Σ. By the van Kampen Theorem, the image of $\pi_1(\Sigma)$ normally generates $\pi_1(W)$, so we obtain a contradiction by Theorem 4.4. □

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Department of Mathematics & Statistics
McMaster University
Hamilton, ON L8S 4K1, Canada
E-mail address: `hambleton@mcmaster.ca`

Department of Mathematics & Statistics
McMaster University
Hamilton, ON L8S 4K1, Canada
E-mail address: `anvarin@math.mcmaster.ca`