TOPOLOGICAL 4-MANIFOLDS WITH RIGHT-ANGLED ARTIN FUNDAMENTAL GROUPS: CORRIGENDUM

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Abstract. We correct the stability bound in our classification of closed, spin topological 4-manifolds with fundamental group $\pi$ of cohomological dimension $\leq 3$ (up to $s$-cobordism), after stabilization by connected sum with at most $b_3(\pi)$ copies of $S^2 \times S^2$. If $\pi$ is a right-angled Artin group whose defining graphs have no 4-cliques, then the new stability bound is $r \geq \max(b_3(\pi), 6)$. The other results of the paper are not affected (see [4]).

1. Introduction

We correct the stability bounds used in the statements of Theorem A and Theorem 11.2 in our paper [4]. We are indebted to Daniel Kasprowski for pointing out a gap in the last step of our arguments, and to Diarmuid Crowley for a very useful conversation. To explain and repair the error we need to briefly describe the setting.

A standard approach to the classification of topological 4-manifolds uses the theory of “modified surgery” due to Matthias Kreck [6, §6]. We briefly recall some of the features of modified surgery in our setting (see [6, Theorem 4, p. 735] for the notation):

• Let $M$ and $N$ be closed, oriented topological 4-manifolds with the same Euler characteristic, which admit normal 1-smoothings in a fibration $B \to \text{BSTOP}$. If $W$ is a normal $B$-bordism between these two 1-smoothings, with normal $B$-structure $\tilde{\nu}$, then there exists an obstruction $\Theta(W, \tilde{\nu}) \in \ell_5(\pi_1(B))$ which is elementary if and only if $(W, \tilde{\nu})$ is $B$-bordant relative to the boundary to an $s$-cobordism.

• Let $\pi := \pi_1(B)$ and $\Lambda := \mathbb{Z}[\pi]$ denote the integral group ring of the fundamental group. The elements of $\ell_5(\pi)$ are represented by pairs $(H(\Lambda'), V)$, where $V$ is a half-rank direct summand of the hyperbolic form $H(\Lambda')$.

• In a pair $(H(\Lambda'), V)$, if the quadratic form vanishes on $V$, then the element $\Theta(W, \tilde{\nu})$ lies in the image of $L_5(\mathbb{Z}[\pi]) \to \ell_5(\pi)$ (see [6, Proposition 8, p. 739] or [6, p. 734] for criteria to ensure that this will happen).

In our applications, we assumed the following conditions.

Definition 1.1. A group $\pi$ satisfies properties (W-A) whenever

(i) The Whitehead group $\text{Wh}(\pi)$ vanishes.

(ii) The assembly map $A_5: H_5(\pi; \mathbb{L}_0) \to L_5(\mathbb{Z}[\pi])$ is surjective.

If, in addition, the assembly map $A_4: H_4(\pi; \mathbb{L}_0) \to L_4(\mathbb{Z}[\pi])$ is injective, we say that $\pi$ satisfies properties (W-AA).

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These properties hold whenever the group \( \pi \) is torsion-free and satisfies the Farrell-Jones isomorphism conjectures in \( K \)-theory and \( L \)-theory. These conjectures have been verified for many classes of groups, and in particular for all right-angled Artin groups (see [2], [1]).

If \( \pi \) satisfies properties (W-A), then the action of elements in the image \( \text{Im} \, A_5(M) \subseteq L_5(\mathbb{Z}[\pi]) \) of the assembly map on \( \Theta(W, \tilde{\nu}) \in \ell_5(\pi_1(B)) \) can be defined geometrically by the action of degree 1 normal maps on the \( B \)-bordism \((W, \tilde{\nu})\). Here

\[
A_5(M) : H_5(M; \mathbb{L}_0) = H_1(\pi; \mathbb{Z}) \oplus H_3(\pi; \mathbb{Z}/2) \to L_5(\mathbb{Z}[\pi])
\]
is defined by the surgery obstructions of degree 1 normal maps

\[
F : (\partial_0 U, \partial_1 U) \to (M \times I, M \times 0, M \times 1).
\]

By definition, \( \partial_0 U = \partial_1 U = M \), and \( F \) restricted to both boundary components is a homeomorphism. Such inertial normal cobordisms can be glued to \((W, \tilde{\nu})\) to produce a new \( B \)-bordism \((W', \tilde{\nu})\) between \( M \) and \( N \), with surgery obstruction \( \Theta(W', \tilde{\nu}) = \Theta(W, \tilde{\nu}) + \sigma(F) \) (see the proof of [5, Theorem 2.6]).

This is the argument we proposed for the final step to eliminate the obstruction \( \Theta(W, \tilde{\nu}) \), and thus obtain an \( s \)-cobordism between \( M \) and \( N \) under the assumptions of Theorem 11.2 (and its application to Theorem A).

**The Error:** By assumption, the assembly map \( A_5 \) in condition (W-A) is surjective, and its domain:

\[
A_5 : H_5(\pi; \mathbb{L}_0) = H_1(\pi; \mathbb{Z}) \oplus H_3(\pi; \mathbb{Z}/2) \to L_5(\mathbb{Z}[\pi]),
\]
is expressed in terms of the group homology of \( \pi \). However, the above construction can only realize the action of elements in the image of the partial assembly map

\[
H_5(M; \mathbb{L}_0) = H_1(\pi; \mathbb{Z}) \oplus H_3(\pi; \mathbb{Z}/2) \to H_5(\pi; \mathbb{L}_0) \to L_5(\mathbb{Z}[\pi])
\]
from the homology of \( M \). Since the reference map \( M \to B \) is 2-connected, the summand \( H_1(M; \mathbb{Z}) \cong H_1(\pi; \mathbb{Z}) \). However, if the map \( H_3(M; \mathbb{Z}/2) \to H_3(\pi; \mathbb{Z}/2) \) is not surjective, we will not be able to realize all possible obstructions by this construction.

**Remark 1.2.** The statements of [5, Theorems 2.2 & 2.6] are a bit misleading, since they appear (incorrectly) to be stated for arbitrary fundamental groups. However, the goal of [5] was to study fundamental groups \( \pi \) of geometric (and hence cohomological) dimension at most two. In these cases, \( H_3(\pi; \mathbb{Z}/2) = 0 \) so the domain of \( A_5 \) is just \( H_1(\pi; \mathbb{Z}) \), and the problem above does not arise. In contrast, if \( \text{cd} \, \pi = 3 \) and \( \pi_1(M) = \pi \), then by Poincaré duality:

\[
\begin{array}{ccc}
H^1(M; \mathbb{Z}/2) & \xleftarrow{\cong} & H^1(\pi; \mathbb{Z}/2) \\
\cong c_*[M] & \downarrow{\cap[M]} & c_*[M] \\
H_3(M; \mathbb{Z}/2) & \longrightarrow & H_3(\pi; \mathbb{Z}/2)
\end{array}
\]

and the map \( H_3(M; \mathbb{Z}/2) \to H_3(\pi; \mathbb{Z}/2) \) is zero since \( 0 = c_*[M] \in H_4(\pi; \mathbb{Z}/2) \).
2. A stable range for $L$-theory

For any finitely presented group $\pi$, the odd dimensional surgery obstruction groups are defined as $L_5(\mathbb{Z}[\pi]) = SU(\Lambda)/RU(\Lambda)$, in the notation of Wall [8, Chap. 6]. Here $SU(\Lambda)$ is the limit of the automorphism groups $SU_r(\Lambda)$ of the hyperbolic (quadratic) form $H(\Lambda^r)$ under certain injective maps

$$\ldots SU_r(\Lambda) \to SU_{r+1}(\Lambda) \to \ldots \to SU(\Lambda),$$

and $RU(\Lambda)$ is a suitable subgroup determined by the surgery data, so that $L_5(\mathbb{Z}[\pi])$ is an abelian group. To define a stable range, we will assume that the fundamental groups have type $F_3$, meaning that there is a model for the classifying space $B\pi$ with finite 3-skeleton. In particular, groups of type $F_3$ are finitely presented, but not conversely.

**Definition 2.1.** For an element $x \in L_5(\mathbb{Z}[\pi])$, we denote its stable $L_5$-range by:

$$sr(x) = \min\{r \geq 0 : x \text{ is represented by an matrix in } SU_r(\Lambda)\}.$$  

The stable range of a finitely presented group $\pi$ is defined as:

$$sr_3(\pi) = \max\{sr(A_5(\alpha)) : \text{where } \alpha \text{ varies over a } \mathbb{Z}/2\text{-basis for } H_3(\pi; \mathbb{Z}/2)\}.$$

**Remark 2.2.** If $\pi$ has type $F_3$ then the stable range will be finite. In general, $sr_3(\pi)$ could be infinite, since there are finitely presented groups with $H_3(\pi; \mathbb{Z}/2)$ of infinite rank (see Stallings [7]).

**Lemma 2.3.** Let $\pi$ be a right-angled Artin group. Then $sr_3(\pi) \leq 6$.

**Proof.** Every right-angled Artin group has type $F_3$ since it is defined by a finite graph. The homology group $H_3(\pi; \mathbb{Z}/2)$ has $\mathbb{Z}/2$-rank $b_3(\pi)$, which is equal to the number of 3-cliques in the defining graph for $\pi$. Moreover, since each 3-clique determines a subgroup $\mathbb{Z}^3 \subseteq \pi$, the group $H_3(\pi; \mathbb{Z}/2)$ is generated by the images of the fundamental classes under all the induced maps $H_3(T^3; \mathbb{Z}/2) \to H_3(\pi; \mathbb{Z}/2)$. It is therefore enough to determine the stable range for $\rho = \mathbb{Z}^3$.

By definition of the assembly map, we need to determine the minimum representative in $SU_r(\Lambda)$ for the surgery obstruction of the degree one normal map

$$g := (\text{id} \times f) : N \times T^2 \to N \times S^2$$

given by the the product of the Arf invariant one normal map $f : T^2 \to S^2$ with the identity on $N = T^3$. After surgery on the generators of

$$K_1(g) = \ker\{H_1(N \times T^2; \Lambda) \to H_1(N \times S^2; \Lambda)\} = \mathbb{Z} \oplus \mathbb{Z}$$

we get a 2-connected normal map with $K_2(g') = I(\rho) \oplus I(\rho)$, where $I(\rho) := \ker\{\mathbb{Z}[\rho] \to \mathbb{Z}\}$ is the augmentation ideal of the group ring $\mathbb{Z}[\rho]$. According to the recipe provided by Wall [8, Chap. 6, pp. 58-59], the surgery obstruction is represented in $SU_r(\Lambda)$, where $r \geq 6$ since an epimorphism $\Lambda^r \to I(\rho)$ requires $r \geq 3$. 

We will use a stable range condition to realize the action of $L_5(\mathbb{Z}[\pi])$ on a $B$-bordism, after a suitable stabilization. The following statement is an application of this result in the setting of Kreck [6, Theorem 4].
Proposition 2.4. Let \( \pi \) be a discrete group of type \( F_3 \) satisfying properties (W-A). Let \( M \) and \( N \) be closed, oriented topological 4-manifolds with the same Euler characteristic, which admit normal 1-smoothings in a fibration \( B \to \text{BSTOP} \). Suppose that \( (W, \tilde{\nu}) \) is a normal \( B \)-bordism between these two 1-smoothings. If \( r \geq \text{sr}_3(\pi) \), then for any \( x \in L_5(\mathbb{Z}[\pi]) \) there exists a \( B \)-bordism \( (W', \tilde{\nu}) \) between the stabilized 1-smoothings \( M' := M \# r(S^2 \times S^2) \) and \( N' := N \# r(S^2 \times S^2) \), with \( \Theta(W', \tilde{\nu}) = \Theta(W, \tilde{\nu}) + x \in \ell_5(\pi) \).

Proof. By property (W-A), the assembly map \( A_5 : H_1(\pi; \mathbb{Z}) \oplus H_3(\pi; \mathbb{Z}/2) \to L_5(\mathbb{Z}[\pi]) \) is surjective. The elements \( x \in L_5(\mathbb{Z}[\pi]) \) in the image of \( H_1(\pi; \mathbb{Z}) \cong \pi_1(M; \mathbb{Z}) \) are realized as above (without stabilization). For the elements \( x = A_5(\alpha) \in L_5(\mathbb{Z}[\pi]) \) in the image of \( \alpha \in H_3(\pi; \mathbb{Z}/2) \), we use the stabilized version of Wall realization due to Cappell and Shaneson [3, Theorem 3.1].

Any element is the image of a finite sum \( \alpha = \sum \alpha_i \) of basis elements of \( H_3(\pi; \mathbb{Z}/2) \), which all have stable \( L \)-range at most \( \text{sr}_3(\pi) \). The realization construction can be done (for each term \( \alpha_i \) of the finite sum) in small disjoint intervals

\[
M' \times [t_{i-1}, t_i] \subset M' \times [0, 1],
\]

with \( 0 = t_0 < t_1 < \cdots < t_k = 1 \), to produce degree one normal maps

\[
F_i : (U_i, \partial_0 U_i, \partial_1 U_i) \to (M' \times [t_{i-1}, t_i], M' \times t_{i-1}, M \times t_i), \quad 1 \leq i \leq k,
\]
such that \( \partial_0 U_i = \partial_1 U_i = M' = M \# r(S^2 \times S^2) \). The restrictions of \( F_i \) to the boundary components have the property that \( F_i |_{\partial_0 U_i} = \text{id}, \) and \( F_i |_{\partial_1 U} : f_i \) is a simple homotopy equivalence. In other words, this construction produces elements of the structure set \( S(M') \) represented by self-equivalences of \( M' \).

These normal bordisms can be glued (at disjoint levels) into a collar \( M' \times [0, 1] \) attached to the stabilization \( W_2 \# r(S^2 \times S^2 \times I) \) of the given \( B \)-bordism, and the reference map to \( B \) extended through \( M \). After including all these bordisms, the induced homotopy equivalence with target \( M' \times 1 \) is the composite \( f := f_1 \circ f_2 \circ \cdots \circ f_k \). The surgery obstruction over the collar \( M' \times [0, 1] \) is \( x = A_5(\alpha) = \sum A_5(\alpha_i) \), and the result follows. \( \square \)

The following application of the theory in Kreck [6, §6] may be useful in cases where a potentially harder bordism calculation is feasible.

Corollary 2.5. If \( M \) and \( N \) are closed, oriented topological 4-manifolds which admit \( B \)-bordant normal 2-smoothings in the same fibration \( B \to \text{BSTOP} \), then they are s-cobordant after at most \( \text{sr}_3(\pi) \) stabilizations, provided their fundamental group has type \( F_3 \) and satisfies properties (W-A).

Proof. For normal 2-smoothings of \( M \) and \( N \), the reference maps are 3-connected. In this case, Kreck [6, p. 734] shows that the surgery obstruction \( \Theta(W, \tilde{\nu}) \) of a \( B \)-bordism \( (W, \tilde{\nu}) \) lies in the image of \( L_5(\mathbb{Z}[\pi]) \to \ell_5(\pi) \). The result now follows from Proposition 2.4. \( \square \)

3. The main results corrected

To correct the statements of Theorem A and Theorem 11.2 in [4], we use the new stable range conditions.

Theorem A. Let \( \pi \) be a right-angled Artin group defined by a graph \( \Gamma \) with no 4-cliques. Suppose that \( M \) and \( N \) are closed, spin\(^+ \), topological 4-manifolds with fundamental group
\[\pi.\] Then any isometry between the quadratic 2-types of \(M\) and \(N\) is stably realized by an \(s\)-cobordism between \(M \# r(S^2 \times S^2)\) and \(N \# r(S^2 \times S^2)\), whenever \(r \geq \max\{b_3(\pi), 6\} \).

As shown in the paper [4], this is a consequence of our main result:

**Theorem 11.2.** Let \(\pi\) be a discrete group of type \(F_3\) with \(\text{cd}\ \pi \leq 3\) satisfying the properties (W-AA). If \(M\) and \(N\) are closed, oriented, spin\(^+\), \(\text{TOP} 4\)-manifolds with fundamental group \(\pi\), then any isometry between the quadratic 2-types of \(M\) and \(N\) is stably realized by an \(s\)-cobordism between \(M \# r(S^2 \times S^2)\) and \(N \# r(S^2 \times S^2)\), for \(r \geq \max\{b_3(\pi), s_{r_3}(\pi)\}\).

**Remark 3.1.** Note that we obtain \(s\)-cobordisms after connected sum with a *uniformly bounded* number of copies of \(S^2 \times S^2\), where the bound depends only on the fundamental group. In contrast, “stable classification” results (such as [4, Theorem B]) might require an unbounded number of stabilizations as the manifolds \(M\) and \(N\) vary.

**References**


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