TOPOLOGICAL 4-MANIFOLDS WITH RIGHT-ANGLED ARTIN FUNDAMENTAL GROUPS

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Abstract. We classify closed, topological spin 4-manifolds with a right-angled Artin group π of cohomological dimension ≤ 3 as fundamental group (up to s-cobordism), after stabilization by connected sum with at most $b_3(\pi)$ copies of $S^2 \times S^2$.

1. Introduction

Freedman [13] classified simply connected, topological 4-manifolds up to homeomorphism, and established a framework for studying non-simply connected 4-manifolds.

For a non-simply connected 4-manifold M, the basic homotopy invariants are the fundamental group $\pi := \pi_1(M)$, the second homotopy group $\pi_2(M)$, the equivariant intersection form s_M, and the first k-invariant, $k_M \in H^3(\pi; \pi_2(M))$. These invariants give the *quadratic* 2*-type*

$$Q(M) := [\pi_1(M), \pi_2(M), k_M, s_M]$$

which has been shown to determine the classification up to s-cobordism of TOP 4-manifolds with geometrically 2-dimensional fundamental groups (see [15]). For manifolds with finite fundamental groups, it is likely that additional invariants are needed (see [14]).

Question. Are closed, oriented, spin 4-manifolds with isometric quadratic 2-types and torsion-free fundamental groups always s-cobordant ?

In this paper, we study spin 4-manifolds with fundamental groups belonging to the interesting class of *right-angled Artin groups* (or RAAGs). We build on the methods of [15] and [23], but new difficulties appear since our fundamental groups have cohomological dimension > 2 in general. Recall that a right-angled Artin group is defined by a presentation associated to a finite graph Γ (see Section 2). An *i-clique* in Γ is a complete subgraph of Γ with i vertices, and we let $b_i(\pi)$ denote the number of i-cliques in Γ.

Theorem A. *Let π be a right-angled Artin group defined by a graph Γ with no 4-cliques. Suppose that M and N are closed, oriented, spin, TOP 4-manifolds with fundamental group π. Then any isometry between the quadratic 2-types of M and N is stably realized by an s-cobordism between $M \# r(S^2 \times S^2)$ and $N \# r(S^2 \times S^2)$, whenever $r \geq b_3(\pi)$.*

We will recall below the notion of (stable) *isometry* for quadratic 2-types. Our results actually cover the large class of fundamental groups with cohomological dimension $\operatorname{cd} \pi \leq 3$ which satisfy certain assembly map conditions (see Definition 11.1 and Theorem 11.2).

Date: Feb. 5, 2015.

Research partially supported by NSERC Discovery Grant A4000.

This extends the results of [15] which handled the class of geometrically 2-dimensional groups. If the fundamental group π happens to be a "good" group for topological surgery [10], [11], [12], then we show that any isometry can be realized by a homeomorphism.

For a right-angled Artin group π these assembly conditions hold [1], and π has cohomological dimension $\operatorname{cd}\pi \leq 3$ if and only if the defining graph has no 4-cliques (or equivalently if $H_4(\pi;\mathbb{Z}) = 0$). Thus Theorem A applies to an infinite number of right-angled Artin groups with cohomological dimension $\operatorname{cd}\pi = 3$, including $\pi = \mathbb{Z}^3$ (see the examples following Proposition 3.6).

Here is a brief summary of the definitions in [14] and [15].

Definition 1.1. For an oriented 4-manifold M, the *equivariant intersection form* is the triple $(\pi_1(M, x_0), \pi_2(M, x_0), s_M)$, where $x_0 \in M$ is a base point and

$$s_M\colon \pi_2(M, x_0) \otimes_{\mathbb{Z}} \pi_2(M, x_0) \to \Lambda,$$

where $\Lambda := \mathbb{Z}[\pi_1(M, x_0)]$. This pairing is derived from the cup product on $H^2_c(\widetilde{M};\mathbb{Z})$, where $\widetilde{M}$ is the universal cover of M; we identify $H^2_c(\widetilde{M};\mathbb{Z})$ with $\pi_2(M)$ via Poincaré duality and the Hurewicz Theorem, and so s_M is defined by

$$s_M(x, y) = \sum_{g\in\pi} \varepsilon_0(\tilde{x} \cup \tilde{y}g^{-1}) \cdot g \in \mathbb{Z}[\pi],$$

where $\tilde{x}, \tilde{y} \in H^2_c(\widetilde{M};\mathbb{Z})$ are the images of $x, y \in \pi_2(M)$ under the composite isomorphism $\pi_2(M) \to H_2(\widetilde{M};\mathbb{Z}) \to H^2_c(\widetilde{M};\mathbb{Z})$ and ε_0 is given by $\varepsilon_0\colon H^4_c(\widetilde{M};\mathbb{Z}) \to H_0(\widetilde{M};\mathbb{Z}) = \mathbb{Z}$.

Unless otherwise mentioned, our modules are *right* Λ-modules. This pairing is Λ-hermitian, in the sense that for all $\lambda \in \Lambda$, we have

$$s_M(x, y \cdot \lambda) = s_M(x, y) \cdot \lambda \quad \text{and} \quad s_M(y, x) = \overline{s_M(x, y)},$$

where $\lambda \mapsto \bar{\lambda}$ is the involution on Λ given by the orientation character of M. This involution is determined by $\bar{g} = g^{-1}$ for $g \in \pi_1(M, x_0)$. For later reference, we note that when M is spin the term $\varepsilon_0(\tilde{x}, \tilde{y}) \equiv 0 \pmod 2$, so s_M is an *even* hermitian form.

Definition 1.2. An *isometry* between quadratic 2-types $Q(M)$ and $Q(M')$ is a pair (α, β), where $\alpha\colon \pi_1(M, x_0) \to \pi_1(M', x_0')$ is an isomorphism of fundamental groups and

$$\beta\colon (\pi_2(M, x_0), s_M) \to (\pi_2(M', x_0'), s_{M'})$$

is an α-invariant isometry of the equivariant intersection forms, such that $(\alpha^*, \beta_*^{-1})(k_{M'}) = k_M$. In addition, the following diagram

$$\begin{tikzcd}
0 \arrow[r] & H^2(\pi;\Lambda) \arrow[r] \arrow[d, equal] & H^2(M';\Lambda) \arrow[r, "\operatorname{ad} s_{M'}"] \arrow[d, "\cong"', "\beta^*"] & \operatorname{Hom}_\Lambda(H_2(M';\Lambda), \Lambda) \arrow[r] \arrow[d, "\cong"', "\beta^*"] & H^3(\pi;\Lambda) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H^2(\pi;\Lambda) \arrow[r] & H^2(M;\Lambda) \arrow[r, "\operatorname{ad} s_M"] & \operatorname{Hom}_\Lambda(H_2(M;\Lambda), \Lambda) \arrow[r] & H^3(\pi;\Lambda) \arrow[r] & 0
\end{tikzcd}$$

arising from the universal coefficient spectral sequence *commutes*, with maps induced by β after identifying $\pi := \pi_1(M, x_0) \cong \pi_1(M', x_0')$ via α. By a *stable* isometry, we mean an isometry of quadratic 2-types after adding a hyperbolic form $H(\Lambda^r)$ to both sides.

We will assume throughout that our manifolds are connected, so that a change of base points leads to isometric intersection forms. For this reason, we will omit the base points from the notation.

Remark 1.3. We restrict to *spin* 4-manifolds for simplicity, but expect that analogous results hold by including the w_2-type in the data (see [15]). To shorten notation in later sections, we will let $\Omega_*^{Spin}(B)$ denote the *topological* spin bordism group of a space B. Note that the Kirby-Siebenmann invariant [20] is determined by $\operatorname{sign}(M) \pmod{16}$ for spin manifolds.

2. Cohomology of right-angled Artin groups

A *right-angled Artin group* π is a finitely generated group whose relators consist solely of commutators between generators. Right-angled Artin groups are also called *graph groups* since each generator of π can be represented by a vertex of a graph $\Gamma = \Gamma(\pi)$, and pairs of commuting generators in π are represented by edges in Γ between the corresponding vertices.

If π has a presentation with g generators and r relators, we construct a handlebody with fundamental group π using one 0-handle, g 1-handles, and r 2-handles attached to reflect the relations of π. In the case that π is a right-angled Artin group, this handlebody is homotopy equivalent to the 2-skeleton K of a standard classifying space for π, known as the *Salvetti complex* (see Charney [4, §3.6]).

The integral homology and cohomology ring of π are calculated in [19] and [5]. In [5], it is shown that the ith homology group and ith cohomology group are both isomorphic to the ith group of cellular chains of the Salvetti complex, and thus are free abelian groups. In fact, the rank $b_i(\pi)$ of $H_i(\pi; \mathbb{Z})$ is equal to the number of i-cliques (complete subgraphs on i vertices) in the defining graph Γ for π. Thus the (co)homological dimension $\operatorname{cd}\pi$ of a right-angled Artin group π equals the maximum number of i-cliques in Γ.

In [18], Jensen and Meier calculate the cohomology of right-angled Artin groups with *group ring coefficients* using a simplicial complex $\hat{\Gamma}$ induced from the defining graph Γ of π. In [7], Davis and Okun give a different formulation of the same theorem. We first give some necessary definitions before the statement of their results.

Let Γ be a simplicial graph. Then the *flag* complex $\hat{\Gamma}$ generated by Γ is the minimal simplicial complex in which every complete subgraph in Γ spans a simplex. The *link* $\operatorname{Lk}(\sigma)$ of a simplex σ in $\hat{\Gamma}$ is the collection of simplices $\tau \in \hat{\Gamma}$ disjoint from σ, such that σ and τ are sub-complexes of a higher dimensional simplex in $\hat{\Gamma}$. By definition, the link of the empty simplex is the entire flag complex $\hat{\Gamma}$ and by convention, $\dim \emptyset = -1$.

Definition 2.1. For a simplex $\sigma \in \hat{\Gamma}$, we define the subgroup $\pi_\sigma \leq \pi$ to be the right-angled Artin group generated by the subgraph of Γ spanned by the vertices of σ. By convention, $\pi_\emptyset = 1$.

In general, an induced subgraph of Γ defines a subgroup of π which is also a right-angled Artin group, but in this case, since simplices in $\hat{\Gamma}$ are in bijection with complete subgraphs in Γ, we see that π_σ is a free abelian group of rank equal to $\dim \sigma + 1$.

Theorem 2.2 (Jensen-Meier [18], Davis-Okun [7, Theorem 3.3])**.** *Let Γ be the defining graph for a right-angled Artin group π, and let $\mathcal{S}$ be the set of simplices in the induced flag complex $\hat{\Gamma}$. There is a spectral sequence converging to $H^*(\pi;\Lambda)$ whose associated graded groups are given by*

$$\operatorname{Gr} H^*(\pi;\Lambda) = \bigoplus_{\sigma\in\mathcal{S}} \left(\widetilde{H}^{*-\dim\sigma-2}(\operatorname{Lk}(\sigma))\otimes\mathbb{Z}[\pi/\pi_\sigma]\right)$$

when $\hat{\Gamma}$ is not a single simplex. If $\hat{\Gamma}$ is a single simplex, then $\pi\cong\mathbb{Z}^n$, and $H^n(\pi;\Lambda)=\mathbb{Z}$ is the only non-vanishing cohomology group.

Example 2.3. We describe the associated graded group for $H^2(\pi;\Lambda)$ using Theorem 2.2. In the corresponding filtration, the empty simplex is the bottom of the filtration, so we have the following filtration subgroups (indexed so that the top index matches the cohomology degree in question):

$$\begin{tikzcd} 0 \arrow[r, hook] & \mathcal{F}_0 \arrow[r, hook] \arrow[d, equal] & \mathcal{F}_1 \arrow[r, hook] \arrow[d] & \mathcal{F}_2 = H^2(\pi;\Lambda) \arrow[d] \\ & \mathcal{F}_0 & \mathcal{F}_1/\mathcal{F}_0 & \mathcal{F}_2/\mathcal{F}_1 \end{tikzcd}$$

Note that $\widetilde{H}^{-1}(\emptyset)=\mathbb{Z}$ is the only non-trivial cohomology group of the empty simplex. The filtration quotients are given by:

(i) $\mathcal{F}_0 \cong H^1(\hat{\Gamma})\otimes\mathbb{Z}[\pi]$,

(ii) $\mathcal{F}_1/\mathcal{F}_0 \cong \bigoplus_{\sigma\in Vert(\hat{\Gamma})}(\widetilde{H}^0(\operatorname{Lk}(\sigma))\otimes\mathbb{Z}[\pi/\pi_\sigma])$, and

(iii) $\mathcal{F}_2/\mathcal{F}_1 \cong \bigoplus_{\sigma\in Edge(\hat{\Gamma})}(\widetilde{H}^{-1}(\operatorname{Lk}(\sigma))\otimes\mathbb{Z}[\pi/\pi_\sigma])$.

The Λ-action on the right-hand side is given by the identity on the integral cohomology of the links, tensored over $\mathbb{Z}$ with the natural action on the induced modules $\mathbb{Z}[\pi/\pi_\sigma]$.

3. Right-angled Artin groups with tame cohomology

In this section we investigate certain group cohomological conditions arising in the study of 4-manifolds. Recall that finitely presented groups have either 1, 2, or infinitely many ends, where the number of ends of a group G is equal to the number of ends of any Cayley graph of G. In [24], Stallings proved that a finitely presented group G has more than one end if and only if it decomposes as a non-trivial amalgamated product or an HNN extension over a finite subgroup.

Since a right-angled Artin group π is infinite and torsion-free, π will have more than one end if and only if it decomposes as a free product (see Dunwoody [9]). Thus right-angled Artin groups are 1-ended, or equivalently $H^1(\pi;\Lambda)=0$, if and only if their defining graphs are connected (see [4, §3.7]). Note that the group cohomology $H^i(\pi;\Lambda)$ of any group π with group ring coefficients is also a Λ-module since Λ is a Λ-Λ bimodule. We say that A is a *torsion* Λ-module whenever $\operatorname{Hom}_\Lambda(A,\Lambda)=0$.

Definition 3.1. A finitely presented group π has *tame cohomology* if the following conditions hold:

(i) $\operatorname{Hom}_\Lambda(H^2(\pi;\Lambda),\Lambda) = 0$
(ii) $\operatorname{Hom}_\Lambda(H^3(\pi;\Lambda),\Lambda) = 0$
(iii) $\operatorname{Ext}^1_\Lambda(H^3(\pi;\Lambda),\Lambda) = 0$

In Section 6 we show that when the fundamental group $\pi_1(M)$ of a closed 4-manifold M has tame cohomology, or in the special case when $\operatorname{cd}\pi_1(M) \leq 3$, then the dual of its equivariant intersection form s_M is non-singular. We expect this property to have a key role in extending our classification results to right-angled Artin groups of higher cohomological dimension.

Remark 3.2. In the first version of this paper, we asked whether all right-angled Artin groups have tame cohomology. In response, Jonathan Hillman kindly provided an example: $H^2(\pi;\Lambda) \cong \Lambda$, for $\pi = (\mathbb{Z}^2 * \mathbb{Z}^2)^2$. However, we do not know the answer for right-angled Artin groups with no 4-cliques in the associated graph.

We now apply the results of Jensen-Meier and Davis-Okun cited in Theorem 2.2 to study the low-dimensional cohomology of right-angled Artin groups. For any Λ-module A, we use the notation $A^* = \operatorname{Hom}_\Lambda(A,\Lambda)$ for the *dual module*. We begin with the following observation. Let $\pi_{n,m}$ denote the free product of n copies of $\mathbb{Z}^2$ and m copies of $\mathbb{Z}$.

Lemma 3.3. *If π is a right-angled Artin group, then there exists a subgroup $\pi_{n.m} \leq \pi$ such that $H^1(\pi;\Lambda) \cong H^1(\pi_{n.m};\operatorname{Res}\Lambda)$ under restriction. If π is a finitely generated free group, then $H^1(\pi;\Lambda)^* = 0$.*

Proof. If the defining graph $\Gamma = \Gamma(\pi)$ is connected, then $H^1(\pi;\Lambda) = 0$ and there is nothing to prove (take the empty free product). Otherwise, we can write Γ as a disjoint union of non-empty connected graphs, and let $\Gamma(n,m) \subset \Gamma$ denote a subgraph consisting of all the m singleton vertex components together with one edge from each of the other n connected components. The right-angled Artin group defined by $\Gamma(n.m)$ is isomorphic to $\pi_{n.m}$. Since the filtration terms for computing $H^1(\pi;\Lambda)$ consist of $\mathcal{F}_0 \cong \widetilde{H}^0(\hat{\Gamma}) \otimes \mathbb{Z}[\pi]$ and

$$\mathcal{F}_1/\mathcal{F}_0 \cong \bigoplus_{\sigma \in Vert(\hat{\Gamma})} (\widetilde{H}^{-1}(\operatorname{Lk}(\sigma)) \otimes \mathbb{Z}[\pi/\pi_\sigma]),$$

it follows that $H^1(\pi;\Lambda)$ is mapped isomorphically under restriction to $H^1(\pi_{n.m};\operatorname{Res}\Lambda)$.

Suppose now that $n = 0$ so that π is a free group of rank m. Let N be the connected sum of m copies of $S^1 \times S^3$. An easy argument using the universal coefficient spectral sequence now shows that $H^1(\pi;\Lambda)^* = 0$. $\square$

The next result gives an inductive criterion for the cohomology groups to be torsion Λ-modules.

Lemma 3.4. *If $H^i(\hat{\Gamma}) = 0$ then* $\operatorname{Hom}_\Lambda(H^{i+1}(\pi;\Lambda),\Lambda) = 0$.

Proof. The case $i = 0$ is trivial, so consider the case when $i = 1$. The filtrations for $H^2(\pi;\Lambda)$ are given in Example 2.3, with $\mathcal{F}_0 = 0$ by the assumption that $H^1(\hat{\Gamma}) = 0$. Consequently, $\mathcal{F}_1 = \mathcal{F}_1/\mathcal{F}_0$. By dualizing the short exact sequence

$$0 \to \mathcal{F}_1/\mathcal{F}_0 \to \mathcal{F}_2 \to \mathcal{F}_2/\mathcal{F}_1 \to 0,$$

we get the exact sequence

$$0 \to (\mathcal{F}_2/\mathcal{F}_1)^* \to (\mathcal{F}_2)^* \to (\mathcal{F}_1/\mathcal{F}_0)^* \to \operatorname{Ext}^1_\Lambda(\mathcal{F}_2/\mathcal{F}_1, \Lambda) \to \cdots .$$

We claim $(\mathcal{F}_1/\mathcal{F}_0)^* = 0$ and $(\mathcal{F}_2/\mathcal{F}_1)^* = 0$: Hom splits over a finite direct sum, and any nonzero summands in $\mathcal{F}_1/\mathcal{F}_0$ or $\mathcal{F}_2/\mathcal{F}_1$ will have torsion elements in the tensor product, and are killed by the Hom functor. For example,

$$\begin{aligned}(\mathcal{F}_1/\mathcal{F}_0)^* &= \operatorname{Hom}_\Lambda\left(\bigoplus\nolimits_{\sigma\in Vert(\hat{\Gamma})} \widetilde{H}^0(\operatorname{Lk}(\sigma))\otimes \mathbb{Z}[\pi/\pi_\sigma], \Lambda\right)\\ &= \bigoplus\nolimits_{\sigma\in Vert(\hat{\Gamma})} \operatorname{Hom}_\Lambda(\widetilde{H}^0(\operatorname{Lk}(\sigma))\otimes \mathbb{Z}[\pi/\pi_\sigma], \Lambda),\end{aligned}$$

where the direct sum is taken over all vertices σ in $\hat{\Gamma}$. By [2, Corollary 2.8.4] (the Eckmann-Shapiro Lemma),

$$\operatorname{Hom}_{\mathbb{Z}[\pi]}(\mathbb{Z}[\pi/\pi_\sigma], \mathbb{Z}[\pi]) \cong \operatorname{Hom}_{\mathbb{Z}[\pi_\sigma]}(\mathbb{Z}, \operatorname{Res}^\pi_{\pi_\sigma}(\mathbb{Z}[\pi])) \tag{3.4}$$

is zero as long as $\pi_\sigma \neq 1$, or equivalently, σ is not the empty simplex. Thus from the long exact sequence, we see $(\mathcal{F}_2)^* = H^2(\pi;\Lambda)^* = 0$.

For any i, the assumption that $H^i(\hat{\Gamma}) = 0$ implies that the bottom filtration term $\mathcal{F}_0 = 0$. By the argument used in (3.4), the subquotients $\mathcal{F}_k/\mathcal{F}_{k-1}$, $1 \le k \le i+1$, in the filtration for $H^{i+1}(\pi;\Lambda)$ are all torsion modules.

$$\begin{tikzcd} 0 = \mathcal{F}_0 \arrow[r, hook] & \mathcal{F}_1 \arrow[r, hook] \arrow[d, two heads] & \cdots \arrow[r, hook] & \mathcal{F}_{i-1} \arrow[r, hook] \arrow[d, two heads] & \mathcal{F}_i \arrow[r, hook] \arrow[d, two heads] & \mathcal{F}_{i+1} = H^{i+1}(\pi;\Lambda) \arrow[d, two heads] \\ & \mathcal{F}_1/\mathcal{F}_0 & & \mathcal{F}_{i-1}/\mathcal{F}_{i-2} & \mathcal{F}_i/\mathcal{F}_{i-1} & \mathcal{F}_{i+1}/\mathcal{F}_i \end{tikzcd}$$

By dualizing the short exact sequences

$$0 \to \mathcal{F}_{k-1} \to \mathcal{F}_k \to \mathcal{F}_k/\mathcal{F}_{k-1} \to 0,$$

we see that $(\mathcal{F}_{i+1})^* = H^{i+1}(\pi;\Lambda)^* = 0$. □

Lemma 3.5. *Let π be a right-angled Artin group with defining graph Γ. Suppose that the induced flag complex $\hat{\Gamma}$ is 2-connected and that $H^1(\operatorname{Lk}(\sigma))$ is zero for every vertex σ. Then π has tame cohomology.*

Proof. Since $\hat{\Gamma}$ is 2-connected, $H^i(\hat{\Gamma}) = 0$ for $i = 1, 2$. By Lemma 3.4, $H^{i+1}(\pi;\Lambda)^* = 0$ for $i = 2, 3$. Thus the first two conditions for π to have tame cohomology are satisfied. For the last condition, we claim that $\operatorname{Ext}^1_\Lambda(H^3(\pi;\Lambda), \Lambda) = 0$. The filtration quotients are described below (with $\mathcal{F}_3 = H^3(\pi;\Lambda)$):

(i) $\mathcal{F}_0 = H^2(\hat{\Gamma}) \otimes \mathbb{Z}[\pi]$,

(ii) $\mathcal{F}_1/\mathcal{F}_0$ is a direct sum of $H^1(\mathrm{Lk}(\sigma)) \otimes \mathbb{Z}[\pi/\pi_\sigma]$ over vertices,
(iii) $\mathcal{F}_2/\mathcal{F}_1$ is a direct sum of $\widetilde{H}^0(\mathrm{Lk}(\sigma)) \otimes \mathbb{Z}[\pi/\pi_\sigma]$ over edges, and
(iv) $\mathcal{F}_3/\mathcal{F}_2$ is a direct sum of $\widetilde{H}^{-1}(\mathrm{Lk}(\sigma)) \otimes \mathbb{Z}[\pi/\pi_\sigma]$ over faces.

Consider the dual of the the short exact sequence $0 \to \mathcal{F}_2 \to \mathcal{F}_3 \to \mathcal{F}_3/\mathcal{F}_2 \to 0$, recalling that $(\mathcal{F}_2)^* = 0$:

$$0 \to \mathrm{Ext}^1_\Lambda(\mathcal{F}_3/\mathcal{F}_2, \Lambda) \to \mathrm{Ext}^1_\Lambda(\mathcal{F}_3, \Lambda) \to \mathrm{Ext}^1_\Lambda(\mathcal{F}_2, \Lambda) \to \cdots$$

Since $H^1(\mathrm{Lk}(\sigma)) = 0$ for every vertex σ, $\mathcal{F}_1/\mathcal{F}_0 = 0$. This implies that $\mathcal{F}_1 = 0$ and so $\mathcal{F}_2 \cong \mathcal{F}_2/\mathcal{F}_1$. The functor Ext^1_Λ splits over a finite direct sum, so the first and third nonzero terms in the above sequence are of the form

$$\mathrm{Ext}^1_\Lambda(\mathcal{F}_{i+1}/\mathcal{F}_i, \Lambda) = \bigoplus \widetilde{H}^{1-i}(\mathrm{Lk}(\sigma)) \otimes \mathrm{Ext}^1_\Lambda(\mathbb{Z}[\pi/\pi_\sigma], \Lambda),$$

and we claim that the Ext term on the right-hand side is zero, hence $\mathrm{Ext}^1_\Lambda(\mathcal{F}_3, \Lambda) = 0$. The simplices in question are either faces (involved in the first term of the sequence) or edges (involved in the third term). By the Eckmann-Shapiro Lemma,

$$\mathrm{Ext}^1_{\mathbb{Z}[\pi]}(\mathbb{Z}[\pi/\pi_\sigma], \mathbb{Z}[\pi]) \cong \mathrm{Ext}^1_{\mathbb{Z}[\pi_\sigma]}(\mathbb{Z}, \mathrm{Res}^\pi_{\pi_\sigma} \mathbb{Z}[\pi]).$$

The restriction of $\mathbb{Z}[\pi]$ to π_σ is just an infinite direct sum of $\mathbb{Z}[\pi_\sigma]$'s, and

$$\mathrm{Ext}^1_{\mathbb{Z}[\pi_\sigma]}(\mathbb{Z}, \mathbb{Z}[\pi_\sigma]) \cong H^1(\pi_\sigma; \mathbb{Z}[\pi_\sigma]) = 0$$

when σ is an edge or a face, since π_σ is a 1-ended group ($\pi_\sigma = \mathbb{Z}^2$ and $\mathbb{Z}^3$). □

The condition that π has tame cohomology is perhaps restrictive (see Remark 3.2) but we have many examples, including an algorithm that can produce infinitely many right-angled Artin groups with tame cohomology.

Proposition 3.6. *Let Γ be the simplicial graph obtained by carrying out finitely many of the following operations, starting with a single vertex:*

(i) Attach a new 1-simplex or a new 2-simplex to the previous graph Γ_0 by identifying one of its vertices with any (but only one) vertex of Γ_0.
(ii) Attach a new 2-simplex to the previous graph Γ_0 by identifying one of its 1-simplices with any (but only one) 1-simplex of Γ_0.

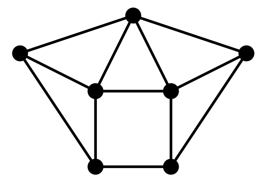
Then the right-angled Artin group defined by Γ will have tame cohomology.

Proof. At each step we add a new simplex to the previously constructed graph Γ_0, which can always be contracted to Γ_0 in its flag complex. The algorithm provided by repeating steps (i) and (ii) produces a simplicial graph Γ with contractible flag complex $\hat{\Gamma}$, which is clearly 2-connected.

Furthermore, this algorithm guarantees that $H^1(\mathrm{Lk}(\sigma)) = 0$ for the link of every vertex σ. We may then apply Lemma 3.5 to conclude that the right-angled Artin group defined by Γ has tame cohomology. □

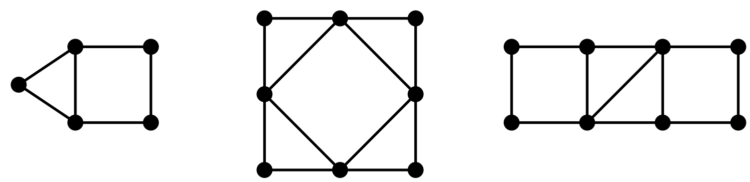
In the setting of Theorem A, we restrict attention to right-angled Artin groups with no 4-cliques in their defining graphs. It is possible that all such right-angled Artin groups have tame cohomology, but we do not yet know if the first condition is always satisfied.

Example 3.7. Does the right-angled Artin group π with the following defining graph have $H^2(\pi;\Lambda)^* = 0$?



The following examples provide graphs with non-simply connected flag complexes whose associated right-angled Artin groups have $H^2(\pi;\Lambda)^* = 0$. This shows at least that Lemma 3.5 is not the best possible result.

Example 3.8. The right-angled Artin groups defined by the following graphs all have $H^2(\pi;\Lambda)^* = 0$.



(i) (ii) (iii)

The argument involves dualizing the short exact sequences derived from the Mayer-Vietoris sequence for π expressed as the amalgamated product of two subgroups. We denote by π_i the right-angled Artin group associated to a graph Γ_i. We first remark that if Γ is the union of two subgraphs Γ_1 and Γ_2 which intersect at Γ_0, then π is the amalgamated product of π_1 and π_2 over π_0: this can be seen by comparing the presentations for π_1 and π_2 with the relations of π_0. Secondly, if Γ_i is the induced subgraph on a collection of vertices from Γ, the groups π_i are subgroups of π.

Proposition 3.9. *If $\Gamma = \Gamma(\pi)$ satisfies the following conditions, then $H^2(\pi;\Lambda)^* = 0$.*

(i) Γ is the union of two graphs Γ_1 and Γ_2 which intersect in Γ_0, and all three subgraphs induce subgroups of π.

(ii) Γ_1 and Γ_2 are connected and have 2-connected flag complexes.

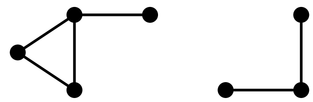
(iii) Γ_0 is a disjoint union of vertices.

Proof. Since Γ_1 and Γ_2 are connected and Γ_0 is 0-dimensional, we get the short exact sequence

$$0 \to H^1(\pi_0;\Lambda) \to H^2(\pi;\Lambda) \to H^2(\pi_1;\Lambda) \oplus H^2(\pi_2;\Lambda) \to 0$$

from Mayer-Vietoris. Since $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are 2-connected, $H^2(\pi_1;\Lambda)^*$ and $H^2(\pi_2;\Lambda)^*$ are both zero by Lemma 3.4. Additionally, $H^1(\pi_0;\Lambda)^* = 0$ by Lemma 3.3. Thus dualizing the short exact sequence above gives the desired result. □

We can apply this proposition to the graph Γ in Example 3.8 (i) with the following two subgraphs below, to be denoted by Γ_1 and Γ_2, respectively:



Together, their union $\Gamma_1 \cup \Gamma_2$ is Γ and their intersection Γ_0 is two disjoint vertices. Both Γ_1 and Γ_2 are connected and have 2-connected flag complexes. Thus Proposition 3.9 shows the graph in Example 3.8 (i) satisfies the first condition for tame cohomology.

4. Finitely presented groups with $\operatorname{cd}\pi \leq 3$

In this section we will show that for a right-angled Artin group π with no 4-cliques in its defining graph Γ, conditions (ii) and (iii) for π to have tame cohomology are satisfied. This follows from a more general result.

Proposition 4.1. *Let π be a finitely presented group with $\operatorname{cd}\pi \leq 3$. Then $H^3(\pi;\Lambda)^* = 0$ and $\operatorname{Ext}^1_\Lambda(H^3(\pi;\Lambda),\Lambda) = 0$.*

Proof. By Wall [25, Theorem E], a finitely presented group π with $\operatorname{cd}\pi \leq 3$ has geometric dimension ≤ 3. In other words, there exists a 3-dimensional, finite, aspherical complex L with $\pi_1(L) = \pi$. Let $N = N(L)$ be an 8-dimensional thickening of L, and let M be the double of N. The long exact sequence of the pair $(N, \partial N)$ with Λ-coefficients gives isomorphisms $H_{i-1}(\partial N) \to H_{i-1}(N)$ for $i \leq 4$, since $H_i(N,\partial N) \cong H^{8-i}(N) \cong H^{8-i}(K)$. Furthermore, $H_{i-1}(N) = 0$ for $i > 1$. Then by the Mayer-Vietoris sequence, $H_i(M;\Lambda) = 0$ for $1 < i \leq 4$, and $H_1(M;\Lambda) = 0$ since $\widetilde{M}$ is simply connected.

We use the universal coefficient spectral sequence which has E_2-page

$$E_2^{p,q} = \operatorname{Ext}^p_\Lambda(H_q(M;\Lambda),\Lambda).$$

Since, $H_q(M;\Lambda) = 0$ for $q = 1,\dots,4$, the $E_2^{p,q}$-terms are all zero for $q = 1,\dots,4$, and since $H^p(\pi;\Lambda) = 0$ for all $p > 3$, the $E_2^{p,q}$-terms are zero for $p > 3$. Thus no differentials affect the $E_2^{0,5}$- and $E_2^{1,5}$-terms, which are $E_2^{0,5} \cong H^1(\pi;\Lambda)^*$ and $E_2^{1,5} \cong \operatorname{Ext}^1_\Lambda(H^3(\pi;\Lambda),\Lambda)$ via the isomorphisms $H_5(M;\Lambda) \cong H^3(M;\Lambda) \cong H^3(\pi;\Lambda)$. Since $H^5(M;\Lambda) \cong H_3(M;\Lambda) = 0$ and $H^6(M;\Lambda) \cong H_2(M;\Lambda) = 0$, these two terms must be zero. □

In the universal coefficient spectral sequence computing $H^*(M;\Lambda)$, since $H^7(M;\Lambda) = H_1(M;\Lambda) = 0$, the differential

$$d_2\colon E_2^{0,6} = H^2(\pi;\Lambda)^* \xrightarrow{\cong} E_2^{2,5} = \operatorname{Ext}^2_\Lambda(H^3(\pi;\Lambda),\Lambda) \tag{4.2}$$

is an isomorphism. In addition, $\operatorname{Ext}^1_\Lambda(H^2(\pi;\Lambda))$ injects into $\operatorname{Ext}^3_\Lambda(H^3(\pi;\Lambda),\Lambda)$.

Consider the universal coefficient sequence

$$0 \to H^2(\pi;\Lambda) \to H^2(K;\Lambda) \to \pi_2(K)^* \to H^3(\pi;\Lambda) \to 0 \tag{4.3}$$

for the 2-skeleton K of L, where the last nonzero map is a surjection since $H^3(K;\Lambda) = 0$. This sequence can be obtained by splicing together the short exact sequences:

$$0 \to H^2(\pi;\Lambda) \to H^2(K;\Lambda) \to V \to 0 \quad \text{and} \quad 0 \to V \to \pi_2(K)^* \to H^3(\pi;\Lambda) \to 0.$$

After dualizing, the associated long exact cohomology sequences give two connecting homomorphisms $\delta_1\colon H^2(\pi;\Lambda)^* \to \operatorname{Ext}^1(V,\Lambda)$ and $\delta_2\colon \operatorname{Ext}^1(V,\Lambda) \to \operatorname{Ext}^1(H^3(\pi;\Lambda),\Lambda)$.

Remark 4.4. The spectral sequence above, combined with Lemma 3.3, shows that $H^1(\pi;\Lambda)^* \neq 0$ for a right-angled Artin group π if and only if its defining graph is disconnected and contains at least one edge. The simplest example is $\pi = \mathbb{Z}^2 * \mathbb{Z}$. For the groups $\pi = \pi_{n.m}$, we have an exact sequence

$$0 \to H^1(\pi;\Lambda)^* \to \operatorname{Ext}^2(H^2(\pi;\Lambda),\Lambda) \to \mathbb{Z} \to \operatorname{Ext}^1(H^1(\pi;\Lambda),\Lambda) \to 0.$$

A further calculation with the filtration sequences shows that $\operatorname{Ext}^2(H^2(\pi;\Lambda),\Lambda)$ is a direct sum of n copies of $\operatorname{Ext}^2(\mathbb{Z}[\pi/\pi_\sigma],\Lambda)$, where σ is an edge. If $n > 0$ this is enough to show that $H^1(\pi;\Lambda)^* \neq 0$. If $m = 0$ as well, then the exact sequence above reduces to

$$0 \to H^1(\pi;\Lambda)^* \to (\mathbb{Z}[\pi/\pi_\sigma])^n \to \mathbb{Z} \to 0,$$

and $H^1(\pi;\Lambda)^* \cong \Lambda^{n-1}$. For each of the n edges, $\pi_\sigma \cong \mathbb{Z}^2$.

Lemma 4.5. *Let π be a finitely presented group with $\operatorname{cd}\pi \leq 3$. The differential (4.2) induces a natural isomorphism $d_2\colon H^2(\pi;\Lambda)^* \cong \operatorname{Ext}^2_\Lambda(H^3(\pi;\Lambda),\Lambda)$, where $d_2 = \delta_2 \circ \delta_1$.*

Proof. The description of the differential d_2 follows from [3, Section 3.4]. □

Remark 4.6. These results, which apply to all right-angled Artin groups with $H_4(\pi;\mathbb{Z}) = 0$, are sharper in this special case than our conclusions about tame cohomology from Section 3. For example, we no longer require the associated flag complex $\hat{\Gamma}$ to be simply connected.

5. The minimal model for π

For any finitely presented group π, one can construct a 4-manifold M with fundamental group π by doubling a thickening of a finite 2-complex K with $\pi_1(K) = \pi$. If K has minimal Euler characteristic, then the double of K will have minimal Euler characteristic over all double constructions.

If π is a right-angled Artin group, we can take K to be the 2-skeleton of the Salvetti complex mentioned in Section 2. The Salvetti complex has minimal Euler characteristic over all possible $K(\pi, 1)$ since its chain complex gives a minimal resolution for π.

Definition 5.1. We say that a 4-manifold X is *minimal for* π if its Euler characteristic is minimal over all closed, oriented 4-manifolds with fundamental group π.

The Euler characteristic of a minimal 4-manifold for π is the Hausmann-Weinberger invariant [16]. It has been determined for free abelian groups by Kirk and Livingston [22], but is still unknown for most classes of finitely presented groups.

Theorem 5.2 (Hildum [17, Theorem 1.2])**.** *Let π be a right-angled Artin group with $H_4(\pi;\mathbb{Z}) = 0$. Let K be the 2-skeleton of the Salvetti complex with fundamental group π, and let $N(K)$ denote a spin 4-dimensional thickening of K. Then the double $M_0 := N(K) \cup N(K)$ is a minimal spin 4-manifold for π.*

The double construction allows us to determine the structure of $\pi_2(M_0)$ as a Λ-module.

Lemma 5.3. *Let K be a finite 2-complex with fundamental group π, and let $N(K)$ denote a 4-dimensional thickening of K. Then*

$$\pi_2(M_0) = H_2(K;\Lambda) \oplus H^2(K;\Lambda)$$

as a Λ-module, for the double $M_0 = N(K) \cup N(K)$.

Proof. Let $N = N(K)$ and notice that $M_0 = \partial(N \times I)$. We start with the long exact sequence in homology for the pair (M_0, N):

$$\cdots \to H_3(M_0, N;\Lambda) \to H_2(N;\Lambda) \to H_2(M_0;\Lambda) \to H_2(M_0, N;\Lambda) \to H_1(N;\Lambda) \to \cdots$$

The inclusion of N into M_0 induces a split injective map $H_i(N;\Lambda) \to H_i(M_0;\Lambda)$ in every dimension. Thus the maps from $H_i(M_0, N;\Lambda)$ to $H_i(N;\Lambda)$ are all zero maps. In addition, using excision properties as well as Poincaré-Lefschetz duality, we have the isomorphisms $H_2(M_0, N;\Lambda) \cong H_2(N, \partial N;\Lambda) \cong H^2(N;\Lambda)$. This gives the split short exact sequence

$$0 \to H_2(K;\Lambda) \to H_2(M_0;\Lambda) \to H^2(K;\Lambda) \to 0.$$

This, along with the Hurewicz isomorphism $H_2(M_0;\Lambda) \cong H_2(\widetilde{M_0};\mathbb{Z}) \cong \pi_2(M_0)$ yields the desired result. □

Remark 5.4. The split short exact sequence exists for any coefficient module, not only for the group ring Λ. For π a right-angled Artin group we can take K to be the 2-skeleton of the Salvetti complex.

We need more information about the summands of $\pi_2(M_0)$ in our setting.

Lemma 5.5. *Let π be a finitely presented group with $\operatorname{cd}\pi \leq 3$. If K is a finite 2-complex with fundamental group π, then $H_2(K;\Lambda)$ is stably free as a Λ-module.*

Proof. Let X be an aspherical 3-complex model for $K(\pi, 1)$, and let K be the 2-skeleton of X. Consider the chain complex $C_*(X) := C_*(X;\Lambda)$ with Λ-module coefficients:

$$0 \longrightarrow C_3(X) \longrightarrow C_2(X) \longrightarrow C_1(X) \longrightarrow C_0(X) \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{5.6}$$

This chain complex is exact since $\widetilde{X}$ is contractible, and so $H_2(K;\Lambda) \cong Z_2(K) = Z_2(X) = C_3(X)$. Since $C_i(X) \cong \Lambda^{b_i(\pi)}$, we have $H_2(K;\Lambda) \cong \Lambda^{b_3(\pi)}$, where $b_i(\pi)$ denotes the number of i-cells in X. □

Remark 5.7. The k-invariant of the double $M_0 = N(K) \cup N(K)$ is the image (induced by the inclusion) of the k-invariant of K.

Corollary 5.8. *Let π be a right-angled Artin group with $H_4(\pi;\mathbb{Z}) = 0$. Then*

$$\operatorname{rk}_{\mathbb{Z}}(\pi_2(M_0) \otimes_\Lambda \mathbb{Z}) = b_2(\pi) + b_3(\pi).$$

Proof. The spectral sequence converging to $H_*(M_0)$ yields the following exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & H_3(\pi) \arrow[r, "d^3"] \arrow[d, "\cong"] & \pi_2(M_0) \otimes_\Lambda \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & H_2(M_0) \arrow[r] \arrow[d, "\cong"] & H_2(\pi) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathbb{Z}^{b_3(\pi)} \arrow[r] & \pi_2(M_0) \otimes_\Lambda \mathbb{Z} \arrow[r] & \mathbb{Z}^{2b_2(\pi)} \arrow[r] & \mathbb{Z}^{b_2(\pi)} \arrow[r] & 0
\end{tikzcd}$$

The second term in the exact sequence above is the $(0,2)$ term in the spectral sequence, $H_0(\pi;\pi_2(M_0)) \cong \pi_2(M_0)\otimes_\Lambda \mathbb{Z}$. By exactness, the $\mathbb{Z}$-rank of $\pi_2(M_0)\otimes_\Lambda \mathbb{Z}$ is $b_2(\pi)+b_3(\pi)$. □

There is not much we can say in general about $H^2(K;\Lambda)$ beyond the 4-term exact sequence (4.3):

$$0 \to H^2(\pi;\Lambda) \to H^2(K;\Lambda) \to \mathrm{Hom}_\Lambda(H_2(K;\Lambda),\Lambda) \to H^3(\pi;\Lambda) \to 0$$

In certain cases, we can obtain more specific calculations for $H^2(K;\Lambda)$.

Example 5.9. Consider the case of the right-angled Artin group $\pi = \mathbb{Z}^3$ in which the associated graph Γ is a 3-clique and the flag complex $\hat{\Gamma}$ is a single 2-simplex. Therefore, $H^2(\pi;\Lambda) = 0$ and $H^3(\pi;\Lambda) = \mathbb{Z}$ by Theorem 2.2. This example indicates that even though $H_2(K;\Lambda)$ is a free Λ-module, $H^2(K;\Lambda)$ may not be free. In this case, (4.3) becomes

$$0 \to H^2(K;\Lambda) \to \Lambda \to \mathbb{Z} \to 0,$$

which shows that $H^2(K;\Lambda)$ is the augmentation ideal $I(\pi)$. Thus, $\pi_2(M_0) = \Lambda \oplus I(\pi)$ for $\pi = \mathbb{Z}^3$, and the module $I(\pi)$ is not free since $H_1(\pi; I(\pi)) \cong H_2(\pi;\mathbb{Z}) \cong \mathbb{Z}^3$.

Proposition 5.10. *Let N be a closed, oriented, spin, TOP 4-manifold with fundamental group π. If $\mathrm{cd}\,\pi \le 3$, then there exists a simply connected, closed 4-manifold X such that*

$$N' \# r(S^2\times S^2) \approx M' \# s(S^2\times S^2),$$

for some $r, s \ge 0$, where $N' := N \# \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ and $M' := M_0 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2 \# X$.

Proof. The double model M_0 represents the zero bordism element in $\Omega_4^{STOP}(K(\pi,1)) = \mathbb{Z}$. Since the signature detects elements in this bordism group, it follows that $N' := N \# \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ is stably homeomorphic to $M' := M_0 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2 \# X$ for some closed, simply connected 4-manifold X with $\mathrm{sign}(N') = \mathrm{sign}(X)$. For example, if M_0 is minimal we can take X to be a further connected sum of copies of $\mathbb{C}P^2$ and $\overline{\mathbb{C}P}^2$ so that N' and M' have the same signature and Euler characteristic. It follows that $N' \# r(S^2\times S^2) \approx M' \# r(S^2\times S^2)$, for some $r \ge 0$ (see [23, Theorem C] and [10, §9.1]). □

Corollary 5.11. *Let N be a closed, oriented, spin, TOP 4-manifold with fundamental group π. If $\mathrm{cd}\,\pi \le 3$, then $\pi_2(N)$ is stably isomorphic to $\pi_2(M_0)$ as a Λ-module.*

Proof. By Proposition 5.10, the modules $\pi_2(N)$ and $\pi_2(M_0)$ become isomorphic after stabilization by direct sum with free Λ-modules. □

6. The reduced equivariant intersection form

In this section, let π denote a finitely presented group with $\mathrm{cd}\,\pi \le 3$. Let M denote a closed, oriented, spin 4-manifold with fundamental group π. As above, we can construct a model 4-manifold M_0 by taking the double of a thickening $N(K)$, where K denotes the 2-skeleton of a minimal aspherical 3-complex L with $\pi_1(L) = \pi$.

For fundamental groups of geometric dimension ≤ 2, the quotient

$$\pi_2(M)^\dagger := \pi_2(M)/H^2(\pi;\Lambda) \cong \mathrm{Hom}_\Lambda(H_2(M;\Lambda),\Lambda)$$

is stably free since $H^3(\pi;\Lambda) = 0$ (see [15, 4.4]). In that context, the "reduced" equivariant intersection form ${s_M}^\dagger$ was a non-singular form on $\pi_2(M)^\dagger$. In our setting, where π can have geometric dimension 3, the quotient $\pi_2(M)^\dagger$ is finitely generated but may not be stably free as a Λ-module. Instead, we define

$$L_M := \pi_2(M)^* = \mathrm{Hom}_\Lambda(H_2(M;\Lambda),\Lambda),$$

and obtain a non-singular hermitian form $h_M\colon L_M \times L_M \to \Lambda$, which we also call the *reduced equivariant intersection form.*

Lemma 6.1. *Let $\pi_1(M)$ be a finitely presented group with $\mathrm{cd}\,\pi \leq 3$. Then*

(i) $(\mathrm{ad}\, s_M)^*$ is an isomorphism of finitely generated, stably free Λ-modules, and
(ii) the inverse of $(\mathrm{ad}\, s_M)^*$ is the adjoint of a nonsingular hermitian form

$$h_M\colon L_M \times L_M \to \Lambda,$$

on $L_M = \pi_2(M)^*$.

Proof. By Corollary 5.11, we may assume that $M = M_0$ is our model manifold with the given fundamental group. Consider the 4-term exact sequence (4.3) for the 2-skeleton K of an aspherical 3-complex:

$$0 \to H^2(\pi;\Lambda) \to H^2(K;\Lambda) \to \mathrm{Hom}_\Lambda(H_2(K;\Lambda),\Lambda) \to H^3(\pi;\Lambda) \to 0$$

The third nonzero term is a free Λ-module, since $H_2(K;\Lambda) \cong \Lambda^{b_3(\pi)}$. For the following argument, we define $F := \mathrm{Hom}_\Lambda(H_2(K;\Lambda),\Lambda) \cong \Lambda^{b_3(\pi)}$. Recall from Section 4 that the above sequence is obtained by splicing together

$$0 \to H^2(\pi;\Lambda) \to H^2(K;\Lambda) \to V \to 0$$

and

$$0 \to V \to F \to H^3(\pi;\Lambda) \to 0,$$

where $V := \mathrm{coker}(H^2(\pi;\Lambda) \to H^2(K;\Lambda)) = \ker(F \to H^3(\pi;\Lambda))$. Taking the dual of both short exact sequences yields

$$0 \to V^* \to H^2(K;\Lambda)^* \to H^2(\pi;\Lambda)^* \xrightarrow{\delta_1} \mathrm{Ext}^1(V,\Lambda) \to \cdots$$

and

$$0 \to H^3(\pi;\Lambda)^* \to F^* \to V^* \xrightarrow{\delta_2} \mathrm{Ext}^1_\Lambda(H^3(\pi;\Lambda),\Lambda) \to \cdots .$$

By Proposition 4.1, we have $F^* \cong V^*$, and since F is a free Λ-module the connecting map $\delta_2\colon \mathrm{Ext}^1(V,\Lambda) \to \mathrm{Ext}^1(H^3(\pi;\Lambda),\Lambda)$ is an isomorphism. Now Lemma 4.5 shows that the connecting map $\delta_1\colon H^2(\pi;\Lambda) \to \mathrm{Ext}^1(V,\Lambda)$ is also an isomorphism. It follows that $V^* \cong H^2(K;\Lambda)^*$. Together these facts give an isomorphism $H^2(K;\Lambda)^* \cong F^*$ to a free Λ-module.

Since the dual of $\pi_2(M)$ is the dual of $\pi_2(K) \oplus H^2(K;\Lambda)$ by Lemma 5.3, we have $\pi_2(M)^* \cong \Lambda^{2b_3(\pi)}$. Furthermore, performing the same trick for the 4-term exact sequence

$$0 \to H^2(\pi;\Lambda) \to \pi_2(M) \to \pi_2(M)^* \to H^3(\pi;\Lambda) \to 0 \tag{6.2}$$

yields the isomorphism $(\pi_2(M)^*)^* \cong \pi_2(M)^*$, or $L_M^* \cong L_M$, which is given by $(\operatorname{ad} s_M)^*$. Thus we can define a non-singular hermitian form $h_M\colon L_M \times L_M \to \Lambda$ by the formula

$$h_M(x, y) = ((\operatorname{ad} s_M)^*)^{-1}(x)(y) \in \Lambda,$$

whose adjoint is the inverse of $(\operatorname{ad} s_M)^*\colon (\pi_2(M)^*)^* \to \pi_2(M)^*$. □

Remark 6.3. If $\pi = \pi_1(M)$ is any finitely presented group with tame cohomology, then the same argument shows that the dual of the equivariant intersection form s_M induces a well-defined, non-singular hermitian form $h_M\colon L_M \times L_M \to \Lambda$ on $\pi_2(M)^*$.

7. The reduced 2-type P

In order to prove Theorem A, we will follow the strategy of [15, Section 2] involving the reduced normal 2-type and the "modified" surgery theory developed by Kreck [23]. Since we have restricted our attention to spin manifolds, the reduced normal 2-type for a given 4-manifold M is a fibration $B \to BSTOP$, with total space

$$B := B(M) = P \times BTOPSPIN$$

defined by the second factor projection $P \times BTOPSPIN \to BTOPSPIN$ and the natural map $BTOPSPIN \to BSTOP$. It remains to describe the reduced 2-type $P = P(M)$.

Definition 7.1. Let M be a closed, oriented, spin 4-manifold with fundamental group π. We define the *reduced* 2*-type of* M as follows: let $P = P(M)$ be the two-stage Postnikov system with $\pi_1(P) = \pi_1(M) = \pi$ and $\pi_2(P) = L_M = \pi_2(M)^*$. The total space P is defined by a fibration over $K(\pi, 1)$ with fiber $K(L_M, 2)$ and classified by k_P in $H^3(\pi; L_M)$:

$$\begin{array}{ccc} K(L_M, 2) \longrightarrow & P & \\ & \downarrow & \\ & K(\pi, 1) & \longrightarrow H^3(\pi; L_M) \end{array}$$

We have a reference map $c_M\colon M \to P$ which factors through the algebraic 2-type of M (classified by $\pi_1(M) = \pi$, $\pi_2(M)$, and $k_M \in H^3(\pi; \pi_2(M))$). The k-invariant k_P is given by the image of k_M under the map induced by $\pi_2(M) \to \pi_2(P)$.

The map $\pi_2(M) \to \pi_2(P)$ is neither an injection nor a surjection in general because P is given by the union of cells of dimension ≥ 2 with M. In particular, the map c_M is 1-connected but not 2-connected.

Definition 7.2. For M a closed, spin, TOP 4-manifold, a map $c\colon M \to P$ to a space P is called a *reduced 3-equivalence* if c induces an isomorphism $c_*\colon \pi_1(M) \cong \pi_1(P)$ and an isomorphism $c^*\colon \pi_2(P)^* \cong \pi_2(M)^*$. If c is a reduced 3-equivalence, then $c \times \nu_M\colon M \to P \times BTOPSPIN$ is called a *reduced normal 2-smoothing.*

We have a result similar to [15, 2.11].

Proposition 7.3. *Let M and M' be closed, oriented, spin, topological 4-manifolds with fundamental group π. Suppose that there is an isometry $Q(M) \cong Q(M')$ given by isomorphisms $\alpha\colon \pi_1(M) \to \pi_1(M')$ and $\beta\colon \pi_2(M) \to \pi_2(M')$. Then there is a 3-coconnected* *fibration $P \times BTOPSPIN \to BSTOP$ admitting two reduced normal 2-smoothings $M \to P \times BTOPSPIN$ and $M' \to P \times BTOPSPIN$ that induce (α, β).*

Proof. After identifying $\pi_1(M) = \pi$ and $\pi_1(M')$ via the given isomorphism α, the map $\beta\colon \pi_2(M) \to \pi_2(M')$ is π-equivariant, and $(\operatorname{ad} s_M) = \beta^* \circ (\operatorname{ad} s_{M'}) \circ \beta$. If we let $P = P(M)$ and $P' = P(M')$ denote the reduced 2-types for M and M' respectively, then we have a fibre homotopy equivalence $P \to P'$ over $K(\pi, 1)$ induced by (α, β). □

8. The cohomology of P

In order to prove Theorem A, we need to calculate the spin bordism group $\Omega_4^{Spin}(P)$, and this requires some information about $H^*(P; \Lambda)$ and $H_*(P; \mathbb{Z})$.

If $c\colon M \to P$ is a reduced 3-equivalence, we have the same 4-term exact sequence for $P = P(M)$ as we do for K and M. The following diagram commutes:

(8.1)
$$\begin{tikzcd}
0 \arrow[r] & H^2(\pi;\Lambda) \arrow[r] \arrow[d, equal] & H^2(P;\Lambda) \arrow[r, "eval"] \arrow[d, "\cong"', "c^*"] & \operatorname{Hom}_\Lambda(\pi_2(P),\Lambda) \arrow[r, "d_3"] \arrow[d, "\cong"', "c^*"] & H^3(\pi;\Lambda) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H^2(\pi;\Lambda) \arrow[r] & H^2(M;\Lambda) \arrow[r, "eval"] & \operatorname{Hom}_\Lambda(\pi_2(M),\Lambda) \arrow[r] & H^3(\pi;\Lambda) \arrow[r] & 0
\end{tikzcd}$$

and we see that $H^2(P; \Lambda) \cong H^2(M; \Lambda)$. Here is a partial calculation of the cohomology of the reduced 2-type $P = P(M)$.

Lemma 8.2. *$H^1(P; \Lambda) \cong H^1(\pi; \Lambda)$, $H^2(P; \Lambda)$ is an extension of $\ker d_3 \subseteq \pi_2(P)^*$ by $H^2(\pi; \Lambda)$, and $H^3(P; \Lambda) = 0$.*

Proof. We look at the universal coefficient spectral sequence $E_2^{p,q} \cong \operatorname{Ext}^p_\Lambda(H_q(P; \Lambda), \Lambda)$ to compute $H^*(P; \Lambda)$. For $q = 0$ we have $E_2^{p,0} \cong \operatorname{Ext}^p_\Lambda(\mathbb{Z}, \Lambda) = H^p(\pi; \Lambda)$. When q is odd, $H_q(P; \Lambda) \cong H_q(\widetilde{P}; \mathbb{Z}) = 0$ since $\widetilde{P}$ is a product of copies of $\mathbb{C}P^\infty$. Thus the $E_2^{p,1}$- and $E_2^{p,3}$-terms are zero. Additionally, since $H_2(P; \Lambda) \cong \pi_2(P)$ is stably Λ-free (by Corollary 5.11 and Lemma 6.1), the terms $\operatorname{Ext}^p_\Lambda(H_2(P; \Lambda), \Lambda) = 0$ for $p > 0$. The $E_2^{1,2}$-term of the spectral sequence will therefore be zero. The only nonzero differential is the d_3 map in Figure 1, which is surjective by (8.1). Hence the $E_\infty^{3,0}$-term is zero, and so $H^3(P; \Lambda) = 0$. □

Lemma 8.3. *For a closed, oriented, TOP 4-manifold M with reduced 2-type P,*

$$H^3(P, M; \Lambda) = 0.$$

Proof. We use the long exact sequence in cohomology of the pair (P, M) with Λ-coefficients. We have the isomorphisms $H^3(M; \Lambda) \cong H_1(M; \Lambda) \cong H_1(\widetilde{M}; \mathbb{Z}) = 0$, so the long exact sequence becomes

$$\cdots \to H^2(P; \Lambda) \to H^2(M; \Lambda) \to H^3(P, M; \Lambda) \to H^3(P; \Lambda) \to 0.$$

From (8.1), the isomorphism $H^2(P; \Lambda) \cong H^2(M; \Lambda)$ forces the middle map in the above sequence to be zero. This yields the isomorphism $H^3(P, M; \Lambda) \cong H^3(P; \Lambda)$, which is zero by Lemma 8.2. □

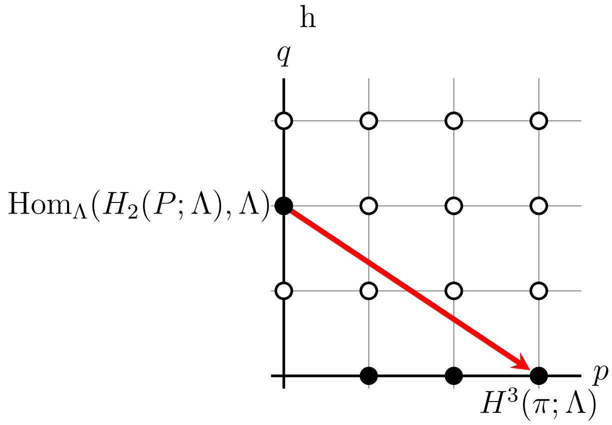


FIGURE 1. The E_2-page of the spectral sequence $E_2^{p,q} = \mathrm{Ext}^p_\Lambda(H_q(P;\Lambda),\Lambda)$ converging to $H^*(P;\Lambda)$. The solid dots represent possible nonzero terms.

9. THE HOMOLOGY OF P

In this section, let M be a closed, spin, TOP 4-manifold such that $\pi_1(M) = \pi$ is a finitely presented group with $\operatorname{cd}\pi \leq 3$. To compute the homology of the reduced 2-type $P = P(M)$, we use the Serre spectral sequence for the fibration $\widetilde{P} \to P \to K(\pi,1)$. This spectral sequence with integral coefficients has the E^2-page

$$E^2_{p,q} = H_p(\pi; H_q(\widetilde{P}))$$

and we only need the homology of P up to dimension 5. Note that $\widetilde{P}$ is a product of copies of $\mathbb{C}P^\infty$, so $H_q(\widetilde{P}) = 0$ for q odd. We have already seen that $H_2(\widetilde{P}) = \pi_2(P)$ is a stably free Λ-module. Therefore, by [14, Lemma 2.2][1], we see that $H_4(\widetilde{P}) = \Gamma(\pi_2(P))$ is also stably free. In summary:

(i) $E^2_{p,0} \cong H_p(\pi;\mathbb{Z})$, which is zero when $p \geq 4$.
(ii) $E^2_{0,q} \cong H_0(\pi; H_q(\widetilde{P})) \cong H_q(\widetilde{P}) \otimes_\Lambda \mathbb{Z}$.
(iii) $E^2_{p,q}$ is zero for odd q since $\widetilde{P}$ is a product of copies of $\mathbb{C}P^\infty$.
(iv) $E^2_{p,2} = H_p(\pi; H_2(\widetilde{P})) = 0$, for $p > 0$.
(v) $E^2_{p,4} = H_p(\pi; H_4(\widetilde{P})) = H_p(\pi; \Gamma(\pi_2(P))) = 0$, for $p > 0$,

In the E^2-page, all d^2 maps that affect $H_i(P)$, $i \leq 5$, are zero. In the spectral sequence, the only possibly nonzero differential is $d^3\colon H_3(\pi) \to H_0(\pi;\pi_2(P))$.

Proposition 9.1. *$d^3\colon H_3(\pi) \to H_0(\pi;\pi_2(P))$ is injective.*

The first step of the proof is to establish this result for the minimal model M_0.

Lemma 9.2. *Let $P = P(M_0)$ be the reduced 2-type of the minimal model. Then the map $d^3\colon H_3(\pi) \to H_0(\pi;\pi_2(P))$ is injective.*

[1]The proof given for this lemma applies without change to infinite groups.

Proof. The injectivity argument comes from comparing the same d^3 maps in three spectral sequences, the first of which is for $H_*(K)$. In the spectral sequence converging to $H_*(K)$, since $H_3(K)$ surjects onto the $E^\infty_{3,0}$ term and $H_3(K) = 0$, the differential

$$d^3\colon H_3(\pi) \to H_0(\pi; \pi_2(K))$$

must be injective. Recall that M_0 is the double of a thickening $N := N(K)$ of K, and we view M_0 as the boundary of $N \times I$; this gives a map $M_0 \hookrightarrow N \times I \simeq K$. The reduced 2-type P is constructed by attaching cells of dimension 2 and higher to M_0. We define

$$P^{(3)} := M_0 \cup \bigcup_\alpha e^2_\alpha \cup \bigcup_\beta e^3_\beta,$$

as the union of M_0 with only the 2-cells and 3-cells from P. Since $H^3(P, M_0; \pi_2(K)) = 0$ by Lemma 8.3, obstruction theory tells us the map $M_0 \to K$ extends over $P^{(3)}$, and we obtain an induced map $H_0(\pi; \pi_2(P^{(3)})) \to H_0(\pi; \pi_2(K))$. By commutativity of the diagram below, the d^3 map in the spectral sequence converging to $H_*(P^{(3)})$ must also be injective.

$$\begin{tikzcd}
H_0(\pi;\pi_2(K)) & H_0(\pi;\pi_2(P^{(3)})) \arrow[l] \\
H_3(\pi) \arrow[u, hook, "d^3"] \arrow[r, equal] & H_3(\pi) \arrow[u, "d^3"']
\end{tikzcd}$$

It remains to compare the d^3 differentials for $H_*(P)$ and $H_*(P^{(3)})$. We claim that $\pi_2(P) \cong \pi_2(P^{(3)})$: the relative homologies $H^i(P, P^{(3)}; \Lambda)$ vanish in dimension $i = 2, 3$, so the isomorphism is given by the long exact sequence of the pair. The injectivity of $d_3\colon H_3(\pi) \to H_0(\pi; \pi_2(P^{(3)}))$ implies that $d^3\colon H_3(\pi) \to H_0(\pi; \pi_2(P_0))$ is also injective. □

The following lemma is used in the proof of Proposition 9.1.

Lemma 9.3. *Let M be a closed, oriented, TOP 4-manifold with $\pi_1(M) = \pi$, and let X be a closed, simply connected 4-manifold. Then the map $d^3\colon H_3(\pi) \to H_0(\pi; \pi_2(P(M)))$ is injective if and only if the map $d^3\colon H_3(\pi) \to H_0(\pi; \pi_2(P(M\#X)))$ is injective.*

Proof. We begin by comparing M and $M\#X$. By removing the top dimensional cells of M and $M\#X$, we get an inclusion $M^o \hookrightarrow (M\#X)^o$, the latter of which is just M^o wedged with a collection of n 2-spheres arising from X^o. This inclusion induces a split injection $\pi_2(M^o) \to \pi_2((M\#X)^o)$, and so $\pi_2(M)$ is stably isomorphic to $\pi_2(M\#X)$:

$$\pi_2(M\#X) \cong \pi_2((M\#X)^o) \cong \pi_2(M^o) \oplus \Lambda^n \cong \pi_2(M) \oplus \Lambda^n,$$

where $n = b_2(X)$. If $P(M)$ is the reduced 2-type of M, and $P(M\#X)$ is the reduced 2-type of $M\#X$, then it follows that $\pi_2(P(M))$ is stably isomorphic to $\pi_2(P(M\#X))$, and therefore $H_0(\pi; \pi_2(P(M))) \cong H_0(\pi; \pi_2(P(M\#X)))$. The conclusion about injectivity for the maps d^3 now follows by naturality of the spectral sequences with respect to the map $P(M) \to P(M\#X)$. □

The proof of Proposition 9.1. By Proposition 5.10, we have

$$M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}\#r(S^2\times S^2)\approx M'\#s(S^2\times S^2).$$

where $M' = M_0\#p\mathbb{C}P^2\#q\overline{\mathbb{C}P^2}$ is a suitable stabilization of the minimal model M_0.

This homeomorphism allows us to equate their second homotopy groups, and thus identify their reduced 2-types. By applying Lemma 9.3 several times, the d^3 map for $P(M)$ is injective if and only if the d^3 map for $P(M_0)$ is injective, and Lemma 9.2 completes the proof. □

Remark 9.4. The same d^3 map in the spectral sequence converging to $H_*(M)$ is injective as well, given by naturality of the spectral sequences under the map $M\to K$.

Proposition 9.5. *The integral homology $H_i(P)$, for $i\leq 5$, is given as follows:*

(i) $H_0(P)=\mathbb{Z}$ *and* $H_1(P)=H_1(\pi)$.
(ii) $H_2(P)$ *is an extension of* $H_2(\pi)$ *by* $H_2(\widetilde{P})\otimes_\Lambda\mathbb{Z}$.
(iii) $H_3(P)=H_5(P)=0$.
(iv) $H_4(P)\cong H_4(\widetilde{P})\otimes_\Lambda\mathbb{Z}$.

Proof. The table summarizes the calculations above. For right-angled Artin groups the extension in (ii) is split. □

10. The spin bordism group $\Omega_*^{Spin}(P)$

The James spectral sequence is used to compute the spin bordism groups of the reduced 2-type $P=P(M)$. Specifically, we are interested in $\Omega_4^{Spin}(P)$. The E^2-page of the spectral sequence is given by $H_p(P;\Omega_q^{Spin}(\mathrm{pt}))$, and the relevant Spin bordism groups of a point are given below:

$$\Omega_q^{Spin}(\mathrm{pt})=\mathbb{Z},\mathbb{Z}/2,\mathbb{Z}/2,0,\mathbb{Z}\quad\text{for } q=0,1,2,3,4$$

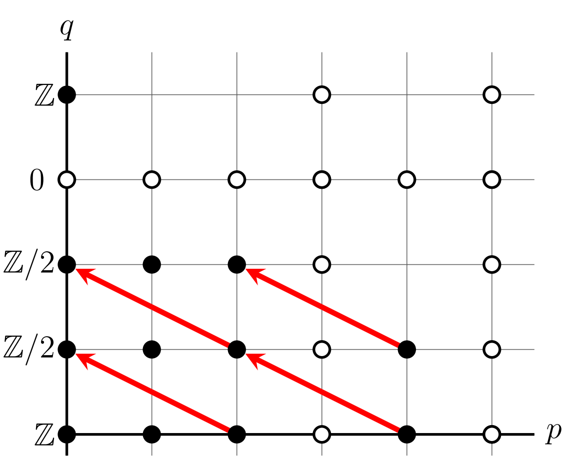


Figure 2. The E^2-page of the spectral sequence converging to $\Omega_*^{Spin}(P)$.

In Figure 2, we have included the information about $H_*(P)$ from the last section. The two d^2 maps on the left in the E^2-page are both zero, otherwise $\Omega_*^{Spin}(\mathrm{pt})$ would not split off in the E^∞-page of $\Omega_*^{Spin}(P)$. The other two d^2 maps are the duals of the Sq^2 maps composed with reduction mod 2.

Consider the commutative diagram below that arises from the fibration $\widetilde{P} \to P \to K(\pi, 1)$. We take homology with $\mathbb{Z}/2$-coefficients.

$$\begin{tikzcd}
H_4(\widetilde{P}) \arrow[r, two heads] \arrow[d, two heads] & H_2(\widetilde{P}) \arrow[d] & & \\
H_4(P) \arrow[r, "DSq^2"] \arrow[d] & H_2(P) \arrow[r] \arrow[d, two heads] & \mathrm{coker}(DSq^2) \arrow[r] \arrow[d, dotted] & 0 \\
0 = H_4(\pi) \arrow[r] & H_2(\pi) \arrow[r, equal] \arrow[d] & H_2(\pi) & \\
& 0 & &
\end{tikzcd}$$

The above map labeled DSq^2 is the dual of $Sq^2\colon H^2(P) \to H^4(P)$. Since $\widetilde{P}$ is a product of copies of $\mathbb{C}P^\infty$, $H^2(\widetilde{P})$ injects into $H^4(\widetilde{P})$, and thus $H_4(\widetilde{P})$ surjects onto $H_2(\widetilde{P})$. But $H_4(\pi) = 0$ by assumption, so by exactness of the middle vertical sequence, $\mathrm{coker}(DSq^2) \cong H_2(\pi; \mathbb{Z}/2)$. Since $H_5(P; \mathbb{Z}) = 0$, the term $H_2(\pi; \mathbb{Z}/2)$ survives to $\Omega_4^{Spin}(P)$.

Proposition 10.1. *The spin bordism groups of the reduced normal* 2*-type* $P = P(M)$ *for* M *are detected by an injection*

$$\Omega_4^{Spin}(P) \subseteq \mathbb{Z} \oplus H_2(\pi; \mathbb{Z}/2) \oplus H_4(P; \mathbb{Z}).$$

The invariants are the signature, an invariant in $H_2(\pi; \mathbb{Z}/2)$*, and the fundamental class* $c_*[M] \in H_4(P; \mathbb{Z})$*.*

Proof. This follows from our discussion of the differentials in the spectral sequence. □

We will show that the bordism invariant in $H_2(\pi; \mathbb{Z}/2)$ is determined by the other invariants. The method follows [15, §5], where the authors define a subset $\Omega_4(P)_M \subset \Omega_4^{Spin}(P)$, called the *normal structures*, consisting of the spin bordism classes (N, f) with $f_*[N] = c_*[M]$ and $\mathrm{sign}(N) = \mathrm{sign}(M)$. There is a map

$$\theta_M\colon \Omega_4(P)_M \to L_4(\mathbb{Z}[\pi])$$

defined as follows: after preliminary surgeries we may assume that the map f is 2-connected, and let

$$V := \ker(f_*\colon \pi_2(N) \to \pi_2(P)).$$

After applying $\mathrm{Hom}_\Lambda(-, \Lambda)$, we obtain a short exact sequence

$$0 \to \pi_2(P)^* \xrightarrow{f^*} \pi_2(N)^* \to V^* \to 0.$$

On $\pi_2(N)^*$ we have the reduced (even) equivariant intersection form h_N whose adjoint is the inverse of $(\operatorname{ad} s_N)^*$.

Lemma 10.2. *The reduced (even) intersection form $(\pi_2(N)^*, h_N)$ restricted to the image of $f^*\colon \pi_2(P)^* \to \pi_2(N)^*$ is isometric to the dual of h_M.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
\pi_2(M) \arrow[d, "c_*"] & H^2(M;\Lambda) \arrow[l, "\approx", "\cap[M]"'] \arrow[r, "eval"] & \pi_2(M)^* \\
\pi_2(P) & H^2(P;\Lambda) \arrow[u, "\approx", "c^*"'] \arrow[l, "\cap c_*[M]"'] \arrow[r, "eval"] \arrow[d, "f^*"] & \pi_2(P)^* \arrow[u, "c^*", "\approx"'] \arrow[d, "f^*"] \\
\pi_2(N) \arrow[u, "f_*"] & H^2(N;\Lambda) \arrow[l, "\approx", "\cap[N]"'] \arrow[r, "eval"] & \pi_2(N)^*
\end{tikzcd}$$

The composite on the top row defines $\operatorname{ad} s_M$, and the composite on the bottom row defines $\operatorname{ad} s_N$. After dualizing each term in the diagram, each square still commutes, and all the maps become isomorphisms. The top and bottom rows of the dualized diagram give $(\operatorname{ad} s_M)^*$ and $(\operatorname{ad} s_N)^*$, respectively. One can check by a diagram chase that the dualized composite in the middle row is isometric via (c^*, c^{**}) to $(\operatorname{ad} s_M)^*$. This composite is also isometric to the pull-back of $(\operatorname{ad} s_N)^*$ via (f^*, f^{**}). $\square$

Since h_M is non-singular and even, we obtain an isometric splitting

$$(\pi_2(N)^*, h_N) = (\pi_2(P)^*, (h_M)^*) \oplus (V, \lambda_{N,f}),$$

where $(V, \lambda_{N,f})$ is a non-singular even hermitian form on a finitely generated, stably free Λ-module. We define

$$\theta_M(N, f) = (V, \lambda_{N,f}) \in \tilde{L}_4(\mathbb{Z}[\pi]),$$

where $\tilde{L}_4(\mathbb{Z}[\pi])$ denotes the reduced surgery obstruction group (represented by quadratic forms with signature zero).

Lemma 10.3. *The map θ_M is well-defined. In addition, $\theta_M(N, f) = 0$ if f is a reduced 3-equivalence.*

Proof. If (N, g) and (N', g') represent the same bordism element in $\Omega_4(P)_M$, then N and N' are stably homeomorphic over P, hence θ_M is well-defined (compare with [15, Lemma 5.9]). If $f\colon N \to P$ is a reduced 3-equivalence, then $f^*\colon \pi_2(P)^* \to \pi_2(N)^*$ is an isomorphism and $V^* = 0$. $\square$

The next step is to define a map

$$\rho_M\colon \Omega_4(P)_M \to H_2(\pi; \mathbb{Z}/2)$$

on an element $[N, f]$ by the projection of the difference $[N, f] - [M, c]$ from

$$\ker(\Omega_4(P) \to H_4(P; \mathbb{Z}))$$

to the subquotient $E^{2,2}_\infty = H_2(\pi; \mathbb{Z}/2)$ in the James spectral sequence.

Lemma 10.4. $\rho_M\colon \Omega_4(P)_M \to H_2(\pi;\mathbb{Z}/2)$ *is a bijection.*

Proof. This follows from Proposition 10.1 and the definition of the subset $\Omega_4(P)_M$. □

Similarly, we define $\Omega_4(M)_M$ and obtain a bijection $\hat{\rho}_M\colon \Omega_4(M)_M \xrightarrow{\approx} H_2(M;\mathbb{Z}/2)$. The map $\hat{\theta}_M\colon \Omega_4(M)_M \to \tilde{L}_4(\mathbb{Z}[\pi])$ is defined by $\hat{\theta}_M(N,g) = \theta_M(N, c\circ g)$, where $g\colon N \to M$ represents a bordism element in $\Omega_4(M)_M$. Recall that there is a "universal" assembly map homomorphism

$$\kappa_2\colon H_2(\pi;\mathbb{Z}/2) \to L_4(\mathbb{Z}[\pi])$$

defined for any group (see [21, §3]). We have a version of [15, Lemma 5.11] in our setting.

Lemma 10.5. $\theta_M = \kappa_2 \circ \rho_M$.

Proof. The following diagram commutes:

$$\begin{tikzcd}
& & L_4(\mathbb{Z}[\pi]) \\
\Omega_4(M)_M \arrow[r, "c_*"] \arrow[d, "\hat{\rho}_M"] \arrow[urr, bend left, "\hat{\theta}_M"] & \Omega_4(P)_M \arrow[ur, "\theta_M"] \arrow[d, "\rho_M"] & \\
H_2(M;\mathbb{Z}/2) \arrow[r, "c_*"] & H_2(\pi;\mathbb{Z}/2) \arrow[uur, bend right, "\kappa_2"'] &
\end{tikzcd}$$

The outer composition $\hat{\theta}_M = \kappa_2 \circ c_* \circ \hat{\rho}_M$ holds by the same argument given in the proof of [15, Lemma 5.11]. The elements in $\Omega_4(M)_M$ are represented by degree 1 normal maps (N, g) covered by a bundle map $\nu_N \to \nu_M$, since $g^*(\nu_M) \cong \nu_N$ by [8, 20]. The required formula now follows from Wall's characteristic class formula for surgery obstructions (see Davis [6]).

Since the map $c_*\colon H_2(M;\mathbb{Z}/2) \to H_2(\pi;\mathbb{Z}/2)$ is surjective and both $\hat{\rho}_M$ and ρ_M are bijections, the formula $\theta_M = \kappa_2\circ\rho_M$ follows from the commutivity of the inner square. □

Corollary 10.6. *Suppose that* $[N,f]$ *is an element in* $\Omega_4(P)$, *with* f *a reduced* 3*-equivalence such that* $\operatorname{sign}(N) = \operatorname{sign}(M)$. *If* $f_*[N] = c_*[M]$ *and* $\kappa_2\colon H_2(\pi;\mathbb{Z}/2) \to L_4(\mathbb{Z}[\pi])$ *is injective, then* $[N,f] = [M,c] \in \Omega_4(P)$.

Proof. This is a version of [15, Corollary 5.12] and the proof is analogous. The bordism group $\Omega_4^{Spin}(P)$ is detected by Proposition 10.1, and the difference $[M,c]-[N,f]$ projects to zero in $H_4(P;\mathbb{Z})$ since $f_*[N] = c_*[M]$. By definition, the map $\rho_M(N,f)$ is the projection of the difference $[M,c]-[N,f]$ to the subquotient $H_2(\pi;\mathbb{Z}/2)$. Since f is a reduced 3-equivalence, $\theta_M(N,f) = 0$ by Lemma 9.3, and since κ_2 is injective, $\rho_M(N,f)$ must be zero by Lemma 9.5. Furthermore, since $\operatorname{sign}(N) = \operatorname{sign}(M)$, the elements $[N,f]$ and $[M,c]$ are bordant in $\Omega_4(P)$. □

11. The proof of Theorem A

Our main result, Theorem A, is an immediate consequence of a more general statement. For a finitely presented group π with $\operatorname{cd}\pi \leq 3$, let $b_3(\pi)$ denote the minimum number of generators for $H^3(\pi;\Lambda)$ as a Λ-module. Note that $b_3(\pi$ is bounded above by the Λ-rank of $C_3(L)$, where L is a minimal aspherical 3-complex with $\pi_1(L) = \pi$. We recall a definition from [15].

Definition 11.1. A group π satisfies properties (W-AA) whenever

(i) The Whitehead group $\mathrm{Wh}(\pi)$ vanishes.
(ii) The assembly map $A_5\colon H_5(\pi;\mathbb{L}_0) \to L_5(\mathbb{Z}[\pi])$ is surjective.
(iii) The assembly map $A_4\colon H_4(\pi;\mathbb{L}_0) \to L_4(\mathbb{Z}[\pi])$ is injective.

These properties hold whenever the group π satisfies the Farrell-Jones isomorphism conjectures in K-theory and L-theory. These conjectures have been verified for many classes of groups, and in particular for all right-angled Artin groups (see [1]).

Theorem 11.2. *Let π be a finitely presented group with $\operatorname{cd}\pi \leq 3$ satisfying the properties* (W-AA)*. If M and N are closed, oriented, spin, TOP 4-manifolds with fundamental group π, then any isometry between the quadratic 2-types of M and N is stably realized by an s-cobordism between $M \# r(S^2 \times S^2)$ and $N \# r(S^2 \times S^2)$, for $r \geq b_3(\pi)$.*

Remark 11.3. Note that the main theorem of [15] applies to groups π with $\operatorname{cd}\pi = 2$, unless π has geometric dimension 3. Any such group would be a counterexample to the famous Eilenberg-Ganea conjecture.

We divide the proof of Theorem 11.2 into the following steps.

(i) Two reduced 3-equivalences $c_M\colon M \to P$ and $c_N\colon N \to P$ satisfy
$$(c_M)_*[M] = (c_N)_*[N] \in H_4(P;\mathbb{Z})$$
if and only if the composite $((c_M)^*)^{-1} \circ (c_N)^*\colon \pi_2(N)^* \to \pi_2(M)^*$ induces an isometry of reduced intersection forms. This is the corresponding result to [15, Theorem 5.13], and the proof carries over without change.
(ii) The assumption in (W-AA) that the assembly map $A_4\colon H_4(\pi;\mathbb{L}_0) \to L_4(\mathbb{Z}[\pi])$ is injective implies that the map $\kappa_2\colon H_2(\pi;\mathbb{Z}/2) \to L_4(\mathbb{Z}[\pi])$ is injective.
(iii) Suppose that M and N are closed, oriented, spin, TOP 4-manifolds. If M and N have isometric quadratic 2-types, then there are reduced normal 2-smoothings $M \to P$ and $N \to P$ which are bordant in $\Omega_4(P)$ (this follows as in [15, Corollary 5.14]).
(iv) We show how to apply [15, Theorem 2.2] of Kreck's modified surgery theory to obtain an s-cobordism between M and N, after a specified stabilization.

We have now established step (iii) by applying steps (i)-(ii) and Corollary 10.6. The difficulty in the last step (iv) is that our reduced normal 2-smoothings are not given by 2-connected reference maps $M \to P$ and $N \to P$, so the modified surgery result does not apply directly. We now show how a limited amount of stabilization can be used to get 2-connected reference maps, and thus complete the proof of Theorem 11.2.

First we construct an abstract diagram using an exact sequence

$$0 \to A \to B \xrightarrow{g} C \to D \to 0$$

and a factorization $B \xrightarrow{j} V \to C$ of the map $g\colon B \to C$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r, "j"] \arrow[d] \arrow[dr, "g"] & V \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & B \oplus F \arrow[r, "f"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[d] & F \arrow[r, "\bar{\phi}"] \arrow[d] \arrow[ur, "\phi"] & D \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Let $f(b,x) = g(b) + \phi(x)$, for $b \in B$ and $x \in F$. The map $\phi\colon F \to C$ is a lifting of a surjective map $\bar{\phi}\colon F \to D$ from a free Λ-module $F \cong \Lambda^r$. Let $E = \ker \bar{\phi}$. We will apply this diagram to the universal coefficient sequence

$$0 \to H^2(\pi;\Lambda) \to \pi_2(M) \to \pi_2(M)^* \to H^3(\pi;\Lambda) \to 0$$

from (8.1). The map $g = \operatorname{ad} s_M$, and in our application $F \to H^3(\pi;\Lambda)$ will be a given by a (minimal) set of r generators for $H^3(\pi;\Lambda)$ as a Λ-module.

Let $\theta = -\phi^*(h_M)$ denote the hermitian form on F pulled back from *minus* the reduced intersection form $h_M\colon L_M \times L_M \to \Lambda$ on $L_M = \pi_2(M)^*$, so that the map ϕ is an isometry with respect to $-h_M$. We can embed the form (F, θ) isometrically as a direct summand $\ell\colon (F,\theta) \to H(\Lambda^r)$ in the hyperbolic form $H(\Lambda^r)$ by an explicit map of based modules. If $\{e_1, \dots, e_r, f_1, \dots, f_r\}$ is a standard hyperbolic base for $H(\Lambda^r)$, and $\{a_1, \dots, a_r\}$ is a base for F, then we let $\ell(a_i) = e_i + \sum_j \alpha_{ij} f_j$, where $\theta = (\alpha_{ij})$ in matrix form. Let

$$k\colon H(\Lambda^r) = \Lambda^r \oplus \Lambda^r \to F$$

be the retraction defined by $k(e_i) = a_i$ and $k(f_j) = 0$, so that $k \circ \ell = id_F$.

Claim: *The form on $K = \ker f$ induced by $(B \oplus F, s_M \oplus \theta)$ is identically zero.*

To check this, we let $\gamma\colon B \to (B^*)^*$ denote the map defined by the evaluation

$$\langle \gamma(b), v \rangle = \overline{v(b)}, \text{ for } v \in B^*$$

and verify that we have a map of right Λ-modules:

$$\langle \gamma(b\lambda), v \rangle := \gamma(b\lambda)(v) = \overline{v(b\lambda)} = \overline{(v\bar{\lambda})(b)} = \gamma(b)(v\bar{\lambda}) = (\gamma(b)\cdot\lambda)(v),$$

where as usual, the right Λ-module structure on $B^* = \operatorname{Hom}_\Lambda(B,\Lambda)$ is given by the formula $(v\lambda)(b) = v(b\bar{\lambda})$, for all $v \in B^*$ and $\lambda \in \Lambda$.

After substituting for B and C, we have a diagram:

$$\begin{tikzcd} & (\pi_2(M)^*)^* \arrow[d, "(\mathrm{ad}\, s_M)^*"] \\ \pi_2(M) \arrow[ur, "\gamma"] \arrow[r, "\mathrm{ad}\, s_M"'] & \pi_2(M)^* \end{tikzcd}$$

We claim that this diagram commutes. The relation

$$(\mathrm{ad}\, s_M)^* \circ \gamma = \mathrm{ad}\, s_M$$

follows from the calculation

$$\langle ((\mathrm{ad}\, s_M)^* \circ \gamma)(b), b' \rangle = \langle \gamma(b), \mathrm{ad}\, s_M(b') \rangle = \overline{\mathrm{ad}\, s_M(b')(b)} = \overline{s_M(b', b)} = s_M(b, b'), \tag{11.4}$$

since s_M is hermitian symmetric. However, $\mathrm{ad}\, h_M = ((\mathrm{ad}\, s_M)^*)^{-1}$, so we have the relation

$$\gamma = \mathrm{ad}\, h_M \circ \mathrm{ad}\, s_M.$$

Now we compute the hermitian form $\omega := s_M \oplus \theta$ on elements $z = (b, x)$ and $z' = (b', x')$ of K via

$$\omega(z, z') = s_M(b, b') + \theta(x, x') = s_M(b, b') - h_M(\phi(x), \phi(x')) = s_M(b, b') - \mathrm{ad}\, h_M(\phi(x))(\phi(x')).$$

But $\mathrm{ad}\, s_M(b) = -\phi(x)$ and $\mathrm{ad}\, s_M(b') = -\phi(x')$, since our elements lie in $K = \ker f$, so we obtain

$$\omega(z, z') = s_M(b, b') - \mathrm{ad}\, h_M(\mathrm{ad}\, s_M(b))(\mathrm{ad}\, s_M(b')) = s_M(b, b') - \langle \gamma(b), \mathrm{ad}\, s_M(b') \rangle = 0$$

by the formula in (11.4).

The proof of Theorem 11.2. To apply this algebra to our geometric setting, we form

$$M' = M \# r(S^2 \times S^2)$$

by performing surgery on null-homotopic circles in M. Since the free module $F \cong \Lambda^r$ is required to map surjectively onto $H^3(\pi; \Lambda)$, we may take any $r \geq b_3(\pi)$. The reference map $c_M \colon M \to P$ induced by $\mathrm{ad}\, s_M \colon \pi_2(M) \to \pi_2(M)^* = \pi_2(P)$ can be extended to a 2-connected reference map $c'_M \colon M' \to P$ with induced map

$$(c'_M)_* \colon \pi_2(M') = \pi_2(M) \oplus H(\Lambda^r) \xrightarrow{id \oplus k} \pi_2(M) \oplus F \xrightarrow{f} \pi_2(P)$$

using the map $f \colon B \oplus F \to C$ from the diagram above. Recall that $B = \pi_2(M)$ and $C = \pi_2(P)$. Note that (M', c'_M) and (M, c_M) are spin bordant over P.

The form on M' is given by

$$(\pi_2(M'), s_{M'}) = (\pi_2(M) \oplus F, s_M \oplus \theta)\ .$$

Claim: *The form $s_{M'}$ restricted to* $\ker (c'_M)_* \subset \pi_2(M')$ *is identically zero.*

Since the map $k \colon H(\Lambda^r) \to F$ has kernel $0 \oplus \Lambda^r = \{f_1, \dots, f_r\}$, we see that $\ker (c'_M)_*$ is the orthogonal direct sum of K and the lagrangian summand $0 \oplus \Lambda^r \subset H(\Lambda^r)$. By the first claim above, the intersection form $s_{M'}$ restricted to K is identically zero, so the second claim is verified.

By a similar construction, we can extend the reference map $c_N\colon N \to P$ to a 2-connected reference map $c'_N\colon N' \to P$, where $N' = N \# r(S^2 \times S^2)$. The elements (M', c'_M) and (N', c'_N) are spin bordant over P, and we have the setting to apply [23, Theorem 4, p. 735]. The second claim implies that the modified surgery obstruction is zero, by [23, Prop. 8, p. 739]. The role of the first two properties in (W-AA) is explained in [15, Theorem 2.6]. This completes step (iv) and the proof of Theorem 11.2. □

The proof of Theorem A. For π a right-angled Artin group with $\operatorname{cd}\pi \leq 3$, the conditions (W-AA) were established by Bartels and Lück [1]. This shows that Theorem A follows from Theorem 11.2. □

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