# RECOGNIZING PRODUCTS OF SURFACES AND SIMPLY CONNECTED 4-MANIFOLDS

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Abstract. We give necessary and sufficient conditions for a closed smooth 6-manifold $N$ to be diffeomorphic to a product of a surface $F$ and a simply connected 4-manifold $M$ in terms of basic invariants like the fundamental group and cohomological data. Any isometry of the intersection form of $M$ is realized by a self-diffeomorphism of $M \times F$.

## 1. Introduction

Simply-connected closed 6-manifolds were classified by Wall [13], Jupp [6], and Žubr [14]. However, if the fundamental group is non-trivial, such complete information is not within reach of current techniques except in special cases.

In this paper we consider the following problem: given a closed, oriented 6-manifold $N$, can we identify a closed, oriented surface $F$ and a simply-connected, closed 4-manifold $M$ such that $N$ is diffeomorphic to $M \times F$ ? Since simply connected 6-manifolds are already classified, we assume from now on that $F \neq S^2$ has genus $\geq 1$, but the results remain true in the simply connected case. First we discuss some of the necessary conditions.

**Condition 1.** *The fundamental group $\pi_1(N)$ is isomorphic to the fundamental group of a closed, oriented surface $F$.*

We choose a base-point preserving classifying map $u\colon N \to F$ for the universal covering. Up to homotopy and choice of base points this is equivalent to choosing an isomorphism $\alpha\colon \pi_1(N) \to \pi_1(F)$, where $u_\# = \alpha$.

The next condition concerns the second homology group of the universal covering $H_2(\widetilde{N})$, which for the product of $F$ with a simply connected 4-manifold $M$ is a trivial module over $\pi_1(N)$ and so we require this:

**Condition 2.** *$H_2(\widetilde{N})$ is a trivial $\pi_1(N)$-module.*

Under this condition, the Serre spectral sequence for the fibration over $F = K(\pi_1(N), 1)$ with fibre $\widetilde{N}$, implies that we have an exact sequence

$$0 \to H^2(F) \xrightarrow{u^*} H^2(N) \xrightarrow{p^*} H^2(\widetilde{N}) \to 0,$$

where $p$ is the universal covering projection. It follows that $H_2(\widetilde{N})$ is a finitely-generated free abelian group.

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The key to our recognition result is the observation that the cohomology algebra of $N$ provides a candidate for the intersection form of a closed, simply-connected topological 4-manifold $M$. To identify this candidate, suppose that Condition 1 holds. Then the trilinear cup product form on $H^2(N)$ induces a well-defined symmetric bilinear form

$$I(N) : H^2(N)/u^*H^2(F) \times H^2(N)/u^*H^2(F) \to \mathbb{Z}$$

by mapping $x$ and $y$ to $\langle u^*([F]) \cup x \cup y, [N]\rangle$, where $[F] \in H^2(F)$ is the cohomology fundamental class. If Condition 2 holds, then $V := H^2(N)/u^*H^2(F) \cong H^2(\widetilde{N})$ is a finitely-generated free abelian group.

Under the assumption that $N \approx M \times F$, this form $I(N)$ is *unimodular* and the finitely-generated free abelian group $V$ is isomorphic to $H^2(M)$. Moreover, the form $I(N)$ and the intersection form

$$s_M \colon H^2(M) \times H^2(M) \to \mathbb{Z}$$

are isometric. We recall that vanishing of the Kirby-Siebenmann invariant of $M$, denoted $KS(M)$ is a necessary and sufficient condition for $M \times F$ to be smoothable (see [7, Theorem 5.14, p. 318]). If $M$ is a spin manifold, then this condition is assured by requiring $\operatorname{sign}(M) \equiv 0 \pmod{16}$. Thus we have:

**Condition 3.** *The symmetric bilinear form $I(N)$ is unimodular, and* $\operatorname{sign} I(N) \equiv 0 \pmod{16}$ *if $N$ is a spin manifold.*

If $I(N)$ is unimodular there exists a closed, simply-connected topological 4-manifold $M$ with this intersection form, by the foundational results of Freedman [4, Theorem 1.5]. If $M$ is non-spin, then Freedman shows that we may assume $KS(M) = 0$. In either case, if $KS(M) = 0$ the manifold $M$ is *uniquely* determined (up to homeomorphism) by its intersection form $s_M$. Moreover, the smooth structures on $M \times F$ are determined by lifts of its stable topological tangent bundle $\tau_{M\times F}$ (see [7, Theorem 10.1, p. 194] for the precise statement).

**Definition 1.1.** The *standard smooth structure* on $M \times F$ is the one determined by product of the unique lift of $\tau_M \colon M \to BTOP$ to $BO$, together with $\tau_F \colon F \to BO$. The lift of $\tau_M$ is unique because $TOP/O \simeq TOP/PL = K(\mathbb{Z}/2, 3)$ in this range of dimensions.

We then fix the standard smooth structure on $M \times F$ and take the product orientation with respect to given orientations on $M$ and $F$. This is our candidate for recognizing $N$ as the product $M \times F$.

Finally, we need some more information about the *oriented* integral cohomology ring of $N$ and the Pontrjagin class $p_1(N) \in H^4(N)$. Let $q_1 \colon M \times F \to M$ and $q_2 \colon M \times F \to F$ denote the first and second factor projection maps. Note that the integral cohomology of $M \times F$ is $\mathbb{Z}$-torsion free, so any map $H^*(M \times F) \to H^*(N)$ of integral cohomology rings reduced mod 2 induces a map on $\mathbb{Z}/2$-cohomology.

**Condition 4.** *Let $M$ be a closed, oriented, simply-connected topological 4-manifold with $s_M \cong I(N)$ and $KS(M) = 0$. There exists an isomorphism*

$$\phi \colon H^*(M \times F) \to H^*(N)$$

*of oriented integral cohomology rings. We assume that*

(i) $\phi([M] \times [F]) = [N] \in H^6(N)$,
(ii) $\phi \circ q_2^* = u^* \colon H^*(F) \to H^*(N)$, *and*
(iii) $\phi$ *preserves the second Stiefel-Whitney class:*

$$\phi(w_2(M \times F)) = w_2(N) \in H^2(N; \mathbb{Z}/2).$$

(iv) *Moreover, the relation*

$$\langle \phi(x) \cup p_1(N), [N] \rangle = \begin{cases} 3\,\mathrm{sign}(M) & \textit{if } x = q_2^*([F]) \in H^2(M \times F), \\ 0 & \textit{if } x = q_1^*(y) \end{cases}$$

*holds for all* $y \in H^2(M)$.

**Example 1.2.** Unless $M = S^4$, the cohomology cohomology ring determines the Steenrod operations, and so $\phi$ preserves the second Stiefel-Whitney class. On the other hand, consider an oriented 4-sphere bundle $N$ over $F$ with $w_2(N) \neq 0$. Then $N$ has the same cohomology ring as $S^4 \times F$ but is not diffeomorphic to the product.

Now we are ready to formulate our main result.

**Theorem A.** *Let $N$ be a closed, oriented smooth 6-manifold, and $\alpha \colon \pi_1(N) \cong \pi_1(F)$ for some closed, oriented surface $F$, such that Conditions 1-3 hold. Suppose that*

(i) *$M$ is the closed, simply-connected topological 4-manifold $M$, such that $s_M \cong I(N)$, with $KS(M) = 0$, and*
(ii) *$\phi \colon H^*(M \times F) \xrightarrow{\approx} H^*(N)$ is a ring isomorphism satisfying Condition 4.*

*Then, there is an orientation and base-point preserving diffeomorphism $f \colon N \to M \times F$ such that $f_\# = \alpha$ and $f^* = \phi$.*

We can also ask which automorphisms of the second cohomology of $M \times F$ are induced by self-diffeomorphisms. In particular, we consider automorphisms of $H^2(M)$, and extend them by the identity on $H^2(F)$ via the identification:

$$(q_1^*, q_2^*) \colon H^2(M) \oplus H^2(F) \cong H^2(M \times F).$$

From the ring structure in cohomology, a necessary condition is that the automorphism on $H^2(M)$ is an isometry of the intersection form.

**Corollary B.** *Let $M$ be a closed topological 4-manifold with $KS(M) = 0$ and $F$ a closed oriented surface. Then each isometry of the intersection form of $M$ is induced by a self-diffeomorphism of $M \times F$.*

*Proof.* There is an automorphism $\phi$ of $H^*(M \times F)$, which on $H^2(M \times F)$ is the given isometry on $H^2(M)$ extended by the identity on $H^2(F)$. By Theorem A there is a self-diffeomorphism of $M \times F$ inducing $\phi$, and therefore the given isometry on $H^2(M)$. □

**Remark 1.3.** In the case where $M$ is itself smooth, Donaldson theory (see [5, Theorem 6]) provides examples of isometries of $H^2(M)$ which cannot be realized by self-diffeomorphisms of $M$. We also remark that an alternate argument can be given for Corollary B by using further results of Freedman and Kirby-Siebenmann. By [4, p. 371] there is a homeomorphism $h\colon M \to M$ realizing the given isometry. Consider the $s$-cobordism

$$W^5 := (M \times I) \cup_h (M \times I)$$

obtained by gluing two cylinders $M \times I$ via $h$. Since $H^4(W, \partial W; \mathbb{Z}/2) = H^3(M; \mathbb{Z}/2) = 0$, we can pick a lift of $\tau_W \colon W \to BO$ extending the standard lift of $\tau_M$ on both boundary components. Taking the product of the $s$-cobordism with $F$, we obtain an $s$-cobordism $W \times F$ with the product lift of $\tau_{W\times F}$ over $BO$. By Kirby-Siebenmann [7, Theorem 10.1, p. 194] there is a smooth structure on $W \times F$ which restricts to the standard smooth structure on both ends. The $s$-cobordism theorem then gives a self-diffeomorphism of the standard smooth structure on $M \times F$, realizing the given isometry.

Finally, we note that the smooth structure on $M \times F$ is actually *unique* up to diffeomorphism.

**Corollary C.** *Let $M$ be a closed, simply-connected topological 4-manifold with $KS(M) = 0$, and let $F$ be a closed oriented surface. Then $M \times F$ has a unique smooth structure.*

*Proof.* We can apply Theorem A to the topological manifold $M \times F$ equipped with two different smooth structures. By Novikov [10, Theorem 1], we have Condition 4 with $\phi = id$. □

**Remark 1.4.** The results of Kirby and Siebenmann [7, Theorem 5.4, p. 318] show that the set of distinct smoothings of $M \times F$ is in bijection with

$$[M \times F, TOP/O] = [M \times F, TOP/PL] = H^3(M \times F; \mathbb{Z}/2),$$

since in this dimension every $PL$ manifold admits a unique smooth structure. Theorem A shows that Homeo$(M \times F)$ acts transitively on the set of smoothings. It would be interesting to construct a corresponding homeomorphism for each $\alpha \in H^3(M \times F; \mathbb{Z}/2)$.

Here a *smoothing* is a pair $(N, h)$, where $N^6$ is a smooth 6-manifold and $h\colon N \to M \times F$ is a homeomorphism; two smoothings $(N, h)$ and $(N', h')$ are equivalent if there exists a diffeomorphism $\varphi\colon N \to N'$ such that $h$ and $h' \circ \varphi$ are topologically isotopic.

**Remark 1.5.** The effectiveness of our recognition result in practice will depend on the difficulty of verifying Conditions 3 and 4, but most of this is linear algebra. After obtaining Conditions 1 and 2, one might proceed by showing that $H^*(\widetilde{N})$ is isomorphic to a 4-dimensional algebra $\Lambda^*$, with $\Lambda^0 = \Lambda^4 = \mathbb{Z}$, $\Lambda^1 = \Lambda^3 = 0$, carrying the symmetric bilinear form $I(N)\colon \Lambda^2 \otimes \Lambda^2 \to \mathbb{Z}$ on a free abelian group $\Lambda^2 \cong \mathbb{Z}^r$. This gives the Euler characteristic equation $\chi(N) = \chi(F) \cdot (r + 2)$, and shows that $H^3(N) \cong H^1(F) \otimes H^2(\widetilde{N})$ is torsion-free. Now Poincaré duality for $H^3(N)$ shows that $I(N)$ is unimodular. After that, it will be necessary to check that $H^*(N) \cong \Lambda^* \otimes H^*(F)$ as graded algebras, and proceed to construct a cohomology ring isomorphism $\phi\colon H^*(M \times F) \to H^*(N)$ with the required conditions on $w_2(N)$ and $p_1(N)$.

However complicated the process, at least the conditions depend only on the primary algebraic topology of $N$ and do not involve determining the full homotopy type of $N$. For example, we do not assume anything about $\pi_3(N)$.

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## 2. The normal 2-type and normal 2-smoothings

For the proof we use the methods from [8] and assume that the reader is familiar with the basic concepts and theorems although we repeat the relevant definitions briefly.

We abbreviate $V := H^2(N)/u^*H^2(F) \cong H^2(\widetilde{N})$, and let $H = \pi_2(N) = H_2(\widetilde{N})$. We have $H^2(\widetilde{N}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_2(\widetilde{N}), \mathbb{Z})$, so that $H \cong \mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Z}) = V^*$. The following remark is immediate from the definitions.

**Lemma 2.1.** *If $N$ satisfies Condition 4 with respect to $M \times F$, then $H^2(M) \cong V$.*

We start by determining the normal 2-type of $N$. By definition, this is a fibration $B$ over $BSO$ where the homotopy groups of the fibre vanish in degree $\geq 3$ and such there is a lift of the normal Gauss map of $N$ over $B$, which is a 3-equivalence. We have to distinguish two cases, where the symmetric bilinear form $I(N) \colon V \times V \to \mathbb{Z}$ is a even or odd. In the first case, the normal 2-type is

$$p_{even} \colon B = K(H, 2) \times F \times BSpin \to BSO,$$

where the map is the composition of the projection to $BSpin$ and the canonical projection to $BSO$. If the form $I(N)$ is odd, one chooses a primitive characteristic element $v \in V$, and a complex line bundle $L_v$ over $K(H, 2)$ with first Chern class $v$. Then the normal 2-type is

$$p_{odd} \colon B = K(H, 2) \times F \times BSpin \to BSO,$$

where $p_{odd}$ is the map given by the projection to $K(H, 2) \times BSpin$ composed by the map given by the Whitney sum of line bundle $L_v$ and the canonical map to $BSO$ (of course, we have to replace this map by a fibration).

**Lemma 2.2.** *The normal 2-types of $M \times F$ and $N$ are given by $(B, p_{even})$, if $M$ is spin, or $(B, p_{odd})$ if $M$ is non-spin.*

*Proof.* We first look at the second stage of the Postnikov tower of $N$, this is a fibration over $K(\pi_1(N), 1)$ with fibre $K(\pi_2(N), 2)$, where in our situation $\pi_2(N) = H$. These fibrations are classified by the action of $\pi_1(N)$ on $\pi_2(N)$ and the $k$-invariant $k \in H^3(\pi_1(N); \pi_2(N))$. This group is zero, and so the action of $\pi_1(N)$ on $\pi_2(N)$ determines the Postnikov tower. If the $\pi_1(N)$-action is trivial, then we have the trivial fibration. Next, we use our data to construct a 3-equivalence

$$c_{M\times F} := g_{M\times F} \times h_{M\times F} \colon M \times F \to K(H, 2) \times F,$$

and a 3-equivalence

$$c_N := g_N \times h_N \colon N \to K(H, 2) \times F,$$

which is compatible with our data $\alpha$ and $\phi$. For this we consider the map

$$g_{M\times F} \colon M \times F \to K(H, 2)$$

such that $(g_{M\times F})^* \colon V \to H^2(M\times F) = H^2(M) \oplus H^2(F) = V \oplus H^2(F)$ is the inclusion onto the first summand (see Lemma 2.1), and choose a base point preserving map $g_N \colon N \to$

$K(H,2)$ such that $(g_N)^* = \phi \circ (g_{M\times F})^*$. Then we consider the projection $h_{M\times F} = q_2\colon M \times F \to F$ and $h_N = u\colon N \to F$. From Conditions 1 - 4 it is clear that the maps $c_{M\times F}$ and $c_N$ are 3-equivalences, with $(c_N)^* = \phi \circ (c_{M\times F})^*$.

If $N$ is Spin-manifold, then by assumption $M \times F$ is a Spin-manifold and we equip both manifolds with an arbitrary Spin structure $\omega_N$ and $\omega_{M\times F}$. If $N$ and so $M \times F$ are not Spin-manifolds, then we choose a primitive class $v \in H^2(M \times F;\mathbb{Z})$, such that its component in $H^2(F;\mathbb{Z})$ is zero, which reduces to $w_2(M \times F)$ and a spin structure $\omega_{M\times F}$ on $\nu(M \times F) \oplus L_v$, where $L_v$ is the complex line bundle classified by $v$. Similarly, we choose a Spin structure $\omega_N$ on $\nu(M) \oplus L_{\phi(v)}$. The maps $c_{M\times F}$ and $c_N$ together with the (twisted) Spin-structures are normal 2-smoothings in $(B, p_{odd/even})$. $\square$

## 3. The bordism groups

The next step in the proof of Theorem A is to show that, under the given conditions, the normal 2-smoothings constructed in Section 2 are bordant in $\Omega_6(B,\xi)$, where $\xi$ is the bundle classified by $p_{odd}$ or $p_{even}$ depending on the normal 2-type.

The method of proof is based on detecting elements in the bordism group by explicit invariants. We have $H \cong \mathbb{Z}^r$ so that $K := K(H,2) = (\mathbb{CP}^\infty)^r$. Let $Dp_1(N) \in H_2(N)$ denote the Poincaré dual of the first Pontrjagin class.

**Proposition 3.1.** *There is an injection $\Omega_6(B,\xi) \to \mathbb{Z} \oplus H_6(K) \oplus H_4(K) \oplus H_2(K)$, given by $\operatorname{sign} I(N)$, and the images of $[N]$, $[N] \cap u^*([F])$, $Dp_1(N)$ under the reference maps $c_N\colon N \to B$ for the normal 2-types.*

To compute the bordism groups we consider the functor associating to a space $X$ the bordism group of $p_{odd/even}\colon X \times K(H,2) \times BSpin \to BSO$, where the maps are defined as above in the case $X = F$. This is a homology theory denoted by $h_k(X)$ and so we can use the Mayer-Vietoris sequence to compute it, by writing a surface of genus $g$ as $D_2 \cup Y$, where $Y$ is a wedge of $2g$ circles. Then we obtain an exact sequence

$$\tilde{h}_7(S^2) \to \tilde{h}_6(Y) \to \tilde{h}_6(F) \to \tilde{h}_6(S^2) \to \tilde{h}_5(Y),$$

or, if we apply the suspension isomorphism, the exact sequence:

$$h_5(pt) \to \sum_{2g} h_5(pt) \to \tilde{h}_6(F) \to h_4(pt) \to . \tag{3.2}$$

The map from $h_6(F)$ to $h_4(pt)$ is defined by sending $[N, c_N] \mapsto [Q, c_Q]$, where $c_N\colon N \to B$ is a lift of the normal Gauss map, and $Q \subset N$ is the pre-image of a regular value of the composition of the map to $B$ with the projection to $F$. The reference map $c_Q\colon Q \to B$ is given by the restriction of $c_N$ to $K := K(H,2)$, together with the induced bundle and (twisted) Spin-structure.

To proceed further we need information about $h_k(pt) = \Omega_k^{\mathrm{Spin}}((\mathbb{CP}^\infty)^r)$, for $p_{even}$, and $h_k(pt) = \Omega_k^{\mathrm{Spin}}((\mathbb{CP}^\infty)^r, L)$, for $p_{odd}$. We begin with the case $r = 1$.

**Lemma 3.3.** *Let $L$ denote the Hopf bundle over $\mathbb{CP}^\infty$.*

(i) *The map $\Omega_4^{\mathrm{Spin}}(\mathbb{CP}^\infty) \to \mathbb{Z} \oplus \mathbb{Z}$ given by the signature and the image of the fundamental class is injective.*

(ii) $\Omega_6^{\mathrm{Spin}}(\mathbb{CP}^\infty) = \mathbb{Z} \oplus \mathbb{Z}$, *detected by the image of the fundamental class and the image* $Dp_1(N)$,

(iii) $\Omega_4^{\mathrm{Spin}}(\mathbb{CP}^\infty; L) \cong \mathbb{Z} \oplus \mathbb{Z}$, *detected by the image of the fundamental class and the image of* $Dp_1(N)$.

*Proof.* The $E^2$-term of the Atiyah-Hirzebruch spectral sequence computing $\Omega_4^{\mathrm{Spin}}(\mathbb{CP}^\infty)$ gives $\mathbb{Z}$ in position $(0,4)$ and $(4,0)$, and $\mathbb{Z}/2$ in position $(2,2)$. The differential

$$d\colon H_4(\mathbb{CP}^\infty; \mathbb{Z}) \to H_2(\mathbb{CP}^\infty; \mathbb{Z}/2)$$

is the reduction mod 2 composed by the dual of $Sq^2$ [12, Proposition 1, p. 750] and so is nontrivial. This implies that

$$\Omega_4^{\mathrm{Spin}}(\mathbb{CP}^\infty) \to \mathbb{Z} \oplus \mathbb{Z}$$

given by the signature and the image of the fundamental class is injective.

Analyzing the Atiyah-Hirzebruch spectral sequence for $\Omega_6^{\mathrm{Spin}}(\mathbb{CP}^\infty)$ gives an entry $\mathbb{Z}$ at position $(2,4)$ and $(6,0)$ and $\mathbb{Z}/2$ at position $(4,2)$. This time the differential vanishes and so the bordism group is either $\mathbb{Z} \oplus \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$. It was proven in [9, p. 258] that

$$\Omega_6^{\mathrm{Spin}}(\mathbb{CP}^\infty) = \mathbb{Z} \oplus \mathbb{Z},$$

detected by the image of the fundamental class and the image of $Dp_1(N)$.

Now we consider the bordism groups twisted by the line bundle $L$. We reduce the 4-th bordism group to the untwisted case by using the isomorphism given by taking the transversal preimage of $\mathbb{CP}^{N-1}$, where we replace $\mathbb{CP}^\infty$ by $\mathbb{CP}^N$ for a large $N$:

$$\Omega_6^{\mathrm{Spin}}(\mathbb{CP}^\infty) \cong \Omega_4^{\mathrm{Spin}}(\mathbb{CP}^\infty; L)$$

(here we use that $\Omega_6^{\mathrm{Spin}} = \Omega_5^{\mathrm{Spin}} = 0$) implying that $\Omega_4^{\mathrm{Spin}}(\mathbb{CP}^\infty; L) \cong \mathbb{Z} \oplus \mathbb{Z}$ again detected by the image of the fundamental class and the signature.

Finally the computation of $\Omega_4^{\mathrm{Spin}}(\mathbb{CP}^\infty; L) \cong \mathbb{Z} \oplus \mathbb{Z}$, again detected by the image of the fundamental class and the image of $Dp_1(N)$, follows from the Atiyah-Hirzebruch spectral sequence. This time the $E^\infty$-term is torsion free in the 6-line, since the differential $d\colon H_6(\mathbb{CP}^\infty; \mathbb{Z}) \to H_4(\mathbb{CP}^\infty; \mathbb{Z}/2)$ is the reduction mod 2 composed by the dual of $Sq^2$ plus $c_1(L) \cup \ldots$) (see again [12, Proposition 1, p. 750]) and so is trivial. □

**Lemma 3.4.** $h_5(pt)$ *is zero. The map* $h_6(pt) \to H_6(K) \oplus H_2(K)$ *given by the image of the fundamental class and the image of* $Dp_1(N)$ *is injective. The map given by the signature and the image of the fundamental class is an injection* $h_4(pt) \to \mathbb{Z} \oplus H_4(K)$.

*Proof.* Now we consider the general case. If we show that the bordism groups are again torsion free, then the statements follow from the Atiyah-Hirzebruch spectral sequence. We first note that by applying an appropriate isomorphism of $H \cong \mathbb{Z}^r$ we can assume in the twisted case that $c_1(L) = (0, \ldots, 0, 1)$. With this we write $(\mathbb{CP}^\infty)^r = X \times \mathbb{CP}^\infty$ and compute $\Omega_k^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty)$ and $\Omega_k^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L)$ for $k = 4$ and 6, where $X = (\mathbb{CP}^\infty)^{r-1}$ and $L$ is the Hopf bundle over the last factor. We assume inductively that $\Omega_k(X)$ is torsion free for $k = 4$ and $k = 6$. Using again the transversal preimage of $\mathbb{CP}^{N-1}$, where we replace $\mathbb{CP}^\infty$ by $\mathbb{CP}^N$ for a large $N$, we have an exact Gysin sequence (see [2, Section I.6, p. 315], [11]):

$$\Omega_5^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L)\to\Omega_6^{\mathrm{Spin}}(X)\to\Omega_6^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty)\to\Omega_4^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L).$$

Since the odd dimensional groups are by the Atiyah-Hirzebruch spectral sequence torsion, we see that $\Omega_6^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty)$ is torsion free, if $\Omega_4^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L)$ is torsion free. For this we consider the corresponding exact Gysin sequence (again, see [2, Section I.6]):

$$\Omega_3^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L\oplus L)\to\Omega_4^{\mathrm{Spin}}(X)\to\Omega_4^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L)\to\Omega_2^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L\oplus L).$$

The Atiyah-Hirzebruch spectral sequence implies that

$$\Omega_2^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L\oplus L)\cong H_2(X\times\mathbb{CP}^\infty)\oplus\mathbb{Z}/2.$$

Now we compare this exact sequence with that for $X$ a point:

$$\Omega_3^{\mathrm{Spin}}(\mathbb{CP}^\infty;L\oplus L)\to\Omega_4^{\mathrm{Spin}}\to\Omega_4^{\mathrm{Spin}}(\mathbb{CP}^\infty;L)\to\Omega_2^{\mathrm{Spin}}(\mathbb{CP}^\infty;L\oplus L).$$

We have maps from the first to the second exact sequence given by the projection from $X$ to a point. Now suppose that $\Omega_4^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L)$ contains a torsion element. Then, since by assumption $\Omega_4^{\mathrm{Spin}}(X)$ is torsion free, this maps to the non-trivial torsion element in $\Omega_2^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L\oplus L)$. But then the image in $\Omega_4^{\mathrm{Spin}}(\mathbb{CP}^\infty;L)$ is again a non-trivial torsion element, since in $\Omega_2^{\mathrm{Spin}}(\mathbb{CP}^\infty;L\oplus L)$ it maps to the non-trivial element. But this is a contradiction to what we have shown above that $\Omega_4^{\mathrm{Spin}}(\mathbb{CP}^\infty;L)$ is torsion free.

Now we have shown half of our statements, namely that $\Omega_6^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty)$ is torsion free as well as $\Omega_4^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L)$. We prove the other cases by a similar argument using this time the exact Gysin sequences:

$$\Omega_5^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L^{\oplus2})\to\Omega_6^{\mathrm{Spin}}(X)\to\Omega_6^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L)\to\Omega_4^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L^{\oplus2})$$

and

$$\Omega_3^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L^{\oplus3})\to\Omega_4^{\mathrm{Spin}}(X)\to\Omega_4^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L^{\oplus2})\to\Omega_2^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L^{\oplus3}).$$

This case is easier since $\Omega_2^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L^{\oplus3})$ is torsion free, the torsion in the $E^2$ term is killed by the $d_2$-differential.

Finally we show that $\Omega_4^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty)$ is torsion free using the exact sequence:

$$\Omega_3^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L)\to\Omega_4^{\mathrm{Spin}}(X)\to\Omega_4^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty)\to\Omega_2^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L).$$

By the same argument as above the group $\Omega_2^{\mathrm{Spin}}(X\times\mathbb{CP}^\infty;L)$ is torsion free finishing the argument.

Now we show that $h_5(pt)=0$. On the line corresponding to $h_5(pt)$ the only non-trivial entry in the $E_2$-term is $H_4(K;\mathbb{Z}/2)$. If $I(N)$ is even, the differentials are even given by the dual of $Sq^2$. If $I(N)$ is odd, where we had to use twisted Spin-structures, the differentials are given by the dual of $Sq^2$ plus $x\mapsto Sq^2x+w_2\cup x$, where $w_2$ is the reduction of $c$ mod 2. It is an easy exercise to show that the $E^3$-term is zero in both cases. $\square$

With this information we show that the bordism classes of $N$ and $M \times F$, equipped with the normal 2-smoothings constructed in Section 2, agree when identified via the maps $\alpha$ and $\phi$. By the exact sequence (3.2) and Lemma 3.4, this amounts to showing (i) the bordism classes in $h_6(pt)$ agree, and (ii) that the classes in $h_4(pt)$ agree, which we obtain as transversal preimages of a regular value of the map to $F$ given by composing our normal 2-smoothings with the projection to $F$.

By Lemma 3.4, the first invariant is given by two invariants, the image of the fundamental class in $H_6(K(H,2))$ and the image of $Dp_1(N)$ in $H_2(K(H,2))$. The image of the fundamental class in $H_6(K(H,2))$ is (by the cohomological structure of $K(H,2)$) equivalent to the triple product $g^*(x) \cup g^*(y) \cup g^*(z)$ for classes $x, y, z$ in $H^2(K(H,2))$. But these products vanish for $M \times F$ with $g = g_{M\times F}$, and for $N$ with $g = g_N$, since $\phi$ is an isometry of the cohomology rings and $(g_N)^* = \phi \circ (g_{M\times F})^*$. The image of $Dp_1(N)$ in $H_2(K(H,2))$ is determined by the products $g^*(x) \cup p_1$ for all $x \in H^2(K(H,2))$ and vanishes for $M \times F$ and for $N$ by Condition 4.

Thus we are left with the invariant in $h_4(pt)$. Let $Q \subset N$ be the transversal preimage of a regular value of the map $u\colon N \to F$. By Lemma 3.4, bordism classes in $h_4(pt)$ are determined by the signature of the underlying 4-manifold, and the image of the fundamental class $[Q]$ in $H_4(K(H,2)$. For a class $\beta \in H^4(N)$ we have the adjunction formula

$$\langle u^*([F]) \cup \beta, [N]\rangle = \langle i^*(\beta), [Q]\rangle,$$

where $i\colon Q \to N$ is the inclusion. Applying this to $\beta = p_1(N)$ we obtain:

$$\langle p_1(N) \cup u^*([F]), [N]\rangle = \langle p_1(Q), [Q]\rangle,$$

since the normal bundle of $Q$ is trivial. The signature theorem for $Q$ and Condition 4 (iv) imply that

$$\langle p_1(N) \cup u^*([F]), [N]\rangle = 3\,\mathrm{sign}(Q) = 3\,\mathrm{sign}(M),$$

proving the equality for the first invariant in $h_4(pt)$.

For the second invariant we note that the image of the fundamental class of $Q$ in $H_4(K(H,2)$ is determined by the numbers

$$\langle i^*g^*(x) \cup i^*g^*(y), [Q]\rangle.$$

We apply again the adjunction formula for $\beta = g^*(x) \cup g^*(y)$, and get

$$\langle g^*(x) \cup g^*(y) \cup u^*([F]), [N]\rangle = \langle i^*g^*(x) \cup i^*g^*(y), [Q]\rangle,$$

where $g = g_N$. A similar formula holds for $M \times F$ and $g = g_{M\times F}$. The left side agrees for $N$ and $M \times F$, since $\phi$ is an isomorphism of the cohomology ring. Thus also the second invariant for the element in $h_4(pt)$ agrees. Summarizing, we have shown:

**Proposition 3.5.** *If the conditions of Theorem A are fulfilled, then the bordism classes*

$$[N, c_N] = [M \times F, c_{M\times F}] \in \Omega_6(B, \xi),$$

*for the normal* 2*-smoothings on* $N$ *and* $M \times F$ *constructed in Lemma* 2.2.

## 4. The proof of Theorem A

We consider $N$ and $M \times F$ equipped with normal 2-smoothings compatible with $\alpha$ and $\phi$. By Proposition 3.5, the corresponding bordism classes are equal. Choose a $B$-bordism $W$ between these two normal 2-smoothings. Since the Euler characteristics of $N$ and $M \times F$ agree, there is an obstruction $\theta(W) \in l_7(\pi_1(F))$ which is elementary if and only if $W$ is $B$-bordant to an s-cobordism. We first note that the Whitehead group for $\pi_1(F)$ vanishes by a result of Farrell-Hsiang [3], so that we can ignore decorations in the $l$-monoids and $L$-groups. Next we note that the intersection form on $\pi_3(M \times F) \cong \pi_3(M)$ with values in the group ring vanishes identically (since $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G) = 0$ for $G$ an infinite group). By [8, Proposition 8, p. 739], this implies that $\theta(W)$ sits in the ordinary $L$-group $L_7(\pi_1(F))$. But by Cappell [1, Theorem 18], there is a closed 7-manifold with $B$-structure so that after taking the disjoint union of $W$ with this manifold the obstruction in $L_7(\pi_1(F))$ vanishes. This completes the proof.

## References


[1] S. E. Cappell, *Mayer-Vietoris sequences in hermitian K-theory*, Algebraic K-theory, III: Hermitian K-theory and geometric applications (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972) (Berlin), Lecture Notes in Math., vol. 343, Springer, 1973, pp. 478–512.

[2] J.-P. Dax, *Étude homotopique des espaces de plongements*, Ann. Sci. École Norm. Sup. (4) **5** (1972), 303–377.

[3] F. T. Farrell and W.-C. Hsiang, *A formula for* $K_1 R_\alpha [T]$, Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 192–218.

[4] M. H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), 357–453.

[5] R. Friedman and J. W. Morgan, *On the diffeomorphism types of certain algebraic surfaces. I*, J. Differential Geom. **27** (1988), 297–369.

[6] P. E. Jupp, *Classification of certain* 6*-manifolds*, Proc. Cambridge Philos. Soc. **73** (1973), 293–300.

[7] R. C. Kirby and L. C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Princeton University Press, Princeton, N.J., 1977, With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88.

[8] M. Kreck, *Surgery and duality*, Ann. of Math. (2) **149** (1999), 707–754.

[9] ______, *Simply connected asymmetric manifolds*, J. Topol. **2** (2009), 249–261.

[10] S. P. Novikov, *On manifolds with free abelian fundamental group and their application*, Izv. Akad. Nauk SSSR Ser. Mat. **30** (1966), 207–246.

[11] H. A. Salomonsen, *Bordism and geometric dimension*, Math. Scand. **32** (1973), 87–111.

[12] P. Teichner, *On the signature of four-manifolds with universal covering spin*, Math. Ann. **295** (1993), 745–759.

[13] C. T. C. Wall, *Classification problems in differential topology. V. On certain* 6*-manifolds*, Invent. Math. 1 (1966), 355-374; corrigendum, ibid **2** (1966), 306.

[14] A. V. Žubr, *Classification of simply connected six-dimensional spin manifolds*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), 839–859.



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