FREE ACTIONS OF EXTRASPECIAL p-GROUPS ON $S^n \times S^n$

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Abstract. Let p be an odd regular prime, and let G_p denote the extraspecial p–group of order p^3 and exponent p. We show that G_p acts freely and smoothly on $S^{2p-1} \times S^{2p-1}$. For $p = 3$ we explicitly construct a free smooth action of a Lie group $\widetilde{G}_3$ containing G_3 on $S^5 \times S^5$. In addition, we show that any finite odd order subgroup of the exceptional Lie group $\mathfrak{G}_2$ admits a free smooth action on $S^{11} \times S^{11}$.

Introduction

Conner [10] and Heller [15] proved that any finite group G acting freely on a product of two spheres must have $\operatorname{rank} G \leqq 2$. In other words, the maximal rank of an elementary abelian subgroup of G has to be less than or equal to 2. The survey article by A. Adem [1] describes recent progress on the existence problem (see also [17]), and it is now known that all rank two finite p-groups, for p odd, admit a free smooth action on some product $S^n \times S^m$ (see [2] for $p > 3$, [36] for $p = 3$). If the two spheres have the same dimension, then there are additional restrictions, at least for groups of composite order (see Oliver [26]), but it is not yet known even which rank two p-groups act freely on $S^n \times S^n$.

The work of G. Lewis [23] shows that the 2-Sylow subgroup of a finite group G which acts freely on a finite CW-complex homotopy equivalent to $S^n \times S^n$, is abelian provided $n \equiv 1 \bmod 4$. For p odd, Lewis proves that the p-Sylow subgroup is abelian if $p \nmid n + 1$, n odd. Lewis also points out [23, p. 538] that any metacyclic group can act freely and smoothly on some $S^n \times S^n$, but the existence of free actions by other non-abelian p-groups has been a long-standing open question.

Theorem A. *Let $p \geq 3$ be an odd prime, and let G be the non-abelian p-group of order p^3 and exponent p. Then there exists a finitely-dominated G-CW complex $X \simeq S^{2p-1} \times S^{2p-1}$. If p is a regular prime, then G acts freely and smoothly on $S^{2p-1} \times S^{2p-1}$.*

Our existence result contradicts claims made in [3], [4], [35], and [39] that these actions do not exist. It was later shown by Benson and Carlson [6] that such actions could not be ruled out by cohomological methods.

It is also interesting to ask about the existence of free actions in other dimensions. Lewis [24] proved that the only possible dimensions for a free G_p-action on $S^n \times S^n$ are $n = 2pr - 1$, $r \geq 1$. Let $\mathfrak{G}_2$ denote the exceptional Lie group of dimension 14.

Theorem B. *All odd order finite subgroups of $\mathfrak{G}_2$ act freely and smoothly on $S^{11} \times S^{11}$.*

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Note that the extraspecial 3-group of order 27 is a subgroup of $\mathfrak{G}_2$. Therefore the existence problem for the extraspecial p-groups, for p an odd regular prime, is now settled for $r = 1$, and for $r = 2$ if $p = 3$, but the other cases for $r > 1$ are open.

As an alternative proof, and an extension of Theorem A for $p = 3$, in Section 11 we provide an explicit construction of a free smooth $\widetilde{G}_3$-action, where $\widetilde{G}_3$ is a certain infinite Lie group containing the extraspecial 3-group of order 27 (see Section 3 for the definition).

Theorem C. *The group $\widetilde{G}_3$ acts freely and smoothly on $S^5 \times S^5$.*

We can expect an even more complicated structure for the 2-Sylow subgroup of a finite group acting freely on some $S^n \times S^n$, since this is already the case for free actions on S^n. We can take products of periodic groups $G_1 \times G_2$ and obtain a variety of actions of non-metacyclic groups on $S^n \times S^n$ (see [13] for the existence of these examples, generalizing the results of Stein [30]). Here the 2-groups are all metabelian, so one might hope that this is the correct restriction on the 2-Sylow subgroup. However, there are non-metabelian 2-groups which are subgroups of $Sp(2)$, hence by generalizing the notion of fixity in [2] to quaternionic fixity, one can construct free actions of these non-metabelian 2-groups on $S^7 \times S^7$ (see [36]).

We will always assume that our actions on $S^n \times S^n$ are homologically trivial, implying that n is odd.

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Contents

1. An exact sequence of G. Lewis

Suppose G is a finite group, and let X be a finite, free G-CW complex homotopy equivalent to $S^n \times S^n$, with $n > 1$. For any subgroup $H \subset G$, we let $X_H = X/H$ denote the quotient complex. By [23, Prop. 2.1] there is an exact sequence

$$\cdots \to \widehat{H}^{i+n}(G;\mathbf{Z}) \to \widehat{H}^{i-n-1}(G;\mathbf{Z}) \to \widehat{H}^{i}(G;\mathbf{Z}\oplus\mathbf{Z}) \to \widehat{H}^{i+n+1}(G;\mathbf{Z}) \to \dots$$

in Tate cohomology, for any $i \in \mathbf{Z}$. It is useful to recall that $\widehat{H}^{-i} = \widehat{H}^{i}$ and that $\widehat{H}^{i}(G;A) = \widehat{H}_{-i-1}(G;A)$ for any G-module A. If the coefficients in Tate cohomology are not mentioned, then $\mathbf{Z}$-coefficients are understood.

We denote by $\Omega^k\mathbf{Z}$ the kernel after k steps in a resolution of $\mathbf{Z}$ by finitely generated projective $\mathbf{Z}G$-modules. This $\mathbf{Z}G$-module is stably well-defined, up to direct sum with f.g. projectives, by Schanuel's Lemma. The dual module is denoted $S^k\mathbf{Z}$. The Lewis exact sequence arises from chain duality.

Proposition 1.1 ([14, Prop. 2.4])**.** *Let G be a finite group, and let X_G be a finitely dominated, oriented Poincaré $2n$-complex, with $\pi_1(X_G, y_0) = G$ and universal covering space X. If X is $(n-1)$-connected, $n \geq 2$, then there is an exact sequence*

$$0 \to \Omega^{n+1}\mathbf{Z} \to \pi_n(X_G) \oplus P \to S^{n+1}\mathbf{Z} \to 0$$

for some f.g. projective $\mathbf{Z}G$-module P. There is a natural identification

$$\operatorname{Ext}^1_{\mathbf{Z}G}(S^{n+1}\mathbf{Z}, \Omega^{n+1}\mathbf{Z}) \cong H_{2n}(G;\mathbf{Z})$$

under which the class of this extension corresponds to the image of the fundamental class $c_*[X_G] \in H_{2n}(G;\mathbf{Z})$, where $c\colon X_G \to BG$ classifies the universal covering $X \to X_G$.

Proof. The proof given in [14] for $n = 2$ generalizes easily. □

We also have the identifications

$$\widehat{H}^{2n+1}(G;\mathbf{Z}) \cong \widehat{H}^{-2n-1}(G;\mathbf{Z}) = H_{2n}(G;\mathbf{Z})\ .$$

Let $\lambda_G(X) \in \widehat{H}^{2n+1}(G;\mathbf{Z})$ denote the class corresponding to $c_*[X_G]$.

Corollary 1.2. *The connecting map*

$$\widehat{H}^{i+n}(G;\mathbf{Z}) \to \widehat{H}^{i-n-1}(G;\mathbf{Z}) = \widehat{H}^{n+1-i}(G;\mathbf{Z})$$

in the Lewis exact sequence sends $a \mapsto \lambda_G(X) \cup a$.

Since the Lewis exact sequence is natural under restriction to a subgroup $H \subset G$ and $\operatorname{Res}_H(1_G) = 1_H$, we obtain

$$\operatorname{Res}_H(\lambda_G(X)) = \lambda_H(X)\ .$$

Corollary 1.3. *Suppose that X is a finite, free G-CW complex homotopy equivalent to $S^n \times S^n$, and $H \lhd G$ is a normal subgroup. Then $\operatorname{Res}_H(c_*[X_G]) = c_*[X_H] \in H_{2n}(H;\mathbf{Z})$ is invariant under the action of G by conjugation.*

Lewis points out that this exact sequence can be used to give some restrictions on groups acting freely on products $S^n \times S^n$. The invariance of the fundamental class leads to further restrictions.

2. An overview of the proof

Given a finite group G, and two cohomology classes $\theta_1,\ \theta_2 \in H^{n+1}(G;\mathbf{Z})$, we can construct an associated space B_G as the total space of the induced fibration

$$\begin{array}{ccc} K(\mathbf{Z}\oplus\mathbf{Z},n) \longrightarrow & B_G & \\ & \big\downarrow & \\ & BG & \xrightarrow{\theta_1,\theta_2} K(\mathbf{Z}\oplus\mathbf{Z},n) \end{array}$$

where BG denotes the classifying space of G. If we also have a stable oriented bundle $\nu_G\colon B_G \to BSO$, then we can consider the bordism groups

$$\Omega_{2n}(B_G,\nu_G)$$

defined as in [31, Chap. II]. The objects are commutative diagrams

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \nu_G \\ \big\downarrow & & \big\downarrow \\ M & \xrightarrow{f} & B_G \end{array}$$

where M^{2n} is a closed, smooth $2n$-dimensional manifold with stable normal bundle ν_M, $f\colon M \to B_G$ is a reference map, and $b\colon \nu_M \to \nu_G$ is a stable bundle map covering f. The bordism relation is the obvious one consistent with the normal data and reference maps.

For an extraspecial p-group $G = G_p$, p odd, and $n = 2p-1$, we will choose suitable cohomology classes θ_1, θ_2 and a suitable bundle ν_G over B_G. The space B_G will then be a model for the n-type of the orbit space of a possible free G-action on a finite dimensional G-CW complex $X \simeq S^n \times S^n$, and ν_G will be a candidate for its stable normal bundle. The choice of this data was not obvious (at least to us), but depended on a detailed preliminary study of the constraints on possible G-actions imposed by the subgroup structure of G.

After selecting the n-type B_G and the bundle ν_G, our existence proof proceeds in the following steps:

(i) We construct a subset $T_G \subseteq H_{2n}(B_G;\mathbf{Z})$, depending on the data $(G,\theta_1,\theta_2,\nu_G)$, containing the images of fundamental classes of some possible free G-actions on $S^n \times S^n$.

(ii) We show that there is a bordism element $[M,f] \in \Omega_{2n}(B_G,\nu_G)$ whose image under the Hurewicz map $\Omega_{2n}(B_G,\nu_G) \to H_{2n}(B_G;\mathbf{Z})$ lies in T_G.

(iii) We prove that we can obtain $\widetilde{M} = S^n \times S^n$ by surgery on $[M,f]$ within its bordism class (at least if p is an odd regular prime).

This approach to the problem follows the general outline of Kreck's "modified surgery" program (see [20]). To carry out this outline, we lift the data for our problem to an extension group

$$1 \to S^1 \to \widetilde{G}_p \to \mathbf{Z}/p \times \mathbf{Z}/p \to 1$$

of $\mathbf{Z}/p \times \mathbf{Z}/p$ by the circle S^1. Then, by construction, $\widetilde{G}_p$ contains G_p as a subgroup, and the induced map on classifying spaces gives a circle bundle

$$S^1 \to B_{G_p} \to B_{\widetilde{G}_p} .$$

The cohomology of the groups $\widetilde{G}_p$ is much simpler than that of the extra-special p-groups, so the computations of Step (ii) are done with the lifted data. We define a lifted subset

$$T_{\widetilde{G}_p} \subseteq H_{2n-1}(B_{\widetilde{G}_p}; \mathbf{Z})$$

which maps to T_{G_p} under the S^1-bundle transfer

$$trf\colon H_{2n-1}(B_{\widetilde{G}_p}; \mathbf{Z}) \to H_{2n}(B_{G_p}; \mathbf{Z})$$

induced by the fibration of classifying spaces. The definition of $T_{\widetilde{G}_p}$ depends on the existence of a free smooth action on $S^n \times S^n$ by the circle subgroup of $\widetilde{G}_p$, whose stable normal bundle is compatible with the chosen normal data $\nu_{\widetilde{G}_p}$.

We then prove that any element $\gamma \in T_{\widetilde{G}_p}$ lies in the image of the Hurewicz map

$$\Omega_{2n-1}(B_{\widetilde{G}_p}, \tilde{\nu}_G) \to H_{2n-1}(B_{\widetilde{G}_p}; \mathbf{Z}),$$

and construct the element $[M, f]$ required for Step (ii) as the total space of the pulled-back S^1-bundle. Step (iii) now proceeds by surgery theory.

3. Representations and cohomology of the extension group

§3A. Definition of $\widetilde{G}_p$ and some of its subgroups. Let p be an odd prime and G_p be the extraspecial p–group of order p^3 and exponent p. Consider the following presentation:

$$G_p = \langle a, b, c \mid a^p = b^p = c^p = [a, c] = [b, c] = 1,\ [a, b] = c \rangle$$

where $[x, y]$ denotes $x^{-1}y^{-1}xy$. Let $\widetilde{G}_p$ be the group obtained from $S^1 \times G_p$ by amalgamating $\langle c \rangle$ the subgroup of G_p generated by c to the cyclic subgroup of S^1 of order p. Hence we have a commutative diagram of central extensions where $Q_p \cong \mathbf{Z}/p \times \mathbf{Z}/p$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \langle c \rangle & \longrightarrow & G_p & \longrightarrow & Q_p & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & S^1 & \longrightarrow & \widetilde{G}_p & \longrightarrow & Q_p & \longrightarrow & 1
\end{array}$$

Considering G_p as a subgroup of $\widetilde{G}_p$, let d_t denote the element ab^t in $\widetilde{G}_p$, for $t = 0, \dots, p-1$, and let $d_p = b$ in $\widetilde{G}_p$. Let $\widetilde{H}_t = \langle d_t, S^1 \rangle$ denote the subgroup of $\widetilde{G}_p$ generated by d_t and S^1 for $t = 0, \dots, p$. In particular, we will sometimes write $\widetilde{H}$ instead of $\widetilde{H}_p$ for the subgroup of $\widetilde{G}_p$ generated by b and S^1. Similarly, we denote some subgroups of G_p as follows: $D_t = \langle d_t \rangle$, $H_t = \langle d_t, c \rangle$ for $t = 0, \dots, p$, and $H = H_p$.

Remark 3.1. The automorphism group $\mathrm{Aut}(\widetilde{G}_p)$ surjects onto $\mathrm{Aut}(Q_p) = GL_2(p)$. More explicitly, the action of a matrix $A \in GL_2(p)$ can be lifted to $\phi \in \mathrm{Aut}(\widetilde{G}_p)$ by specifying $\phi(z) = z^{\det A}$, for all $z \in S^1$. In particular, $SL_2(p)$ is a subgroup of $\mathrm{Aut}(\widetilde{G}_p)$.

§3B. Representations of $\widetilde{G}_p$ and some of its subgroups. First, we define a 1–dimensional representation Φ_t of $\widetilde{G}_p$ for every $t \in \{0, \dots, p\}$, so that

$$\Phi_t \colon \widetilde{G}_p \to U(1)$$

where Φ_t has kernel $\widetilde{H}_{p-t}$ and sends d_t to $e^{2\pi i/p}$. For G any subgroup of $\widetilde{G}_p$ and t in $\{0, \dots, p\}$ we define $\Phi_{t,G}$ as the 1-dimensional representation of G

$$\Phi_{t,G} = \operatorname{Res}_G^{\widetilde{G}_p}(\Phi_t) : G \to U(1)$$

Second, we define a 1–dimensional representation Φ'_t of $\widetilde{H}_t$ for every $t \in \{0, \dots, p\}$, so that

$$\Phi'_t \colon \widetilde{H}_t \to U(1)$$

where $\Phi'_t(d_t) = 1$ and $\Phi'_t(z) = z$ for z in S^1. For G any subgroup of $\widetilde{H}_t$ and t in $\{0, \dots, p\}$ we define $\Phi_{t,G}$ as the 1-dimensional representation of G

$$\Phi'_{t,G} = \operatorname{Res}_G^{\widetilde{H}_t}(\Phi'_t) : G \to U(1)$$

Finally, we define a p–dimensional irreducible representation Ψ of $\widetilde{G}_p$ as follows:

$$\Psi = \operatorname{Ind}_{\widetilde{H}}^{\widetilde{G}_p}(\Phi) \colon \widetilde{G}_p \to SU(p)$$

where $\Phi = \Phi'_p$. For G any subgroup of $\widetilde{G}_p$ we define Ψ_G as the p-dimensional representation of G

$$\Psi_G = \operatorname{Res}_G^{\widetilde{G}_p}(\Psi) : G \to SU(p)$$

§3C. Cohomology of $\widetilde{G}_p$ and some of its subgroups. We will use the notations and results of Leary [21] for the integral cohomology ring of $\widetilde{G}_p$.

Theorem 3.2 (Theorem 2 in [21]). *$H^*(B\widetilde{G}_p; \mathbf{Z})$ is generated by elements $\alpha, \beta, \chi_1, \chi_2, \dots, \chi_{p-1}, \zeta$, with*

$$\deg(\alpha) = \deg(\beta) = 2, \quad \deg(\zeta) = 2p, \quad \deg(\chi_i) = 2i,$$

subject to some relations.

In the statement of Theorem 3.2, the elements $\alpha = \Phi_0 \colon \widetilde{G}_p \to U(1)$ and $\beta = \Phi_p \colon \widetilde{G}_p \to U(1)$, by considering $H^2(B\widetilde{G}_p; \mathbf{Z}) = \operatorname{Hom}(\widetilde{G}_p, S^1)$, and ζ is the p^{th} Chern class of the p–dimensional irreducible representation Ψ of $\widetilde{G}_p$. The mod p cohomology ring of $\widetilde{G}_p$ is also given by Leary:

Theorem 3.3 (Theorem 2 in [22]). *$H^*(B\widetilde{G}_p; \mathbf{Z}/p)$ is generated by elements y, y', x, x', $c_2, c_3, \dots, c_{p-1}, z$, with*

$$\begin{aligned} \deg(y) &= \deg(y') = 1, \ \deg(x) = \deg(x') = 2, \\ \deg(z) &= 2p, \quad \text{and} \quad \deg(c_i) = 2i, \end{aligned}$$

subject to some relations.

Let π_* stand for the projection map from $H^*(B\widetilde{G}_p;\mathbf{Z})$ to $H^*(B\widetilde{G}_p;\mathbf{Z}/p)$, and δ_p for the Bockstein from $H^*(B\widetilde{G}_p;\mathbf{Z}/p)$ to $H^{*+1}(B\widetilde{G}_p;\mathbf{Z})$ then $\delta_p(y)=\alpha$, $\delta_p(y')=\beta$, $\pi_*(\alpha)=x$, $\pi_*(\beta)=x'$, $\pi_*(\chi_i)=c_i$, and $\pi_*(\zeta)=z$. Here are some facts about the cohomology of a few more groups.

Remark 3.4. Considering $H^2(BG,\mathbf{Z})=\mathrm{Hom}(G,S^1)$

(1) $H^*(BS^1;\mathbf{Z})=\mathbf{Z}[\tau]$ where $\tau=\Phi'_{t,S^1}$. So τ is the identity map on S^1.
(2) $H^*(B\widetilde{H}_t;\mathbf{Z})=\mathbf{Z}[\tau',v'\,|\,pv'=0]$ where $\tau'=\Phi'_t$ and $v'=\Phi_{t,\widetilde{H}_t}$
(3) $H^*(B\widetilde{H}_t,\mathbf{Z}/p)=\mathbf{F}_p[\overline{\tau}]\otimes(\Lambda(u)\otimes\mathbf{F}_p[v])$ where $\overline{\tau}$ and v are mod p reductions of τ' and v' respectively and $\beta(u)=v$.

We calculate some restriction maps.

Lemma 3.5. *The restriction map $H^2(B\widetilde{G}_p;\mathbf{Z})\to H^2(B\widetilde{H}_t;\mathbf{Z})$ maps α to v' when $0\le t\le p-1$ and to 0 when $t=p$ and maps β to tv' when $0\le t\le p-1$ and to v' when $t=p$.*

Proof. It is easy to see that $\Phi_{0,\widetilde{H}_t}$ maps $S^1\subseteq\widetilde{H}_t$ to 1 and maps d_t to $e^{2\pi i/p}$ when $0\le t\le p-1$, and to 1 when $t=p$. Hence $\Phi_{0,\widetilde{H}_t}=\Phi_{t,\widetilde{H}_t}$ when $0\le t\le p-1$ and $\Phi_{0,\widetilde{H}_t}$ is the trivial representation when $t=p$. Considering $H^2(B\widetilde{H}_t;\mathbf{Z})=\mathrm{Hom}(\widetilde{H}_t,S^1)$, we see that the restriction map from $H^2(B\widetilde{G}_p;\mathbf{Z})$ to $H^2(B\widetilde{H}_t;\mathbf{Z})$ maps α to v' when $0\le t\le p-1$ and to 0 when $t=p$ and the rest of the result of this lemma follows as $\Phi_{p,\widetilde{H}_t}$ maps $S^1\subseteq\widetilde{H}_t$ to 1 and maps d_t to $e^{2\pi it/p}$ when $0\le t\le p-1$ and to $e^{2\pi i/p}$ when $t=p$. □

Lemma 3.6. *The restriction $H^{2p}(B\widetilde{G}_p;\mathbf{Z})\to H^{2p}(B\widetilde{H}_t;\mathbf{Z})$ maps $\alpha^p-\alpha^{p-1}\beta+\beta^p$ to $(v')^p$.*

Proof. By Lemma 3.5 we see that $Res^{\widetilde{G}_p}_{\widetilde{H}_t}(\alpha^p-\alpha^{p-1}\beta+\beta^p)=(v')^p-t(v')^p+(tv')^p=(v')^p$, for $0\le t\le p-1$, and equals $0-0+(v')^p=(v')^p$ for $t=p$. □

4. The associated space $B_{\widetilde{G}_p}$

We now construct the space $B_{\widetilde{G}_p}$ needed as a model for the $(2p-1)$-type of our action. Take the element:

$$k=\theta_1\oplus\theta_2=\zeta\oplus\left(\alpha^p-\alpha^{p-1}\beta+\beta^p\right)$$

in $H^{2p}(\widetilde{G}_p;\mathbf{Z}\oplus\mathbf{Z})=H^{2p}(\widetilde{G}_p;\mathbf{Z})\oplus H^{2p}(\widetilde{G}_p;\mathbf{Z})$. For G any subgroup of $\widetilde{G}_p$ define

$$k_G=\mathrm{Res}^{\widetilde{G}_p}_G(k)\in H^{2p}(G;\mathbf{Z}\oplus\mathbf{Z}),$$

and define π_G as the fibration classified by k_G:

$$\begin{array}{ccccc} K(\mathbf{Z}\oplus\mathbf{Z},2p-1) & \longrightarrow & B_G & & \\ & & \Big\downarrow{\scriptstyle \pi_G} & & \\ & & BG & \xrightarrow{k_G} & K(\mathbf{Z}\oplus\mathbf{Z},2p)\ . \end{array}$$

Note that the natural map $BG \to B\widetilde{G}_p$, induced by the inclusion, gives a diagram

$$\begin{array}{ccc} B_G & \longrightarrow & B_{\widetilde{G}_p} \\ \downarrow{\scriptstyle \pi_G} & & \downarrow{\scriptstyle \pi_{\widetilde{G}_p}} \\ BG & \longrightarrow & B\widetilde{G}_p \end{array}$$

which is a pull-back square.

5. The bundle data for $B_{\widetilde{G}_p}$

For any subgroup $G \subseteq \widetilde{G}_p$ we will define two bundles $\widehat{\psi}_G$ and ξ_G over BG, which will pull back by the classifying map to the stable tangent and normal bundle respectively of the quotient of a possible G-action on $S^n \times S^n$. The pullbacks of these bundles over BG to bundles over B_G will be denoted by τ_G and ν_G respectively.

§5A. Tangent bundles. We have the representations $\Psi_G \colon G \to SU(p)$ and $\phi_{t,G} \colon G \to U(1)$. Let ψ_G denote the p–dimensional complex vector bundle classified by

$$\psi_G = B\Psi_G \colon BG \to BSU(p),$$

and let $\phi_{t,G}$ denote the complex line bundle classified by

$$\phi_{t,G} = B\Phi_{t,G} \colon BG \to BU(1) = BS^1 \ .$$

We define a $3p$–dimensional complex vector bundle $\widehat{\psi}_G$ on BG by the Whitney sum

$$\widehat{\psi}_G = \psi_G \oplus \phi_{0,G}^{\oplus p} \oplus \phi_{p,G}^{\oplus p}$$

and use the same notation for the stable vector bundle $\widehat{\psi}_G \colon BG \to BSO$. We now identify our candidate τ_G for the stable tangent bundle.

Definition 5.1. Let τ_G denote the stable vector bundle on B_G classified by the composition

$$\tau_G \colon B_G \xrightarrow{\pi_G} BG \xrightarrow{\widehat{\psi}_G} BSO \ .$$

§5B. Normal bundles. First we show that there is an stable inverse of the vector bundle $\widehat{\psi}_G$ over BG, when restricted to a finite skeleton of BG.

Lemma 5.2. *For any subgroup $G \subseteq \widetilde{G}_p$, there exists a stable bundle $\xi_G \colon BG \to BSO$, such that $\xi_G \oplus \widehat{\psi}_G = \varepsilon$, the trivial bundle, when restricted to the $(4p-1)$-skeleton of BG.*

Proof. Take $N = 4p-1$ and let $\widehat{\psi}_{\widetilde{G}_p}|_{B\widetilde{G}_p^{(N)}}$ denote the pull-back of $\widehat{\psi}_{\widetilde{G}_p}$ to $B\widetilde{G}_p^{(N)}$, the N–th skeleton of $B\widetilde{G}_p$, by the inclusion map of $B\widetilde{G}_p^{(N)}$ in $B\widetilde{G}_p$. Then there exists a vector bundle $\xi_{\widetilde{G}_p}$ over $B_{\widetilde{G}_p}^{(N)}$ such that the bundle $\xi_{\widetilde{G}_p} \oplus (\widehat{\psi}_{\widetilde{G}_p}|_{B\widetilde{G}_p^{(N)}})$ is trivial over $B\widetilde{G}_p^{(N)}$, since $B\widetilde{G}_p^{(N)}$ is a finite CW–complex. Stably this vector bundle is classified by a map $\xi_{\widetilde{G}_p} \colon B\widetilde{G}_p^{(N)} \to BU$

and there is no obstruction to extending this classifying map to a map $B\widetilde{G}_p \to BU$, as the obstructions to doing so lie in the cohomology groups

$$H^{*+1}(B\widetilde{G}_p, B\widetilde{G}_p^{(N)}; \pi_*(BU)) = 0 \ .$$

We will use the same notation $\xi_{\widetilde{G}_p}$ to denote the stable vector bundle classified by any extension map $B\widetilde{G}_p \to BU \to BSO$. We then define

$$\xi_G \colon BG \to BSO$$

by composition with the map $BG \to B\widetilde{G}_p$ induced by $G \subseteq \widetilde{G}_p$. □

We now identify our candidates ν_G for the stable normal bundle.

Definition 5.3. Let ν_G denote the stable vector bundle on B_G classified by the composition

$$\nu_G \colon B_G \xrightarrow{\pi_G} BG \xrightarrow{\xi_G} BSO \ .$$

§5C. Characteristics classes. Let ξ be a bundle over B. The k^{th} Chern class (see [25, p. 158]) of the bundle ξ will be denoted as follows

$$\mathrm{c}_k(\xi) \in H^{2k}(B; \mathbf{Z})$$

and the total Chern class of the bundle ξ will be denoted as follows

$$\mathrm{c}(\xi) = \mathrm{c}_0(\xi) + \mathrm{c}_1(\xi) + \mathrm{c}_2(\xi) + \dots$$

The k^{th} Pontrjagin class (see [25, p. 174]) of the bundle ξ is denoted as follows

$$p_k(\xi) \in H^{4k}(B; \mathbf{Z})$$

and see [25, p. 228] for the definition of the following characteristic classes

$$q_k(\xi) \in H^{2(p-1)k}(B; \mathbf{Z}/p)$$

§5D. Calculations of characteristics classes. In this section we calculate some of these characteristic classes for the bundles $\widehat{\psi}_{\widetilde{H}_t}$ and $\xi_{\widetilde{H}_t}$ over $B\widetilde{H}_t$.

Lemma 5.4. *The total Chern class of $\widehat{\psi}_{\widetilde{H}_t}$ is*

$$\mathrm{c}(\widehat{\psi}_{\widetilde{H}_t}) = \mathrm{c}(\psi_{\widetilde{H}_t})\, \mathrm{c}(\phi_{0,\widetilde{H}_t}^{\oplus p} \oplus \phi_{p,\widetilde{H}_t}^{\oplus p})$$

where

(1) $\mathrm{c}(\psi_{\widetilde{H}_t}) = 1 - (v')^{p-1} + ((\tau')^p - (v')^{p-1}\tau')$
(2) $\mathrm{c}(\phi_{0,\widetilde{H}_t}^{\oplus p} \oplus \phi_{p,\widetilde{H}_t}^{\oplus p}) = 1 + (1+t)(v')^p + t(v')^{2p}$

Proof. Given two 1-dimensional representation $\Phi \colon G \to S^1$ and $\Phi' \colon G \to S^1$ and a natural number k, we will write $\Phi^k(g) = (\Phi(g))^k$ and $(\Phi\Phi')(g) = \Phi(g)\Phi'(g)$. It is easy to see that

$$\Psi_{\widetilde{H}_t} = \Phi'_t \oplus \Phi_{t,\widetilde{H}_t}\Phi'_t \oplus \Phi^2_{t,\widetilde{H}_t}\Phi'_t \oplus \dots \oplus \Phi^{p-1}_{t,\widetilde{H}_t}\Phi'_t \ .$$

Hence the total Chern class of $\psi_{\widetilde{H}_t}$ is

$$(1+\tau')(1+v'+\tau')(1+2v'+\tau')\dots(1+(p-1)v'+\tau') = 1 - (v')^{p-1} + (\tau')^p - (v')^{p-1}\tau'$$

since $pv' = 0$.

By Lemma 3.5, $\mathrm{c}(\phi^{\oplus p}_{0,\widetilde{H}_t}) = (1+v')^p$ when $0 \le t \le p-1$ (1 when $t = p$), and $\mathrm{c}(\phi^{\oplus p}_{p,\widetilde{H}_t}) = (1+tv')^p$ when $0 \le t \le p-1$ (but $(1+v')^p$ when $t = p$). Hence the total Chern class of $\phi^{\oplus p}_{0,\widetilde{H}_t} \oplus \phi^{\oplus p}_{p,\widetilde{H}_t}$ is equal to

$$(1+v')^p(1+tv')^p = (1+(v')^p)(1+(tv')^p) = (1+(1+t)(v')^p + t(v')^{2p})$$

when $0 \le t \le p-1$ and it is equal to

$$(1+v')^p = (1+(v')^p) = (1+(1+t)(v')^p + t(v')^{2p})$$

when $t = p$. □

Now we will calculate the total Chern class of bundle over $B\widetilde{H}_t$ that pulls backs to the normal bundle.

Lemma 5.5. *The total Chern class of* $\xi_{\widetilde{H}_t}$ *is*

$$\mathrm{c}(\xi_{\widetilde{H}_t}) = 1 + (v')^{p-1} + \text{ \textit{higher terms}}$$

Proof. By Lemma 5.4 we know that the total Chern class of $\widehat{\psi}_{\widetilde{H}_t}$ is

$$\mathrm{c}(\widehat{\psi}_{\widetilde{H}_t}) = 1 - (v')^{p-1} + \text{ higher terms}$$

By the construction of $\xi_{\widetilde{H}_t}$, we know that $\xi_{\widetilde{H}_t} \oplus \widehat{\psi}_{\widetilde{H}_t}$ is a trivial bundle over $B\widetilde{H}_t^{(4p-1)}$. Hence the result follows. □

For the rest of this section set $r = \frac{p-1}{2}$.

Lemma 5.6. *The first few Pontrjagin classes of the bundle* $\xi_{\widetilde{H}_t}$ *are as follows*

$$p_k(\xi_{\widetilde{H}_t}) = \begin{cases} 1 & \text{if } \ k = 0, \\ 0 & \text{if } \ 0 < k < r, \\ (-1)^r 2(v')^{p-1} & \text{if } \ k = r \ . \end{cases}$$

Proof. This is direct calculation given Lemma 5.5 and the fact that

$$p_k(\xi_{\widetilde{H}_t}) = \mathrm{c}_k(\xi_{\widetilde{H}_t})^2 - 2\,\mathrm{c}_{k-1}(\xi_{\widetilde{H}_t})\,\mathrm{c}_{k+1}(\xi_{\widetilde{H}_t}) + - \cdots \mp 2\,\mathrm{c}_1(\xi_{\widetilde{H}_t})\,\mathrm{c}_{2k-1}(\xi_{\widetilde{H}_t}) \pm 2\,\mathrm{c}_{2k}(\xi_{\widetilde{H}_t})$$

□

The main result of this section is the following:

Lemma 5.7. $q_1(\xi_{\widetilde{H}_t}) = v^{p-1} \in H^{2(p-1)}(B\widetilde{H}_t; \mathbf{Z}/p)$

Proof. Let $\{K_n\}$ be the multiplicative sequence belonging to the polynomial $f(t) = 1+t^r$. A result of Wu shows (see Theorem 19.7 in [25]) that

$$q_1(\xi_{\widetilde{H}_t}) = K_r(p_1(\xi_{\widetilde{H}_t}), \dots, p_r(\xi_{\widetilde{H}_t})) \bmod p \ .$$

By Lemma 5.6 we know that $p_1(\xi_{\widetilde{H}_t}), \dots, p_{r-1}(\xi_{\widetilde{H}_t})$ are all zero, hence we are only interested in the coefficient of x_r in the polynomial $K_r(x_1, \dots, x_r)$. By Problem 19-B in [25] this coefficient is equal to $s_r(0,0,\dots,0,1) = (-1)^{r+1}r$ (see [25, p. 188]) Hence we have

$$q_1(\xi_{\widetilde{H}_t}) = (-1)^{r+1} r \overline{p}_r(\xi_{\widetilde{H}_t}) = (-1)^{r+1} r (-1)^r 2 v^{p-1} = (-1)(p-1)v^{p-1} = v^{p-1}$$

where $\overline{p}_r(\xi_{\widetilde{H}_t})$ denotes the mod p reduction of $p_r(\xi_{\widetilde{H}_t})$. □

6. The subset $T_{\widetilde{G}_p} \subseteq H_{2n-1}(B_{\widetilde{G}_p};\mathbf{Z})$

We first describe a useful construction, leading to some examples of manifolds with certain fundamental classes and the right bundle information. Given an m–dimensional representation $\Phi\colon G \to U(m)$ of a group G, we have an induced G–action on $\mathbb{C}^m$, and the space $S(\Phi) = S^{2m-1}$ will be the G–equivariant unit sphere in $\mathbb{C}^m$. Let Φ_1, Φ_2, ..., Φ_k be representations of a group G then

$$S(\Phi_1, \Phi_2, \dots, \Phi_k) = S(\Phi_1) \times S(\Phi_2) \times \cdots \times S(\Phi_k)$$

is a product of k unit spheres, and

$$L(\Phi_1, \Phi_2, \dots, \Phi_n) = S(\Phi_1, \Phi_2, \dots, \Phi_k)/G$$

is the quotient space. When the action of G on $S(\Phi_1, \Phi_2, ..., \Phi_k)$ is free, we have the following pull-back diagram

$$\begin{array}{ccc} S(\Phi_1, \Phi_2, ..., \Phi_k) & \longrightarrow & EG \\ \downarrow & & \downarrow \\ L(\Phi_1, \Phi_2, ..., \Phi_k) & \xrightarrow{c} & BG \end{array}$$

We will write c for the bottom map, called the classifying map, whenever the pull-back diagram we are talking about is clear.

Example 6.1. We will need two main examples of this construction.

(1) For $G = \widetilde{H}_t$ and $t \in \{0, \dots, p\}$, define

$$M_t = L(\Psi_{\widetilde{H}_t},\ (\Phi_{t,\widetilde{H}_t})^{\oplus p}) = (S^{2p-1} \times S^{2p-1})/\widetilde{H}_t$$

by choosing the representations $\Phi_1 = \Psi_{\widetilde{H}_t}$ and $\Phi_2 = (\Phi_{t,\widetilde{H}_t})^{\oplus p}$ in the construction above.

(2) For $G = D_t$ and $t \in \{0, \dots, p\}$, define

$$N_t = L(\Phi_{t,D_t} \oplus \Phi^2_{t,D_t} \oplus \cdots \oplus \Phi^{p-1}_{t,D_t} \oplus (\Phi_{t,D_t})^{\oplus p}) = S^{4p-3}/D_t$$

where, for a 1-dimensional representation Φ, we set Φ^k to be the k^{th} power of Φ induced by the multiplication in S^1. In other words, if $\Phi(v) = \lambda v$ then we set $\Phi^k(v) = \lambda^k v$.

§6A. Definition of the subset $T_{\widetilde{G}_p} \subseteq H_{4p-3}(B_{\widetilde{G}_p})$. First note that for all t the universal cover of M_t is $\mathbf{CP}^{p-1} \times S^{2p-1}$ and the universal cover of $B_{\widetilde{H}_t}$ is B_{S^1}. Hence, we can assume that we have the following pull-back diagram where the map c does not depend on t.

$$\begin{array}{ccc} \mathbf{CP}^{p-1} \times S^{2p-1} & \xrightarrow{c} & B_{S^1} \\ \downarrow & & \downarrow \\ M_t & \xrightarrow{c_t} & B_{\widetilde{H}_t} \end{array}$$

We define an element γ_{S^1} in $H_{4p-3}(B_{S^1};\mathbf{Z})$ as the image of the fundamental class of $\mathbf{CP}^{p-1}\times S^{2p-1}$ under the map c defined in the above diagram. In other words

$$\gamma_{S^1}=c_*\big[\mathbf{CP}^{p-1}\times S^{2p-1}\big]\in H_{4p-3}(B_{S^1};\mathbf{Z})\ .$$

Similarly, we define $\gamma_{\widetilde{H}_t}\in H_{4p-3}(B_{\widetilde{H}_t};\mathbf{Z})$ as the image of the fundamental class of M_t for t in $\{0,\dots,p\}$ in other words

$$\gamma_{\widetilde{H}_t}=(c_t)_*\big[M_t\big]\in H_{4p-3}(B_{\widetilde{H}_t};\mathbf{Z})$$

We define

$$T_{\widetilde{G}_p}=\{\gamma\in H_{4p-3}(B_{\widetilde{G}_p};\mathbf{Z})\mid\ p\cdot(\mathrm{tr}(\gamma)-\gamma_{S^1})=0\}$$

where $\mathrm{tr}\colon H_{4p-3}(B_{\widetilde{G}_p};\mathbf{Z})\to H_{4p-3}(B_{S^1};\mathbf{Z})$ denotes the transfer map. One of our main tasks is to show that this subset $T_{\widetilde{G}_p}$ is non-empty !

§6B. The $(2p-1)$-type of M_t. We first establish some notation for the Postnikov tower of a connected CW-complex X. We have a diagram of fibrations

$$\begin{array}{ccccccccc} & & X & & & & & & \\ & & \downarrow{\scriptstyle i_n} & \searrow{\scriptstyle i_{n-1}} & & & & & \\ \cdots & \longrightarrow & X^{[n]} & \longrightarrow & X^{[n-1]} & \longrightarrow & \cdots & \longrightarrow & X^{[0]} \end{array}$$

and the k–invariants of X that classify these fibrations are denoted as follows:

$$\begin{array}{ccc} K(\pi_n(X),n) \longrightarrow & X^{[n]} & \\ & \downarrow & \\ & X^{[n-1]} & \xrightarrow{k_n(X)} K(\pi_n(X),n+1) \end{array}$$

The space $X^{[n]}$ is called the n-type of X.

Lemma 6.2. *$M_t^{[2p-2]}\simeq B\widetilde{H}_t$ and the composition $M_t\xrightarrow{i_{2p-2}}M_t^{[2p-2]}\xrightarrow{\simeq}B\widetilde{H}_t$ is homotopy equivalent to the classifying map $c_t\colon M_t\to B\widetilde{H}_t$.*

Proof. This follows as $S(\Psi_{\widetilde{H}},(\Phi_{t,\widetilde{H}_t})^{\oplus p})=S^{2p-1}\times S^{2p-1}$ is $(2p-2)$–connected and the action of $\widetilde{H}_t$ on $S(\Psi_{\widetilde{H}_t},(\Phi_{t,\widetilde{H}_t})^{\oplus p})$ is free. □

Lemma 6.3. *$M_t^{[2p-1]}\simeq B_{\widetilde{H}_t}$.*

Proof. By Lemma 6.2 the following equality proves the result:

$$\begin{aligned} k_{2p-1}(M_t) &= \mathrm{c}_p(\Psi_{\widetilde{H}_t})\oplus\mathrm{c}_p((\Phi_{t,\widetilde{H}_t})^{\oplus p}) \\ &= \mathrm{Res}^{\widetilde{G}_p}_{\widetilde{H}_t}(\mathrm{c}_p(\Psi))\oplus(v')^p \\ &= \mathrm{Res}^{\widetilde{G}_p}_{\widetilde{H}_t}(\zeta)\oplus\mathrm{Res}^{\widetilde{G}_p}_{\widetilde{H}_t}(\alpha^p-\alpha^{p-1}\beta+\beta^p) \\ &= \mathrm{Res}^{\widetilde{G}_p}_{\widetilde{H}_t}(\zeta\oplus\alpha^p-\alpha^{p-1}\beta+\beta^p) \\ &= \mathrm{Res}^{\widetilde{G}_p}_{\widetilde{H}_t}(k) \\ &= k_{\widetilde{H}_t} \end{aligned}$$

where $c_p(\Phi)$ denotes the p^{th} Chern class of a representation Φ. □

§6C. The tangent bundle of M_t.

Lemma 6.4. *The tangent bundle of M_t is stably equivalent to the pull-back of $\tau_{\widetilde{H}_t}\colon B_{\widetilde{H}_t} \to BSO$ (see* Definition 5.1*).*

Proof. The tangent bundle $T(M_t)$ of M_t clearly fits into the following pull-back diagram

$$\begin{array}{ccc} T(M_t)\oplus\varepsilon & \longrightarrow & E\widetilde{H}_t \times_{\widetilde{H}_t} (\mathbb{C}^p \times \mathbb{C}^p) \\ \downarrow & & \downarrow \pi \\ M_t & \xrightarrow{c_t} & B\widetilde{H}_t \end{array}$$

where ε is a trivial bundle over M_t and the action of $\widetilde{H}_t$ on $\mathbb{C}^p \times \mathbb{C}^p$ is given by $\Psi_{\widetilde{H}_t}$ and $(\Phi_{t,\widetilde{H}_t})^{\oplus p}$ respectively. Hence we have $c_t^*([\pi]) = c_t^*([\psi_{\widetilde{H}_t}] + p[\phi_{t,\widetilde{H}_t}])$ in complex K-theory $\widetilde{K}(M_t)$. However, M_t fits into the following pull-back diagram:

$$\begin{array}{ccc} M_t & \longrightarrow & ED_t \times_{D_t} \mathbf{CP}^{p-1} \\ \downarrow & & \downarrow \\ L(\Phi_{t,D_t}^{\oplus p}) & \longrightarrow & BD_t \end{array}$$

where the action of D_t on $\mathbf{CP}^{p-1}$ is induced by action of D_t on $\mathbb{C}^p$ given by Ψ_{D_t}. Hence $\widetilde{K}(M_t)$ is a $\widetilde{K}(L(\Phi_{t,D_t}^{\oplus p}))$–module by Proposition 2.13 in Chapter IV in [19] and the exponent of $\widetilde{K}(L(\Phi_{t,D_t}^{\oplus p}))$ is p (see Theorem 2 in [18]). Hence the exponent of $\widetilde{K}(M_t)$ is p and $c_t^*([\pi]) = c_t^*([\psi_{\widetilde{H}_t}]) = c_t^*([\widehat{\psi}_{\widetilde{H}_t}])$. This means the tangent bundle of M_t is stably equivalent to the pull-back of the bundle $\widehat{\psi}_{\widetilde{H}_t}$ over $B\widetilde{H}_t$ by the classifying map. However, by Lemma 6.2 and Lemma 6.3, we know that the classifying map is homotopy equivalent to the compositon $M_t \xrightarrow{i_{2p-1}} M_t^{[2p-1]} \xrightarrow{\simeq} B_{\widetilde{H}_t} \xrightarrow{\pi_{\widetilde{H}_t}} B\widetilde{H}_t$. Hence, the tangent bundle of M_t is stably equivalent to the pull-back of $\tau_{\widetilde{H}_t}$. □

§6D. The tangent bundle of N_t.

Lemma 6.5. *The tangent bundle of N_t is stably equivalent to the pull-back of $\widehat{\psi}_{D_t}\colon BD_t \to BSO$.*

Proof. The tangent bundle $T(N_t)$ of N_t clearly fits into the following pull-back diagram

$$\begin{array}{ccc} T(N_t)\oplus\varepsilon & \longrightarrow & ED_t \times_{D_t} \mathbb{C}^{2p-1} \\ \downarrow & & \downarrow \\ N_t & \longrightarrow & BD_t \end{array}$$

where ε is a trivial bundle over N_t and the action of D_t on $\mathbb{C}^{2p-1}$ is given by $\Phi_{t,D_t} \oplus \Phi^2_{t,D_t} \oplus ... \oplus \Phi^{p-1}_{t,D_t} \oplus (\Phi_{t,D_t})^{\oplus p}$, where $\mathit{\Phi}^k(g) = (\mathit{\Phi}(g))^k$ for a 1-dimensional representation $\mathit{\Phi}\colon G \to S^1$. Now it is easy to see that

$$\widehat{\psi}_{D_t} = 1 \oplus \Phi_{t,D_t} \oplus \Phi^2_{t,D_t} \oplus \cdots \oplus \Phi^{p-1}_{t,D_t} \oplus (\Phi_{t,D_t})^{\oplus p} .$$

Hence it is clear that the tangent bundle of N_t is the stably equivalent to the pull-back of $\widehat{\psi}_{D_t}$. □

7. Calculation of the cohomology of B_G

For any subgroup $G \subseteq \widetilde{G}_p$, to compute the cohomology of B_G in the range we need, we will use the the cohomology Serre spectral sequence of the fibration

$$K \longrightarrow B_G \xrightarrow{\pi_G} BG$$

where $K = K(\mathbf{Z} \oplus \mathbf{Z}, 2p-1)$. Before starting the calculations of these spectral sequences, we need some information about the cohomology of the fiber K of these fibrations.

§7A. The cohomology of K. Assume R is an abelian group. In this section we compute the cohomology groups of K in the range we need, using the results of Cartan [8], [9]. The group homomorphism

$$\mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z}/p \oplus \mathbf{Z}/p$$

given by the mod p reduction, induces the following fibration

$$K \xrightarrow{\times p} K \longrightarrow K_p$$

where the base space $K_p = K(\mathbf{Z}/p \oplus \mathbf{Z}/p, 2p-1)$ and the fiber is K itself but the inclusion map of the fiber in the total space K is the map induced by the the group homomorphism

$$\mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z}$$

given by multiplication by p. First we give some information about the mod p cohomology of the base space K_p in this fibration. For $j = 1, 2$ let

$$\iota_j\colon K_p \to K(\mathbf{Z}/p, 2p-1)$$

be the map induced by the projection onto the j^{th} factor, where we consider $K_p = K(\mathbf{Z}/p, 2p-1) \times K(\mathbf{Z}/p, 2p-1)$. The map

$$P^1\colon H^i(K_p; \mathbf{Z}/p) \to H^{i+2(p-1)}(K_p; \mathbf{Z}/p)$$

is the first mod p Steenrod operation and the map

$$\beta\colon H^i(K_p; \mathbf{Z}/p) \to H^{i+1}(K_p; \mathbf{Z}/p)$$

is the Bockstein homomorphism.

Lemma 7.1.

(1) $H^0(K_p; \mathbf{Z}/p) = \mathbf{Z}/p$.
(2) $H^i(K_p; \mathbf{Z}/p) = 0$ *for* $1 \leq i \leq 2p-2$.
(3) $H^{2p-1}(K_p; \mathbf{Z}/p) = \langle \iota_1, \iota_2 \rangle = \mathbf{Z}/p \oplus \mathbf{Z}/p$.
(4) $H^{2p}(K_p; \mathbf{Z}/p) = \langle \beta(\iota_1), \beta(\iota_2) \rangle = \mathbf{Z}/p \oplus \mathbf{Z}/p$.

(5) $H^i(K_p;\mathbf{Z}/p) = 0$ *for* $2p+1 \le i \le 4p-4$.
(6) $H^{4p-3}(K_p;\mathbf{Z}/p) = \langle P^1(\iota_1), P^1(\iota_2)\rangle = \mathbf{Z}/p \oplus \mathbf{Z}/p$.
(7) $H^{4p-2}(K_p;\mathbf{Z}/p)$ *is*

$$\langle \iota_1 \cup \iota_2, \beta P^1(\iota_1), P^1\beta(\iota_1), \beta P^1(\iota_2), P^1\beta(\iota_2)\rangle = (\mathbf{Z}/p)^{\oplus 5}$$

(8) $H^{4p-1}(K_p;\mathbf{Z}/p)$ *is*

$$\langle \iota_1 \cup \beta(\iota_2), \iota_2 \cup \beta(\iota_1), \beta P^1\beta(\iota_1), \iota_1 \cup \beta(\iota_1), \beta P^1\beta(\iota_2), \iota_2 \cup \beta(\iota_2)\rangle = (\mathbf{Z}/p)^{\oplus 6}$$

Proof. See Theorem 4 in [8]. □

Let A be an abelian group. We will write

$$_{(p)}A := A/\langle \text{torsion prime to } p\rangle\ .$$

For $j = 1, 2$ let

$$z_j\colon K \to K(\mathbf{Z}, 2p-1)$$

be the map induced by the projection onto the j^{th} factor where $K = K(\mathbf{Z}, 2p-1) \times K(\mathbf{Z}, 2p-1)$.

Lemma 7.2.

(1) $H^0(K;R) = R$.
(2) $H^i(K;R) = 0$ *for* $1 \le i \le 2p-2$.
(3) $H^{2p-1}(K;\mathbf{Z}) = \langle z_1, z_2\rangle = \mathbf{Z} \oplus \mathbf{Z}$.
(4) *Let* $\overline{z}_1$ *and* $\overline{z}_2$ *denote the mod* p *reductions of* z_1 *and* z_2 *then we have*

$$H^{2p-1}(K;\mathbf{Z}/p) = \langle \overline{z}_1, \overline{z}_2\rangle = \mathbf{Z}/p \oplus \mathbf{Z}/p\ .$$

(5) $H^i(K;R)$ *is a torsion group for* $2p \le i \le 4p-3$.
(6) $_{(p)}H^i(K;R) = 0$ *for* $2p \le i \le 4p-4$.
(7) $_{(p)}H^{4p-3}(K;\mathbf{Z}) = 0$.
(8) $H^{4p-3}(K;\mathbf{Z}/p) = \langle P^1(\overline{z}_1), P^1(\overline{z}_2)\rangle = \mathbf{Z}/p \oplus \mathbf{Z}/p$.
(9) *Let* $\delta\colon H^{4p-3}(K;\mathbf{Z}/p) \to H^{4p-2}(K;\mathbf{Z})$ *be the Bockstein map. Then*

$$_{(p)}H^{4p-2}(K;\mathbf{Z}) = \langle z_1 \cup z_2, \delta(P^1(\overline{z}_1)), \delta(P^1(\overline{z}_2))\rangle = \mathbf{Z} \oplus \mathbf{Z}/p \oplus \mathbf{Z}/p\ .$$

(10) *Let* $\beta\colon H^{4p-3}(K;\mathbf{Z}/p) \to H^{4p-2}(K;\mathbf{Z}/p)$ *be the Bockstein map. Then*

$$H^{4p-2}(K;\mathbf{Z}/p) = \langle \overline{z}_1 \cup \overline{z}_2, \beta(P^1(\overline{z}_1)), \beta(P^1(\overline{z}_1))\rangle = \mathbf{Z}/p \oplus \mathbf{Z}/p \oplus \mathbf{Z}/p\ .$$

(11) $H^{4p-1}(K;\mathbf{Z})$ *has no* p*-torsion.*

Proof. The facts (1), (2), (3), and (4) are well known. The fact (5) is a consequence of Theorem 7 in [8] and the universal coefficient theorem. By the previous Lemma we know that $H^i(K_p;\mathbf{Z}/p) = 0$ for $1 \le i \le 2p-2$, and by Fact (2) we have

$$E_r^{n,m} = 0, \quad \text{for } 1 \le n, m \le 2p-2 \text{ and } 2 \le r$$

in the spectral sequence for the fibration $K \xrightarrow{\times p} K \longrightarrow K_p$. Hence we have

$$E_r^{0,r-1} = E_2^{0,r-1} = H^{r-1}(K;\mathbf{Z}/p) \ \text{ when } 2 \le r \le 4p-3$$

and

$$E_r^{r,0} = E_2^{r,0} = H^r(K_p; \mathbf{Z}/p) \text{ when } 2 \leq r \leq 4p-2 \ .$$

Moreover the map $\times p$ induces multiplication by p on cohomology. Therefore

$$E_\infty^{0,m} = pE_2^{0,m} = 0 \text{ when } m \geq 1 \ .$$

Hence the following differential is injective

$$d_r \colon E_r^{0,r-1} \to E_2^{r,0} \text{ when } 2 \leq r \leq 4p-4,$$

and

$$E_r^{r,0}/d_r(E_r^{0,r-1}) \cong E_r^{0,r} \text{ when } 2 \leq r \leq 4p-3 \ .$$

As in the Lemma, we will write

$$H^{2p-1}(K; \mathbf{Z}/p) = \langle \overline{z}_1, \overline{z}_2 \rangle = \mathbf{Z}/p \oplus \mathbf{Z}/p \ .$$

Now from the facts just given about the spectral sequence $\{E_r^{n,m}, d_r\}$ it is easy to see that

$$d_{2p}(\overline{z}_1) = \beta(\iota_1) \text{ and } d_{2p}(\overline{z}_2) = \beta(\iota_2),$$

and

$$H^i(K; \mathbf{Z}/p) = 0 \text{ when } 2p \leq i \leq 4p-4,$$

and also

$$H^{4p-3}(K; \mathbf{Z}/p) = \mathbf{Z}/p \oplus \mathbf{Z}/p \ .$$

The Steenrod operations commute with transgression, and hence we have

$$d_{4p-2}(P^1(\overline{z}_1)) = P^1\beta(\iota_1) \text{ and } d_{2p}(P^1(\overline{z}_2)) = P^1\beta(\iota_2)$$

and

$$H^{4p-3}(K; \mathbf{Z}/p) = \langle P^1(\overline{z}_1), P^1(\overline{z}_2) \rangle = \mathbf{Z}/p \oplus \mathbf{Z}/p \ .$$

Commutativity with Steenrod operation will also give us

$$d_{4p-2}(\beta P^1(\overline{z}_1)) = \beta P^1\beta(\iota_1) \text{ and } d_{2p}(\beta P^1(\overline{z}_2)) = \beta P^1\beta(\iota_2),$$

and the multiplicative structure gives us

$$d_{2p}(\iota_1 \cdot \overline{z}_1) = \iota_1 \cdot \beta(\iota_1) \text{ and } d_{2p}(\iota_1 \overline{z}_2) = \iota_1\beta(\iota_2)$$

and

$$d_{2p}(\iota_2 \cdot \overline{z}_1) = \iota_2 \cdot \beta(\iota_1) \text{ and } d_{2p}(\iota_2 \overline{z}_2) = \iota_2\beta(\iota_2) \ .$$

Counting the dimensions we see that

$$d_{4p-1}(\overline{z}_1 \cdot \overline{z}_2) = 0 \ .$$

Hence Fact (10) is proved: in other words

$$H^{4p-2}(K; \mathbf{Z}/p) = \langle \overline{z}_1 \cup \overline{z}_2, \beta(P^1(\overline{z}_1)), \beta(P^1(\overline{z}_1)) \rangle = \mathbf{Z}/p \oplus \mathbf{Z}/p \oplus \mathbf{Z}/p \ .$$

Now the consider the Bockstein long exact sequence for $i \geq 2p-1$:

$$\dots \xrightarrow{\rho} H^{i-1}(K; \mathbf{Z}/p) \xrightarrow{\delta} H^i(K; \mathbf{Z}) \xrightarrow{\times p} H^i(K; \mathbf{Z}) \xrightarrow{\rho} H^i(K; \mathbf{Z}/p) \xrightarrow{\delta} \dots$$

We will write $\twoheadrightarrow$ to indicate that the map is surjective, and we will write $\rightarrowtail$ when the map is injective, and write $\xrightarrow{0}$ when we want to say that the map is zero. For $i = 2p-1$ we have

$$\dots \xrightarrow{\rho} 0 \xrightarrow{\delta} \underbrace{\langle z_1, z_2\rangle}_{\mathbf{Z}^{\oplus 2}} \overset{\times p}{\rightarrowtail} \underbrace{\langle z_1, z_2\rangle}_{\mathbf{Z}^{\oplus 2}} \overset{\rho}{\twoheadrightarrow} \underbrace{\langle \overline{z}_1, \overline{z}_2\rangle}_{(\mathbf{Z}/p)^{\oplus 2}} \xrightarrow{0} H^{2p}(K;\mathbf{Z}) \xrightarrow{\times p} \dots$$

For $i = 2p$ we have

$$\dots \overset{\rho}{\twoheadrightarrow} \underbrace{\langle \overline{z}_1, \overline{z}_2\rangle}_{(\mathbf{Z}/p)^{\oplus 2}} \xrightarrow{0} H^{2p}(K;\mathbf{Z}) \overset{\times p}{\rightarrowtail} H^{2p}(K;\mathbf{Z}) \xrightarrow{\rho} 0 \xrightarrow{\delta} H^{2p+1}(K;\mathbf{Z}) \xrightarrow{\times p} \dots$$

Hence we have ${}_{(p)}H^{2p}(K;\mathbf{Z}) = 0$. Now considering the Bockstein long exact sequence for $2p \leq i \leq 4p-4$ we see that ${}_{(p)}H^{i}(K;\mathbf{Z}) = 0$ for $2p \leq i \leq 4p-4$. Hence we can prove the Fact (6) by the universal coefficient theorem. For $i = 4p-3$ we have

$$0 \xrightarrow{\delta} H^{4p-3}(K;\mathbf{Z}) \overset{\times p}{\rightarrowtail} H^{4p-3}(K;\mathbf{Z}) \xrightarrow{\rho} \underbrace{\langle P^1(\overline{z}_1), P^1(\overline{z}_2)\rangle}_{(\mathbf{Z}/p)^{\oplus 2}} \xrightarrow{\delta} H^{4p-2}(K;\mathbf{Z})$$

But by Fact (5) we know that $H^{4p-3}(K;\mathbf{Z})$ is a torsion group. Hence we have ${}_{(p)}H^{4p-3}(K;\mathbf{Z}) = 0$, which proves Fact (7). For $i = 4p-2$ we have

$$\dots \xrightarrow{0} \underbrace{\langle P^1(\overline{z}_1), P^1(\overline{z}_2)\rangle}_{(\mathbf{Z}/p)^{\oplus 2}} \overset{\delta}{\rightarrowtail} H^{4p-2}(K;\mathbf{Z}) \overset{\times p}{\rightarrowtail} H^{4p-2}(K;\mathbf{Z})$$
$$\xrightarrow{\rho} \underbrace{\langle \overline{z}_1 \cup \overline{z}_2, \beta P^1(\overline{z}_1), \beta P^1(\overline{z}_2)\rangle}_{(\mathbf{Z}/p)^{\oplus 3}} \xrightarrow{\delta} H^{4p-1}(K;\mathbf{Z}) \xrightarrow{\times p} \dots$$

Hence Fact (9) is proved. Since the last map above in injective, Fact (11) is proved. □

§7B. The cohomology Serre spectral sequence. Let $\{E_r^{n,m}(G,R), d_r\}$ denote the cohomology Serre spectral sequence with R-coefficients of the fibration

$$K \longrightarrow B_G \xrightarrow{\pi_G} BG$$

for $K = K(\mathbf{Z}\oplus\mathbf{Z}, 2p-1)$, and $G \subseteq \widetilde{G}_p$ subgroup. The second page of this spectral sequence is given by:

$$E_2^{n,m}(G,R) = H^n(BG; H^m(K;R)),$$

and the spectral sequence converges to $H^*(B_G;R)$ with the filtration given by

$$F^nH^{n+m}(B_G;R) = \ker\left\{H^{n+m}(B_G;R) \to H^{n+m}(B_G^{\{n-1\}};R)\right\}$$

where $B_G^{\{n\}}$ stands for the inverse image of the n^{th} skeleton of BG under the map π_G. In other words

$$B_G^{\{n\}} = \pi_G^{-1}(BG^{(n)})$$

Note that we have

$$E_\infty^{n,m}(G,R) = F^nH^{n+m}(B_G;R)/F^{n+1}H^{n+m}(B_G;R)$$

We will only write $\{E_r^{n,m}(G), d_r\}$ for $R = \mathbf{Z}$ coefficients. Hence

$$\{E_r^{n,m}(G), d_r\} = \{E_r^{n,m}(G, \mathbf{Z}), d_r\}$$

The cohomology groups for

$$E_2^{*,0}(G, R) = H^*(BG; R)$$

are given in Theorem 3.2, Theorem 3.3, and Remark 3.4, and the calculation of

$$E_2^{0,*}(G, R) = H^*(K; R)$$

is given in Lemma 7.2. Assume

$$d_{2p}(z_1) = \theta_1 \quad \text{and} \quad d_{2p}(z_2) = \theta_2$$

where d_{2p} denotes the differential in the Serre spectral sequence $\{E_r^{n,m}(\widetilde{G}_p), d_r\}$.

§7C. The cohomology of $B_{\widetilde{G}_p}$. The following are some calculations and definitions for $\{E_r^{n,m}(\widetilde{G}_p), d_r\}$

1. *All the low differentials in this spectral sequence are zero.* More precisely, d_r is zero for $2 \leq r \leq 2p-1$. This is due to the fact that

$$d_r(z_i) = 0$$

for $2 \leq r \leq 2p-1$ and $1 \leq i \leq 2$, and every element in this spectral sequence can be written as a linear combination of multiples of elements in the algebra $E_r^{0,*}(\widetilde{G}_p)$, which is generated by 1, z_1, and z_2 as a module over the Steenrod algebra. Hence, we have

$$E_2^{*,*}(\widetilde{G}_p) = E_{2p}^{*,*}(\widetilde{G}_p) \ .$$

2. *We have $d_{2p}(z_1) = \zeta$ and $d_{2p}(z_2) = \alpha^p - \alpha^{p-1}\beta + \beta^p$.* This is due to the definition of θ_1 and θ_2 in Section 4.

3. *We calculate $d_{2p}\colon E_{2p}^{2p-2,2p-1}(\widetilde{G}_p) \to E_{2p}^{4p-2,0}(\widetilde{G}_p)$.* From Lemma 7.2, Part 3, we have $H^{2p-1}(K; \mathbf{Z}) = \langle z_1, z_2 \rangle$ and by Theorem 3.2 there are no relations among the elements α^{p-1}, $\alpha^{p-2}\beta$, ..., β^{p-1}, and χ_{p-1}. Hence we have

$$E_{2p}^{2p-2,2p-1}(\widetilde{G}_p) = (\mathbf{Z}/p)^{\oplus 2p} \oplus \mathbf{Z}^{\oplus 2}$$

given by

$$\langle z_1, z_2 \rangle \cdot \langle \alpha^{p-1}, \alpha^{p-2}\beta, \ldots, \beta^{p-1}, \chi_{p-1} \rangle \ .$$

Now by Theorem 3.2 we know that $\alpha^p\beta = \alpha\beta^p$. Hence we have

$$E_{2p}^{4p-2,0}(\widetilde{G}_p) = (\mathbf{Z}/p)^{\oplus 2p+1} \oplus \mathbf{Z}$$

given by

$$\langle \alpha^{2p-1}, \alpha^{2p-2}\beta, \ldots, \alpha^p\beta^{p-1}, \beta^{2p-1} \rangle \oplus \zeta \cdot \langle \alpha^{p-1}, \alpha^{p-2}\beta, \ldots, \beta^{p-1}, \chi_{p-1} \rangle \ .$$

The map

$$d_{2p}\colon E_{2p}^{2p-2,2p-1}(\widetilde{G}_p) \to E_{2p}^{4p-2,0}(\widetilde{G}_p)$$

is surjective because the following list of images of d_{2p} will span $E_{2p}^{4p-2,0}(\widetilde{G}_p)$ considered as above:

- Clearly the image of
$$z_1 \cdot \langle \alpha^{p-1}, \alpha^{p-2}\beta, \ldots, \beta^{p-1}, \chi_{p-1}\rangle$$
under the differential d_{2p} is equal to
$$\zeta \cdot \langle \alpha^{p-1}, \alpha^{p-2}\beta, \ldots, \beta^{p-1}, \chi_{p-1}\rangle$$
- $d_{2p}(z_2 \cdot \alpha^s \beta^{p-1-s}) = (\alpha^p - \alpha^{p-1}\beta + \beta^p)\alpha^s\beta^{p-1-s} = \alpha^{s+p}\beta^{p-1-s}$ for $1 \leq s \leq p-1$.
- $d_{2p}(z_2 \cdot \beta^{p-1}) = (\alpha^p - \alpha^{p-1}\beta + \beta^p)\beta^{p-1} = \alpha^p\beta^{p-1} + \alpha^{2p-2}\beta + \beta^{2p-1}$
- $d_{2p}(z_2 \cdot \chi_{p-1}) = (\alpha^p - \alpha^{p-1}\beta + \beta^p)\chi_{p-1} = \alpha^{2p-1} + \alpha^{2p-2}\beta + \beta^{2p-1}$

This means we have
$$E_{2p+1}^{2p-2,2p-1}(\widetilde{G}_p) = \langle pz_2 \cdot \chi_{p-1}\rangle = \mathbf{Z}$$
as the kernel of the above differential d_{2p} is $\langle pz_2 \cdot \chi_{p-1}\rangle$.

4. *We define an element* $\Gamma_{\widetilde{G}_p} \in H^{4p-3}(B_{\widetilde{G}_p}; \mathbf{Z})$*.* Since there are no possible differentials coming in or out of the position $(2p-2, 0)$ in this spectral sequence we have $E_\infty^{2p-2,0}(\widetilde{G}_p) = H^{2p-2}(B\widetilde{G}_p; \mathbf{Z}) = \langle \alpha^{p-1}, \alpha^{p-2}\beta, \ldots, \beta^{p-1}, \chi_{p-1}\rangle$ Hence we have
$$0 \neq \pi^*_{\widetilde{G}_p}(\chi_{p-1}) \in H^{2p-2}(B_{\widetilde{G}_p}; \mathbf{Z})$$
Moreover, there exists
$$z_{\widetilde{G}_p} \in H^{2p-1}(B_{\widetilde{G}_p}; \mathbf{Z})$$
such that $i^*(z_{\widetilde{G}_p}) = pz_2 \in H^{2p-1}(K; \mathbf{Z})$, where $i\colon K \to B_{\widetilde{G}_p}$ is the inclusion map. This is because $E_\infty^{0,2p-1}(\widetilde{G}_p) = \langle pz_2\rangle \subseteq H^{2p-1}(K; \mathbf{Z})$. Now, define
$$\Gamma_{\widetilde{G}_p} = z_{\widetilde{G}_p} \cup \pi^*_{\widetilde{G}_p}(\chi_{p-1}) \in H^{4p-3}(B_{\widetilde{G}_p}; \mathbf{Z}).$$

5. *The element represented by* $\Gamma_{\widetilde{G}_p}$ *in the spectral sequence.* First note that
$$E_\infty^{2p-2,2p-1}(\widetilde{G}_p) = E_{2p}^{2p-2,2p-1}(\widetilde{G}_p) = \langle pz_2 \cdot \chi_{p-1}\rangle$$
as there are no other possible differentials that could affect this point. Second, note that due to the definition of $\Gamma_{\widetilde{G}_p}$ it is clear that
$$\Gamma_{\widetilde{G}_p} \in F^{2p-2}H^{4p-3}(B_{\widetilde{G}_p}; \mathbf{Z})$$
represents the following generator of the quotient
$$pz_2 \cdot \chi_{p-1} \in E_\infty^{2p-2,2p-1}(\widetilde{G}_p) = F^{2p-2}H^{4p-3}(B_{\widetilde{G}_p}; \mathbf{Z})/F^{2p-1}H^{4p-3}(B_{\widetilde{G}_p}; \mathbf{Z})\ .$$

6. *The filtration term* $F^{2p-2}H^{4p-3}(B_{\widetilde{G}_p};\mathbf{Z}) = \langle \Gamma_{\widetilde{G}_p}\rangle = \mathbf{Z}$. By Theorem 3.2 the integral cohomology of $B\widetilde{G}_p$ is zero in odd degrees. Hence

$$E_\infty^{4p-3,0}(\widetilde{G}_p) = 0 \ .$$

Moreover, by Lemma 7.2 Part 2 we have $H^i(K;\mathbf{Z}) = 0$ for $1 \leq i \leq 2p-2$, and hence we have

$$E_\infty^{4p-3-i,i}(\widetilde{G}_p) = 0$$

for $1 \leq i \leq 2p-2$. This means

$$F^{2p-2}H^{4p-3}(B_{\widetilde{G}_p};\mathbf{Z}) = E_\infty^{2p-2,2p-1}(\widetilde{G}_p) = \langle pz_2 \cdot \chi_{p-1}\rangle = \mathbf{Z}$$

7. *The quotient* $H^{4p-3}(B_{\widetilde{G}_p};\mathbf{Z})/\langle \Gamma_{\widetilde{G}_p}\rangle$. Consider the short exact sequence

$$0 \to \langle \Gamma_{\widetilde{G}_p}\rangle \to H^{4p-3}(B_{\widetilde{G}_p};\mathbf{Z}) \to T \to 0$$

where

$$T = H^{4p-3}(B_{\widetilde{G}_p};\mathbf{Z})/F^{2p-2}H^{4p-3}(B_{\widetilde{G}_p};\mathbf{Z}) \ .$$

Now by Lemma 7.2, Part 5 and Part 6, we see that

$$E_\infty^{i,4p-3-i}(\widetilde{G}_p)$$

is a torsion group with no p-torsion for $0 \leq i \leq 2p-3$. Hence T is a torsion group with no p-torsion.

§7D. The cohomology of $B_{\widetilde{H}_t}$. The calculation for $E_*^{*,*}(\widetilde{H}_t)$ are similar (but easier).

1. *All the low differentials in this spectral sequence are zero.* More precisely, d_r is zero for $2 \leq r \leq 2p-1$. The proof is same as above.

2. *We show that* $d_{2p}(z_1) = (\tau')^p - (v')^{p-1}\tau'$ *and* $d_{2p}(z_2) = (v')^p$. This is due to the restriction maps from $H^{2p}(B\widetilde{G}_p;\mathbf{Z})$ to $H^{2p}(B\widetilde{H}_t;\mathbf{Z})$. We know that for $t \in \{0,1,\dots,p\}$, $\alpha^p - \alpha^{p-1}\beta + \beta^p$ maps to $(v')^p$. This proves that

$$d_{2p}(z_2) = (v')^p \ .$$

By Lemma 5.4, we know that the p^{th} Chern class of $\psi_{\widetilde{H}_t}$ is $(\tau')^p - (v')^{p-1}\tau'$. This proves

$$d_{2p}(z_1) = (\tau')^p - (v')^{p-1}\tau' \ .$$

3. *We calculate* $d_{2p}\colon E_{2p}^{2p-2,2p-1}(\widetilde{H}_t) \to E_{2p}^{4p-2,0}(\widetilde{H}_t)$. We have

$$E_{2p}^{2p-2,2p-1}(\widetilde{H}_t) = (\mathbf{Z}/p)^{\oplus 2p} \oplus \mathbf{Z}^{\oplus 2}$$

given by

$$\langle z_1, z_2\rangle \cdot \langle (v')^{p-1}, (v')^{p-2}\tau', \ldots, (\tau')^{p-1}\rangle \ .$$

and we have

$$E_{2p}^{4p-2,0}(\widetilde{H}_t) = (\mathbf{Z}/p)^{\oplus 2p+1} \oplus \mathbf{Z}$$

given by

$$\langle (v')^{2p-1}, (v')^{2p-2}\tau', \ldots, (\tau')^{2p-1}\rangle \ .$$

The map

$$d_{2p}\colon E_{2p}^{2p-2,2p-1}(\widetilde{H}_t) \to E_{2p}^{4p-2,0}(\widetilde{H}_t)$$

is surjective because the following list of images of d_{2p} will span $E_{2p}^{4p-2,0}(\widetilde{H}_t)$ considered as above:

- $d_{2p}(z_2 \cdot (v')^s(\tau')^{p-1-s}) = (v')^{p+s}(\tau')^{p-1-s}$ for $0 \leq s \leq p-1$
- $d_{2p}(z_1 \cdot (v')^s(\tau')^{p-1-s}) = (v')^s(\tau')^{2p-1-s} + (v')^{p-1+s}(\tau')^{p-s}$ for $0 \leq s \leq p-1$

This means we have

$$E_{2p+1}^{2p-2,2p-1}(\widetilde{H}_t) = \langle pz_2 \cdot (\tau')^{p-1}\rangle$$

as the kernel of the above differential d_{2p} is $\langle pz_2 \cdot (\tau')^{p-1}\rangle$

4. *We define an element* $\Gamma_{\widetilde{H}_t} \in H^{4p-3}(B_{\widetilde{H}_t}; \mathbf{Z})$. As above $E_\infty^{2p-2,0}(\widetilde{H}_t) = H^{2p-2}(B\widetilde{H}_t; \mathbf{Z}) = \langle (v')^{p-1}, (v')^{p-2}\tau', \ldots, (\tau')^{p-1}\rangle$ Hence we have

$$0 \neq \pi^*_{\widetilde{H}_t}((\tau')^{p-1}) \in H^{2p-2}(B_{\widetilde{H}_t}; \mathbf{Z})$$

Moreover, there exists

$$z_{\widetilde{H}_t} \in H^{2p-1}(B_{\widetilde{H}_t}; \mathbf{Z})$$

such that $i^*(z_{\widetilde{H}_t}) = pz_2 \in H^{2p-1}(K; \mathbf{Z})$, as above. Define

$$\Gamma_{\widetilde{H}_t} = z_{\widetilde{H}_t} \cup \pi^*_{\widetilde{H}_t}((\tau')^{p-1}) \in H^{4p-3}(B_{\widetilde{H}_t}; \mathbf{Z})$$

5. *The element represented by* $\Gamma_{\widetilde{H}_t}$ *in the spectral sequence.* As above

$$\Gamma_{\widetilde{H}_t} \in F^{2p-2}H^{4p-3}(B_{\widetilde{H}_t}; \mathbf{Z})$$

and represents the following generator of the quotient

$$pz_2 \cdot (\tau')^{p-1} \in E_\infty^{2p-2,2p-1}(\widetilde{H}_t) = F^{2p-2}H^{4p-3}(B_{\widetilde{H}_t}; \mathbf{Z})/F^{2p-1}H^{4p-3}(B_{\widetilde{H}_t}; \mathbf{Z})$$

6. *The filtration term* $F^{2p-2}H^{4p-3}(B_{\widetilde{H}_t}; \mathbf{Z}) = \langle \Gamma_{\widetilde{H}_t}\rangle = \mathbf{Z}$.

7. *The quotient* $H^{4p-3}(B_{\widetilde{H}_t}; \mathbf{Z})/\langle \Gamma_{\widetilde{H}_t}\rangle$ *is a torsion group with no p-torsion.*

8. *The group* ${}_{(p)}H^{4p-2}(B_{\widetilde{H}_t}; \mathbf{Z}) = \mathbf{Z}/p \oplus \mathbf{Z}/p$. This is because $E_\infty^{r,4p-2-r}(\widetilde{H}_t)$ has no p-torsion for $1 \leq r \leq 4p-2$, and the p-torsion part of $E_\infty^{0,4p-2}(\widetilde{H}_t)$ is $\mathbf{Z}/p \oplus \mathbf{Z}/p$.

§7E. **The cohomology of** B_{S^1}**.** We also need the corresponding information about $E_*^{*,*}(S^1)$.

1. *All the low differentials in this spectral sequence are zero.* More precisely, d_r is zero for $2 \leq r \leq 2p-1$. The proof is same as above.

2. *We have $d_{2p}(z_1) = \tau^p$ and $d_{2p}(z_2) = 0$.* This is due to the restriction maps from $H^{2p}(B\widetilde{G}_p; \mathbf{Z})$ to $H^{2p}(BS^1; \mathbf{Z})$.

3. *We calculate $d_{2p}\colon E_{2p}^{2p-2,2p-1}(S^1) \to E_{2p}^{4p-2,0}(S^1)$.* We have

$$E_{2p}^{2p-2,2p-1}(S^1) = \langle z_1 \cdot \tau^{p-1}, z_2 \cdot \tau^{p-1} \rangle = \mathbf{Z}^{\oplus 2}$$

and

$$E_{2p}^{4p-2,0}(S^1) = \langle \tau^{2p-1} \rangle = \mathbf{Z}\ .$$

The map

$$d_{2p}\colon E_{2p}^{2p-2,2p-1}(S^1) \to E_{2p}^{4p-2,0}(S^1)$$

is surjective because $d_{2p}(z_1 \cdot \tau^{p-1}) = \tau^{2p-1}$ spans $E_{2p}^{4p-2,0}(S^1)$. This means we have

$$E_{2p+1}^{2p-2,2p-1}(S^1) = \langle z_2 \cdot \tau^{p-1} \rangle$$

as the kernel of the above differential d_{2p} is $\langle z_2 \cdot \tau^{p-1} \rangle$.

4. *We define an element $\Gamma_{S^1} \in H^{4p-3}(B_{S^1}; \mathbf{Z})$.* As above $E_\infty^{2p-2,0}(S^1) = H^{2p-2}(BS^1; \mathbf{Z}) = \langle \tau^{p-1} \rangle$. Hence we have

$$0 \neq \pi_{S^1}^*(\tau^{p-1}) \in H^{2p-2}(B_{S^1}; \mathbf{Z})$$

Moreover, there exists

$$z_{S^1} \in H^{2p-1}(B_{S^1}; \mathbf{Z})$$

such that $i^*(z_{S^1}) = z_2 \in H^{2p-1}(K; \mathbf{Z})$. Define

$$\Gamma_{S^1} = z_{S^1} \cup \pi_{S^1}^*(\tau^{p-1}) \in H^{4p-3}(B_{S^1}; \mathbf{Z})\ .$$

5. *The element represented by Γ_{S^1} in the spectral sequence.* As above

$$\Gamma_{S^1} \in F^{2p-2}H^{4p-3}(B_{S^1}; \mathbf{Z})$$

and represents the following generator of the quotient

$$z_2 \cdot \tau^{p-1} \in E_\infty^{2p-2,2p-1}(S^1) = F^{2p-2}H^{4p-3}(B_{S^1}; \mathbf{Z})/F^{2p-1}H^{4p-3}(B_{S^1}; \mathbf{Z})$$

6. *The filtration term $F^{2p-2}H^{4p-3}(B_{S^1}; \mathbf{Z}) = \langle \Gamma_{S^1} \rangle = \mathbf{Z}$.*

7. *The quotient $H^{4p-3}(B_{S^1}; \mathbf{Z})/\langle \Gamma_{S^1} \rangle$ is a torsion group with no p-torsion.*

8. *The group ${}_{(p)}H^{4p-2}(B_{S^1}; \mathbf{Z}) = \mathbf{Z}/p \oplus \mathbf{Z}/p$.*

8. Transfer maps and the subset $T_{\widetilde{G}_p}$

The subset $T_{\widetilde{G}_p} \subseteq H_{4p-3}(B_{\widetilde{G}_p}; \mathbf{Z})$ was defined in **§6A** in terms of a transfer map on homology. The goal of this section is to show that

$$T_{\widetilde{G}_p} = \{\gamma \in H_{4p-3}(B_{\widetilde{G}_p}; \mathbf{Z}) \mid\ p \cdot (\mathrm{tr}(\gamma) - \gamma_{S^1}) = 0\}$$

is non-empty. We start by studying the transfers in cohomology.

§8A. **Transfers on certain elements.** Here we use the cohomology calculations in the previous section, and all through this section we use the elements $\Gamma_{\widetilde{G}_p}$, $\Gamma_{\widetilde{H}_t}$, and $\Gamma_{S^1} \in H^{4p-3}$ as defined in Section **§7C**, Section **§7D**, and Section **§7E** respectively.

Lemma 8.1.

(1) *Let* tr_1 *denote the transfer map for the p–covering* $B_{\widetilde{H}_t} \xrightarrow{\pi_1} B_{\widetilde{G}_p}$ *then*
$$\mathrm{tr}_1(\Gamma_{\widetilde{H}_t}) = \Gamma_{\widetilde{G}_p} \text{ and } \pi_1^*(\Gamma_{\widetilde{G}_p}) = p\Gamma_{\widetilde{H}_t}$$

(2) *Let* tr_2 *denote the transfer map for the p–covering* $B_{S^1} \xrightarrow{\pi_2} B_{\widetilde{H}_t}$ *then*
$$\mathrm{tr}_2(\Gamma_{S^1}) = \Gamma_{\widetilde{H}_t} \text{ and } \pi_2^*(\Gamma_{\widetilde{H}_t}) = p\Gamma_{S^1}$$

(3) *Let* tr *denote the transfer map for the* p^2*–covering* $B_{S^1} \xrightarrow{\pi} B_{\widetilde{G}_p}$ *then*
$$\mathrm{tr}(\Gamma_{S^1}) = \Gamma_{\widetilde{G}_p} \text{ and } \pi^*(\Gamma_{\widetilde{G}_p}) = p^2\Gamma_{S^1}$$

Proof. The proof consists of several transfer calculations.

(1) $\mathrm{tr}_1(\Gamma_{\widetilde{H}_t}) = \mathrm{tr}_1(z_{\widetilde{H}_t} \cup \pi^*_{\widetilde{H}_t}((\tau')^{p-1}))$
$= \mathrm{tr}_1(\pi_1^*(z_{\widetilde{G}_p}) \cup \pi^*_{\widetilde{H}_t}((\tau')^{p-1}))$ where $\pi_1 \colon B_{\widetilde{H}_t} \to B_{\widetilde{G}_p}$ is the natural map
$= z_{\widetilde{G}_p} \cup \pi^*_{\widetilde{G}_p}(\mathrm{tr}_1((\tau')^{p-1}))$ by the transfer formula
$= z_{\widetilde{G}_p} \cup \pi^*_{\widetilde{G}_p}(\chi_{p-1} - \alpha^{p-1})$ by Theorem 3.2
$= \Gamma_{\widetilde{G}_p}$.

(2) $\mathrm{tr}_2(\Gamma_{S^1}) = \mathrm{tr}_2(z_{S^1} \cup \pi^*_{S^1}(\tau^{p-1}))$
$= \mathrm{tr}_2(z_{S^1} \cup \pi^*_{S^1}(\pi_2^*((\tau')^{p-1}))$ where $\pi_2 \colon B_{S^1} \to B_{\widetilde{H}_t}$ is the natural map
$= \mathrm{tr}_2(z_{S^1}) \cup \pi^*_{\widetilde{H}_t}((\tau')^{p-1})$ by the transfer formula
$= z_{\widetilde{H}_t} \cup \pi^*_{\widetilde{H}_t}((\tau')^{p-1})$
$= \Gamma_{\widetilde{H}_t}$.

(3) Follows from the above. □

These results lead to some important properties of the homology classes γ_{S^1} and $\gamma_{\widetilde{H}_t}$ defined in Section **§6A**.

Proposition 8.2. $\mathrm{tr}(\gamma_{\widetilde{H}_t}) = \gamma_{S^1}$ *where* $\mathrm{tr} \colon H_{4p-3}(B_{\widetilde{H}_t}; \mathbf{Z}) \to H_{4p-3}(B_{S^1}; \mathbf{Z})$ *denotes the transfer map.*

Proof. Note that $\mathbf{CP}^{p-1} \times S^{2p-1}$ is the universal covering of M_t. Hence the transfer map from $H_{4p-3}(M_t; \mathbf{Z})$ to $H_{4p-3}(\mathbf{CP}^{p-1} \times S^{2p-1}; \mathbf{Z})$ maps the fundamental class of M_t to the fundamental class of $\mathbf{CP}^{p-1} \times S^{2p-1}$. In other words we have $\mathrm{tr}([M_t]) = [\mathbf{CP}^{p-1} \times S^{2p-1}]$. Now by considering the following pull-back diagram

$$\begin{array}{ccc} \mathbf{CP}^{p-1} \times S^{2p-1} & \xrightarrow{c} & B_{S^1} \\ \downarrow & & \downarrow \\ M_t & \xrightarrow{c_t} & B_{\widetilde{H}_t} \end{array}$$

we see that $\mathrm{tr}(\gamma_{\widetilde{H}_t}) = \gamma_{S^1}$, since $\gamma_{\widetilde{H}_t}$ and γ_{S^1} are respectively the images of the fundamental classes of M_t and $\mathbf{CP}^{p-1} \times S^{2p-1}$ under the maps c_* and $(c_t)_*$. □

Proposition 8.3. *Γ_{S^1} is a primitive element in $H^{4p-3}(B_{S^1};\mathbf{Z})$ and*

$$\langle \Gamma_{\widetilde{H}_t}, \gamma_{\widetilde{H}_t} \rangle = 1 \quad \textit{and} \quad \langle \Gamma_{S^1}, \gamma_{S^1} \rangle = 1$$

Proof. Considering the map $c\colon \mathbf{CP}^{p-1} \times S^{2p-1} \to B_{S^1}$ we have

$$c^*(\Gamma_{S^1}) = A \times B$$

where A is the cohomology fundamental class of $\mathbf{CP}^{p-1}$ and B is the cohomology fundamental class of S^{2p-1}. This proves the first result of this Lemma namely Γ_{S^1} is a primitive element in $H^{4p-3}(B_{S^1};\mathbf{Z})$ as $A \times B$ is a primitive element in $H^{4p-3}(\mathbf{CP}^{p-1} \times S^{2p-1};\mathbf{Z})$. The rest of the results follow from the following equalities:

$$\langle \Gamma_{\widetilde{H}_t}, \gamma_{\widetilde{H}_t} \rangle = \langle \mathrm{tr}(\Gamma_{S^1}), \gamma_{\widetilde{H}_t} \rangle = \langle \Gamma_{S^1}, \mathrm{tr}(\gamma_{\widetilde{H}_t}) \rangle = 1$$

and

$$\langle \Gamma_{S^1}, \gamma_{S^1} \rangle = \langle \Gamma_{S^1}, c_*([\mathbf{CP}^{p-1} \times S^{2p-1}]) \rangle = \langle A \times B, [\mathbf{CP}^{p-1}] \times [S^{2p-1}] \rangle = 1 \ .$$

□

Corollary 8.4. *$\Gamma_{\widetilde{G}_p}$, $\Gamma_{\widetilde{H}_t}$, and Γ_{S^1} are primitive elements in $H^{4p-3}(B_{\widetilde{G}_p};\mathbf{Z})$, $H^{4p-3}(B_{\widetilde{H}_t};\mathbf{Z})$, and $H^{4p-3}(B_{S^1};\mathbf{Z})$ respectively*

Proof. This result follows by Proposition 8.2 and Proposition 8.3. □

Corollary 8.5. *$\gamma_{\widetilde{H}_t}$, and γ_{S^1} are primitive elements in $H_{4p-3}(B_{\widetilde{H}_t};\mathbf{Z})$ and $H_{4p-3}(B_{S^1};\mathbf{Z})$ respectively*

Proof. This result follows by Universal Coefficient Theorem, Proposition 8.3 and Corollary 8.4. □

Corollary 8.6. *If $\gamma \in T_{\widetilde{G}_p}$, then $\langle \Gamma_{\widetilde{G}_p}, \gamma \rangle = 1$.*

Proof. From the formulas above, $\langle \Gamma_{\widetilde{G}_p}, \gamma \rangle = \langle \mathrm{tr}(\Gamma_{S^1}, \gamma \rangle = \langle \Gamma_{S^1}, \mathrm{tr}(\gamma) \rangle = \langle \Gamma_{S^1}, \gamma_{S^1} \rangle = 1$, since $\mathrm{tr}(\gamma) - \gamma_{S^1}$ is a torsion element. □

§8B. The homology spectral sequence. For any subgroup G of $\widetilde{G}_p$, let

$$\{E^r_{n,m}(G), d^r\}$$

be the homology Serre spectral sequence of the fibration

$$K \longrightarrow B_G \xrightarrow{\pi_G} BG$$

where $K = K(\mathbf{Z} \oplus \mathbf{Z}, 2p-1)$. The second page of this spectral sequence is given by:

$$E^2_{n,m}(G) = H_n(BG; H_m(K;\mathbf{Z}))$$

and it converges to $H_*(B_G;\mathbf{Z})$ with the filtration $F_*H_*(B_G;\mathbf{Z})$ of $H_*(B_G;\mathbf{Z})$ given by:

$$F_nH_{n+m}(B_G;\mathbf{Z}) = \mathrm{Im}\left\{H_{n+m}(B_G^{\{n\}};\mathbf{Z}) \to H_{n+m}(B_G;\mathbf{Z})\right\}$$

where $B_G^{\{n\}}$ stands for the inverse image of the n^{th} skeleton of BG under the map π_G in other words

$$B_G^{\{n\}} = \pi_G^{-1}(BG^{(n)})$$

Note that this means we have

$$E^{\infty}_{n,m}(G) = F_n H_{n+m}(B_G; \mathbf{Z})/F_{n-1}H_{n+m}(B_G; \mathbf{Z})$$

We will need some information about the homology of the fiber of these fibrations (see also [8], [9]).

Lemma 8.7. *Let $K = K(\mathbf{Z} \oplus \mathbf{Z}, 2p-1)$ and R be an abelian group.*

(1) $H_0(K; R) = R$
(2) $H_i(K; R)$ *is* 0 *for* $0 < i < 2p-1$,
(3) $H_{2p-1}(K; \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$,
(4) $H_{2p-1}(K, ; \mathbf{Z}/p) = \mathbf{Z}/p \oplus \mathbf{Z}/p$,
(5) ${}_{(p)}H_i(K; R)$ *is* 0 *for* $2p-1 < i < 4p-3$,
(6) ${}_{(p)}H_{4p-3}(K; \mathbf{Z}) = \mathbf{Z}/p \oplus \mathbf{Z}/p$,
(7) $H_{4p-3}(K; \mathbf{Z}/p) = \mathbf{Z}/p \oplus \mathbf{Z}/p$.

Proof. This Lemma is proved by the Universal Coefficient Theorem and Lemma 7.2. □

§8C. The homology of $B_{\widetilde{G}_p}$ and $B_{\widetilde{G}_p}^{\{k\}}$. To do our calculations we will use the cohomology (resp. homology) Serre spectral sequence

$$\{E_r^{n,m}(\widetilde{G}_p), d^r\} \text{ (resp. } \{E^r_{n,m}(\widetilde{G}_p), d^r\}\text{)}$$

defined as in Section **§7B** (resp. Section **§8B**) associated to the following fibration

$$K \longrightarrow B_{\widetilde{G}_p} \xrightarrow{\pi_{\widetilde{G}_p}} B\widetilde{G}_p$$

and we will also use the cohomology (resp. homology) Serre spectral sequence

$$\{E_r^{n,m}(k), d^r\} \text{ (resp. } \{E^r_{n,m}(k), d^r\}\text{)}$$

associated to the following fibration

$$K \longrightarrow B_{\widetilde{G}}^{\{k\}} \xrightarrow{\pi_k} B\widetilde{G}^{(k)}$$

where $0 \leq k \leq 4p-3$. Note that we will consider

$$B_{\widetilde{G}}^{\{0\}} = K$$

For any $0 \leq k \leq 2p-1$ the latter two spectral sequences both collapse as there are no possible differentials for $r \geq 2$. In other words, we have

$$E_\infty^{n,m}(k) = E_2^{n,m}(k)$$

and

$$E^\infty_{n,m}(k) = E^2_{n,m}(k)$$

Denote the inclusion maps as follows

$$i_k \colon B_{\widetilde{G}}^{\{k\}} \to B_{\widetilde{G}_p}$$

where $0 \leq k \leq 4p-3$. Note that for all $0 \leq k \leq 2p-1$ and $0 \leq n \leq k-1$, $(i_k)^*$ induces an isomorphism

$$E_2^{n,m}(k) = E_2^{n,m}(\widetilde{G}_p)$$

and $(i_k)_*$ induces an isomorphism

$$E^2_{n,m}(k) = E^2_{n,m}(\widetilde{G}_p)$$

Now we start our calculations:

1. $\mathrm{Im}((\pi_{\widetilde{G}_p})_*\colon H_{4p-3}(B_{\widetilde{G}_p};\mathbf{Z}) \to H_{4p-3}(B\widetilde{G}_p;\mathbf{Z})) \cong \mathbf{Z}/p$. First note that $H^{4p-3}(B_{\widetilde{G}_p};\mathbf{Z})$ has no p-torsion because the groups $E_\infty^{4p-3-r,r}(\widetilde{G}_p)$ have no p-torsion: (i) for $r=0$ because $H^*(B\widetilde{G}_p;\mathbf{Z})$ is zero in odd degrees, (ii) for $1 \leq r \leq 2p-2$ or $2p \leq r \leq 4p-3$ by Lemma 7.2, and (iii) for $r = 2p-1$ because $E_{2p+1}^{2p-2,2p-1}(\widetilde{G}_p) = \langle pz_2 \cdot \chi_{p-1}\rangle = \mathbf{Z}$ (see Section **§7C**, Part **3**).

This also shows that $H_{4p-4}(B_{\widetilde{G}_p};\mathbf{Z})$ has no p-torsion. However, by the universal coefficient theorem and Part **3** in Section **§7C** again we have

$$E^2_{2p-3,2p-1}(\widetilde{G}_p) = E^{2p}_{2p-3,2p-1}(\widetilde{G}_p) = (\mathbf{Z}/p)^{\oplus 2p}$$

Hence

$$E^{2p+1}_{2p-3,2p-1}(\widetilde{G}_p) = 0\ .$$

In other words

$$d^{2p}\colon E^{2p}_{4p-3,0}(\widetilde{G}_p) \to E^{2p}_{2p-3,2p-1}(\widetilde{G}_p)$$

is surjective. Again by the universal coefficient theorem and Part **3** in Section **§7C** we have

$$E^2_{4p-3,0}(\widetilde{G}_p) = E^{2p}_{4p-3,0}(\widetilde{G}_p) = (\mathbf{Z}/p)^{\oplus 2p+1}\ .$$

Hence the kernel of the above differential is a single $\mathbf{Z}/p$ which has to live to the E^∞-term, so we get

$$E^\infty_{4p-3,0}(\widetilde{G}_p) = \mathbf{Z}/p\ .$$

This proves that

$$\mathbf{Z}/p \cong \mathrm{Im}((\pi_{\widetilde{G}_p})_*\colon H_{4p-3}(B_{\widetilde{G}_p};\mathbf{Z}) \to H_{4p-3}(B\widetilde{G}_p;\mathbf{Z}))\ .$$

2. *For any element $\gamma \in T_{\widetilde{G}_p}$, then $(\pi_{\widetilde{G}_p})_*(\gamma)$ generates the image of $(\pi_{\widetilde{G}_p})_*$ in $H_{4p-3}(B\widetilde{G}_p;\mathbf{Z})$.* By the above we have $E_\infty^{2p-2,0}(2p-1) = E_2^{2p-2,0}(\widetilde{G}_p) = H^{2p-2}(B\widetilde{G}_p;\mathbf{Z})$ Hence we have

$$0 \neq \pi^*(\chi_{p-1}) \in H^{2p-2}(B^{\{2p-1\}}_{\widetilde{G}};\mathbf{Z})$$

Moreover, there exists

$$z_1', z_2' \in H^{2p-1}(B^{\{2p-1\}}_{\widetilde{G}};\mathbf{Z})$$

such that z_1' maps to $z_1 \in H^{2p-1}(K;\mathbf{Z})$ and z_2' maps to $z_2 \in H^{2p-1}(K;\mathbf{Z})$ under the map $K \to B^{\{2p-1\}}_{\widetilde{G}}$ induced by the inclusion. This holds because $E_\infty^{0,2p-1}(2p-1) = \langle z_1, z_2\rangle =$

$H^{2p-1}(K;\mathbf{Z})$. Moreover we can assume that under restriction $pz_2' = i^*(z_{\widetilde{G}_p})$ where $z_{\widetilde{G}_p}$ is defined as in Section **§7C**. Now, define

$$\Gamma = z_2' \cup \pi^*_{\widetilde{G}_p}(\chi_{p-1}) \in H^{4p-3}(B^{\{2p-1\}}_{\widetilde{G}_p};\mathbf{Z}).$$

Then we have

$$(i_{2p-1})^*(\Gamma_{\widetilde{G}_p}) = p\Gamma$$

where $\Gamma_{\widetilde{G}_p}$ is defined as in Section **§7C**.By Corollary 8.6, an element $\gamma \in T_{\widetilde{G}_p}$ can not be in the image of the map

$$(i_{2p-1})_*\colon H_*(B^{\{2p-1\}}_{\widetilde{G}_p};\mathbf{Z}) \to H_*(B_{\widetilde{G}_p};\mathbf{Z}) \ .$$

It is clear that

$$E^2_{2p-2+r,2p-1-r}(\widetilde{G}_p) = 0$$

for $1 \le r \le 2p-2$. Hence $\gamma \in T_{\widetilde{G}_p}$ is non zero in $E^\infty_{4p-3,0}(\widetilde{G}_p)$, which proves the result by the previous fact.

3. *$F_{2p-3}H_{4p-3}(B_{\widetilde{G}_p};\mathbf{Z})$ is the torsion part of $H_{4p-3}(B_{\widetilde{G}_p};\mathbf{Z})$.*

4. *${}_{(p)}F_0H_{4p-3}(B_{\widetilde{H}_t};\mathbf{Z}) = \mathbf{Z}/p \oplus \mathbf{Z}/p$ is the p-torsion part of $H_{4p-3}(B_{\widetilde{H}_t};\mathbf{Z})$.*

§8D. The homology of B_{S^1} and $B^{\{k\}}_{S^1}$.

1. *$F_{2p-3}H_{4p-3}(B_{S^1};\mathbf{Z})$ is the torsion part of $H_{4p-3}(B_{S^1};\mathbf{Z})$.*

2. *The natural map $\pi_*\colon H_{4p-3}(B_{S^1};\mathbf{Z}) \to H_{4p-3}(B_{\widetilde{G}_p};\mathbf{Z})$ is a injection on the torsion elements.* This implies that $\pi_*\colon F_{2p-3}H_{4p-3}(B_{S^1};\mathbf{Z}) \to F_{2p-3}H_{4p-3}(B_{\widetilde{G}_p};\mathbf{Z})$ is a bijection.

3. *$F_0H_{4p-3}(B_{S^1};\mathbf{Z})$ is the p-torsion subgroup of $H_{4p-3}(B_{S^1};\mathbf{Z})$. .*

§8E. The subset $T_{\widetilde{G}_p}$ is non-empty. Let $\mathrm{tr}\colon H_{4p-3}(B_{\widetilde{G}_p};\mathbf{Z}) \to H_{4p-3}(B_{S^1};\mathbf{Z})$ denote the transfer map.

Lemma 8.8. *Let γ' in $H_{4p-3}(B_{\widetilde{G}_p};\mathbf{Z})$ be an element such that $\langle \Gamma_{S^1}, \mathrm{tr}(\gamma')\rangle = 1$. Then there exists an integer $N_{\gamma'}$ such that $p(1-p^2N_{\gamma'})(\mathrm{tr}(\gamma') - \gamma_{S^1}) = 0$.*

Proof. First note that

$$\langle \Gamma_{S^1}, \mathrm{tr}(\gamma') - \gamma_{S^1}\rangle = 0$$

by Proposition 8.3. Hence $\mathrm{tr}(\gamma') - \gamma_{S^1}$ is a torsion element, by the universal coefficient theorem applied to the calculations in Part **7** of Section **§7E**. Hence it is enough to prove that the order of $p(\mathrm{tr}(\gamma')-\gamma_{S^1})$ is relatively prime to p. But this is clear as the p-torsion of part of $H_{4p-3}(B_{S^1};\mathbf{Z})$ is same as the p-torsion part of $H^{4p-2}(B_{S^1};\mathbf{Z})$, which is $\mathbf{Z}/p \oplus \mathbf{Z}/p$ by Part **8** of Section **§7E**. □

Theorem 8.9. *The set $T_{\widetilde{G}_p}$ is not empty. Any $\gamma \in T_{\widetilde{G}_p}$ is a primitive element of infinite order in $H_{4p-3}(B_{\widetilde{G}_p};\mathbf{Z})$.*

Proof. The Universal Coefficient Theorem with Lemma 8.1 and Corollary 8.4 tells us that there exists a primitive element γ' in $H_{4p-3}(B_{\widetilde{G}_p};\mathbf{Z})$ such that $\mathrm{tr}(\gamma')$ is a primitive element in $H_{4p-3}(B_{S^1};\mathbf{Z})$ and

$$\langle \Gamma_{\widetilde{G}_p}, \gamma' \rangle = 1 \text{ and } \langle \Gamma_{S^1}, \mathrm{tr}(\gamma') \rangle = 1 \ .$$

Let π_1 and π_2 be as in Lemma 8.1, and take $N_{\gamma'}$ as in Lemma 8.8. Define

$$\gamma_{\widetilde{G}_p} = \gamma' - N_{\gamma'}(\pi_2 \circ \pi_1)_*(\mathrm{tr}(\gamma') - \gamma_{S^1}) \ .$$

Then $\gamma_{\widetilde{G}_p}$ is in $T_{\widetilde{G}_p}$, because by Lemma 8.8 we have

$$p(\mathrm{tr}(\gamma_{\widetilde{G}_p}) - \gamma_{S^1}) = p(\mathrm{tr}(\gamma' - N_{\gamma'}(\pi_2 \circ \pi_1)_*(\mathrm{tr}(\gamma') - \gamma_{S^1})) - \gamma_{S^1}) = p(1 - p^2 N_{\gamma'})(\mathrm{tr}(\gamma') - \gamma_{S^1}) = 0$$

Now, take any γ in $T_{\widetilde{G}_p}$. Suppose that $\gamma = r\gamma_1$, for some γ_1 in $H_{4p-3}(B_{\widetilde{G}_p};\mathbf{Z})$. Then we would have $p \cdot (\gamma_{S^1} - r \cdot \mathrm{tr}(\gamma_1)) = 0$. But $\langle \Gamma_{S^1}, \gamma_{S^1} \rangle = 1$ by Proposition 8.3 implies $r = \pm 1$. □

Proposition 8.10. *Let* $\mathrm{tr}\colon H_{4p-3}(B_{\widetilde{G}_p};\mathbf{Z}) \to H_{4p-3}(B_{\widetilde{H}_t};\mathbf{Z})$ *denote the transfer map. Then any* γ *in* $T_{\widetilde{G}_p}$ *satisfies the following equation*

$$p(\mathrm{tr}(\gamma) - \gamma_{\widetilde{H}_t}) = 0$$

Proof. For any γ in $T_{\widetilde{G}_p}$ the image of $p(\mathrm{tr}(\gamma) - \gamma_{\widetilde{H}_t})$ under the transfer map from $H_{4p-3}(B_{\widetilde{H}_t};\mathbf{Z})$ to $H_{4p-3}(B_{S^1};\mathbf{Z})$ is 0 by definition of $T_{\widetilde{G}_p}$ and Proposition 8.2. Note that the kernel of the above transfer map is included in the p-torsion part of $H_{4p-3}(B_{\widetilde{H}_t};\mathbf{Z})$, as $B_{S^1} \to B_{\widetilde{H}_t}$ is p-covering. But the p-torsion part of $H_{4p-3}(B_{\widetilde{H}_t};\mathbf{Z})$ is $\mathbf{Z}/p \oplus \mathbf{Z}/p$ (which has exponent p). This proves the result. □

9. The construction of the bordism element

The next step in our argument is to study the bordism groups $\Omega_{4p-3}(B_{\widetilde{G}_p}, \nu_{\widetilde{G}_p})$ of our normal $(2p-1)$-type. The main result of this section is Theorem 9.9, which proves that the image of the Hurewicz map

$$\Omega_{4p-3}(B_{\widetilde{G}_p}, \nu_{\widetilde{G}_p}) \to H_{4p-3}(B_{\widetilde{G}_p};\mathbf{Z})$$

contains the non-empty subset $T_{\widetilde{G}_p}$. The main difficulty in computing the bordism groups is dealing with p-torsion. We will primarily use the James spectral sequence associated to the fibration

$$* \longrightarrow B_{\widetilde{G}_p} \longrightarrow B_{\widetilde{G}_p}$$

with E^2-term

$$E^2_{n.m}(\nu_{\widetilde{G}_p}) = H_n(B_{\widetilde{G}_p};\Omega^{fr}_m(*))$$

where the coefficients $\Omega^{fr}_m(*) = \pi^S_m$ are the stable homotopy groups of spheres. In our range, the p-torsion in π^S_m occurs only for π^S_{2p-3} and π^S_{4p-5}, where the p-primary part is $\mathbf{Z}/p$ (see [29, p. 5] and Example 9.3). This means that, after localizing at p, there are only two possibly non-zero differentials with source at the $(4p-3,0)$-position, namely d_{2p-2} and d_{4p-4}. To show that these differentials are in fact both zero, and to prove that all other differentials starting at the $(4p-3,0)$-position also vanish on $T_{\widetilde{G}_p}$, we use two techniques:

(i) For the differentials d^r with $2 \leq r \leq 4p-5$, and d^{4p-3}, we compare the James spectral sequence for $\Omega_{4p-3}(B_{\widetilde{G}_p}, \nu_{\widetilde{G}_p})$ to the ones for $\Omega_{4p-3}(B_{\widetilde{H}_t}, \nu_{\widetilde{H}_t})$ via transfer, and use naturality.

(ii) For the differential d^{4p-4} we compare the James spectral sequence for $\Omega_{4p-3}(B_{\widetilde{G}_p}, \nu_{\widetilde{G}_p})$ to the James spectral sequences for the fibrations $B\widetilde{H}_t \to B\widetilde{G}_p \to B(\widetilde{G}_p/\widetilde{H}_t)$, and use naturality again.

In carrying out the second step, we will need to use the Adams spectral sequence to prove that the natural map from the p–component of $\Omega^{fr}_{4p-5}(*)$ to $\Omega_{4p-5}(B\widetilde{H}_t, \xi_{\widetilde{H}_t})$ is injective (see Theorem 9.5).

§9A. **The James Spectral Sequence.** Let $\{E^r_{n,m}(\nu)\}$ denote the James spectral sequence (see [34]) associated to a vector bundle ν over a base space B and the fibration

$$* \longrightarrow B \longrightarrow B$$

and denote the differentials of this spectral sequence by d^r. We know that the second page is given by

$$E^2_{n,m}(\nu) = H_n(B, \Omega^{fr}_m(*))$$

and the spectral sequence converges to

$$E^{\infty}_{n,m}(\nu) = F_n\Omega_{n+m}(B,\nu)/F_{n-1}\Omega_{n+m}(B,\nu)$$

where $B^{(n)}$ stands for the n^{th} skeleton of B and

$$F_n\Omega_{n+m}(B,\nu) = \mathrm{Im}(\Omega_{n+m}(B^{(n)}, \nu|_{B^{(n)}}) \to \Omega_{n+m}(B,\nu))$$

For $0 \leq t \leq p$, let

$$\mathrm{tr}_t \colon E^r_{n,m}(\nu_{\widetilde{G}_p}) \to E^r_{n,m}(\nu_{\widetilde{H}_t})$$

denote the transfer map.

§9B. **Calculation of d^r when $2 \leq r \leq 4p-5$.** Here we employ our first technique. We first need some information about the James spectral sequences for $\Omega_{4p-3}(B_{\widetilde{H}_t}, \nu_{\widetilde{H}_t})$.

Lemma 9.1. *For $2 \leq r \leq 4p-5$, the differential*

$$d^r \colon E^r_{4p-3,0}(\nu_{\widetilde{H}_t}) \to E^r_{4p-3-r,r-1}(\nu_{\widetilde{H}_t})$$

is zero on $\mathrm{tr}_t(T_{\widetilde{G}_p})$, where $T_{\widetilde{G}_p}$ is considered as subgroup of $E^r_{4p-3,0}(\nu_{\widetilde{G}_p})$.

Proof. By Proposition 8.10, and the fact that $d^r(\gamma_{\widetilde{H}_t}) = 0$, it is enough to show that $d^r \colon E^r_{4p-3,0}(\nu_{\widetilde{H}_t}) \to E^r_{4p-3-r,r-1}(\nu_{\widetilde{H}_t})$ is zero on the p-torsion subgroup

$$I_t = \mathrm{Im}({}_{(p)}H_{4p-3}(K;\mathbf{Z}) \to H_{4p-3}(B_{\widetilde{H}_t};\mathbf{Z}))$$

for $2 \leq r \leq 4p-5$ (using Part **4**, Section **§8C**). We consider the cases (i) $2 \leq r \leq 2p-3$, (ii) $r = 2p-2$, and (iii) $2p-1 \leq r \leq 4p-5$ separately.

Case (i). Consider the natural map

$$i_*\colon E^r_{4p-3-r,r-1}(\nu_{\widetilde{H}_t}|_K) \to E^r_{4p-3-r,r-1}(\nu_{\widetilde{H}_t}) \ .$$

Since ${}_{(p)}E^2_{4p-3,0}(\nu_{\widetilde{H}_t}|_K)$ is all p-torsion, and the group $E^2_{4p-3-r,r-1}(\nu_{\widetilde{H}_t}|_K)$ is p-torsion free for $2 \leq r \leq 2p-3$, it follows that the differential

$$d^r\colon E^r_{4p-3,0}(\nu_{\widetilde{H}_t}|_K) \to E^r_{4p-3-r,r-1}(\nu_{\widetilde{H}_t}|_K)$$

is zero on ${}_{(p)}E^r_{4p-3,0}(\nu_{\widetilde{H}_t}|_K)$. However, the image of $i_*\colon {}_{(p)}E^2_{4p-3,0}(\nu_{\widetilde{H}_t}|_K) \to E^2_{4p-3,0}(\nu_{\widetilde{H}_t})$ contains I_t. Hence the differential

$$d^r\colon E^r_{4p-3,0}(\nu_{\widetilde{H}_t}) \to E^r_{4p-3-r,r-1}(\nu_{\widetilde{H}_t})$$

is zero on I_t, for $2 \leq r \leq 2p-3$.

Case (ii). Next we observe that the map $i_*\colon E^2_{2p-1,2p-3}(\nu_{\widetilde{H}_t}|_K) \to E^2_{2p-1,2p-3}(\nu_{\widetilde{H}_t})$, restricted to p-torsion, is just the natural map $i_*\colon H_{2p-1}(K;\mathbf{Z}/p) \to H_{2p-1}(B_{\widetilde{H}_t};\mathbf{Z}/p)$, which is zero (see Section **§7D**, Part **2**, reduced mod p, which shows that the dual map is zero). Hence, the differential

$$d^{2p-2}\colon E^{2p-2}_{4p-3,0}(\nu_{\widetilde{H}_t}) \to E^{2p-2}_{2p-1,2p-3}(\nu_{\widetilde{H}_t})$$

is zero on I_t by naturality.

Case (iii). Finally, we note that I_t is all p–torsion, but for $2p-1 \leq r \leq 4p-5$, the group $E^2_{4p-3-r,r-1}(\nu_{\widetilde{H}_t})$ is p–torsion free. Hence, for $2p-1 \leq r \leq 4p-5$, the differential

$$d^r\colon E^r_{4p-3,0}(\nu_{\widetilde{H}_t}) \to E^r_{4p-3-r,r-1}(\nu_{\widetilde{H}_t})$$

is zero on I_t. □

Lemma 9.2. *For $2 \leq r \leq 4p-5$, the differential*

$$d^r\colon E^r_{4p-3,0}(\nu_{\widetilde{G}_p}) \to E^r_{4p-3-r,r-1}(\nu_{\widetilde{G}_p})$$

is zero on $T_{\widetilde{G}_p}$, where $T_{\widetilde{G}_p}$ is considered as a subgroup of $E^r_{4p-3,0}(\nu_{\widetilde{G}_p})$.

Proof. Assume $2 \leq r \leq 4p-5$. By Lemma 9.1 we know that

$$d^r\colon E^r_{4p-3,0}(\nu_{\widetilde{H}_t}) \to E^r_{4p-3-r,r-1}(\nu_{\widetilde{H}_t})$$

is zero for all $t \in \{0,1,\dots,p\}$. Hence it is enough to show that

$$\bigoplus\nolimits_t \mathrm{tr}_t\colon E^r_{4p-3-r,r-1}(\widetilde{G}_p) \to \bigoplus\nolimits_t E^r_{4p-3-r,r-1}(\widetilde{H}_t)$$

is injective. The map tr_0 is clearly injective for $r \neq 2p-2$ because the p–component of $\Omega^*_{r-1}(*)$ is 0 and $B_{\widetilde{H}_t} \to B_{\widetilde{G}_p}$ is a p-covering map. Hence $\bigoplus_t \mathrm{tr}_t$ is injective for $r \neq 2p-2$. Now we know that $\widetilde{B_{\widetilde{G}_p}}$ and $\widetilde{B_{\widetilde{H}_t}}$ are $2p-2$ connected. Hence for $r = 2p-2$ the map tr_t is the usual transfer map $H_{2p-1}(B\widetilde{G}_p;\mathbf{Z}/p) \to H_{2p-1}(B\widetilde{H}_t;\mathbf{Z}/p)$. Hence, it is enough to show that the map

$$\bigoplus\nolimits_t \mathrm{tr}_t\colon H_{2p-1}(B\widetilde{G}_p;\mathbf{Z}/p) \to \bigoplus\nolimits_t H_{2p-1}(B\widetilde{H}_t;\mathbf{Z}/p)$$

is injective. Dually, this is equivalent to showing that

$$\bigoplus_t \mathrm{tr}_t \colon \bigoplus_t H^{2p-1}(B\widetilde{H}_t; \mathbf{Z}/p) \to H^{2p-1}(B\widetilde{G}_p; \mathbf{Z}/p)$$

is surjective. By Theorem 3.3, we know that

$$H^{2p-1}(B\widetilde{G}_p; \mathbf{Z}/p) = \langle x^{p-1}y, x^{p-2}x'y, \ldots, (x')^{p-1}y, (x')^{p-1}y' \rangle \ .$$

Under the Bockstein homomorphism, this can be identified with

$$V_{p+1} = \langle \alpha^p, \alpha^{p-1}\beta, \ldots, \alpha\beta^{p-1}, \beta^p \rangle \subseteq H^{2p}(B\widetilde{G}_p; \mathbf{Z})$$

and this identifcation is natural with respect to the action of the automorphisms $\mathrm{Aut}(\widetilde{G}_p)$ acting through its quotient group $GL_2(p)$. This module V_{p+1} is known to be an indecomposable $GL_2(p)$-module (see [12, 5.7]), and there is a short exact sequence

$$0 \to V_2 \to V_{p+1} \to V_{p-1} \to 0$$

of $GL_2(p)$-modules, where $V_2 = \langle \alpha^p, \beta^p \rangle$ has dimension 2 and V_{p-1} is irreducible.

Now the image of the map $\bigoplus_t \mathrm{tr}_t$ is invariant under all automorphisms of the group $\widetilde{G}_p$. Hence it is enough to show that $\mathrm{Im}(\bigoplus_t \mathrm{tr}_t)$ projects non-trivially into V_{p-1}. However, the calculations of [22, p. 67] show that

$$\mathrm{tr}_p(\mathrm{Res}_{\widetilde{H}_p}(y') \cdot \bar{\tau}^{p-1}) = y' \cdot \mathrm{tr}_p(\bar{\tau}^{p-1}) = y'(c_{p-1} + x^{p-1}) = -(x')^{p-1}y' + y'x^{p-1} \ .$$

After applying the Bockstein, this shows that the element $\beta^p - \beta\alpha^{p-1}$ is contained in the image of the transfer. Since this element is not contained in the submodule V_2, we are done. □

The remaining possibly non-zero differentials are d^{4p-4} and d^{4p-3}. The first one is handled by comparison with the fibrations

$$B\widetilde{H}_t \to B\widetilde{G}_p \to B(\widetilde{G}_p/\widetilde{H}_t)$$

but first we must show that the induced map on coefficients at the $(0, 4p-3)$-position is injective on the p-component. For this we use the Adams spectral sequence.

§9C. **The Adams spectral sequence.** Let X be a connective spectrum of finite type we will write

$$X = \{X_n, i_n\}_{n \geq 0}$$

where each X_n is a space with a basepoint and $i_n \colon \Sigma X_n \to X_{n+1}$ is a basepoint preserving map. We will denote the Adams spectral sequnce for X as follows:

$$\{E_r^{n,m}(X), d_r\}$$

The second page of this spectral sequence is given by

$$E_2^{n,m}(X) = \mathrm{Ext}_{\mathcal{A}_p}^{n,m}(H^*(X; \mathbf{Z}/p), \mathbf{Z}/p)$$

where $\mathcal{A}_p$ is the mod-p Steenrod algebra and $H^*(X; \mathbf{Z}/p)$ is considered as an $\mathcal{A}_p$-module. The differentials of this spectral sequence are as follows:

$$d_r \colon E_r^{n,m} \to E_r^{n+r, m+r-1}$$

for $r \geq 2$, and it converges to

$$_{(p)}\pi_*^S(X) = \pi_*^S(X)/\langle \text{torsion prime to } p\rangle$$

with the filtration

$$\cdots \subseteq F^{2,*+2}(X) \subseteq F^{1,*+1}(X) \subseteq F^{0,*}(X) = {}_{(p)}\pi_*^S(X)$$

defined as follows:

$$F^{n,m}(X) = {}_{(p)}\operatorname{Im}\{\pi_m^S(X_n) \to \pi_{m-n}^S(X)\}$$

In other words

$$E_\infty^{n,m}(X) = F^{n,m}(X)/F^{n+1,m+1}(X) \ .$$

First we will go over the technique that we will use to calculate the Adams spectral sequence. We take a minimal $\mathcal{A}_p$–free resolution of $H^*(X; \mathbf{Z}/p)$

$$\ldots \xrightarrow{\partial_3} F_2^X \xrightarrow{\partial_2} F_1^X \xrightarrow{\partial_1} F_0^X \xrightarrow{\partial_0} H^*(X; \mathbf{Z}/p) \ .$$

Then all the boundary maps in the dual complex

$$\operatorname{Hom}_{\mathcal{A}_p}(F_0^X, \mathbf{Z}/p) \xrightarrow{\delta_1} \operatorname{Hom}_{\mathcal{A}_p}(F_1^X, \mathbf{Z}/p) \xrightarrow{\delta_2} \operatorname{Hom}_{\mathcal{A}_p}(F_2^X, \mathbf{Z}/p) \xrightarrow{\delta_3} \ldots$$

are zero, hence we have

$$\operatorname{Ext}_{\mathcal{A}_p}^{n,m}(H^*(X; \mathbf{Z}/p), \mathbf{Z}/p) = \operatorname{Hom}_{\mathcal{A}_p}^m(F_n^X, \mathbf{Z}/p)$$

where F_n^X is a graded module over the graded algebra $\mathcal{A}_p$ with degrees ≥ 0, and $\mathbf{Z}/p$ is considered as a graded module over the graded algebra $\mathcal{A}_p$ with only non-zero elements in degree 0. In addition, $\operatorname{Hom}_{\mathcal{A}_p}^m(F_n^X, \mathbf{Z}/p)$ is the group of all $\mathcal{A}_p$-homomorphisms from F_n^X to $\mathbf{Z}/p$ which shift the degree by $-m$. Therefore, if we select a basis $\mathcal{B}(X, n)$ for the free $\mathcal{A}_p$-module F_n^X and write

$$\mathcal{B}(X, n, m) = \{\ b \in \mathcal{B}(X, n) \,|\, \text{degree of } b \text{ is } m\ \}$$

Then

$$\operatorname{Hom}_{\mathcal{A}_p}^m(F_n^X, \mathbf{Z}/p) = \bigoplus_{b \in \mathcal{B}(X,n,m)} \operatorname{Hom}_{\mathcal{A}_p}(\mathcal{A}_p, \mathbf{Z}/p) = \bigoplus_{b \in \mathcal{B}(X,n,m)} \mathbf{Z}/p$$

To simplfy our notations we will write

$$\mathcal{B}(X, n, \leq k) = \bigcup_{0 \leq m \leq k} \mathcal{B}(X, n, m)$$

and for $n = 0, 1, 2$ respectively we will denote the elements of $\mathcal{B}(X, n, m)$ by

$$\iota_{m,*}^X,\ \alpha_{m,*}^X, \text{ and } \beta_{m,*}^X$$

where $*$ ranges over some indexing set $I_{n,m}$. Moreover if $\mathcal{B}(X, n, m)$ has only one element then we will forget the indexing and only write

$$\iota_m^X,\ \alpha_m^X, \text{ and } \beta_m^X$$

respectively, for elements in $\mathcal{B}(X, 0, m)$, $\mathcal{B}(X, 1, m)$, and $\mathcal{B}(X, 2, m)$. In order to demonstrate the technique that we will use to calculate the Adams spectral sequence, we will go over an already well known calculation which will, in fact, be used in our calculations later on.

Example 9.3. Take an $\mathcal{A}_p$–free resolution $F_*^{\mathbb{S}}$ of the sphere spectrum $\mathbb{S}$ as follows

$$\dots \xrightarrow{\partial_3} F_2^{\mathbb{S}} \xrightarrow{\partial_2} F_1^{\mathbb{S}} \xrightarrow{\partial_1} F_0^{\mathbb{S}} \xrightarrow{\partial_0} H^*(\mathbb{S};\mathbf{Z}/p)$$

We obtain:
$\mathcal{B}(\mathbb{S},0,\le\infty)=\{\iota_0^{\mathbb{S}}\}$ where

- $\partial_0(\iota_0^{\mathbb{S}})$ is a generator of $H^*(\mathbb{S};\mathbf{Z}/p)=\mathbf{Z}/p$

$\mathcal{B}(\mathbb{S},1,\le 4p-4)=\{\alpha_0^{\mathbb{S}},\alpha_{2p-3}^{\mathbb{S}}\}$ where

- $\partial_1(\alpha_0^{\mathbb{S}})=\beta(\iota_0^{\mathbb{S}})$
- $\partial_1(\alpha_{2p-3}^{\mathbb{S}})=P^1(\iota_0^{\mathbb{S}})$

$\mathcal{B}(\mathbb{S},2,\le 4p-5)=\{\beta_0^{\mathbb{S}},\beta_{4p-5}^{\mathbb{S}}\}$ where

- $\partial_2(\beta_0^{\mathbb{S}})=\beta(\alpha_0^{\mathbb{S}})$
- $\partial_2(\beta_{4p-5}^{\mathbb{S}})=P^2(\alpha_0^{\mathbb{S}})-P^1\beta(\alpha_{2p-3}^{\mathbb{S}})+2\beta P^1(\alpha_{2p-3}^{\mathbb{S}})$

$\mathcal{B}(\mathbb{S},n,\le 4p-6)$ has only one element w_n when $n\ge 3$ where

- $\partial_3(w_3)=\beta(\beta_0^{\mathbb{S}})$
- $\partial_n(w_n)=\beta(w_{n-1})$

In the Adams spectral sequence that converges to the p–component of $\pi_*^S(\mathbb{S})=\Omega_*^{fr}(*)$, as there are no possible differentials, the elements $\iota_0^{\mathbb{S}}$, $\alpha_0^{\mathbb{S}}$, $\alpha_{4p-3}^{\mathbb{S}}$, $\beta_0^{\mathbb{S}}$, and $\beta_{4p-5}^{\mathbb{S}}$ must survive to the E^∞–term. Hence we have the following

1. ${}_{(p)}\Omega_0^{fr}(*)=\mathbf{Z}=\langle\iota_0^{\mathbb{S}}\rangle$ where $p\iota_0^{\mathbb{S}}=\alpha_0^{\mathbb{S}}$, $p\alpha_0^{\mathbb{S}}=\beta_0^{\mathbb{S}}$, ...
2. ${}_{(p)}\Omega_{2p-3}^{fr}(*)=\mathbf{Z}/p=\langle\alpha_{2p-3}^{\mathbb{S}}\rangle$
3. ${}_{(p)}\Omega_{4p-5}^{fr}(*)=\mathbf{Z}/p=\langle\beta_{4p-5}^{\mathbb{S}}\rangle$

§9D. **Cohomology of the Thom spectrum associated to ξ_G.** Now take any $G\subseteq\widetilde{G}_p$ and let $M\xi_G$ denote the Thom spectrum associated to the bundle ξ_G. Since the bundle ξ_G is fixed, for a given G, we will shorten the notation by writing $MG=M\xi_G$. As in the previous section we will denote an $\mathcal{A}_p$–free resolution of $H^*(MG;\mathbf{Z}/p)$ as follows:

$$\dots \xrightarrow{\partial_2} F_1^{MG} \xrightarrow{\partial_1} F_0^{MG} \xrightarrow{\partial_0} H^*(MG;\mathbf{Z}/p)$$

It is clear that, to understand these resolutions, we must first understand the $\mathcal{A}_p$–module structure on the cohomology $H^*(MG;\mathbf{Z}/p)$ of these spectra. Let $U_G\in H^0(MG;\mathbf{Z}/p)$ denote the Thom class of the Thom spectrum MG then we can write

$$H^*(MG;\mathbf{Z}/p)=U_G\cdot H^*(BG;\mathbf{Z}/p)$$

Moreover, for $G=S^1$ we will write

$$H^*(BS^1;\mathbf{Z}/p)=\mathbf{F}_p[\overline{\tau}]$$

for $G=D_t$ we have

$$H^*(BD_t;\mathbf{Z}/p)=(\Lambda(u)\otimes\mathbf{F}_p[v])$$

hence, for $G = \widetilde{H}_t$ we can consider

$$H^*(B\widetilde{H}_t; \mathbf{Z}/p) = H^*(BD_t; \mathbf{Z}/p) \otimes H^*(BS^1; \mathbf{Z}/p) = (\Lambda(u) \otimes \mathbf{F}_p[v]) \otimes \mathbf{F}_p[\overline{\tau}]$$

Lemma 9.4. *For $G = S^1$, D_t, or $\widetilde{H}_t$ we have*

(1) $\beta(U_G) = 0$
(2) $P^1(U_{S^1}) = 0$, $P^1(U_{\widetilde{H}_t}) = U_{\widetilde{H}_t} v^{p-1}$, *and* $P^1(U_{D_t}) = U_{D_t} v^{p-1}$
(3) $P^1\beta(U_G) = 0$ *and* $\beta P^1(U_G) = 0$
(4) $\beta P^1 \beta(U_G) = 0$
(5) $P^2(U_G) = 0$

Proof. The Thom class U_G is the mod p reduction of an integral cohomology class, so $\beta(U_G) = 0$ and Part (4) is clear. By Lemma 5.7, $q_1(\xi_{\widetilde{H}_t}) = v^{p-1}$. Since $P^1(U_{\widetilde{H}_t}) = U_{\widetilde{H}_t} v^{p-1}$, we obtain

$$P^1(U_{S^1}) = 0 \quad \text{and} \quad P^1(U_{D_t}) = U_{D_t} v^{p-1}$$

by restriction to $H^*(BD_t; \mathbf{Z}/p)$ and $H^*(BS^1; \mathbf{Z}/p)$. For $G = D_t$ or $\widetilde{H}_t$ we have

$$\beta P^1(U_G) = \beta(U_G v^{p-1}) = \beta(U) v^{p-1} + U\beta(v^{p-1}) = 0 + 0 = 0$$

and it is clear that $\beta P^1(U_{S^1}) = 0$. By the Adem relations we have $P^2(U_G) = 2P^1P^1(U_G)$. Hence for $G = D_t$ or $\widetilde{H}_t$ we have

$$P^2(U_G) = 2P^1(U_G v^{p-1}) = 2(P^1(U_G)v^{p-1}) + U_G P^1(v^{p-1})) = 2(U_G v^{2p-2} - U_G v^{2p-2}) = 0$$

and it is clear that $P^2(U_{S^1}) = 0$. □

§9E. Calculation of d^{4p-4}. The inclusion of a point induces a natural map from $\Omega^{fr}_{4p-5}(*)$ to $\Omega_{4p-5}(B\widetilde{H}_t, \xi_{\widetilde{H}_t})$ for each of the subgroups $\widetilde{H}_t$, $t = 0, \dots, p$.

Theorem 9.5. *The natural map $\Omega^{fr}_{4p-5}(*) \to \Omega_{4p-5}(B\widetilde{H}_t, \xi_{\widetilde{H}_t})$ is injective on the p–component.*

Proof. The generator of ${}_{(p)}\Omega^{fr}_{4p-5}(*)$ is represented by the class $\beta^{\mathbb{S}}_{4p-5}$ defined above. We will show that this element maps non-trivially in the Adams spectral sequence.

Denote the elements of $H^*(M\widetilde{H}_t; \mathbf{Z}/p)$, $H^*(MS^1; \mathbf{Z}/p)$, and $H^*(MD_t; \mathbf{Z}/p)$ as in Section **§9D**. Take an ideal $\mathcal{I}$ in any of these cohomology rings generated by the elements of degree higher that $4p-3$, so that $\mathcal{I}$ is closed under the Steenrod operations. Note that we can identify

$$U_{\widetilde{H}_t} \cdot (1 \otimes H^*(BD_t; \mathbf{Z}/p))/\mathcal{I} \equiv H^*(MD_t; \mathbf{Z}/p)/\mathcal{I}$$

as $\mathcal{A}_p$-modules, since $P^1(U_{S^1}) = P^2(U_{S^1}) = 0$ and $\widetilde{H}_t = S^1 \times D_t$. Take an $\mathcal{A}_p$–free resolution $F^{\widetilde{H}_t}_*$ of $H^*(M\widetilde{H}_t; \mathbf{Z}/p)/\mathcal{I}$ as follows:

$$\dots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} H^*(M\widetilde{H}_t; \mathbf{Z}/p)/\mathcal{I}$$

and similarly let $F_*^{D_t}$ denote a free $\mathcal{A}_p$-resolution of $H^*(MD_t;\mathbf{Z}/p)/\mathcal{I}$. Consider the chain homotopy commutative diagram

$$\begin{array}{ccccc}
F_*^{D_t} & \dashrightarrow & F_*^{\widetilde{H}_t} & \longrightarrow & F_*^{D_t} \\
\downarrow & & \downarrow & & \downarrow \\
U_{\widetilde{H}_t}\cdot(1\otimes H^*(BD_t;\mathbf{Z}/p))/\mathcal{I} & \hookrightarrow & H^*(M\widetilde{H}_t;\mathbf{Z}/p)/\mathcal{I} & \xrightarrow{i^*} & H^*(MD_t;\mathbf{Z}/p)/\mathcal{I}
\end{array}$$

where the dotted chain map exists by comparison of resolutions, and the map i^* is induced by the subgroup inclusion $D_t \subseteq \widetilde{H}_t$. The composition of the upper two chain maps is chain homotopic to the identity (by uniqueness of lifts).

We now assume that $F_*^{D_t}$ is a minimal resolution. We will define a chain map $F_*^{MD_t} \to F_*^{\mathbb{S}}$ in degrees $\leq 4p-3$ that extends the natural map $H^0(MD_t;\mathbf{Z}/p) \to H^0(\mathbb{S};\mathbf{Z}/p)$, with the additional property that some element $\beta_{4p-5}^{MD_t} \in \mathcal{B}(MD_t, 2, 4p-5)$ maps to $\beta_{4p-5}^{\mathbb{S}}$ in $\mathcal{B}(\mathbb{S}, 2, 4p-5)$. This will show that the class represented by $\beta_{4p-5}^{\mathbb{S}}$ in the E_2-term of the Adams spectral sequence for π_*^S maps non-trivially into the E_2-term for $\pi_*(MD_t)$. We then verify that there is no element in $\mathcal{B}(MD_t, 0, 4p-5)$, and hence no possible differential hitting the class $\beta_{4p-5}^{MD_t}$. This will complete the proof.

It is straightforward to check the following:

$\mathcal{B}(MD_t, 0, \leq 4p-3) = \{\iota_0^{MD_t}, \iota_{2k-1}^{MD_t}, \iota_{4p-3}^{MD_t} | k \in \{1, 2, \ldots, p-1\}\}$ where

- $\partial_0(\iota_0^{MD_t}) = U$
- For $k \in \{1, 2, \ldots, p-1\}$

$$\partial_0(\iota_{2k-1}^{MD_t}) = Uuv^{(k-1)}$$

- $\partial_0(\iota_{4p-3}^{MD_t}) = Uuv^{(2p-2)}$

$\mathcal{B}(MD_t, 1, \leq 4p-4) = \{\alpha_0^{MD_t}, \alpha_{2k-1}^{MD_t}, \alpha_{4p-4}^{MD_t} | k \in \{p-1, p, \ldots, 2p-2\}\}$ where

- $\partial_1(\alpha_0^{MD_t}) = \beta(\iota_0^{MD_t})$
- $\partial_1(\alpha_{2p-3}^{MD_t}) = P^1(\iota_0^{MD_t}) - \beta(\iota_{2p-3}^{MD_t})$
- For $k \in \{p, p+1, \ldots, 2p-2\}$

$$\partial_1(\alpha_{2k-1}^{MD_t}) = (k-p+2)\beta P^1(\iota_{k-p+1}^{MD_t}) - (k-p+1)P^1\beta(\iota_{k-p+1}^{MD_t})$$

- $\partial_1(\alpha_{4p-4}^{MD_t})) = P^2(\iota_1^{MD_t})$

$\mathcal{B}(MD_t, 2, \leq 4p-5) = \{\beta_0^{MD_t}, \beta_{4p-5}^{MD_t}\}$ where

- $\partial_2(\beta_0^{MD_t}) = \beta(\alpha_0^{MD_t})$
- $\partial_2(\beta_{4p-5}^{MD_t}) = P^2(\alpha_0^{MD_t}) - P^1\beta(\alpha_{2p-3}^{MD_t}) + 2\beta P^1(\alpha_{2p-3}^{MD_t}) + 2\beta(\alpha_{4p-5}^{MD_t})$

$\mathcal{B}(MD_t, n, \leq 4p-6)$ has only one element w_n when $n \geq 3$ where

- $\partial_3(w_3) = \beta(\beta_0^{MD_t})$
- $\partial_n(w_n) = \beta(w_{n-1})$

Now we define a part of the chain map $F_*^{MD_t} \to F_*^{\mathbb{S}}$. We send

$$\iota_0^{MD_t} \mapsto \iota_0^{\mathbb{S}}, \quad \iota_{2k-1}^{MD_t} \mapsto 0, \quad 1 \leq k \leq p-1, \quad \text{and} \quad \iota_{4p-3}^{MD_t} \mapsto 0 \ .$$

Since $\beta(\iota_0^{MD_t}) \mapsto \beta(\iota_0^{\mathbb{S}})$ and $P^1(\iota_0^{MD_t}) - \beta(\iota_{2p-3}^{MD_t}) \mapsto P^1(\iota_0^{\mathbb{S}})$ we must have

$$\alpha_0^{MD_t} \mapsto \alpha_0^{\mathbb{S}} \quad \text{and} \quad \alpha_{2p-3}^{MD_t} \mapsto \alpha_{2p-3}^{\mathbb{S}} \ .$$

Define

$$\alpha_{2k-1}^{MD_t} \mapsto 0, \quad \text{for} \quad p \leq k \leq 2p-2, \quad \alpha_{4p-4}^{MD_t} \mapsto 0 \quad \text{and} \quad \beta_0^{MD_t} \mapsto \beta_0^{\mathbb{S}}$$

since $\beta(\alpha_0^{MD_t}) \mapsto \beta(\alpha_0^{\mathbb{S}})$. Finally, we define

$$\beta_{4p-5}^{MD_t} \mapsto \beta_{4p-5}^{\mathbb{S}}$$

and this definition proves the Lemma. □

Remark 9.6. A similar technique can be used to prove that the natural map $\Omega_{10}^{fr}(*) \to \Omega_{10}(BS^1, \xi_{S^1})$ is injective on the 3-component. One constructs a chain map $F_*^{\mathbb{S}} \to F_*^{MS^1}$ in degrees ≤ 11, whose composite with the chain map induced by the natural map $H^*(MS^1; \mathbf{Z}/p) \to H^*(\mathbb{S}; \mathbf{Z}/p)$ is chain homotopic to the identity. The element $\beta_{10}^{\mathbb{S}}$ generating the 3-component of π_{10}^S arises from $P^2(\alpha_3^{\mathbb{S}})$ and the Adem relation $P^2P^1\iota_0^{\mathbb{S}} = 0$.

Lemma 9.7. *$d^{4p-4}\colon E_{4p-3,0}^{4p-4}(\nu_{\widetilde{G}_p}) \to E_{1,4p-5}^{4p-4}(\nu_{\widetilde{G}_p})$ is zero.*

Proof. For $t \in \{0, 1, 2, \dots, p\}$ we consider the following fibration

$$B\widetilde{H}_t \longrightarrow B\widetilde{G}_p \longrightarrow B(\widetilde{G}_p/\widetilde{H}_t)$$

This fibration induces a James spectral sequence $E_{*,*}^*(t)$ with differential denoted by d_t^* so that the second page is given by

$$E_{n,m}^2(t) = H_n(\widetilde{G}_p/\widetilde{H}_t, \Omega_m(B\widetilde{H}_t, \xi_{\widetilde{H}_t}))$$

and the spectral sequence converges to $\Omega_*(B\widetilde{G}_p, \xi_{\widetilde{G}_p})$. Moreover, we have a natural map $E_{*,*}^{4p-4}(\nu_{\widetilde{G}_p}) \to E_{*,*}^{4p-4}(t)$ due to the following map of fibrations.

$$\begin{array}{ccccc} * & \longrightarrow & B_{\widetilde{G}_p} & \longrightarrow & B_{\widetilde{G}_p} \\ \downarrow & & \downarrow & & \downarrow \\ B\widetilde{H}_t & \longrightarrow & B\widetilde{G}_p & \longrightarrow & B(\widetilde{G}_p/\widetilde{H}_t) \end{array}$$

Theorem 9.5 (applied for $t = 0$ and $t = p$), and the detection of $H_1(\mathbf{Z}/p \times \mathbf{Z}/p; \mathbf{Z}/p)$ by cyclic quotients, shows that the following sum of two of these natural maps is injective

$$E_{1,4p-5}^{4p-4}(\nu_{\widetilde{G}_p}) \to E_{1,4p-5}^{4p-4}(0) \oplus E_{1,4p-5}^{4p-4}(p)$$

However, the differential $d_t^{4p-4}\colon E_{4p-3,0}^{4p-4}(t) \to E_{1,4p-5}^{4p-4}(t)$ is zero for both $t = 0$ and $t = p$, since the element $N_{p-t} \to BD_{p-t} \to B\widetilde{G}_p$, for $t = 0, p$ (defined in Example 6.1) is non-zero in $\Omega_{4p-3}(B\widetilde{G}_p, \xi_{\widetilde{G}_p})$. This is because $[N_{p-t}] \in H_{4p-3}(BD_{p-t}; \mathbf{Z})$ is non-zero, and the inclusion $D_{p-t} \subset \widetilde{G}_p$ is split on homology by projection to $\widetilde{G}_p/\widetilde{H}_t \cong D_{p-t}$. □

§9F. Calculation of d^{4p-3}. The last differential doesn't involve p-torsion in the target, and can be handled by one more transfer argument.

Lemma 9.8. *$d^{4p-3}\colon E^{4p-3}_{4p-3,0}(\nu_{\widetilde{G}_p}) \to E^{4p-4}_{0,4p-4}(\nu_{\widetilde{G}_p})$ is zero on $T_{\widetilde{G}_p}$ where $T_{\widetilde{G}_p}$ is considered as a subgroup of $E^{4p-3}_{4p-3,0}(\nu_{\widetilde{G}_p})$.*

Proof. By Lemma 9.7 and the transfer map tr_t we see that the differential

$$d^{4p-4}\colon E^{4p-4}_{4p-3,0}(\nu_{\widetilde{H}_t}) \to E^{4p-4}_{1,4p-5}(\nu_{\widetilde{H}_t})$$

is zero on $\mathrm{tr}_t(T_{\widetilde{G}_p})$ where $T_{\widetilde{G}_p}$ is considered as subgroup of $E^{4p-4}_{4p-3,0}(\nu_{\widetilde{G}_p})$. Now the differential

$$d^{4p-3}\colon E^{4p-3}_{4p-3,0}(\nu_{\widetilde{H}_t}) \to E^{4p-4}_{0,4p-4}(\nu_{\widetilde{H}_t})$$

has to be zero on $\gamma_{\widetilde{H}_t}$ and on the p-torsion group $\mathrm{Im}\{H_{4p-3}(K) \to H_{4p-3}(B_{\widetilde{H}_t};\mathbf{Z})\}$ because $E^{4p-4}_{0,4p-4}(\nu_{\widetilde{H}_t})$ is p-torsion free. Hence the result follows. □

We have now proved the main result of this section.

Theorem 9.9. *The subset $T_{\widetilde{G}_p} \neq \emptyset$ is contained in the image of the Hurewicz map $\Omega_{4p-3}(B_{\widetilde{G}_p}, \nu_{\widetilde{G}_p}) \to H_{4p-3}(B_{\widetilde{G}_p};\mathbf{Z})$.*

Proof. Lemma 9.2, Lemma 9.7 and Lemma 9.8 shows that all the differentials going out of $E^r_{4p-3,0}(\nu_{\widetilde{G}_p})$ in the James spectral sequence for $\nu_{\widetilde{G}_p}$ are zero on $T_{\widetilde{G}_p}$ and the result follows. □

10. Surgery on the bordism element

In this section we fix an odd prime p, the integer $n = 2p-1$, and set $\widetilde{G} = \widetilde{G}_p$, $G = G_p$ to simplify the notation.

We have now completed the first two steps in the proof of Theorem A. We have shown that there is a non-empty subset

$$T_{\widetilde{G}} = \{\gamma \in H_{2n-1}(B_{\widetilde{G}};\mathbf{Z}) \mid\ p \cdot (\mathrm{tr}(\gamma) - \gamma_{S^1}) = 0\},$$

and that this subset is contained in the the image of the Hurewicz map

$$\Omega_{2n-1}(B_{\widetilde{G}}, \nu_{\widetilde{G}}) \to H_{2n-1}(B_{\widetilde{G}};\mathbf{Z})\ .$$

We now define the subset

$$T_G = \{\mathit{trf}(\gamma) \in H_{2n}(B_G;\mathbf{Z}) \mid \gamma \in T_{\widetilde{G}}\},$$

where $\mathit{trf}\colon H_{2n-1}(B_{\widetilde{G}};\mathbf{Z}) \to H_{2n}(B_G;\mathbf{Z})$ denotes the S^1-bundle transfer induced by the fibration $S^1 \to B_G \to B_{\widetilde{G}}$. Here is the progress so far.

Theorem 10.1. *Given an element $\gamma_G \in T_G$, there exists a bordism class $[M^{2n}, f] \in \Omega_{2n}(B_G, \nu_G)$ such that $\gamma_G = f_*[M]$ is the image of the fundamental class.*

Proof. Any $\gamma_G = \mathit{trf}(\gamma)$, for some $\gamma \in T_{\widetilde{G}}$, so we can pull back the S^1-bundle over a manifold (provided by Theorem 9.9) whose fundamental class represents γ under the bordism Hurewicz map. □

We now fix an element $\gamma_G \in T_G$ and let $\theta_G = \pi_*(\gamma_G) \in H_{2n}(G;\mathbf{Z})$. Let $\widetilde{B_G} \to B_G$ denote the universal covering.

Lemma 10.2. *Under the transfer* $\operatorname{tr}\colon H_{2n}(B_G;\mathbf{Z}) \to H_{2n}(\widetilde{B_G};\mathbf{Z})$*, the class* $\operatorname{tr}(\gamma_G)$ *corresponds to the standard hyperbolic form*

$$(\mathbf{Z}\oplus\mathbf{Z}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$$

on $\pi_n(B_G) = \mathbf{Z}\oplus\mathbf{Z}$ *under the identification* $H_{2n}(\widetilde{B_G};\mathbf{Z})/Tors \cong \Gamma(\mathbf{Z}\oplus\mathbf{Z})$ *with Whitehead's* Γ*-functor.*

Proof. We have a commutative diagram

$$\begin{array}{ccccc}
S^1 & \xrightarrow{p} & S^1 & \longrightarrow & B\mathbf{Z}/p \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{B_G} & \longrightarrow & B_G & \longrightarrow & BG \\
\downarrow & & \downarrow & & \downarrow \\
B_{S^1} & \longrightarrow & B_{\widetilde{G}} & \longrightarrow & BQ
\end{array}$$

where $Q = \mathbf{Z}/p \times \mathbf{Z}/p$ and $\widetilde{B_G} = K(\mathbf{Z}\oplus\mathbf{Z}, n)$. This gives a commutative square

$$\begin{array}{ccc}
H_{2n-1}(B_{S^1};\mathbf{Z}) & \xrightarrow{trf} & H_{2n}(\widetilde{B_G};\mathbf{Z}) \\
{\scriptstyle tr}\uparrow & & \uparrow{\scriptstyle tr} \\
H_{2n-1}(B_{\widetilde{G}};\mathbf{Z}) & \xrightarrow{trf} & H_{2n}(B_G;\mathbf{Z})
\end{array}$$

relating the S^1-bundle transfers and the universal covering transfers. The p-torsion subgroup of $H_{2n-1}(B_{S^1};\mathbf{Z})$ maps to zero under the S^1-bundle transfer, since $H_{2n}(K;\mathbf{Z})$ has no p-torsion, by Lemma 7.2, Fact 11. Therefore $tr(\gamma_G) = trf(\gamma_{S^1})$ is just the image of the fundamental class of $S^{2p-1}\times S^{2p-1}$ in $H_{2n}(\widetilde{B_G};\mathbf{Z}) = H_{2n}(K;\mathbf{Z})$. But $H_{2n}(K;\mathbf{Z})/Tors = \mathbf{Z}$ can be naturally identified with $\Gamma(\mathbf{Z}\oplus\mathbf{Z}) = \mathbf{Z}$, and under this identification the fundamental class of $S^{2p-1}\times S^{2p-1}$ corresponds to a generator, represented by the hyperbolic plane. □

Lemma 10.3. *The map* $\pi^*\colon H^{2n+1}(G;\mathbf{Z}) \to H^{2n+1}(B_G;\mathbf{Z})$ *is zero.*

Proof. We first note that the quotient $H^{2n+1}(G;\mathbf{Z})/\langle\theta_1,\theta_2\rangle \cong \mathbf{Z}/p$. This is an easy calculation, given the choice of k-invariants and the results of Lewis [24] (or Leary [21]) on the cohomology of BG. Since $\operatorname{tr}(\gamma_G)$ is a primitive class in $H_{2n}(\widetilde{B_G};\mathbf{Z})$, the image of $i^*\colon H^{2n}(B_G;\mathbf{Z}/p) \to H^{2n}(\widetilde{B_G};\mathbf{Z}/p)$ has index p^3 in the torsion-free quotient. It follows that the cohomology spectral sequence of the covering $\widetilde{B_G} \to B_G$ must have a non-zero differential $d_{2n+1}\colon E_{2n}^{0,2n} \to E_{2n}^{2n+1,0} = \mathbf{Z}/p$. □

Lemma 10.4. *The element $\theta_G \neq 0$, and generates the image,* $\operatorname{Im}\pi_* \cong \mathbf{Z}/p$, *of the map* $\pi_*\colon H_{2n}(B_G;\mathbf{Z}) \to H_{2n}(G;\mathbf{Z})$.

Proof. Since the map $\pi^*\colon H^{2n+1}(G;\mathbf{Z}) \to H^{2n+1}(B_G;\mathbf{Z})$ is zero, so is the map $\pi_*\colon H_{2n}(B_G;\mathbf{Z}) \to H_{2n}(G;\mathbf{Z})$, when restricted to the torsion subgroup. Consider the commutative diagram:

$$\begin{array}{ccccc}
H_{2n-1}(B_{\widetilde{G}};\mathbf{Z}) & \xrightarrow{trf} & H_{2n}(B_G;\mathbf{Z}) & \longrightarrow & H_{2n}(B_{\widetilde{G}};\mathbf{Z}) \\
\Big\downarrow{\scriptstyle \pi_*} & & \Big\downarrow{\scriptstyle \pi_*} & & \\
H_{2n-1}(B\widetilde{G};\mathbf{Z}) & \xrightarrow{trf} & H_{2n}(G;\mathbf{Z}) & &
\end{array}$$

Any p-torsion element in $H_{2n}(B_{\widetilde{G}};\mathbf{Z})$ is hit by a p-torsion element from $H_{2n}(B_G;\mathbf{Z})$ (these arise from the term $E^2_{1,2n-1}(G)$ in the homology spectral sequence). Since $H_{2n}(G;\mathbf{Z})$ is all p-torsion, it follows that $\operatorname{Im}\pi_*$ equals the image of $trf \circ \pi_*$ in the diagram above.

By Part **2** in Section **§8C**, any element $\gamma \in T_{\widetilde{G}}$ generates the image, $\operatorname{Im}\pi_* \cong \mathbf{Z}/p$, of the map $\pi_*\colon H_{2n-1}(B_{\widetilde{G}};\mathbf{Z}) \to H_{2n-1}(B\widetilde{G};\mathbf{Z})$. We note that $H^{odd}(B\widetilde{G};\mathbf{Z}) = 0$ by [21], and so $H_{odd}(B\widetilde{G};\mathbf{Z})$ is a torsion group. The Chern class of the S^1-bundle

$$S^1 \to B_G \to B_{\widetilde{G}}$$

is given by $c_1 = \pm\chi_1$. We now check (using [21, Theorem 2]) that cup product with χ_1 applied to $H^{even}(B\widetilde{G};\mathbf{Z})$ has kernel just the torsion subgroup. It follows from the Gysin sequence that $H_{2n-1}(B\widetilde{G};\mathbf{Z}) \cong H_{2n}(G;\mathbf{Z})$. □

Remark 10.5. There is a perfect pairing

$$H_{2n}(G;\mathbf{Z}) \otimes H^{2n+1}(G;\mathbf{Z}) \to \mathbf{Q}/\mathbf{Z}$$

and, under this pairing, the quotient homomorphism above corresponds to the subgroup $\operatorname{Im}\pi_* \cong \mathbf{Z}/p \subseteq H_{2n}(G;\mathbf{Z})$.

Lemma 10.6. *There is a chain complex $D_* = C(\theta_1) \otimes_{\mathbf{Z}} C(\theta_2)$ of finitely-generated projective $\mathbf{Z}G$-modules, such that*

(1) $H_*(D;\mathbf{Z}) = H_*(S^n \times S^n;\mathbf{Z})$.
(2) *There is a $\mathbf{Z}G$-module chain map $\psi\colon C_*(\widetilde{B}) \to D_*$ inducing isomorphisms on homology and cohomology in degrees $\leq n$.*
(3) *Under the map*
$$\psi_*\colon H_{2n}(B_G;\mathbf{Z}) \to H_{2n}(D\otimes_{\mathbf{Z}G};\mathbf{Z})$$
induced by the chain map ψ, the image
$$[\bar{D}] := \psi_*(\gamma_G) \in H_{2n}(D\otimes_{\mathbf{Z}G};\mathbf{Z}) \cong \mathbf{Z}$$
is a generator.
(4) *The class θ_G corresponds to an extension*
$$0 \to \Omega^{n+1}\mathbf{Z} \to H_n(D;\mathbf{Z}) \oplus (\mathbf{Z}G)^r \to S^{n+1}\mathbf{Z} \to 0,$$
under the isomorphism $H_{2n}(G;\mathbf{Z}) \cong \operatorname{Ext}^1_{\mathbf{Z}G}(S^{n+1}\mathbf{Z}, \Omega^{n+1}\mathbf{Z})$.

Proof. The chain complex $D = C(\theta_1) \otimes_{\mathbf{Z}} C(\theta_2)$ is constructed in [6] (see Cor. 4.5 and Remark 3, p. 231), and investigated further in [5].

We can also apply the construction of [6, p. 230] to the chain complex $C(\widetilde{B_G})$ of the universal covering of B_G. The same quotient complexes $C(\theta_1)$ and $C(\theta_2)$ arise, and the diagonal map induces a $\mathbf{Z}G$-module chain map $\psi\colon C(\widetilde{B_G}) \to C(\theta_1)\otimes_{\mathbf{Z}} C(\theta_2)$. By construction, this chain map induces isomorphisms on homology in degrees $\leq n$, and cohomology in degrees $\leq n+1$.

The homology of the quotient complex $\bar{D} := D \otimes_{\mathbf{Z}G} \mathbf{Z}$ can be studied by the usual spectral sequence with $E^2_{r,s} = H_r(G; H_s(D;\mathbf{Z}))$. Our choice of k-invariants implies that $E^\infty_{r,s}$, for $r+s = 2n$, is non-zero only for $E^\infty_{2n,0} = \mathbf{Z}/p$, $E^\infty_{n,n} = \mathbf{Z}/p^2$, and $E^\infty_{0,2n} = \mathbf{Z}$. Since $\bar{D}$ satisfies Poincaré duality, $H_{2n}(\bar{D};\mathbf{Z}) = \mathbf{Z}$, so the filtration is non-split at each term. In particular, the subgroup $E^\infty_{0,2n} = \mathbf{Z} \subset H_{2n}(\bar{D};\mathbf{Z})$ has index p^3. Let $[\bar{D}] \in H_{2n}(\bar{D};\mathbf{Z})$ denote a generator. It follows that under the transfer $\operatorname{tr}\colon H_{2n}(\bar{D};\mathbf{Z}) \to H_{2n}(D;\mathbf{Z})$, $[\bar{D}] \mapsto [D]$, which generates $H_{2n}(D;\mathbf{Z}) = \mathbf{Z}$. But the chain map $\psi\colon C(\widetilde{B_G}) \to D$ induces an isomorphism $H_{2n}(\widetilde{B_G};\mathbf{Z})/Tors \cong H_{2n}(D;\mathbf{Z})$. Therefore, by Lemma 10.2, the element $\psi_*(\operatorname{tr}(\gamma_G)) = \pm[D]$. We may assume that $\psi(\gamma_G) = [\bar{D}]$, by choosing the right sign for $[\bar{D}]$.

The relation between $\theta_G = \pi_*([\bar{D}])$ and a stable extension involving $H_n(D;\mathbf{Z})$ follows from Poincaré duality, as in Proposition 1.1. □

We now return to our bordism element $[M,f] \in \Omega_{2n}(B_G, \nu_G)$. Surgery will be used to improve the manifold M within its bordism class. Our first remark is that we may assume f is an n-equivalence (see [20, Cor. 1, p. 719]). In particular, $\pi_1(M) = G$, and $\pi_i(M) = 0$ for $2 \leq i < n$. In addition, the map $f_*\colon \pi_n(M) \to \pi_n(B_G)$ is surjective.

Next we need to determine the structure of $\pi_n(M)$ as a $\mathbf{Z}G$-module. Note that we can always stabilize $\pi_n(M)$ by direct sum with a free $\mathbf{Z}G$-module, without changing the bordism class $[M,f]$.

Lemma 10.7. *There is an isomorphism of* $\mathbf{Z}G$*-modules*

$$\pi_n(M) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus P$$

where P is a finitely-generated projective $\mathbf{Z}G$*-module. Furthermore, the equivariant intersection form*

$$(\pi_n(M), s_M) \cong (\mathbf{Z} \oplus \mathbf{Z}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \perp (P, \lambda)$$

splits orthogonally as a skew-symmetric hyperbolic form on $\mathbf{Z} \oplus \mathbf{Z}$ *and a non-singular skew-hermitian form λ on P.*

Proof. By Proposition 1.1, the image of $f_*[M]$ under the induced map $\pi_*\colon H_{2n}(B_G;\mathbf{Z}) \to H_{2n}(G;\mathbf{Z}) \cong \mathbf{Z}/p$ gives the stable congruence class of the extension

$$0 \to \Omega^{n+1}\mathbf{Z} \to \pi_n(M) \to S^{n+1}\mathbf{Z} \to 0\ .$$

But $\pi_*(f_*[M]) = \pi_*[\gamma_G] = \theta_G$, so the extension for $\pi_n(M)$ is stably congruent to the extension

$$0 \to \Omega^{n+1}\mathbf{Z} \to \pi_n(B_G) \to S^{n+1}\mathbf{Z} \to 0,$$

and hence we have $\pi_n(M) \oplus P' \cong \pi_n(B_G) \oplus P''$ for some finitely generated projective $\mathbf{Z}G$-modules P', P''. But $\pi_n(B_G) = \mathbf{Z} \oplus \mathbf{Z}$ and we can let $P = P'' \oplus Q'$ where $P' \oplus Q'$ is a free $\mathbf{Z}G$-module. After replacing M with an appropriate connected sum $M \# r(S^n \times S^n)$ we obtain $\pi_n(M) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus P$.

To show the splitting of the equivariant intersection form $(\pi_n(M), s_M)$ we consider the relation

$$\langle f^*(z_1) \cup f^*(z_2), [\widetilde{M}] \rangle = \langle z_1 \cup z_2, f_*[\widetilde{M}] \rangle$$

where z_1, z_2 are a symplectic basis for the form on $\pi_n(B_G)$. Therefore, by Lemma 10.2, the map $f^* \colon H^n(\widetilde{B_G}; \mathbf{Z}) \to H^n(\widetilde{M}; \mathbf{Z})$ gives an isometric embedding of the hyperbolic form $\mathbf{H}(\mathbf{Z})$ into s_M. Any such isometric embedding splits (see [16, Lemma 1.4]). □

The next step is to study the projective class $[P] \in \widetilde{K}_0(\mathbf{Z}G)$. Recall that $D(\mathbf{Z}G) = \ker(\widetilde{K}_0(\mathbf{Z}G) \to \widetilde{K}_0(\mathcal{M}))$, where $\mathcal{M}$ denotes a maximal order in $\mathbf{Q}G$ containing $\mathbf{Z}G$. The involution $g \mapsto g^{-1}$ induces an involution $[P] \mapsto [P^*]$ on the projective class group. Since $D(ZG)$ is an abelian p-group, the $(\pm)$-eigenspaces of this involution induce a direct sum splitting

$$D(\mathbf{Z}G) = D(\mathbf{Z}G)^+ \oplus D(ZG)^- \ .$$

Oliver [27] defines a further decomposition of $D(\mathbf{Z}G)^+$ and identifies a summand ${}^0D(\mathbf{Z}G) \subseteq D(\mathbf{Z}G)^+$.

Lemma 10.8. *The order* $|{}^0D(\mathbf{Z}G)| = p^2$.

Proof. We calculate the formula given by Oliver in [27, Theorem 12]. If X denotes a set of conjugacy class representatives for cyclic subgroups $H \subset G$, then

$$\left|{}^0D(\mathbf{Z}G)\right| = \left[\prod_{H \subset G} \frac{|N_G(H)/H|^2}{Z_G(H)|}\right]^{\frac{1}{2}}$$

The non-central subgroups contribute 1 to this product, the trivial subgroup contributes p, and the centre contributes p^3, so after multiplying and taking the square root we get p^2. □

Lemma 10.9. *If $\pi_n(M) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus P$, for some finitely-generated projective $\mathbf{Z}G$-module P, then $[P] = [P^*] \in D(\mathbf{Z}G)^+$. If p is a regular prime, then*

(1) $P \oplus Q \oplus Q^* \cong (\mathbf{Z}G)^r$ *for some integer r*
(2) $\mathbf{Z} \oplus Q \cong \mathbf{Z} \oplus \mathbf{Z}G$, *and*
(3) $[Q] = [Q^*]$

for some finitely generated projective module Q.

Proof. First note that $P \cong P^*$ since $P \subset \pi_n(M)$ supports a non-singular skew-hermitian form. Next we observe that $[P] \in D(\mathbf{Z}G)$. Indeed, the usual Euler characteristic argument tells us that the image of $[P]$ under the Cartan map $\widetilde{K}_0(\mathbf{Z}G) \to \widetilde{G}_0(\mathbf{Z}G)$ vanishes. However, $G_0(\mathbf{Z}G) \cong K_0(\mathcal{M})$ by [33, Cor. 5.14]. Since $P \cong P^*$, the class $[P] \in D(\mathbf{Z}G)^+$.

In [27, Prop. 5], Oliver proves that $D(\mathbf{Z}G)^+ = {}^0D(\mathbf{Z}G)$ for an odd regular prime p. By [11, p. 368] we have ${}^0D(\mathbf{Z}G) \supseteq T(\mathbf{Z}G)$, where $T(\mathbf{Z}G)$ denotes the Swan subgroup of

$\widetilde{K}_0(\mathbf{Z}G)$. This is the subgroup generated by the ideals $\langle r, N\rangle \subset \mathbf{Z}G$, for $(p, |G|) = 1$ and $N = \sum\{g \,|\, g \in G\}$. Since the order $|T(\mathbf{Z}G)| = p^2$, we obtain ${}^0D(\mathbf{Z}G) = T(\mathbf{Z}G)$ from the calculation in Lemma 10.8 (see [11, p. 365]). Now Swan [32, Lemma 6.1] proved that

$$\mathbf{Z} \oplus \langle r, N\rangle \cong \mathbf{Z} \oplus \mathbf{Z}G$$

and we can apply this result to $[Q] \in T(\mathbf{Z}G)$, where $[P] = 2[Q]$.

□

By surgery on a null-homotopic $(n-1)$-sphere in M, we obtain $M' = M\#(S^n \times S^n)$, whose equivariant intersection form is

$$(\pi_n(M'), s_{M'}) \cong (\mathbf{Z} \oplus \mathbf{Z}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \perp (P, \lambda) \perp \mathbf{H}(\mathbf{Z}G)$$

where $\mathbf{H}(\mathbf{Z}G)$ denotes the standard skew-hermitian hyperbolic form on $\mathbf{Z}G$. But the last Lemma implies

$$\mathbf{H}(\mathbf{Z}) \perp \mathbf{H}(\mathbf{Z}G) \cong \mathbf{H}(\mathbf{Z} \oplus \mathbf{Z}G) \cong \mathbf{H}(\mathbf{Z} \oplus Q) \cong \mathbf{H}(\mathbf{Z}) \perp \mathbf{H}(Q)$$

and $\mathbf{H}(Q) \perp (P, \lambda) = (F, \lambda')$ is then a non-singular skew-hermitian form on a finitely-generated free $\mathbf{Z}G$-module. We have proved:

Corollary 10.10. *We may assume that $[M, f] \in \Omega_{2n}(B_G, \nu)$ has equivariant intersection form*

$$(\pi_n(M), s_M) \cong \mathbf{H}(\mathbf{Z}) \perp (F, \lambda)$$

where (F, λ) is a non-singular skew-hermitian form on a finitely-generated free $\mathbf{Z}G$-module.

We next observe that the equivariant intersection form $(\pi_n(M), s_M)$ has a quadratic refinement $\mu\colon \pi_n(M) \to \mathbf{Z}G/\{\nu + \bar{\nu}\}$, in the sense of [38, Theorem 5.2]. Since G has odd order, this follows because the universal covering $\widetilde{M}$ has stably trivial normal bundle. We therefore obtain an element (F, λ, μ) of the surgery obstruction group (see [38, p. 49] for the essential definitions). We need to check the discriminant of this form.

Lemma 10.11. *We obtain an element*

$$(F, \lambda, \mu) \in L'_{2n}(\mathbf{Z}G)$$

of the weakly-simple surgery obstruction group.

Proof. A non-singular, skew-hermitian quadratic form (F, λ, μ) represents an element in $L'_{2n}(\mathbf{Z}G)$ provided that its discriminant lies in $\ker(\mathrm{Wh}(\mathbf{Z}G) \to \mathrm{Wh}(\mathbf{Q}G))$. But the equivariant symmetric Poincaré chain complex $(C(M), \varphi_0)$ is chain equivalent, after tensoring with the rationals $\mathbf{Q}$, to the rational homology (see [28, §4]). Therefore the image of the discriminant of $(\pi_n(M) \otimes \mathbf{Q}, s_M)$ equals the image of the torsion of φ_0, which vanishes in $\mathrm{Wh}(\mathbf{Q}G)$ because closed manifolds have simple Poincaré duality (see [38, Theorem 2.1]). □

The proof of Theorem A. Suppose that $p \geq 3$ is a regular prime. We now have a representative $[M, f]$ for our bordism element in $\Omega_{2n}(B_G, \nu_G)$ whose equivariant intersection form $(\pi_n(M), s_M)$ contains (F, λ, μ) as described above. However, an element in the surgery obstruction group $L'_{2n}(\mathbf{Z}G)$ is zero provided that its multisignature and ordinary Arf invariant both vanish (this is a result of Bak and Wall, see [37, Cor. 2.4.3]). The multisignature invariant is trivial since M is a closed manifold [38, 13B]. The ordinary Arf invariant vanishes since $2n = 4p-2$ is not of the form $2^k - 2$ (a famous result of Browder [7]). We can now do surgery respecting the bordism class in $\Omega_{2n}(B_G, \nu_G)$ to obtain a representative $[M, f]$ which has $\widetilde{M} = S^n \times S^n \# \Sigma$, where Σ is a homotopy $2n$-sphere. Since the p-primary component of Cok J starts in dimension $2p(p-1)-2$ (see [29, p. 5]) we can eliminate this homotopy sphere by equivariant connected sum unless $p = 3$.

In case $p = 3$, we use Remark 9.6 to show that $\widetilde{M} = S^5 \times S^5$. The bordism element $[\widetilde{M}, \widetilde{f}] \in \Omega_{10}^{fr}(K)$ vanishes in $\Omega_{10}(B_{S^1}, \nu_{S^1})$ by the Gysin sequence in bordism. But the difference element $[\widetilde{M}, \widetilde{f}] - [S^5 \times S^5, i_5] \in \Omega_{10}^{fr}(*)$. Since $\Omega_{10}^{fr}(*)$ injects on the 3-component into $\Omega_{10}(B_{S^1}, \nu_{S^1})$, it follows that the difference element is zero. Thus in all cases we can obtain $\widetilde{M} = S^n \times S^n$. This completes the proof of Theorem A if p is a regular prime.

In the above arguments, we needed the assumption that p is a regular prime to eliminate the projective class $[P]$. Without this assumption, we obtain an element in $L^p_{2n}(\mathbf{Z}G)$. Since this L-group is detected by multisignature and the ordinary Arf invariant, we can attach (infinitely many) cells to get a finitely dominated G-CW complex $X \simeq S^n \times S^n$. □

Note that the extraspecial 3-group of order 27 is a subgroup of the exceptional Lie group $\mathfrak{G}_2$ of dimension 14

$$G_3 \xrightarrow{\Psi} SU(3) \hookrightarrow \mathfrak{G}_2 \ .$$

We will now show that G_3 acts freely and smoothly on $S^{11} \times S^{11}$, but we don't know if the corresponding statement is true for G_p, $p \geq 5$, on $S^{2pr-1} \times S^{2pr-1}$ for $r > 1$. This remains an interesting open problem.

The proof of Theorem B. Let E denote any finite odd order subgroup of the exceptional Lie group $\mathfrak{G}_2$. To construct a free E-action on $S^{11} \times S^{11}$, we start with the free E-action on $\mathfrak{G}_2$ given by left multiplication. Now consider the fibre bundle

$$S^3 = SU(2) \to \mathfrak{G}_2 \to \mathfrak{G}_2/SU(2) = V_2(\mathbb{R}^7)$$

with structure group $SU(2)$. This fibre bundle can be identified with the sphere bundle of an associated 2-dimensional complex vector bundle ξ. By construction, the total space

$$E(\xi) = \mathfrak{G}_2 \times_{SU(2)} \mathbb{C}^2$$

so the E-action on the total space $\mathfrak{G}_2$ extends to $E(\xi)$, and freely off the zero-section. Let $S^{11} \to Y \to V_2(\mathbb{R}^7)$ be the sphere bundle of the complex vector bundle $\xi \oplus \xi \oplus \xi$. We therefore obtain a free E-action on the smooth closed manifold Y, and let $X = Y/E$ denote the quotient space. Since Y is 4-connected, we can construct the classfying space BE by adding k-cells to X for $k > 5$. Hence, the successive obstructions to extending the classifying map $\nu_X\colon X \to BSO$ of stable normal bundle of X to a map from $BE^{(k)} \cup X$ to BSO lie in the groups

$$H^k(BE, X; \pi_{k-1}(BSO))$$

for $k \geq 6$. But the Stiefel manifold $V_2(\mathbb{R}^7)$ is stably parallelisable, hence the stable normal bundle of Y is trivial. In addition, Y has the integral homology of $S^{11} \times S^{11}$, except for the groups $H_5(Y; \mathbf{Z}) = H_{16}(Y; \mathbf{Z}) = \mathbf{Z}/2$. The 2-localization of the cohomology obstruction groups is detected by passing to the odd degree universal covering $(\widetilde{BE}, Y)$, so the obstructions vanish 2-locally. The odd localization of the cohomology groups is zero for $k \leq 11$, so the obstructions vanish for $k \leq 11$. Note that $\pi_{11}(BSO) = 0$ so we may extend ν_X over $BE^{(12)} \cup X$. It follows that the restriction of ν_X to the 11-skeleton of X factors through the classifying map $c\colon X \to BE$. This gives a bundle $\eta\colon BE^{(12)} \to BSO$ and a homotopy

$$\eta \circ c\,|_{X^{(11)}} \simeq \nu_X\,|_{X^{(11)}}$$

on restriction to $X^{(11)}$. In this way, we obtain the stable trivializations needed to do surgery on elements of $\pi_k(X)$ up to the middle dimension, starting with $\pi_5(X) = \mathbf{Z}/2$.

We use the short exact sequence

$$0 \to \langle 2, I\rangle \to \mathbf{Z}[E] \to \mathbf{Z}/2 \to 0$$

of $\mathbf{Z}[E]$-modules, where I denotes the augmentation ideal of $\mathbf{Z}[E]$, to keep track of the effect of E-equivariant framed surgery on Y. There is a short exact sequence

$$0 \to \mathbf{Z}[E] \to \langle 2, N\rangle \to \mathbf{Z}/2 \to 0,$$

and Schanuel's Lemma shows that $\langle 2, N\rangle \oplus \langle 2, I\rangle$ is free over $\mathbf{Z}[E]$. It follows that after preliminary surgeries we are again in the situation of Lemma 10.9. Note that the p-component of π^S_{22} is zero for $p \geq 3$ (see [29, p. 5]), so we can get the standard smooth structure on $S^{11} \times S^{11}$. □

11. The Proof of Theorem C

In this section let G denote the group $\widetilde{G}_3$, which contains the extraspecial 3–group of order 27 and exponent 3. Considering $\xi = e^{2\pi i/3} \in S^1 \subseteq \mathbb{C}$, we take the following presentation:

$$G = \left\langle a, b, z \mid z \in S^1,\ a^3 = b^3 = [a, z] = [b, z] = 1,\ [a, b] = \xi \right\rangle$$

and define the following four representation of G:

(1) An irreducible representation $\varphi\colon G \to U(3)$:

$$a \longmapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},\ b \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi^2 \end{bmatrix},\ z \longmapsto \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix}$$

(2) Three representations that pullback from representations of G/S^1:

(a) $\psi_0 \colon G \to U(3)$ given by:

$$a \longmapsto \begin{bmatrix} \xi & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi \end{bmatrix}, b \longmapsto \begin{bmatrix} \xi & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & 1 \end{bmatrix}, z \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $\psi_1 \colon G \to U(3)$ given by:

$$a \longmapsto \begin{bmatrix} \xi & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi \end{bmatrix}, b \longmapsto \begin{bmatrix} \xi & 0 & 0 \\ 0 & \xi^2 & 0 \\ 0 & 0 & \xi^2 \end{bmatrix}, z \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) $\psi_2 \colon G \to U(3)$ given by:

$$a \longmapsto \begin{bmatrix} \xi & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi \end{bmatrix}, b \longmapsto \begin{bmatrix} \xi^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, z \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These representations give an action $\Gamma \colon G \times Y \to Y$ on $Y = S^5$ given by:

$$\Gamma(g, s) = \varphi(g)s,$$

where $s \in S^5$, and for $i = 0$, 1, or 2 an action $\Gamma_i \colon G \times X_i \to X_i$ on $X_i = S^5 \times S^5$ given by:

$$\Gamma_i(g, (s_1, s_2)) = (\varphi(g)s_1, \psi_i(g)s_2),$$

where $s_1, s_2 \in S^5$. To simplify our notations, we let $\Gamma(g, s) = gs$ and $\Gamma_i(g, (s_1, s_2)) = g(s_1, s_2)$, when we know that $s \in Y$ and $(s_1, s_2) \in X_i$. For $i = 0$, 1, or 2, we define a G-equivariant map

$$p_i \colon X_i \to Y \text{ given by } p_i(s_1, s_2) = s_1 \ .$$

Note that p_i is in fact a G-equivariant sphere bundle map. Take $0 < \varepsilon < \frac{1}{4}$ we define three subspaces U_1, U_2, and U_0 of Y as follows:

$$U_1 = \left\{ a^k \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in Y \;\middle|\; \begin{matrix} k \in \{0,1,2\} \\ z_i \in \mathbb{C} \text{ for } i \in \{1,2,3\} \\ |z_2|^2 + |z_3|^2 \le \varepsilon \end{matrix} \right\}, \quad U_2 = PU_1$$

where

$$P = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \xi & 1 \\ 1 & 1 & \xi \\ \xi & 1 & 1 \end{bmatrix} \in U(3) \ .$$

Note that $P\varphi(a)P^{-1} = \varphi(a)$ and $P\varphi(b)P^{-1} = \varphi(a^2 b)$, and let U_0 be the closure of $Y - U_1 \cup U_2$.

Lemma 11.1. *The inclusions $t_i \colon U_i \to Y$ give G-equivariant subspaces of Y.*

Lemma 11.2. *$U_1 \cap U_2 = \emptyset$.*

Proof. Suppose $[z_1, z_2, z_3]^T \in U_1 \cap U_2$. Then there exists $[z_1', z_2', z_3']^T \in U_1$ such that $[z_1, z_2, z_3]^T = P[z_1', z_2', z_3']^T$, since $[z_1, z_2, z_3]^T \in U_2$. So there exists $i \neq j \in \{1,2,3\}$ such that $|z_i'|^2 + \left|z_j'\right|^2 \le \varepsilon$, since $[z_1', z_2', z_3']^T \in U_1$. Let $\{k\} = \{1,2,3\} - \{i,j\}$. Then for any q in $\{1,2,3\}$ we have $|z_q|^2 \ge |z_k'|^2 - |z_i'|^2 - \left|z_j'\right|^2 \ge 1 - 3\varepsilon$. $\square$

Lemma 11.3. $\partial U_0 = \partial U_1 \cup \partial U_2$

Now define a subpace E_i of X_i for $i = 0$, 1, or 2 by the following G-equivariant pulback diagram:

$$\begin{array}{ccc} E_i & \longrightarrow & X_i \\ \downarrow & & \downarrow p_i \\ U_i & \xrightarrow{t_i} & Y \end{array}$$

Lemma 11.4. *The G–action on E_i is free for $i \in \{1, 2, 3\}$.*

Proof. Take two subsets of G as follows:

$$\begin{aligned} A_1 &= \left\{bz, b^2 z \mid z \in S^1\right\} \\ A_2 &= \left\{a^2 bz, a^2 b^2 z \mid z \in S^1\right\} \end{aligned}$$

All elements of G except $A_1 \cup A_2$ act freely on X_0. But all the fixed point sets of elements of A_i are in $p_0^{-1}(U_i - \partial U_i)$ for $i \in \{1, 2\}$. Hence G acts freely on E_0. Now for any $i \in \{1, 2\}$, all elements of G except A_i act freely on U_i, but all the elements of A_i act freely on X_i. Hence G acts freely on E_i. □

Lemma 11.5. *∂E_0 is G-equivariantly isomorphic to $\partial E_1 \cup \partial E_2$ as G-equivariant 5-dimensional sphere bundles over $\partial U_0 = \partial U_1 \cup \partial U_2$ with structure group $U(3)$.*

Proof. For $m = 1$ and 2 we have:

$$\partial U_m = \left\{ P^{m-1}\varphi(a^k) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in Y \;\middle|\; \begin{array}{c} k \in \{0, 1, 2\} \\ z_i \in \mathbb{C} \text{ for } i \in \{1, 2, 3\} \\ |z_2|^2 + |z_3|^2 = \varepsilon \end{array} \right\},$$

and $\partial U_0 = \partial U_1 \cup \partial U_2$. In addition, $\partial E_n = \partial U_n \times S^5$, for $n = 0$, 1, and 2. This means that there is a unique way to write every element of ∂E_0 in the following form

$$\left(P^{m-1}\varphi(a^k) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, s \right)$$

where $m \in \{1, 2\}$, $k \in \{0, 1, 2\}$, $z_i \in \mathbb{C}$ for $i \in \{1, 2, 3\}$, and $|z_2|^2 + |z_3|^2 = \varepsilon$. We define an isomorphism

$$\alpha \colon \partial E_0 \to \partial E_1 \cup \partial E_2$$

given by

$$\alpha \left(P^{m-1}\varphi(a^k) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, s \right) = \left(P^{m-1}\varphi(a^k) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, Z_m s \right),$$

where

$$Z_1 = \frac{1}{\sqrt{\varepsilon(1-\varepsilon)}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \overline{z_1} z_2 & -z_1 \overline{z_3} \\ 0 & \overline{z_1} z_3 & z_1 \overline{z_2} \end{bmatrix} \in SU(3)$$

$$Z_2 = \frac{1}{\sqrt{\varepsilon(1-\varepsilon)}} \begin{bmatrix} \overline{z_1} z_2 & -z_1 \overline{z_3} & 0 \\ \overline{z_1} z_3 & z_1 \overline{z_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in SU(3) .$$

Now it is clear that α is an isomorphism. We just have to check that it is G-equivariant. First, check that α is equivariant under a:

$$\alpha\left(a\left(P^{m-1}\varphi(a^k)\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, s\right)\right) = \alpha\left(\left(\varphi(a)P^{m-1}\varphi(a^k)\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \psi_0(a)s\right)\right)$$

$$= \alpha\left(\left(P^{m-1}\varphi(a^{k+1})\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \psi_m(a)s\right)\right) = \left(P^{m-1}\varphi(a^{k+1})\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, Z_m\psi_m(a)s\right)$$

$$= \left(\varphi(a)P^{m-1}\varphi(a^k)\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \psi_m(a)Z_m s\right) = a\alpha\left(P^{m-1}\varphi(a^k)\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, s\right)$$

Second, check that α is equivariant under b:

$$\alpha\left(b\left(P^{m-1}\varphi(a^k)\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, s\right)\right) = \alpha\left(\left(\varphi(b)P^{m-1}\varphi(a^k)\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \psi_0(b)s\right)\right)$$

$$= \alpha\left(\left(P^{m-1}\varphi(a^{k+2(m-1)})\varphi(b)\varphi(c^{-k})\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \psi_0(b)s\right)\right)$$

$$= \alpha\left(\left(P^{m-1}\varphi(a^{k+2(m-1)})\begin{bmatrix} \xi^{-k} z_1 \\ \xi^{-k+1} z_2 \\ \xi^{-k+2} z_3 \end{bmatrix}, \psi_0(b)s\right)\right) = (*)$$

For $m = 1$ we have

$$(*) = \left(\varphi(a^k)\begin{bmatrix} \xi^{-k} z_1 \\ \xi^{-k+1} z_2 \\ \xi^{-k+2} z_3 \end{bmatrix}, \frac{1}{\sqrt{\varepsilon(1-\varepsilon)}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \overline{z_1}\xi z_2 & z_1\xi^2 z_3 \\ 0 & -\overline{z_1}\xi\overline{z_3} & z_1\xi^2\overline{z_2} \end{bmatrix}\psi_0(b)s\right)$$

$$= \left(\varphi(b)\varphi(a^k)\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, Z_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi^2 \end{bmatrix}\psi_0(b)s\right) = \left(\varphi(b)\varphi(a^k)\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, Z_1\psi_1(b)s\right)$$

$$= \left(\varphi(b)\varphi(a^k) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \psi_1(b) Z_1 s \right) = b\alpha \left(\varphi(a^k) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, s \right)$$

For $m = 2$ we have

$$(*) = \left(P\varphi(a^{k+2}) \begin{bmatrix} \xi^{-k} z_1 \\ \xi^{-k+1} z_2 \\ \xi^{-k+2} z_3 \end{bmatrix}, \frac{1}{\sqrt{\varepsilon(1-\varepsilon)}} \begin{bmatrix} \overline{z_1}\xi z_2 & -\overline{z_1}\xi\overline{z_3} & 0 \\ z_1\xi^2 z_3 & z_1\xi^2\overline{z_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \psi_0(b) s \right)$$

$$= \left(\varphi(b)\varphi(a^k) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \begin{bmatrix} \xi & 0 & 0 \\ 0 & \xi^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} Z_2 \psi_0(b) s \right)$$

$$= \left(\varphi(b)P\varphi(a^k) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \begin{bmatrix} \xi & 0 & 0 \\ 0 & \xi^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \psi_0(b) Z_2 s \right)$$

$$= \left(\varphi(b)P\varphi(a^k) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \psi_2(b) Z_2 s \right) = b\alpha \left(P\varphi(a^k) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, s \right)$$

Third, check that α is equivariant under $z \in S^1$:

$$\alpha \left(z \left(P^{m-1}\varphi(a^k) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, s \right) \right) = \alpha \left(\left(\varphi(z) P^{m-1}\varphi(a^k) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \psi_0(z) s \right) \right)$$

$$= \alpha \left(\left(P^{m-1}\varphi(a^k) \begin{bmatrix} zz_1 \\ zz_2 \\ zz_3 \end{bmatrix}, s \right) \right) = \left(P^{m-1}\varphi(a^k) \begin{bmatrix} zz_1 \\ zz_2 \\ zz_3 \end{bmatrix}, Z_m s \right)$$

$$= \left(\varphi(z) P^{m-1}\varphi(a^k) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \psi_m(z) Z_m s \right) = z\alpha \left(P^{m-1}\varphi(a^k) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, s \right)$$

□

Here is the conclusion of our explicit construction.

Theorem 11.6. *G acts freely and smoothly on $S^5 \times S^5$.*

Proof. Define a new space X by the following pushout diagram

$$\begin{array}{ccc} \partial E_0 \cong \partial E_1 \cup \partial E_2 & \longrightarrow & E_1 \cup E_2 \\ \downarrow & & \downarrow \\ E_0 & \longrightarrow & X \end{array}$$

The above pushout diagram can be considered in the category of G-equivariant 5-dimensional sphere bundles with the structure group $U(3)$. Hence we see that G acts freely on X because the action of G on $E_1 \cup E_2$ and E_0 are both free. In addition, the base spaces of these bundles is given by the following pushout diagram

$$\begin{array}{ccc} \partial U_0 = \partial U_1 \cup \partial U_2 & \longrightarrow & U_1 \cup U_2 \\ \downarrow & & \downarrow \\ U_0 & \longrightarrow & Y \end{array}$$

Hence X is a 5-dimensional sphere bundle over $Y = S^5$ with structure group $U(3)$. But $\pi_4(U(3)) = 0$. Hence $X = S^5 \times S^5$. $\square$

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