GROUP ACTIONS ON SPHERES WITH RANK ONE ISOTROPY

IAN HAMBLETON AND ERGÜN YALÇIN

Abstract. Let G be a rank two finite group, and let $\mathcal{H}$ denote the family of all rank one p-subgroups of G, for which $\operatorname{rank}_p(G) = 2$. We show that a rank two finite group G which satisfies certain group-theoretic conditions admits a finite G-CW-complex $X \simeq S^n$ with isotropy in $\mathcal{H}$, whose fixed sets are homotopy spheres. Our construction provides an infinite family of new examples.

1. Introduction

Let G be a finite group. The unit spheres $S(V)$ in finite-dimensional orthogonal representations of G provide the basic examples of smooth linear G-actions on spheres. However, these actions satisfy a number of special constraints on the dimensions of fixed sets and the structure of the isotropy subgroups arising from character theory, which may not hold for arbitrary smooth actions. Our goal in this series of papers is to construct new examples of smooth *non-linear* finite group actions on spheres.

In the first paper of this series [9], we studied group actions on spheres in the setting of *homotopy representations*, introduced by tom Dieck (see [20, Definition 10.1]). These are finite (or more generally finite dimensional) G-CW-complexes X satisfying the property that for each $H \leq G$, the fixed point set X^H is homotopy equivalent to a sphere $S^{n(H)}$ where $n(H) = \dim X^H$. We introduced algebraic homotopy representations as suitable chain complexes over the orbit category and proved a realization theorem for these algebraic models.

We say that G has *rank* k if it contains a subgroup isomorphic to $(\mathbf{Z}/p)^k$, for some prime p, but no subgroup $(\mathbf{Z}/p)^{k+1}$, for any prime p. In this paper, we use chain complex methods to study the following problem, as the next step towards smooth actions.

Question. For which finite groups G, does there exist a finite G-CW-complex $X \simeq S^n$ with all isotropy subgroups of rank one ?

The isotropy assumption implies that G must have rank ≤ 2, by P. A. Smith theory (see Corollary 6.3). Since every rank one finite group can act freely on a finite complex homotopy equivalent to a sphere (Swan [18]), we will restrict to groups of rank two.

There is another group theoretical *necessary* condition related to fusion properties of the Sylow subgroups. This condition involves the rank two finite group $\mathrm{Qd}(p)$ which is

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the group defined as the semidirect product

$$\mathrm{Qd}(p) = (\mathbf{Z}/p \times \mathbf{Z}/p) \rtimes SL_2(p)$$

with the obvious action of $SL_2(p)$ on $\mathbf{Z}/p \times \mathbf{Z}/p$. In his thesis, Ünlü [21, Theorem 3.3] showed that $\mathrm{Qd}(p)$ does not act on a finite CW-complex $X \simeq S^n$ with rank 1 isotropy. This means that any rank two finite group which includes $\mathrm{Qd}(p)$ as a subgroup cannot admit such actions.

More generally, we say $\mathrm{Qd}(p)$ is *p'-involved in G* if there exists a subgroup $K \leq G$, of order prime to p, such that $N_G(K)/K$ contains a subgroup isomorphic to $\mathrm{Qd}(p)$. The argument given by Ünlü in [21, Theorem 3.3] can be extended easily to obtain the stronger necessary condition (see Proposition 5.3):

($\circledast$). *Suppose that there exists a finite G-CW-complex $X \simeq S^n$ with rank 1 isotropy. Then* $\mathrm{Qd}(p)$ *is not p'-involved in G, for any odd prime p.*

In the other direction, rank two finite groups which do not p'-involve $\mathrm{Qd}(p)$, for p odd, have some interesting complex representations. Such finite groups are sometimes called $\mathrm{Qd}(p)$-free groups, and Jackson [13, Theorem 47] proved that each p-Sylow subgroup G_p of a $\mathrm{Qd}(p)$-free group G has a *p-effective character* $\chi\colon G_p \to U(n)$, meaning that χ respects fusion in G and $\langle \chi|_E, 1_E\rangle = 0$ for each elementary abelian p-subgroup of G with $\operatorname{rank} E = \operatorname{rank}_p G$. We use these characters to reduce the isotropy from p-subgroups to rank one p-subgroups.

Let $\mathcal{F}$ be a family of subgroups of G closed under conjugation and taking subgroups. For constructing group actions on CW-complexes with isotropy in the family $\mathcal{F}$, a good algebraic approach is to consider projective chain complexes over the orbit category relative to the family $\mathcal{F}$ (see [8], [9]).

Let $\mathcal{S}_G$ denote the set of primes p such that $\operatorname{rank}_p(G) = 2$. Let $\mathcal{H}_p$ denote the family of all rank one p-subgroups $H \leq G$, for $p \in \mathcal{S}_G$, and let $\mathcal{H} = \bigcup\{H \in \mathcal{H}_p \,|\, p \in \mathcal{S}_G\}$. Our main result is the following:

Theorem A. *Let G be a rank two finite group satisfying the following two conditions:*

(i) *G does not p'-involve* $\mathrm{Qd}(p)$ *for any odd prime $p \in \mathcal{S}_G$;*

(ii) *if $1 \neq H \in \mathcal{H}_p$, then* $\operatorname{rank}_q(N_G(H)/H) \leq 1$ *for every prime $q \neq p$.*

Then there exists a finite G-CW-complex X with isotropy in $\mathcal{H}$, such that X^H is a homotopy sphere for each $H \in \mathcal{H}$.

Theorem A is an extension of our earlier joint work with Semra Pamuk [8] where we have shown that the first non-linear example, the permutation group $G = S_5$ of order 120, admits a finite G-CW-complex $X \simeq S^n$ with rank one isotropy. Theorem A gives an new proof of this earlier result, by a more systematic method: for $G = S_5$, the set $\mathcal{S}_G$ includes only the prime 2 and the second condition above holds since all p-Sylow subgroups of S_5 for odd primes are cyclic. More generally, we have:

Corollary B. *Let p be a fixed prime and G be a finite group such that* $\operatorname{rank}_p(G) = 2$*, and* $\operatorname{rank}_q(G) = 1$ *for every prime $q \neq p$. If G does not p'-involve* $\mathrm{Qd}(p)$ *when $p > 2$, then there exists a finite G-CW-complex $X \simeq S^n$ with rank one p-group isotropy.*

Our general construction produces new *non-linear* G-CW-complex examples, for certain groups G which do not admit any orthogonal representations V with rank one isotropy on the unit sphere $S(V)$. In particular, we show that the alternating groups A_6 and A_7 admit finite G-CW-complexes $X \simeq S^n$ with rank one isotropy (see Example 6.4 and 6.6).

In fact, our methods apply to most of the rank two simple groups (see the list in [1, p.423]). We remark that $G = PSL_3(q)$, q odd, and $G = PSU_3(q)$, with $9 \mid (q+1)$, are the groups on the list[1] which are not $\mathrm{Qd}(p)$-free at some odd prime. Our next result provides an infinite collection of new non-linear examples.

Theorem C. *Let G be a finite simple group of rank two which is $\mathrm{Qd}(p)$-free, for all odd primes p. Then there exists a finite G-CW-complex $X \simeq S^n$ with rank one isotropy, except possibly for M_{11}, and $PSU_3(q)$, for $9 \nmid (q+1)$ and $(q+1)$ composite.*

For example, we can handle $G = PSU_3(q)$ if $q + 1 = 2^a$, and the groups in the family $PSL_2(q^2)$, $q \geq 3$, with Theorem A. The groups $PSL_2(q)$, $q \geq 5$, are covered by Corollary B. None of the simple groups $PSL_2(q)$, $q > 7$, admit orthogonal representation spheres with rank one isotropy (see Section 7), so the actions we construct provide an infinite family of new examples of non-linear actions.

In Section 6, we give the motivation for condition (ii) in Theorem A on the q-rank of the normalizer quotients. It is used in a crucial way (at the algebraic level) in the construction of our actions, but it is not, in general, a necessary condition for the existence of a finite G-CW-complex $X \simeq S^n$ with rank 1 isotropy (see Example 6.6). Determining the precise list of necessary and sufficient conditions is still an open problem.

We will obtain Theorem A from a more general technical result, Theorem 5.1, which accepts as input a suitable collection of $\mathcal{F}_p$-representations (see Definition 3.1), and produces a finite G-CW complex. Theorem 5.1 is used to construct the action in Example 6.6 for $G = A_7$ with rank one p-group isotropy. In principle, it could be used to construct other interesting non-linear examples for finite groups with specified p-group isotropy.

Here is a brief outline of the paper. We denote the orbit category relative to a family $\mathcal{F}$ by $\varGamma_G = \mathrm{Or}_{\mathcal{F}}\, G$, and construct projective chain complexes over $R\varGamma_G$ for various p-local coefficient rings $R = \mathbf{Z}_{(p)}$. To prove Theorem 5.1, we first introduce *algebraic homotopy representations* (see Definition 2.3), as chain complexes over $R\varGamma_G$ satisfying algebraic versions of the conditions found in tom Dieck's geometric *homotopy representations* (see [20, II.10.1], [6], and Remark 2.7). In Section 2 we summarize the results of [9] which show that the conditions in Definition 2.3 are lead to necessary and sufficient conditions for a chain complex over $R\varGamma_G$ to be homotopy equivalent to a chain complex of a geometric homotopy representation (see Theorem 2.6).

In Section 3, we construct p-local chain complexes where the isotropy subgroups are p-groups. In Section 4, we add homology to these local models so that these modified local complexes $\mathbf{C}^{(p)}$ all have exactly the same dimension function. Results established in [8] are used to glue these algebraic complexes together over $\mathbf{Z}\varGamma_G$, and then to eliminate a finiteness obstruction. In Section 5 we combine these ingredients to give a complete proof for Theorem 5.1 and Theorem A. We end the paper with a discussion about the necessity

[1]This case seems to have been overlooked in [1, p.430]

of the conditions in Theorem A. This discussion and the examples of nonlinear actions for the groups $G = A_6$ and A_7 can be found in Section 6. We discuss the rank two simple groups and prove Theorem C in Section 7.

Remark. One motivation for this project is the work of Adem-Smith [1] and Jackson [13] on the existence of free actions of finite groups on a product of two spheres. There is an interesting set of conditions related to this problem. In the following statements, G denotes a finite group of rank two.

(i) G acts on a finite complex X homotopy equivalent to a sphere, with rank one isotropy.
(ii) G acts with rank one isotropy on a finite dimensional complex X which has a mod p homology of a sphere.
(iii) G does not p'-involve $\mathrm{Qd}(p)$, for p an odd prime.
(iv) Each p-Sylow subgroup G_p of G has a p-effective character.
(v) There exists a spherical fibration $Y \to BG$, such that the total space Y has periodic cohomology.
(vi) G acts freely on a finite complex homotopy equivalent to a product of two spheres.

The implications $(i) \Rightarrow (i{+}1)$ hold for this list, where $(i) \Rightarrow (ii)$ is clear (for each prime p), and $(ii) \Rightarrow (iii)$ is our Proposition 5.3. The implication $(iii) \Leftrightarrow (iv)$ is due to Jackson [13, Theorem 47], using [13, Theorem 44] to show that G_2 always has a 2-effective character.

If condition (iv) holds for all the primes dividing the order of G, then condition (v) holds. This needs some explanation. First, the existence of a spherical fibration $Y \to BG$ classified by $\varphi\colon BG \to BU(n)$, with p-effective Euler class $\beta(\varphi) \in H^n(G;\mathbf{Z})$ for all primes p, was proved by Jackson [12], [13, Theorem 16]. By construction, for each elementary abelian p-subgoup E of G with $\operatorname{rank} E = \operatorname{rank}_p G$, there exists a unitary representation $\lambda\colon E \to U(n)$ such that $\varphi_E = B\lambda$ and $\langle \lambda, 1_E\rangle = 0$ (see [13, Definition 11]). Adem and Smith [1, Definition 4.3] give an equivalent definition of a p-effective cohomology class $\beta \in H^n(G;\mathbf{Z})$ as a class for which the complexity $cx_G(L_\beta \otimes \mathbb{F}_p) = 1$ (see Benson [3, Chap. 5]). It follows from [3, 5.10.4] that $L_{\beta(\varphi)} \otimes \mathbb{F}_p$ is a periodic module, and hence cup product with a periodicity generator α for this module gives the periodicity of $H^*(Y;\mathbb{F}_p)$ in high dimensions. Therefore Y has periodic cohomology in the sense of Adem-Smith [1, Definition 1.1]. Finally, $(v) \Rightarrow (vi)$ follows from the main results of Adem-Smith [1, Theorems 1.2, 3.6].

The reverse implications are mostly unknown. For example, it is not known whether $\mathrm{Qd}(p)$ itself can act freely on a product of two spheres. In [13, Theorem 47] it was claimed that $(iii) \Rightarrow (i)$, but the "proof" seems to confuse homotopy actions with finite G-CW complexes. However, we show in Corollary 3.11 that $(iii) \Rightarrow (ii)$. Finding new criteria for the implication $(iii) \Rightarrow (i)$ is the subject of this paper.

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2. Algebraic homotopy representations

In transformation group theory, a G-CW-complex X is called a geometric *homotopy representation* if it has the property that X^H is homotopy equivalent to the sphere $S^{n(H)}$ where $n(H) = \dim X^H$ for every $H \leq G$ (see tom Dieck [20, Section II.10]).

In this section we summarize the results of [9] which gives the definition and main properties of a suitable algebraic analogue, called *algebraic homotopy representations*.

Let G be a finite group and $\mathcal{F}$ be a family of subgroups of G which is closed under conjugations and taking subgroups. The orbit category $\operatorname{Or}_{\mathcal{F}} G$ is defined as the category whose objects are orbits of type G/K, with $K \in \mathcal{F}$, and where the morphisms from G/K to G/L are given by G-maps:

$$\operatorname{Mor}_{\operatorname{Or}_{\mathcal{F}} G}(G/K, G/L) = \operatorname{Map}_G(G/K, G/L).$$

The category $\varGamma_G = \operatorname{Or}_{\mathcal{F}} G$ is a small category, and we can consider the module category over $\varGamma_G$. Let R be a commutative ring with unity. A (right) $R\varGamma_G$-module M is a contravariant functor from $\varGamma_G$ to the category of R-modules. We denote the R-module $M(G/K)$ simply by $M(K)$ and write $M(f)\colon M(L) \to M(K)$ for a G-map $f\colon G/K \to G/L$. The further details about the properties of modules over the orbit category, such as the definitions of free and projective modules, can be found in [8] (see also Lück [14, §9,§17] and tom Dieck [20, §10-11]).

We will consider chain complexes $\mathbf{C}$ of $R\varGamma_G$-modules, such that $\mathbf{C}_i = 0$ for $i < 0$. We call a chain complex $\mathbf{C}$ *projective* (resp. *free*) if for all $i \geq 0$, the modules $\mathbf{C}_i$ are projective (resp. free). We say that a chain complex $\mathbf{C}$ is *finite* if $\mathbf{C}_i = 0$ for $i > n$, and the chain modules $\mathbf{C}_i$ are all finitely-generated $R\varGamma_G$-modules.

Given a G-CW-complex X, associated to it, there is a chain complex of $R\varGamma_G$-modules

$$\mathbf{C}(X^{?}; R): \quad \cdots \to R[X_n{}^{?}] \xrightarrow{\partial_n} R[X_{n-1}{}^{?}] \to \cdots \xrightarrow{\partial_1} R[X_0{}^{?}] \to 0$$

where X_i denotes the set of i-dimensional cells in X and $R[X_i{}^{?}]$ is the $R\varGamma_G$-module defined by $R[X_i{}^{?}](H) = R[X_i^H]$. We denote the homology of this complex by $H_*(X^{?}; R)$. If the family $\mathcal{F}$ includes the isotropy subgroups of X, then the complex $\mathbf{C}(X^{?}; R)$ is a chain complex of free $R\varGamma_G$-modules.

The *dimension function* of a finite dimensional chain complex $\mathbf{C}$ over $R\varGamma_G$ is defined as the function $\operatorname{Dim} \mathbf{C}\colon \mathscr{S}(G) \to \mathbf{Z}$ given by

$$(\operatorname{Dim} \mathbf{C})(H) = \dim \mathbf{C}(H)$$

for all $H \in \mathcal{F}$. If $\mathbf{C}(H)$ is the zero complex or if H is a subgroup such that $H \notin \mathcal{F}$, then we define $(\operatorname{Dim} \mathbf{C})(H) = -1$. The dimension function $\operatorname{Dim} \mathbf{C}$ is constant on conjugacy classes (a super class function). In a similar way, we can define the *homological dimension function* $\operatorname{hDim} \mathbf{C}\colon \mathcal{F} \to \mathbf{Z}$ of a chain complex $\mathbf{C}$ of $R\varGamma_G$-modules.

We call a function $\underline{n}\colon \mathscr{S}(G) \to \mathbf{Z}$ *monotone* if it satisfies the property that $\underline{n}(K) \leqslant \underline{n}(H)$ whenever $(H) \leq (K)$. We say that a monotone function $\underline{n}$ is *strictly monotone* if $\underline{n}(K) < \underline{n}(H)$, whenever $(H) < (K)$. We have the following:

Lemma 2.1 ([9, Lemma 2.4])**.** *The dimension function of a projective chain complex of $R\varGamma_G$-modules is a monotone function.*

Definition 2.2. We say a chain complex $\mathbf{C}$ of $R\Gamma_G$-modules is *tight at* $H \in \mathcal{F}$ if

$$\operatorname{Dim} \mathbf{C}(H) = \operatorname{hdim} \mathbf{C}(H).$$

We call a chain complex of $R\Gamma_G$-modules *tight* if it is tight at every $H \in \mathcal{F}$.

We are particularly interested in chain complexes which have the homology of a sphere when evaluated at every $K \in \mathcal{F}$. Let $\underline{n}$ be a super class function defined on $\mathcal{F}$, and let $\mathbf{C}$ be a chain complex over $R\Gamma_G$. We say that $\mathbf{C}$ an *R-homology $\underline{n}$-sphere* if $\mathbf{C}$ is an augmented complex with the reduced homology of $\mathbf{C}(K)$ is the same as the reduced homology of an $\underline{n}(K)$-sphere (with coefficients in R) for all $K \in \mathcal{F}$.

In [20, II.10], there is a list of properties that are satisfied by homotopy representations. We will use algebraic versions of these properties to define an analogous notion for chain complexes.

Definition 2.3 ([9, 2.7])**.** Let $\mathbf{C}$ be a finite projective chain complex over $R\Gamma_G$, which is an R-homology $\underline{n}$-sphere. We say $\mathbf{C}$ is an *algebraic homotopy representation* (over R) if

(i) The function $\underline{n}$ is a monotone function.
(ii) If $H, K \in \mathcal{F}$ are such that $n = \underline{n}(K) = \underline{n}(H)$, then for every G-map $f\colon G/H \to G/K$ the induced map $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ is an R-homology isomorphism.
(iii) Suppose $H, K, L \in \mathcal{F}$ are such that $H \leq K, L$ and let $M = \langle K, L\rangle$ be the subgroup of G generated by K and L. If $n = \underline{n}(H) = \underline{n}(K) = \underline{n}(L) > -1$, then $M \in \mathcal{F}$ and $n = \underline{n}(M)$.

Under condition (iii) of Definition 2.3, the isotropy family $\mathcal{F}$ has an important maximality property.

Corollary 2.4 ([9, 2.9])**.** *Let $\mathbf{C}$ be a projective chain complex of $R\Gamma_G$-modules, If condition* (iii) *holds, then the set of subgroups $\mathcal{F}_H = \{K \in \mathcal{F} \mid (H) \leq (K),\ \underline{n}(K) = \underline{n}(H) > -1\}$ has a unique maximal element, up to conjugation.*

In the remainder of this section we will assume that R is a principal ideal domain. The main examples for us are $R = \mathbf{Z}_{(p)}$ or $R = \mathbf{Z}$.

Theorem 2.5 ([9, Theorem A])**.** *Let $\mathbf{C}$ be a finite free chain complex of $R\Gamma_G$-modules which is a homology $\underline{n}$-sphere. Then $\mathbf{C}$ is chain homotopy equivalent to a finite free chain complex $\mathbf{D}$ which is tight if and only if $\mathbf{C}$ is an algebraic homotopy representation.*

When these conditions hold for $R = \mathbb{Z}$, then we apply [8, Theorem 8.10], [17] to obtain a geometric realization result.

Theorem 2.6 ([9, Corollary B])**.** *Let $\mathbf{C}$ be a finite free chain complex of $\mathbf{Z}\Gamma_G$-modules which is a homology $\underline{n}$-sphere. If $\mathbf{C}$ is an algebraic homotopy representation, and $\underline{n}(K) \geq 3$ for all $K \in \mathcal{F}$, then there is a finite G-CW-complex X such that $\mathbf{C}(X^?; \mathbf{Z})$ is chain homotopy equivalent to $\mathbf{C}$ as chain complexes of $\mathbf{Z}\Gamma_G$-modules.*

Remark 2.7. The construction actually produces a finite G-CW-complex X such that all the non-empty fixed sets X^H are simply-connected, and with trivial action of $W_G(H) = N_G(H)/H$ on the homology of X^H. Therefore X will be an *oriented* geometric homotopy representation (in the sense of tom Dieck).

3. Construction of the preliminary local models

Our main technical tool is provided by Theorem 5.1, which gives a method for constructing finite G-CW-complexes, with isotropy in a given family. This theorem will be proved by applying the realization statement of Theorem 2.6. To construct a suitable finite free chain complex $\mathbf{C}$ over $\mathbf{Z}\varGamma_G$, we work one prime at a time to construct local models $\mathbf{C}^{(p)}$, and then apply the glueing method for chain complexes developed in [8, Theorem 6.7].

The main input of Theorem 5.1 is a compatible collection of unitary representations for the p-subgroups of G. We give the precise definition in a more general setting.

Definition 3.1. Let $\mathcal{F}$ be a family of subgroups of G. We say that $\mathbf{V}(\mathcal{F})$ is an $\mathcal{F}$-*representation* for G, if $\mathbf{V}(\mathcal{F}) = \{V_H \in \operatorname{Rep}(H, U(n)) \mid H \in \mathcal{F}\}$ is a compatible collection of (non-zero) unitary H-representations. The collection is *compatible* if $f^*(V_K) \cong V_H$ for every G-map $f \colon G/H \to G/K$.

For any finite G-CW-complex X, we let $\operatorname{Iso}(X) = \{K \leq G \mid X^K \neq \emptyset\}$ denote the *isotropy family* of the G-action on X. This suggests the following notation:

Definition 3.2. Let $\mathbf{V}(\mathcal{F})$ be an $\mathcal{F}$-representation for G. We let

$$\operatorname{Iso}(\mathbf{V}(\mathcal{F})) = \{L \leq H \mid S(V_H)^L \neq \emptyset, \text{ for some } V_H \in \mathbf{V}(\mathcal{F})\}$$

denote the *isotropy family* of $\mathbf{V}(\mathcal{F})$. We note that $\operatorname{Iso}(\mathbf{V}(\mathcal{F}))$ is a sub-family of $\mathcal{F}$.

Example 3.3. Our first example for these definitions will be a compatible collection of representations for the family $\mathcal{F}_p$ of all p-subgroups, with p a fixed prime dividing the order of G. In this case, an $\mathcal{F}_p$-representation $\mathbf{V}(\mathcal{F}_p)$ can be constructed from a suitable representation $V_p \in \operatorname{Rep}(P, U(n))$, where P denotes a p-Sylow subgroup of G. The representations V_H can be constructed for all $H \in \mathcal{F}_p$, by extending V_p to conjugate p-Sylow subgroups and by restriction to subgroups. To ensure a compatible collection $\{V_H\}$, we assume that V_p *respects fusion* in G, meaning that $\chi_p(gxg^{-1}) = \chi_p(x)$ for the corresponding character χ_p, whenever $gxg^{-1} \in P$ for some $g \in G$ and $x \in P$.

We will now specify an isotropy family $\mathcal{J}$ that will be used in the rest of the paper.

Definition 3.4. Let $\{\mathbf{V}(\mathcal{F}_p) \mid p \in \mathcal{S}_G\}$ be a collection of $\mathcal{F}_p$-representations, for a set $\mathcal{S}_G$ of primes dividing the order of G. Let $\mathcal{J}_p = \operatorname{Iso}(\mathbf{V}(\mathcal{F}_p))$ and $\mathcal{J} = \bigcup\{\mathcal{J}_p \mid p \in \mathcal{S}_G\}$ denote the isotropy families.

We note that $\mathcal{J}$ contains no isotropy subgroups of composite order, since each $\mathcal{J}_p$ is a family of p-subgroups. Let $\varGamma_G = \operatorname{Or}_{\mathcal{J}} G$ and $\varGamma_G(p)$ denote the orbit category $\operatorname{Or}_{\mathcal{J}_p} G$ over the family $\mathcal{J}_p$. A chain complex $\mathbf{C}$ over $R\varGamma_G(p)$ can always be considered as a complex of $R\varGamma_G$-modules, by taking the values $\mathbf{C}(H)$ at subgroups $H \notin \mathcal{J}_p$ as zero complexes.

In this section we construct a p-local chain complex $\mathbf{C}^{(p)}(0)$ over $R\varGamma_G(p)$, for $R = \mathbf{Z}_{(p)}$, which we call a *preliminary local model* (see Definition 3.9). From this construction we will obtain a dimension function $\underline{n}^{(p)} \colon \mathcal{J}_p \to \mathbf{Z}$. By taking joins we can assume that these dimension functions have common value at $H = 1$. In the next section, these preliminary local models will be "improved" at each prime p by adding homology as specified by the dimension functions $\underline{n}^{(q)}\colon \mathcal{J}_q \to \mathbf{Z}$, for all $q \in \mathcal{S}_G$ with $q \neq p$. The resulting complexes $\mathbf{C}^{(p)}$ over the orbit category $R\varGamma_G$ will all have the same dimension function

$$\underline{n} = \bigcup \{\underline{n}^{(p)} \,|\, p \in \mathcal{S}_G\}\colon \mathcal{J} \to \mathbf{Z},$$

and satisfy conditions needed for the glueing method.

Proposition 3.5. *Let G be a finite group, and let $\mathbf{V}(\mathcal{F}_p)$ be an $\mathcal{F}_p$-representation for G for some $p \in \mathcal{S}_G$. Then there exists a finite-dimensional G-CW-complex E, with isotropy family equal to $\mathcal{J}_p$, such that for each $H \in \mathcal{J}_p$ the fixed set E^H is simply-connected and p-locally homotopy equivalent to a sphere $S(V_H)$.*

Proof. We recall a result of Jackowski, McClure and Oliver [11, Proposition 2.2]: there exists a simply-connected, finite dimensional G-CW-complex B which is $\mathbf{F}_p$-acyclic and has finitely many orbit types with isotropy in the family of p-subgroups $\mathcal{F}_p$ in G. The quoted result applies more generally to all compact Lie groups and produces a complex with p-toral isotropy (meaning a compact Lie group P whose identity component P_0 is a torus, and P/P_0 is a finite p-group). For G finite, the p-toral subgroups are just the p-subgroups. The property that all fixed sets B^H are simply-connected is established in the proof.

We now apply [22, Proposition 4.3] to this G-CW-complex B and to the given $\mathcal{F}_p$-representation $\mathbf{V}(\mathcal{F}_p)$, to obtain a G-equivariant spherical fibration $E \to B$ with fiber type $S(\mathbf{V}(\mathcal{F}_p)^{\oplus k})$ for some k, such that E is finite dimensional (see [22, Section 2] for necessary definitions). The resulting G-CW-complex E has the required properties. In particular, since B is $\mathbf{F}_p$-acyclic then for each p-subgroup H, the fixed point set B^H will be also $\mathbf{F}_p$-acyclic (and $B^H \neq \emptyset$). This means that the (extended) isotropy family of E is $\mathcal{J}_p = \operatorname{Iso}(\mathbf{V}(\mathcal{F}_p))$ and for every $H \in \mathcal{J}_p$, the mod-p homology of E^H is isomorphic to the mod-p homology of $S(V_H)$. By taking further fiber joins if necessary, we can assume that E^H is simply connected for all $H \in \mathcal{J}_p$. Hence E^H is p-locally homotopy equivalent to a sphere. □

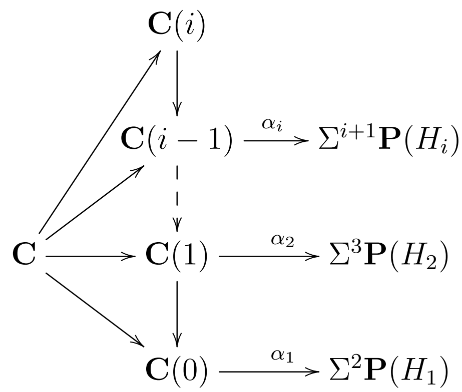
We now let $R = \mathbf{Z}_{(p)}$, and consider the finite dimensional chain complex $\mathbf{C}(E^{?}; R)$ of free $R\varGamma_G(p)$-modules. By taking joins, we may assume that this complex has "homology gaps" of length $> l(\varGamma_G)$ as required for [8, Theorem 6.7], and that all the non-empty fixed sets E^H have $\underline{n}(H) \geq 3$ and trivial action of $W_G(H)$ on homology. Let $\underline{n}^{(p)}\colon \mathcal{J}_p \to \mathbf{Z}$ denote the dimension function $\operatorname{hDim} \mathbf{C}(E^{?}; R)$.

The following result applies to chain complexes over $R\varGamma_G$ with respect to any family $\mathcal{F}$ of subgroups.

Lemma 3.6. *Let R be a noetherian ring and G be a finite group. Suppose that $\mathbf{C}$ is an n-dimensional chain complex of projective $R\varGamma_G$-modules with finitely generated homology groups. Then $\mathbf{C}$ is chain homotopy equivalent to a finitely-generated projective n-dimensional chain complex over $R\varGamma_G$.*

Proof. Note that the chain modules of $\mathbf{C}$ are not assumed to be finitely-generated, but $H_i(\mathbf{C}) = 0$ for $i > n$. We first apply Dold's "algebraic Postnikov system" technique [5, §7], to chain complexes over the orbit category (see [8, §6]).

According to this theory, given a positive projective chain complex $\mathbf{C}$, there is a sequence of positive projective chain complexes $\mathbf{C}(i)$ indexed by positive integers such that $f\colon \mathbf{C} \to \mathbf{C}(i)$ induces a homology isomorphism for dimensions $\leq i$. Moreover, there is a tower of maps



such that $\mathbf{C}(i) = \Sigma^{-1}\mathbf{C}(\alpha_i)$, where $\mathbf{C}(\alpha_i)$ denotes the algebraic mapping cone of α_i, and $\mathbf{P}(H_i)$ denotes a projective resolution of the homology module $H_i = H_i(\mathbf{C})$.

By assumption, since the homology modules H_i are finitely-generated and R is noetherian, we can choose the projective resolutions $\mathbf{P}(H_i)$ to be finitely-generated in each degree. Therefore, at each step the chain complex $\mathbf{C}(i)$ consists of finitely-generated projective $R\varGamma_G$-modules, and $\mathbf{C}(n) \simeq \mathbf{C}$ has homological dimension $\leq n$. Now, since $H^{n+1}(\mathbf{C}(n); M) = H^{n+1}(\mathbf{C}; M) = 0$, for any $R\varGamma_G$-module M, we conclude that $\mathbf{C}(n)$ is chain homotopy equivalent to an n-dimensional finitely-generated projective chain complex by [14, Prop. 11.10]. $\square$

Remark 3.7. See [14, 11.31:ex. 2] or [19, Satz 9] for related background and previous results.

Lemma 3.8. *The chain complex $\mathbf{C}(E^{?}; R)$ is chain homotopy equivalent to an oriented R-homology $\underline{n}^{(p)}$-sphere $\mathbf{C}^{(p)}(0)$, which is an algebraic homotopy representation.*

Proof. The chain complex $\mathbf{C}(E^{?}; R)$ is finite dimensional and free over $R\varGamma_G$, but may not be finitely-generated. However, by the conclusion of Proposition 3.5, the homology groups $H_*(\mathbf{C}(E^{?}; R))$ are finitely generated since $\mathbf{C}(E^{?}; R)$ is an R-homology $\underline{n}$-sphere. The result now follows from Lemma 3.6, which produces a finite length projective chain complex $\mathbf{C}^{(p)}(0)$ of finitely-generated $R\varGamma_G(p)$-modules. Note that $\mathbf{C}(E^{?}; R)$ satisfies the conditions (i)-(iii) in Definition 2.3, so $\mathbf{C}^{(p)}(0)$ also satisfies these conditions (which are chain-homotopy invariant), hence $\mathbf{C}^{(p)}(0)$ is an algebraic homotopy representation. $\square$

Note that $\mathbf{C}^{(p)}(0)$ is an algebraic homotopy representation, meaning that it satisfies the condition (i), (ii), and (iii) in Definition 2.3, even though $\operatorname{Dim} \mathbf{C}^{(p)}$ may not be equal to $\underline{n}^{(p)} = \operatorname{hDim} \mathbf{C}^{(p)}(0)$.

By taking joins, we may assume that there exists a common dimension $N = \underline{n}^{(p)}(1)$, at $H = 1$, for all $p \in \mathcal{S}_G$. Moreover, we may assume that $N + 1$ is a multiple of any given integer m_G (to be chosen below). We now obtain the "global" dimension function

$$\underline{n} = \bigcup \{\underline{n}^{(p)} \mid p \in \mathcal{S}_G\} \colon \mathcal{J} \to \mathbf{Z},$$

where $\underline{n}^{(p)} = \operatorname{hDim} \mathbf{C}^{(p)}(0)$, for all $p \in \mathcal{S}_G$, and $\underline{n}(1) = N$.

Definition 3.9 (*Preliminary local models*)**.** Let $\mathcal{S}_G = \{p \mid \operatorname{rank}_p G \geq 2\}$, and let m_G denote the least common multiple of the q-periods for G (as defined in [18, p. 267]), over all primes q for which $\operatorname{rank}_q G = 1$. We assume that $\underline{n}(1) + 1$ is a multiple of m_G.

(i) We will take the chain complex $\mathbf{C}^{(p)}(0)$ constructed in Lemma 3.8 for our preliminary model at each prime $p \in \mathcal{S}_G$.
(ii) If $\operatorname{rank}_q G = 1$, we take $\mathcal{J}_q = \{1\}$ and $\mathbf{C}^{(q)}(0)$ as the $R\varGamma_G$-chain complex $E_1\mathbf{P}$ where $\mathbf{P}$ is a periodic resolution of R as a RG-module with period $\underline{n}(1) + 1$ (for more details, see the proof of Theorem 4.1 below, or [8, Section 9B]).

This completes the construction of the preliminary local models at each prime dividing the order of G, for a given family of $\mathcal{F}_p$-representations. In the next section we will modify these preliminary models to get p-local chain complexes $\mathbf{C}^{(p)}$ over $R\varGamma_G$ which are R-homology $\underline{n}$-spheres for the dimension function $\underline{n}$ described above.

Example 3.10. In the proof of Theorem A we will be using the setting of Example 3.3. Suppose that G is a rank two finite group which does not p'-involve $\operatorname{Qd}(p)$, for any odd prime p. We let $\mathcal{S}_G$ be the set of primes p where $\operatorname{rank}_p G = 2$. Under this condition, a result of Jackson [13, Theorem 47] asserts that G admits a *p-effective* p-local character V_p. Here p-effective means that when V_p is restricted to an elementary abelian subgroup E of rank 2 then it has no trivial summand. This guarantees that the set of isotropy subgroups $\mathcal{J}_p = \operatorname{Iso}(S(V_p))$ consists of the rank one p-subgroups. In this setting, our preliminary local models arise from the following special case:

Corollary 3.11. *Let G be a finite rank two group with $\operatorname{rank}_p G = 2$. If G does not p'-involve $\operatorname{Qd}(p)$ when $p > 2$, then there exists a simply-connected, finite-dimensional G-CW-complex E with rank one p-group isotropy, which is p-locally homotopy equivalent to a sphere.*

Note that when G is a p-group of rank two, then it has a central element c of order p in G. Using the subgroup generated by c, we can define the induced representation $V = \operatorname{Ind}_{\langle c \rangle}^G \chi$ where χ is a nontrivial one dimensional complex representation of $\langle c \rangle$. Then, the G-action on $S(V)$ will satisfy the conclusion of the above corollary. It is proved by Dotzel-Hamrick [6] that all p-group actions on mod-p homology spheres resemble linear actions on spheres.

4. Construction of the local models: adding homology

Let G be a finite group and let $\mathcal{S}_G = \{p \mid \operatorname{rank}_p G \geq 2\}$. We will use the notation $\mathcal{J}_p = \operatorname{Iso}(\mathbf{V}(\mathcal{F}_p))$, for $p \in \mathcal{S}_G$, as given in Definition 3.4. For $p \notin \mathcal{S}_G$ we have $\mathcal{J}_p = \{1\}$. We will continue to work over the orbit category $\varGamma_G = \operatorname{Or}_{\mathcal{J}} G$ where $\mathcal{J} = \bigcup \{\mathcal{J}_p \mid p \in \mathcal{S}_G\}$. For each prime p dividing the order of G, let $\mathbf{C}^{(p)}(0)$ denote the preliminary p-local model given in Definition 3.9, and denote the homological dimension function of $\mathbf{C}^{(p)}(0)$ by $\underline{n}^{(p)}\colon \mathcal{J}_p \to \mathbf{Z}$ for all primes dividing the order of G.

We now fix a prime q dividing the order of G, and let $R = \mathbf{Z}_{(q)}$. In Theorem 4.1, we will show how to add homology to the preliminary local model $\mathbf{C}^{(q)}(0)$, to obtain an algebraic homotopy representation with dimension function $\underline{n}^{(p)} \cup \underline{n}^{(q)}$ for any prime $p \in \mathcal{S}_G$ such that $p \neq q$. After finitely many such steps, we will obtain our local model $\mathbf{C}^{(q)}$ over $R\varGamma_G$ with dimension function $\operatorname{hDim} \mathbf{C}^{(q)} = \underline{n}$. The main result of this section is the following:

Theorem 4.1. *Let G be a finite group and let $R = \mathbf{Z}_{(q)}$. Suppose that $\mathbf{C}$ is an algebraic homotopy representation over R, such that*

(i) $\mathbf{C}$ *an (oriented) R-homology $\underline{n}^{(q)}$-sphere of projective $R\varGamma_G(q)$-modules;*
(ii) *If $1 \neq H \in \mathcal{J}_p$, then* $\operatorname{rank}_q(N_G(H)/H) \leq 1$, *for every prime $p \neq q$.*

Then there exists an algebraic homotopy representation $\mathbf{C}^{(q)}$ over R, which is an (oriented) R-homology $\underline{n}$-sphere over $R\varGamma_G$.

Remark 4.2. Note that if there exists a q-local model $\mathbf{C}^{(q)}$ with isotropy in $\mathcal{J}_p \cup \mathcal{J}_q$, where $p \in \mathcal{S}_G$, then for every p-subgroup $1 \neq H \in \mathcal{J}_p$, the $RN_G(H)/H$ complex $\mathbf{C}^{(q)}(H)$ is a finite length chain complex of finitely generated modules which has the R-homology of an $\underline{n}(H)$-sphere. Since $R = \mathbf{Z}_{(q)}$, if we take a q-subgroup $Q \leq N_G(H)/H$ with $H \neq 1$, and restrict $\mathbf{C}^{(q)}(H)$ to Q, we obtain a finite dimensional *projective* RQ-complex (see [8, Lemma 3.6]). This means Q has periodic group cohomology and therefore it is a rank one subgroup. So, the condition (ii) in Theorem 4.1 is a necessary condition.

In order to carry out the above construction, we also assume that for every $1 \neq H \in \mathcal{J}_p$, the $\underline{n}(H) + 1$ is a multiple of the q-period of $W_G(H)$ and the gaps between non-zero homology dimensions are large enough: more precisely, for all $K, L \in \mathcal{J}$ with $\underline{n}(K) > \underline{n}(L)$, we have $\underline{n}(K) - \underline{n}(L) \geq l(\varGamma_G)$, where $l(\varGamma_G)$ denotes the length of the longest chain of maps in the category $\varGamma_G$. We can easily guarantee both of these conditions by taking joins of the preliminary local models we have constructed.

The rest of this section is devoted to the proof of Theorem 4.1. We will add the homology specified by the dimension function $\underline{n}^{(p)}$, at a prime $p \neq q$, by an inductive construction using the number of nonzero homology dimensions. The starting point of the induction is the given complex $\mathbf{C}$. Let $n_1 > n_2 > \cdots > n_s$ denote the set of dimensions $\underline{n}(H)$, over all $H \in \mathcal{J}_p$. Note that, since the dimension function $\underline{n}$ comes from a unitary representation, we have $n_s \geq 1$. Let us denote by $\mathcal{F}_i$, the collection of subgroups $H \in \mathcal{J}_p$ such that $\underline{n}(H) = n_i$.

Suppose that we have constructed a finite projective chain complex $\mathbf{C}$ over $R\varGamma_G$, satisfying the conditions (i)-(iii) of Definition 2.3, which has the property that $\operatorname{hDim} \mathbf{C}(H) = \underline{n}(H)$ for all $H \in \mathcal{F}_{\leq k}$ where $\mathcal{F}_{\leq k} = \bigcup_{i \leq k} \mathcal{F}_i$. Our goal is to construct a new finite dimensional projective complex $\mathbf{D}$ which also satisfies the conditions (i)-(iii) of Definition 2.3, and has the property that $\operatorname{hDim} \mathbf{D}(H) = \underline{n}(H)$ for all $H \in \mathcal{F}_i$ with $i \leq k + 1$.

We will construct the complex $\mathbf{D}$ as an extension of $\mathbf{C}$ by a finite projective chain complex whose homology is isomorphic to the homology that we need to add. Note that since the constructed chain complex $\mathbf{D}$ must satisfy the conditions (i)-(iii), the homology we need to add should satisfy the condition that for every $H \leq K$ with $H, K \in \mathcal{F}_{k+1}$, the restriction map on the added homology module is an R-homology isomorphism.

Definition 4.3. Let J_i denote the $R\varGamma_G$-module which has the values $J_i(H) = R$ for all $H \in \mathcal{F}_i$, and otherwise $J_i(H) = 0$. The restriction maps $r_H^K \colon J_i(K) \to J_i(H)$ for every $H, K \in \mathcal{F}_i$ such that $H \leq K$, and the conjugation maps $c^g : J_i(K) \to J_i({}^gK)$ for every $K \in \mathcal{F}$ and $g \in G$, are assumed to be the identity maps.

In this notation, the chain complex $\mathbf{D}$ must have homology isomorphic to J_i in dimension n_i for all $i \leq k+1$, and in dimension zero the homology of $\mathbf{D}$ should be isomorphic to $\underline{R}$ restricted to $\mathcal{F}_{k+1}$. It is in general a difficult problem to find projective chain complexes whose homology is given by a block of R-modules with prescribed restriction maps. But in our situation we will be able to do this using some special properties of the poset of subgroups in $\mathcal{F}_i$ coming from condition (iii) of Definition 2.3. Observe that we have the following property by Corollary 2.4:

Lemma 4.4. *For $1 \leq i \leq s$, each poset $\mathcal{F}_i$ is a disjoint union of components where each component has a unique maximal subgroup up to conjugacy.*

For every $K \in \mathcal{J}_p$, the q-Sylow subgroup of the normalizer quotient $W_G(K) = N_G(K)/K$ has q-rank equal to one, hence it is q-periodic. By our starting assumption, the q-period of $W_G(K)$ divides $\underline{n}(K)+1$. So by Swan [18], there exists a periodic projective resolution $\mathbf{P}$ with

$$0 \to R \to P_n \to \cdots \to P_1 \to P_0 \to R \to 0$$

over the group ring $RW_G(K)$ where $n = \underline{n}(K)$. Note that this statement includes the possibility that q-Sylow subgroup of $W_G(K)$ is trivial since in that case R would be projective as a $RW_G(K)$-module, and we can easily find a chain complex of the above form by adding a split projective chain complex.

Now suppose that $K \in \mathcal{J}_p$ is such that (K) is a maximal conjugacy class in $\mathcal{F}_{k+1}$. Consider the $R\varGamma_G$-complex $E_K\mathbf{P}$ where E_K denotes the extension functor defined in [8, Sect. 2C]. By definition

$$E_K(\mathbf{P})(H) = \mathbf{P} \otimes_{R[W_G(K)]} R[(G/K)^H]$$

for every $H \in \mathcal{F}$. We define the chain complex $E_{k+1}\mathbf{P}$ as the direct sum of the chain complexes $E_K\mathbf{P}$ over all representatives of isomorphism classes of maximal elements in $\mathcal{F}_{k+1}$. Let $\mathbf{N}$ denote the subcomplex of $E_{k+1}(\mathbf{P})$ obtained by restricting $E_K(\mathbf{P})$ to subgroups $H \in \mathcal{F}_{\leq k}$. Let $I_{k+1}\mathbf{P}$ denote the quotient complex $E_{k+1}(\mathbf{P})/\mathbf{N}$. We have the following:

Lemma 4.5. *The homology of $I_{k+1}\mathbf{P}$ is isomorphic to J_{k+1} at dimensions 0 and n_{k+1} and zero everywhere else.*

Proof. The homology of $I_{k+1}\mathbf{P}$ at $H \in \mathcal{F}_{k+1}$ is isomorphic to

$$\bigoplus \{R \otimes_{R[W_G(K)]} R[(G/K)^H] : (K) \text{ maximal in } \mathcal{F}_{k+1}\}$$

at dimensions 0 and n_{k+1} and zero everywhere else. Note that $(G/K)^H = \{gK \colon H^g \leq K\}$. If gK is such that $H^g \leq K$, then $H \leq {}^gK$. Now by condition (iii), we must have $\langle K, {}^gK\rangle \in \mathcal{F}_{k+1}$. But (K) was a maximal conjugacy class in $\mathcal{F}_{k+1}$, so we must have

$K = {}^gK$, hence $g \in N_G(K)$. This gives $1 \otimes gK = 1 \otimes 1$ in $R \otimes_{R[W_G(K)]} R[(G/K)^H]$. Therefore

$$R \otimes_{R[W_G(K)]} R[(G/K)^H] \cong R$$

for every $H \in \mathcal{F}_{k+1}$. Also, by the same argument H can not be included in two different maximal subgroups in $\mathcal{F}_{k+1}$. So we have $I_{k+1}(\mathbf{P})(H) \cong R$ for all $H \in \mathcal{F}_{k+1}$. Since the restriction maps are given by the inclusion map of fixed point sets $(G/H)^U \hookrightarrow (G/H)^V$ for every $U, V \in \mathcal{F}_{k+1}$ with $V \leq U$, we can conclude that all restriction maps are identity maps. This completes the proof of the lemma. □

The above lemma shows that the homology of $I_{k+1}\mathbf{P}$ is exactly the $R\Gamma_G$-module that we would like to add to the homology of $\mathbf{C}$. To construct $\mathbf{D}$ we use an idea similar to the idea used in [8, Section 9B]. Observe that for every $R\Gamma_G$-chain map $f \colon \mathbf{N} \to \mathbf{C}$, there is a push-out diagram of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{N} & \longrightarrow & E_K\mathbf{P} & \longrightarrow & I_{k+1}\mathbf{P} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathbf{C} & \longrightarrow & \mathbf{C}_f & \longrightarrow & I_{k+1}\mathbf{P} & \longrightarrow & 0\ . \end{array}$$

The homology of $\mathbf{N}$ is only nonzero at dimensions 0 and n_{k+1} and at these dimensions the homology is only nonzero at subgroups $H \in \mathcal{F}_{\leq k}$. At these subgroups the homology of $\mathbf{N}(H)$ is isomorphic to the direct sum of modules of the form $R \otimes_{RW_G(K)} R[(G/K)^H]$. Note that for every $H \in \mathcal{F}_{\leq k}$, there is an augmentation map

$$\varepsilon_H \colon R \otimes_{RW_G(K)} R[(G/K)^H] \to R$$

which takes $r \otimes gK$ to r for every $r \in R$. The collection of these maps gives a map of $R\Gamma_G$-modules denoted $\varepsilon_K \colon E_K R \to R$. Taking the sum over all conjugacy classes of maximal subgroups, we get a map $\sum_K \varepsilon_K \colon \oplus_K E_K R \to R$. Repeating the arguments given in [8, Section 9B], it is easy to see that if f is a chain map such that the induced map on zeroth homology $f_* \colon H_0(\mathbf{N}) \to H_0(\mathbf{C})$ is the same map as the sum of augmentation maps $\sum_K \varepsilon_K$, then the chain complex $\mathbf{C}_f$ will have the identity map as the restriction maps on zeroth homology. At dimension n_{k+1} we will have zero map since the homology of $\mathbf{C}$ is zero at dimension n_{k+1} by assumption.

Unfortunately, we can not take $\mathbf{D}$ as $\mathbf{C}_f$ since the complex $I_{k+1}\mathbf{P}$ is not projective in general, and neither is $\mathbf{N}$. We note that finding a chain map $\mathbf{N}$ satisfying the given condition is not an easy task without projectivity (compare [8, Section 9B], where this complex was projective). So we first need to replace the sequence $0 \to \mathbf{N} \to E_{k+1}\mathbf{P} \to I_{k+1}\mathbf{P} \to 0$ with a sequence of projective chain complexes.

Lemma 4.6. *There is a diagram of chain complexes where all the complexes $\mathbf{P}', \mathbf{P}'', \mathbf{P}'''$ are finite projective chain complexes over $R\Gamma_G$ and all the vertical maps induce isomorphisms on homology:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{P}' & \longrightarrow & \mathbf{P}'' & \longrightarrow & \mathbf{P}''' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{N} & \longrightarrow & E_{k+1}\mathbf{P} & \longrightarrow & I_{k+1}\mathbf{P} & \longrightarrow & 0 \ . \end{array}$$

Proof. Since $E_K\mathbf{P}$ is a projective chain complex of length n, $E_{k+1}\mathbf{P}$ is a finite projective chain complex. So, by [14, Lemma 11.6], it is enough to show that $\mathbf{N}$ is weakly equivalent to a finite projective complex. For this first note that $\mathbf{N} = \bigoplus \mathbf{N}_K$ is a direct sum of chain complexes $\mathbf{N}_K$ where $\mathbf{N}_K$ is the restriction of $E_K\mathbf{P}$ to subgroups $H \in \mathcal{F}_{\leq k}$. So it is enough to show that $\mathbf{N}_K$ is weakly equivalent to a finite projective chain complex. To prove this, we will show that for each i, the $R\Gamma_G$-module $\mathbf{N}_i := (\mathbf{N}_K)_i$ has a finite projective resolution. The module $\mathbf{N}_i$ is nonzero only at subgroups $H \in \mathcal{F}_{\leq k}$ and at each such a subgroup, we have

$$\mathbf{N}_i(H) = (E_K\mathbf{P}_i)(H) = \mathbf{P}_i \otimes_{RW_G(K)} R[(G/K)^H].$$

So, as an $RW_G(H)$-module $\mathbf{N}_i(H)$ is a direct summand of $R[(G/K)^H]$ which is isomorphic to

$$\bigoplus \{R\big[W_G(H)/W_{{}^gK}(H)\big] : K\text{-conjugacy classes of subgroups } H^g \leq K\}$$

as an $RW_G(H)$-module. Since K is a p-group, these modules are projective over the ground ring R because R is q-local. So, for each $H \in \mathcal{F}_{\leq k}$, the $RW_G(H)$-module $\mathbf{N}_i(H)$ is projective. Now consider the map

$$\pi\colon \ \oplus_H E_H\mathbf{N}_i(H) \to \mathbf{N}_i$$

induced by maps adjoint to the identity maps at each H. We can take $\oplus_H E_H\mathbf{N}_i(H)$ as the first projective module of the resolution, and consider the kernel $\mathbf{Z}_0$ of $\pi\colon \oplus_H E_H\mathbf{N}_i(H) \to \mathbf{N}_i$. Note that $\mathbf{Z}_0$ has smaller length and it also have the property that at each L, the $W_G(L)$ modules $\mathbf{Z}_0(L)$ are projective. This follows from the fact that $R[(G/H)^L]$ is projective as a $W_G(L)$-module by the same argument we used above. Continuing this way, we can find a finite projective resolution for $\mathbf{N}_i$ of length $\leq l(\Gamma)$. □

Now it remains to show that there is a chain map $f\colon \mathbf{P}' \to \mathbf{C}$ such that the induced map on zeroth homology $f_*\colon H_0(\mathbf{P}') \cong H_0(\mathbf{N}) \to H_0(\mathbf{C})$ is given by the sum of augmentation maps ε_K over the conjugacy classes of maximal subgroups K in $\mathcal{F}_{k+1}$. Then the complex $\mathbf{D}$ will be defined as the push-out complex that fits into the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{P}' & \longrightarrow & \mathbf{P}'' & \longrightarrow & \mathbf{P}''' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathbf{C} & \longrightarrow & \mathbf{D} & \longrightarrow & \mathbf{P}''' & \longrightarrow & 0 \ . \end{array}$$

Since both $\mathbf{C}$ and $\mathbf{P}'''$ are finite projective chain complexes, $\mathbf{D}$ will also be a finite projective complex. The fact that $\mathbf{D}$ has the right homology is explained above in the construction process.

So it remains to show that that there is a chain map $f : \mathbf{P}' \to \mathbf{C}$ which induces the augmentation map $\sum_K e_K$ on zeroth homology. To construct $f : \mathbf{P}' \to \mathbf{C}$, first note that the chain complex $\mathbf{C}$ has no homology below dimension n_k. By assumption on the gaps between nonzero homology dimensions, we can assume that $n_k \geq n_{k+1} + l(\Gamma_G) \geq l(\mathbf{P}')$. So, starting with the sum of augmentation maps $\sum_K \varepsilon_K$ at dimensions zero, we can obtain a chain map as follows:

$$\begin{array}{ccccccccccc}
\longrightarrow & 0 & \longrightarrow & P'_m & \xrightarrow{\partial_m^{P'}} & \cdots & \longrightarrow & P'_0 & \longrightarrow & H_0(\mathbf{N}) & \longrightarrow 0 \\
 & & & \downarrow{\scriptstyle f_m} & & & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \sum_K \varepsilon_K} & \\
\longrightarrow & C_{m+1} & \longrightarrow & C_m & \xrightarrow{\partial_m^{C}} & \cdots & \longrightarrow & C_0 & \longrightarrow & H_0(\mathbf{C}) & \longrightarrow 0
\end{array}$$

where $m = l(\mathbf{P}')$. This completes the proof of Theorem 4.1.

5. The Proof of Theorem A

In this section we establish our main technique for constructing actions on homotopy spheres, based on a given $\mathcal{P}$-representation, where $\mathcal{P} = \bigcup\{\mathcal{F}_p \mid p \in \mathcal{S}_G\}$ denote the family of all p -subgroups of G, for the primes p in a given set $\mathcal{S}_G$ (see Definitions 3.1 and 3.4). Theorem A stated in the introduction will follow from this theorem almost immediately once we use the family of p-effective characters constructed by M. A. Jackson [13]. The main technical theorem is the following:

Theorem 5.1. *Let G be a finite group and let $\mathcal{S}_G = \{p \mid \operatorname{rank}_p G \geq 2\}$. Suppose that*

(i) $\mathbf{V}(\mathcal{F}_p)$ *is a $\mathcal{F}_p$-representation for G, with* $\operatorname{Iso}(\mathbf{V}(\mathcal{F}_p)) = \mathcal{J}_p$*, for each $p \in \mathcal{S}_G$;*

(ii) *If $p \in \mathcal{S}_G$ and $1 \neq H \in \mathcal{J}_p$, then we have* $\operatorname{rank}_q(N_G(H)/H) \leq 1$ *for every $q \neq p$.*

Then there exists a finite G-CW-complex $X \simeq S^n$, with isotropy in $\mathcal{J} = \bigcup\{\mathcal{J}_p \mid p \in \mathcal{S}_G\}$, which is a geometric homotopy representation for G.

Remark 5.2. The construction we give in the proof of Theorem 5.1 gives a simply-connected homotopy representation X, with $\dim X^H \geq 3$, for all $H \in \mathcal{J}$, whenever $X^H \neq \emptyset$. It also relates the dimension function of X to the linear dimension functions $\operatorname{Dim} S(V_H)$, for $V_H \in \bigcup\{\mathbf{V}(\mathcal{F}_p) \mid p \in \mathcal{S}_G\}$ in the following way: for every prime $p \in \mathcal{S}_G$, there exists an integer $k_p > 0$ such that for every $H \in \mathcal{F}_p$, the equality $\dim X^H = \dim S(V_H^{\oplus k_p})^H$ holds.

As we discussed in the previous section, the condition on the q-rank of $N_G(H)/H$ is a necessary condition for the existence of such actions. Recall that this condition is used in an essential way in the proof of Theorem 4.1.

The proof of Theorem 5.1. By the realization theorem (Theorem 2.6), we only need to construct a finite free chain complex of $\mathbf{Z}\Gamma_G$-modules satisfying the conditions (i), (ii) and (iii) of Definition 2.3. If we apply Theorem 4.1 to the preliminary local model constructed in Section 3, we obtain a finite projective complex $\mathbf{C}^{(p)}$, over the orbit category $\mathbf{Z}_{(p)}\Gamma_G$ with respect to the family $\mathcal{J}$, for each prime p dividing the order of G. In addition, $\mathbf{C}^{(p)}$ is an oriented $\mathbf{Z}_{(p)}$-homology $\underline{n}$-sphere, with the same dimension function $\underline{n} = \operatorname{hDim} \mathbf{C}^{(p)}(0)$

coming from the preliminary local models. By construction, the complex $\mathbf{C}^{(p)}$ satisfies the conditions (i), (ii) and (iii) of Definition 2.3 for $R = \mathbf{Z}_{(p)}$.

We may also assume that $\underline{n}(H) \geq 3$ for every $H \in \mathcal{J}$, and that the gaps between non-zero homology dimensions have the following property: for all $K, L \in \mathcal{J}$ with $\underline{n}(K) > \underline{n}(L)$, we have $\underline{n}(K) - \underline{n}(L) \geq l(\varGamma_G)$ where $l(\varGamma_G)$ denotes the length of the longest chain of maps in the category $\varGamma_G$.

To complete the proof of Theorem 5.1, we first need to glue these complexes $\mathbf{C}^{(p)}$ together to obtain an algebraic $\underline{n}$-sphere over $\mathbf{Z}\varGamma_G$. By [8, Theorem 6.7], there exists a finite projective chain complex $\mathbf{C}$ of $\mathbf{Z}\varGamma_G$-modules, which is a $\mathbf{Z}$-homology $\underline{n}$-sphere, such that $\mathbf{Z}_{(p)} \otimes \mathbf{C}$ is chain homotopy equivalent to the local model $\mathbf{C}^{(p)}$, for each prime p dividing the order of G. The complex $\mathbf{C}$ has a (possibly non-zero) finiteness obstruction (see Lueck [14, §10-11]), but this can be eliminated by joins (see [8, §7]).

After applying [8, Theorem 7.6], we may assume that $\mathbf{C}$ is a finite free chain complex of $\mathbf{Z}\varGamma_G$-modules which is a $\mathbf{Z}$-homology $\underline{n}$-sphere. Moreover, $\mathbf{C}$ is an algebraic homotopy representation: it satisfies the conditions (i), (ii) and (iii) of Definition 2.3 for $R = \mathbf{Z}$, since these conditions hold locally at each prime.

We have now established all the requirements for Theorem 2.6. For the family $\mathcal{F}$ used in its statement, we use $\mathcal{F} = \mathcal{J}$. For all $H \in \mathcal{F}$, we have the condition $\underline{n}(H) \geq 3$. Now Theorem 2.6 gives a finite G-CW-complex $X \simeq S^n$ with isotropy $\mathcal{J}$ such that X^H is an homotopy sphere for every $H \in \mathcal{J}$. □

Now we are ready to prove Theorem A.

The proof of Theorem A. Let G be a rank 2 finite group and let $\mathcal{S}_G$ denote the set of primes with $\operatorname{rank}_p G = 2$. Since it is assumed that G does not p'-involve $\operatorname{Qd}(p)$ for any odd prime p, we can apply [13, Theorem 47] and obtain a p-effective representation V_p, for every prime $p \in \mathcal{S}_G$. We apply Theorem 5.1 to the $\mathcal{F}_p$-representations $\mathbf{V}(\mathcal{F}_p)$ given by this collection $\{V_p\}$ (see Example 3.3). Since V_p is p-effective means that all isotropy subgroups in $\mathcal{H}_p$ are rank one p-subgroups (see Example 3.10), the isotropy is contained in the family $\mathcal{H}$ of rank one p-subgroups of G, for all $p \in \mathcal{S}_G$. We therefore obtain a finite G-CW-complex $X \simeq S^n$, with rank 1 isotropy in $\mathcal{H}$, such that X^H is an homotopy sphere (possibly empty) for every $H \in \mathcal{H}$. □

The proof of Corollary B follows easily from Theorem A since if $\operatorname{rank}_q(G) \leq 1$, then for every p-group H, we must have $\operatorname{rank}_q(N_G(H)/H) \leq 1$. So we can apply Theorem A to obtain Corollary B.

Note that the condition about $\operatorname{Qd}(p)$ being not p'-involved in G is a necessary condition for the existence of actions of rank 2 groups on finite CW-complexes $X \simeq S^n$ with rank one isotropy.

Proposition 5.3. *Let p be an odd prime. If G acts with rank one isotropy on a finite dimensional complex X with the mod-p homology of a sphere, then G cannot p'-involve* $\operatorname{Qd}(p)$.

Proof. Suppose that G has a normal p'-subgroup K such that $\operatorname{Qd}(p)$ is isomorphic to a subgroup in $N_G(K)/K$. Let L be subgroup of G such that $K \triangleleft L \leq N_G(K)$ and

$L/K \cong \mathrm{Qd}(p)$. The quotient group $Q = L/K$ acts on the orbit space $Y = X/K$ via the action defined by $(gK)(Kx) = Kgx$ for every $g \in L$ and $x \in X$.

We observe two things about this action. First, by a transfer argument [4, Theorem 2.4, p. 120], the space Y has the mod p homology of a sphere. Second, all the isotropy subgroups of the Q-action on Y have p-rank ≤ 1. To see this, let Q_y denote the isotropy subgroup at $y \in Y$ and let $x \in X$ be such that $y = Kx$. It is easy to see that $Q_y = L_xK/K \cong L_x/(L_x \cap K)$. Since K is a p'-group, this shows that p-subgroups of Q_y are isomorphic to p-subgroups of the isotropy subgroup L_x. Since L acts on X with rank one isotropy, we conclude that $\mathrm{rank}_p(Q_y) \leq 1$ for every $y \in Y$.

Now the rest of the proof follows from the argument given in Ünlü [21, Theorem 3.3]. Let P be a p-Sylow subgroup of $Q \cong \mathrm{Qd}(p)$. Then P is an extra-special p-group of order p^3 with exponent p (since p is odd). Let c denote a central element and a a non-central element in P. Since the P-action on Y has rank 1 isotropy subgroups, we have $Y^E = \emptyset$ for every rank two p-subgroup $E \leq P$. Therefore $Y^{\langle c\rangle} = \emptyset$ by Smith theory, since otherwise $P/\langle c\rangle \cong \mathbf{Z}/p \times \mathbf{Z}/p$ would act freely on $Y^{\langle c\rangle}$ which is a mod p homology sphere. Now consider the subgroup $E = \langle a, c\rangle$. Since $\langle a\rangle$ and $\langle c\rangle$ are conjugate in Q, all cyclic subgroups of E are conjugate. In particular, we have $Y^H = \emptyset$ for every cyclic subgroup H in E. This is a contradiction, since E cannot act freely on Y. □

Remark 5.4. A shorter proof can be given using more group theory. For a finite group L, and a normal p'-subgroup K of L, there is an isomorphism[2] between the p-fusion systems $\mathcal{F}_L(S)$ and $\mathcal{F}_{L/K}(SK/K)$, where S is a p-Sylow subgroup of L. So if $L/K \cong \mathrm{Qd}(p)$, then L has an extra-special p-group P of order p^3 with exponent p such that a central element $c \in P$ is conjugate to a non-central element $a \in P$. This leads to a contradiction in the same way as above.

6. Discussion and examples

We first discuss the rank conditions in the statement of Theorem A. Suppose that X is a finite G-CW-complex. Recall that $\mathrm{Iso}(X) = \{H \mid H \leq G_x \text{ for some } x \in X\}$ denotes the minimal family containing all the isotropy subgroups of the G-action on X. We call this the *isotropy family*. Note that $H \in \mathrm{Iso}(X)$ if and only if $X^H \neq \emptyset$. We say that X has *rank k isotropy* if $\mathrm{rank}\, G_x \leq k$ for all $x \in X$ and there exists a subgroup H with $\mathrm{rank}\, H = k$ and $X^H \neq \emptyset$. Let $\mathcal{P}$ denote the family of all prime-power order subgroups of G.

Lemma 6.1. *Let G be a finite group, and let X be a finite G-CW-complex with $X \simeq S^n$.*

(i) *If H is a maximal p-subgroup in $\mathrm{Iso}(X)$, then $\mathrm{rank}_p(N_G(H)/H) \leq 1$.*
(ii) *If $1 \neq H \in \mathrm{Iso}(X) \subseteq \mathcal{P}$ is a p-subgroup, and X^H is an integral homology sphere, then $\mathrm{rank}_q(N_G(H)/H) \leq 1$, for all primes $q \neq p$.*

Proof. This follows from two basic results of P. A. Smith theory [4, III.8.1]), which state (i) that the fixed set of a p-group action on a finite-dimensional mod p homology sphere

[2]We thank Radha Kessar for this information.

is again a mod p homology sphere (or the empty set), and (ii) that $\mathbf{Z}/p \times \mathbf{Z}/p$ can not act freely on a finite G-CW-complex X with the mod p homology of a sphere.

For any prime p dividing the order of G, let $H \in \operatorname{Iso}(X)$ denote a maximal p-subgroup with $X^H \neq \emptyset$. For any $x \in X^H$, we have $H \leq G_x$ and if $g \cdot x = x$, for some $g \in N_G(H)$ of p-power order, it follows that the subgroup $\langle H, g\rangle \leq G_x$. Since H was a maximal p-subgroup in $\operatorname{Iso}(X)$, we conclude that $g \in H$. Therefore $N_G(H)/H$ acts freely on the fixed set X^H, which is a mod p homology sphere, and hence $\operatorname{rank}_p(N_G(H)/H) \leq 1$.

If $q \neq p$ and H is a p-subgroup in $\operatorname{Iso}(X) \subseteq \mathcal{P}$, then any q-subgroup Q of $N_G(H)/H$ must act freely on X^H (since $x \in X^H$ implies G_x is a p-group). Since X^H is assumed to be an integral homology sphere, Smith theory implies that $\operatorname{rank}_q(Q) \leq 1$. □

Example 6.2. If G is the extra-special p-group of order p^3, then the centre $Z(G) = \mathbf{Z}/p$ can not be a maximal isotropy subgroup in $\operatorname{Iso}(X)$. On the other hand, we know that G acts on a finite complex $X \simeq S^n$ with rank one isotropy: just take the linear sphere $S(\operatorname{Ind}_{Z(G)}^G W)$ for some nontrivial one-dimensional representation W of $Z(G)$. So we can not require that G acts on $X \simeq S^n$ with $\operatorname{Iso}(X)$ containing all rank one subgroups.

For any prime p, we can restrict the G-action on X to a p-subgroup of maximal rank. This gives the following well-known conclusion.

Corollary 6.3. *If X is a finite G-CW-complex with $X \simeq S^n$ and rank k isotropy, then* $\operatorname{rank}_p G \leq k+1$*, for all primes p.*

These results help to explain the rank conditions in Theorem A. If we have rank one isotropy, then we must assume that G has rank two. However, condition (ii) on the q-rank of the normalizer quotient is only necessary for p-subgroups H, with $q \neq p$, for which $X^H \neq \emptyset$ is an integral homology sphere. In order to get a complete list of necessary conditions, we must have more precise control of the structure of the isotropy subgroups.

Now we discuss two applications of Theorem A and Theorem 5.1.

Example 6.4. *The alternating group $G = A_6$ admits a finite G-CW-complex $X \simeq S^n$, with rank one isotropy.* This follows from Theorem A once we verify that G satisfies the necessary conditions. Note that A_6 has order $2^3 \cdot 3^2 \cdot 5 = 360$ so it automatically satisfies the condition about $\operatorname{Qd}(p)$, since it can not include an extra-special p-group of order p^3 for an odd prime p. For the q-rank condition, note that $\mathcal{S}_G = \{2, 3\}$, so we need to check this condition only for primes $p = 2$ and 3. Here are some easily verified facts:

- A 2-Sylow subgroup $P \leq G$ is isomorphic to the dihedral group D_8, so all rank one 2-subgroups are cyclic, and $\mathcal{H}_2 = \{1, C_2, C_4\}$.
- $N_G(C_2) = P$, and $\operatorname{rank}_3(N_G(C_2)/C_2) = 0$.
- $N_G(C_4) = P$ and $\operatorname{rank}_3(N_G(C_4)/C_4) = 0$.

Now, let Q be a 3-Sylow subgroup in G. Then $Q \cong C_3 \times C_3$.

- Any subgroup of order 3 in G is conjugate to $C_3^A = \langle(123)\rangle$ or $C_3^B = \langle(123)(456)\rangle$.
- $|N_G(C_3^A)/C_3^A| = 6$ and $\operatorname{rank}_2(N_G(C_3^A)/C_3^A) = 1$.
- $|N_G(C_3^B)/C_3^B| = 6$ and $\operatorname{rank}_2(N_G(C_3^B)/C_3^B) = 1$

We conclude that condition (ii) of Theorem A holds for this group.

Remark 6.5. Note that by the criteria given in [1, Lemma 5.2], the group A_6 does not have a character which is effective on elementary abelian 2-subgroups. On the other hand, the triple cover of A_6 is a subgroup of $SU(3)$, so it acts linearly on a sphere with rank one isotropy.

We now give an example which does not satisfy the q-rank conditions in Theorem A, but where we can apply Theorem 5.1 directly.

Example 6.6. *The alternating group $G = A_7$ admits a finite G-CW-complex $X \simeq S^n$, with rank one isotropy.* The order of G is $2^3 \cdot 3^2 \cdot 5 \cdot 7$, so this group also automatically satisfies the Qd(p) condition. Here is a summary of the main structural facts:

- The 3-Sylow subgroup $Q \leq G$ is isomorphic to $C_3 \times C_3$.
- Any subgroup of order 3 in G is conjugate to $C_3^A = \langle (123) \rangle$ or $C_3^B = \langle (123)(456) \rangle$.
- The 2-Sylow subgroup of $N_G(C_3^A)$ is isomorphic to D_8.
- $|N_G(C_3^A)/C_3^A| = 24$ and $\mathrm{rank}_2(N_G(C_3^A)/C_3^A) = 2$.
- $N_G(C_3^B) \cong (C_3 \times C_3) \rtimes C_2$ and $\mathrm{rank}_2(N_G(C_3^B)/C_3^B) = 1$.
- $|N_G(C_2)| = 24$, and $\mathrm{rank}_3(N_G(C_2)/C_2) = 1$.
- $N_G(C_4) \cong D_8$ and $\mathrm{rank}_3(N_G(C_4)/C_4) = 0$

We see that the rank condition in Theorem A fails for 3-subgroups, since there is a cyclic 3-subgroup $H = C_3^A$ with $\mathrm{rank}_2(N_G(H)/H) = 2$. On the other hand, by applying Theorem 5.1 directly, we can still find a finite G-CW-complex $X \simeq S^n$, with rank one isotropy in the family generated by $\{1, C_2, C_4, C_3^B\}$.

In this case, we have $\mathcal{S}_G = \{2, 3\}$. For $p = 2$, we can use the $\mathcal{F}_2$-representation V_2 from [13], since A_7 satisfies the rank condition for 2-subgroups. It remains to show that there exists an $\mathcal{F}_3$-representation of G with isotropy subgroups only type B cyclic 3-subgroups. But this is easily constructed by taking V_3 as the direct sum of augmented permutation modules $I(Q/K_1) \oplus I(Q/K_2)$ where $K_1 = \langle (123)(456) \rangle$ and $K_2 = \langle (123)(465) \rangle$. It is clear that this representation respects fusion, and has isotropy given only by the cyclic 3-subgroups of type B lying in $\mathcal{F}_3$.

Remark 6.7. When G is a finite group with a rank two elementary abelian q-Sylow subgroup Q, the representation

$$V_q = \bigoplus \{ I(Q/K_i) : 1 \leq i \leq s \}$$

over some family of rank 1 subgroups K_i, which is closed under G-conjugacy, will give a q-effective representation which respects fusion. But for more general q-Sylow subgroups, the above representation may fail to be q-effective. Note that for V_q to be q-effective one needs to have exactly one double coset in $E \backslash Q / K_i$ for every K_i and for every rank 2 elementary abelian subgroup E of Q. This fails, for example, if $Q = C_{p^2} \times C_{p^2}$ is a non-elementary rank two abelian group.

7. The proof of Theorem C

The finite simple groups of rank two are listed in Adem-Smith [1, p.423] as follows:

$$PSL_2(\mathbf{F}_q), q \geq 5;\ PSL_2(\mathbf{F}_{q^2}), q \text{ odd } ;\ PSL_3(\mathbf{F}_q), q \text{ odd };$$
$$PSU_3(\mathbf{F}_q), q \text{ odd } ;\ PSU_3(\mathbf{F}_4);\ A_7 \text{ and } M_{11}$$

where q denotes a prime. Extensive information about the maximal subgroups of these simple groups is provided in [15], [7]. To prove Theorem C we will consider separate cases. Note that $G = A_7$ is done in Example 6.6.

Case 1: $G = PSL_2(\mathbf{F}_q), q \geq 5$. The order of G is $q(q^2-1)/2$ and the maximal subgroups of G are listed in [7, 6.5.1]. From this list it is easy to see that the 2-Sylow subgroup of G is a dihedral group and for odd primes the Sylow subgroups are cyclic (see also [7, 4.10.5]). It follows that $\mathcal{S}_G = \{2\}$ and G is Qd(p)-free at odd primes, so Corollary B applies. By inspecting the character table of G, and applying the criterion [1, Lemma 5.2], we see that $PSL_2(\mathbf{F}_q)$, $q > 7$, does not admit an orthogonal representation V with rank one isotropy on $S(V)$.

Case 2: $G = PSL_2(\mathbf{F}_{q^2}), q \geq 3$. We did $PSL_2(\mathbf{F}_9) = A_6$ explicitly in Example 6.4. In general, the order of G is $q^2(q^4-1)/2$ and the maximal subgroups are again listed in [7, 6.5.1]. The conditions on the normalizer quotients needed for Theorem A can be checked at the primes $\mathcal{S}_G = \{2, q\}$ using the information in [7], and [10, Chap. II]. The 2-Sylow subgroups are dihedral [7, 4.10.5], and the q-Sylow subgroup Q is elementary abelian of rank two [7, 6.5.1] (with normalizer $N_G(Q)$ represented by the parabolic subgroup of upper triangular matrices). At the other primes $p \neq 2, q$, any p-Sylow subgroup is contained in a dihedral group, and hence cyclic (see [10, II.8.27]).

Case 3: $PSL_3(\mathbf{F}_q), q \geq 3$. We refer to [15, §15] or [7, 6.5.3] for the maximal subgroups. Since G contains Qd(p) for $p = q$, this series of groups is ruled out. An explicit embedding is given by the matrices:

$$\mathrm{Qd}(p) = \left\{ \begin{bmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{bmatrix} \ :\ ad - bc = 1 \right\}$$

with entries in $\mathbf{F}_q$.

Case 4: $G = PSU_3(\mathbf{F}_q), q \geq 3$. The order of G is $(q^3+1)q^3(q^2-1)/d$, where $d = (3, q+1)$, and the maximal subgroups are given in [15, §16] or [7, 6.5.3]. In particular, G contains an abelian subgroup of order $(q+1)^2/d$. If $9 \mid (q+1)$, then G contains Qd(3), hence is ruled out, so we assume that $9 \nmid (q+1)$.

If $3 \mid (q+1)$, then the 3-Sylow subgroup of G is elementary abelian of order 9. If $r > 3$ is an odd prime dividing $q+1$, then the r-Sylow subgroup is abelian of rank two, and order equal to the r-primary part of $(q+1)^2$. Finally, if r is an odd prime not dividing $q+1$, then r divides (q^2-q+1) and the r-Sylow subgroup of G is cyclic (see [7, 6.5.3(c)] and [15, p. 241 (4)]). In summary, $\mathcal{S}_G = \{2, q\} \cup \{r \mid (q+1) :\ r \text{ an odd prime}\}$.

We claim that Theorem A applies to G if and only if q is a Mersenne prime, meaning that $q+1 = 2^a$, for some $a \geq 2$. The discussion in the last paragraph shows that our normalizer rank conditions fail at $p = 2$ for rank one r-subgroups whenever $r \mid (q+1)$, since $2 \mid (q+1)^2/d$. Therefore, in these cases our Theorem A does not apply. Conversely, if $q+1 = 2^a$, then $\mathcal{S}_G = \{2, q\}$ and we must check our normalizer rank condition at these primes.

The 2-Sylow subgroups P are either quasi-dihedral, if $q \equiv 1 \pmod 4$, or wreathed, if $q \equiv -1 \pmod 4$ (a good reference for the facts we need is [2, Chap. I]). In either case, all involutions $x \in G$ are conjugate, and $N = C_G(x)$, modulo a central cyclic subgroup of odd order $d = (q+1, 3)$, is isomorphic to $GU(2, q)$ (see [2, Proposition 4, p. 21]). Therefore, for any rank one 2-subgroup H, we have $N_G(H) \subseteq N$. Since $SU(2, q) \cong SL(2, q)$ [7, p. 69], we see that $\operatorname{rank}_q(N_G(H)/H) = 1$ for any rank one 2-subgroup H.

A q-Sylow subgroup Q of G is contained in a maximal subgroup K of order $q^3(q^2-1)/d$. By [15, p. 241(1)], any such group leaves invariant a line in the projective space $P^2(V)$, where V is a 3-dimensional vector space over the field $\mathbf{F}_{q^2}$. On the invariant line, K acts effectively as a metacyclic group of order $q^2(q^2-1)/d$. It follows that $\operatorname{rank}_2(N_G(H)/H) = 1$ for any rank one q-subgroup of G. We note that this property can also be checked from the explicit matrix description of Q and a cyclic subgroup normalizing Q given by O'Nan [16, §1]. Since $PSU_3(\mathbf{F}_q)$ contains A_6 for primes of the form $q = 15f-1, 15f-4$ (see [15, p. 241 (10)]), there is an infinite sub-family of these unitary groups which do not admit representation spheres with rank one isotropy. However, at present we can only construct finite G-CW complexes for the Mersenne primes q, and it is not known whether infinitely many such primes exist.

Case 5: $G = M_{11}$**.** The order of G is $7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ and $\mathcal{S}_G = \{2, 3\}$. This group is Qd(p)-free, but the 2-rank $\operatorname{rank}_2(N_G(H)/H) = 2$, for H a subgroup of order three (see [7, p. 262]). Since all the subgroups of order three are conjugate, neither Theorem A or the method used in Example 6.6 applies, so this case is open. Note that the normalizer rank conditions is satisfied for H a rank one 2-subgroup by [2, Proposition 4, p. 21], since there is only one conjugacy class of involutions in M_{11}. The character table again shows that M_{11} does not admit a representation sphere with rank one isotropy.

Case 6: $G = PSU_3(\mathbf{F}_4)$**.** The order of G is $65280 = 2^6 \cdot 3 \cdot 5^2 \cdot 13$ and $\mathcal{S}_G = \{2, 5\}$. This group is Qd(p)-free and Theorem A applies. However, this group also acts linearly on $\mathbb{S}^{23}$ with rank one isotropy [1, p. 425].

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Department of Mathematics, McMaster University, Hamilton, Ontario L8S 4K1, Canada
E-mail address: hambleton@mcmaster.ca

Department of Mathematics, Bilkent University, 06800 Bilkent, Ankara, Turkey
E-mail address: yalcine@fen.bilkent.edu.tr