ON CERTAIN 5-MANIFOLDS WITH FUNDAMENTAL GROUP OF ORDER 2

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Abstract. In this paper, an explicit classification result for certain 5-manifolds with fundamental group $\mathbb{Z}/2$ is obtained. These manifolds include total spaces of circle bundles over simply-connected 4-manifolds.

1. Introduction

The classification of manifolds with certain properties is a central topic of topology, and in dimensions ≥ 5 methods from handlebody theory and surgery have been successfully applied to a number of cases. One of the first examples was the complete classification of simply-connected 5-manifolds by Smale [21] and Barden [1] in 1960's. This result has been very useful for studying the existence of other geometric structures on 5-manifolds, such as the existence of Riemannian metrics with given curvature properties. We consider this as a model and motivation for studying the classification of non-simply connected 5-manifolds.

An orientable 5-manifold M is said to be of *fibered type* if $\pi_2(M)$ is a trivial $\mathbb{Z}[\pi_1(M)]$-module. In this paper, we will be concerned with closed, orientable fibered type 5-manifolds M^5 with $\pi_1(M) \cong \mathbb{Z}/2$, and torsion-free $H_2(M;\mathbb{Z})$. The classification of these manifolds in the smooth (or PL) and topological categories is given in Section 3. We give a simple set of invariants, namely the rank of $H_2(M;\mathbb{Z})$ and the $\mathrm{Pin}^\dagger$-bordism ($\mathrm{TopPin}^\dagger$-bordism) class of a characteristic submanifold, which determine the diffeomorphism (homeomorphism) types. Here is the main result in the smooth case.

Theorem 3.1. *Two smooth, closed, orientable fibered type 5-manifolds M and M', with fundamental group $\mathbb{Z}/2$ and torsion-free second homology group, are diffeomorphic if and only if they have the same w_2-type,* $\operatorname{rank} H_2(M) = \operatorname{rank} H_2(M')$, *and* $[P] = [P'] \in \Omega_4^{\mathrm{Pin}^\dagger}/\pm$, *where P and P' are characteristic submanifolds and $\dagger = c, -, +$ for w_2-types* I, II, III *respectively.*

Here $\Omega_4^{\mathrm{Pin}^\dagger}/\pm$ denotes a quotient of the Pin-bordism group by a certain subgroup of order two (see Definition 3.5). The Pin-bordism variants and the w_2-type notation are explained in Section 2.

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The homeomorphism classification is given in Theorem 3.4. We also determine all the relation among these invariants (Theorem 3.6), and give a list of standard forms for these manifolds (Theorem 3.7, Theorem 3.11).

One motivation for this classification problem comes from the study of circle bundles M^5 over simply-connected 4-manifolds, since their total spaces are of fibered type. Duan-Liang [5] gave an explicit geometric description of M^5 for simply-connected total spaces, making essential use of the results of Smale and Barden. As an application of our results, in Section 6 we give an explicit geometric description when the total spaces have fundamental group $\mathbb{Z}/2$.

Theorem 6.5 (type II)**.** *Let X be a closed, simply-connected, topological spin 4-manifold, $\xi\colon S^1 \hookrightarrow M^5 \to X$ be a circle bundle over X with $c_1(\xi) = 2\cdot$(primitive). Then we have*

(1) *if $KS(X) = 0$, then M is smoothable and M is diffeomorphic to*

$$(S^2 \times \mathbb{R}\mathrm{P}^3) \,\sharp_{S^1} ((\sharp_k\, S^2 \times S^2) \times S^1);$$

(2) *if $KS(X) = 1$, then M is non-smoothable and M is homeomorphic to*

$$*(S^2 \times \mathbb{R}\mathrm{P}^3) \,\sharp_{S^1} ((\sharp_k\, S^2 \times S^2) \times S^1).$$

Where $k = \operatorname{rank} H_2(X)/2 - 1$.

In the statement, $*(S^2 \times \mathbb{R}\mathrm{P}^3)$ denotes a non-smoothable manifold homotopy equivalent to $S^2 \times \mathbb{R}\mathrm{P}^3$. The corresponding results for the other w_2-types are given in Theorem 6.7 and Theorem 6.8.

Classification results can also be useful in studying the existence problem for geometric structures on fibered type 5-manifolds. For example, a closed, orientable 5-manifold with $\pi_1 = \mathbb{Z}/2$, such that w_2 vanishes on homology, admits a contact structure by the work of Geiges and Thomas [7]. They showed that all such manifolds can be obtained by surgery on 2-dimensional links from exactly one of ten model manifolds.

The topology of such manifolds of fibered type are described explicitly for the first time by our results, and we note that all the manifolds listed in Theorem 3.7 satisfy the necessary condition $W_3 = 0$ for the existence of contact structures. Our results have already been used by Geiges and Stipsicz [8] to prove new existence theorems for contact structures on 5-manifolds. It may be possible to obtain similar information for fibered type 5-manifolds which admit Sasakian or Einstein metrics by using the work of Boyer and Galicki [2].

The surgery exact sequence of Wall [24] provides a way to classify manifolds within a given (simple) homotopy type. However, in the application to concrete problems, one often faces homotopy theoretical difficulties. In our situation, the setting of the problems is appropriate for the application of the modified surgery methods developed by Kreck [14]. The proofs in Section 4 and Section 5 are based on this theory.

In dimension 5, the smooth category and the PL category are equivalent. By convention, M stands for either a smooth or a topological manifold when not specified.

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2. Preliminaries

§2A. $\mathrm{Pin}^{\dagger}$**-structures on vector bundles.** Recall that the groups $\mathrm{Pin}^{\pm}(n)$ are central extensions of $O(n)$ by $\mathbb{Z}/2$

$$1 \to \mathbb{Z}/2 \to \mathrm{Pin}^{\pm}(n) \to O(n) \to 1,$$

and $\mathrm{Pin}^{c}(n)$ is a central extension of $O(n)$ by $U(1)$

$$1 \to U(1) \to \mathrm{Pin}^{c}(n) \to O(n) \to 1$$

(see [11, §1] and [9, §2]). Let $\dagger \in \{c, +, -\}$. After stabilization we have classifying spaces $B\,\mathrm{Pin}^{\dagger}$ and fibrations $B\,\mathrm{Pin}^{\dagger} \to BO$. A $\mathrm{Pin}^{\dagger}$-structure on a stable vector bundle ξ over a space X is a fiber homotopy class of lifts of a classifying map $c_\xi\colon X \to BO$ to $B\,\mathrm{Pin}^{\dagger}$.

Lemma 2.1. [9, Lemma 1]

(1) *A vector bundle ξ over X admits a $\mathrm{Pin}^{\dagger}$-structure if and only if*

$$\begin{array}{ll} \beta(w_2(\xi)) = 0 & \text{for } \ \dagger = c, \\ w_2(\xi) = 0 & \text{for } \ \dagger = +, \\ w_2(\xi) = w_1(\xi)^2 & \text{for } \ \dagger = -, \end{array}$$

where $\beta\colon H^2(X;\mathbb{Z}/2) \to H^3(X;\mathbb{Z})$ is the Bockstein operator induced from the exact coefficient sequence $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2$.

(2) $\mathrm{Pin}^{\pm}$*-structures are in bijection with $H^1(X;\mathbb{Z}/2)$ and* Pin^{c}*-structures are in bijection with $H^2(X;\mathbb{Z})$.*

$\mathrm{Pin}^{\pm}$-structures on a vector bundle ξ over X are related to Spin-structures on an associated vector bundle:

Lemma 2.2. [11, Lemma 1.7] *Let $\mathcal{S}pin(\xi)$ denote the set of equivalence classes of Spin structures on ξ, and $\mathcal{P}in^{\pm}(\xi)$ denote the set of equivalence classes of* $\mathrm{Pin}^{\pm}$*-structures on ξ. There are bijections*

$$\mathcal{P}in^{-}(\xi) \to \mathcal{S}pin(\xi \oplus \det \xi)$$

$$\mathcal{P}in^{+}(\xi) \to \mathcal{S}pin(\xi \oplus 3 \det \xi).$$

which are natural under the actions of $H^1(X;\mathbb{Z}/2)$.

It is well known that a Spin^c-structure on a vector bundle ξ is the same as a Spin-structure on $\xi \oplus \gamma$, where γ is a complex line bundle with $c_1(\gamma) \equiv w_2(\xi) \pmod 2$ (see [16, Cor. D.4]). Similarly, a Pin^c-structure on a vector bundle ξ may be viewed as a Pin^--structure on $\xi \oplus \gamma$, where γ is a complex line bundle with $c_1(\gamma) \equiv w_1(\xi)^2 + w_2(\xi) \pmod 2$.

§2B. **w_2-types and characteristic submanifolds.** Let M be a closed, orientable 5-manifold with $\pi_1(M) \cong \mathbb{Z}/2$ and universal cover $\widetilde{M}$. The manifold M is said to be of *w_2-type* I if $w_2(\widetilde{M}) \neq 0$, of *$w_2$-type* II if $w_2(M) = 0$, and of *w_2-type* III if $w_2(M) \neq 0$ and $w_2(\widetilde{M}) = 0$. . By the universal coefficient theorem, there is an exact sequence

$$0 \to \mathrm{Ext}(H_1(M), \mathbb{Z}/2) \to H^2(M; \mathbb{Z}/2) \to \mathrm{Hom}(H_2(M), \mathbb{Z}/2) \to 0.$$

Lemma 2.3. *M is of type type* III $\Leftrightarrow$ *$w_2(M) \neq 0$ and $w_2(M) \in \mathrm{Ext}(H_1(M), \mathbb{Z}/2)$.*

Proof. There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_1(M), \mathbb{Z}/2) \arrow[r] \arrow[d] & H^2(M; \mathbb{Z}/2) \arrow[r] \arrow[d] & \mathrm{Hom}(H_2(M), \mathbb{Z}/2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_1(\widetilde{M}), \mathbb{Z}/2) \arrow[r] & H^2(\widetilde{M}; \mathbb{Z}/2) \arrow[r] & \mathrm{Hom}(H_2(\widetilde{M}), \mathbb{Z}/2) \arrow[r] & 0.
\end{tikzcd}$$

Let $p\colon \widetilde{M} \to M$ be the covering map, then $T\widetilde{M} = p^*TM$ and $w_2(\widetilde{M}) = p^*w_2(M)$. By the exact sequence $\pi_2(M) \to H_2(M) \to H_2(\mathbb{Z}/2) \to 0$ and the fact $H_2(\mathbb{Z}/2) = 0$ (cf. [3]), it is seen that the map $H_2(\widetilde{M}) \to H_2(M)$ is surjective, therefore the last vertical map in the diagram $\mathrm{Hom}(H_2(M), \mathbb{Z}/2) \to \mathrm{Hom}(H_2(\widetilde{M}), \mathbb{Z}/2)$ is a monomorphism. Thus $w_2(\widetilde{M}) = 0$ if and only if $w_2(M) \in \mathrm{Ext}(H_1(M), \mathbb{Z}/2)$. $\square$

Remark 2.4. By this Lemma, the type II and type III manifolds are manifolds having second Stiefel-Whitney class equal to zero on homology. The existence of contact structures on these manifolds is shown in [7].

Recall that for a manifold M^n with fundamental group $\mathbb{Z}/2$, a *characteristic submanifold* $P^{n-1} \subset M$ is defined as follows (see [18] and [7, §5]): there is a decomposition $\widetilde{M} = A \cup TA$ such that $\partial A = \partial(TA) = \widetilde{P}$, where T is the deck-transformation. Then $P := \widetilde{P}/T$ is called the characteristic submanifold of M. For example, if $M = \mathbb{R}\mathrm{P}^n$, then $P = \mathbb{R}\mathrm{P}^{n-1}$. In general, let $f\colon M \to \mathbb{R}\mathrm{P}^N$ (N large) be the classifying map of the universal cover, transverse to $\mathbb{R}\mathrm{P}^{N-1}$, then P can be taken as $f^{-1}(\mathbb{R}\mathrm{P}^{n-1})$. By equivariant surgery we may assume that $\pi_1(P) \cong \mathbb{Z}/2$ and that the inclusion $i\colon P \subset M$ induces an isomorphism on π_1. Different characteristic submanifolds of M are bordant, where a bordism is obtained from a homotopy between the relevant classifying maps. The above construction also holds in the topological category by topological transversality [12].

In the smooth category, the division of the manifolds under consideration into three w_2-types corresponds to different $\mathrm{Pin}^\dagger$-structures on their characteristic submanifolds, compare [7, Lemma 9] for $\dagger = \pm$.

Lemma 2.5. *Let M be a smooth, orientable 5-manifold with $\pi_1(M) \cong \mathbb{Z}/2$ and $H_2(M; \mathbb{Z})$ torsion-free. Let $P \subset M$ be a characteristic submanifold (with $\pi_1(P) \cong \pi_1(M)$). Then TP admits a $\mathrm{Pin}^\dagger$-structure, where*

$$\dagger = \begin{cases} c & \textit{if } M \textit{ is of type } \mathrm{I} \\ - & \textit{if } M \textit{ is of type } \mathrm{II} \\ + & \textit{if } M \textit{ is of type } \mathrm{III} \end{cases}$$

More precisely, if M is of type II*, then a Spin-structure on TM gives a* Pin^-*-structure on TP; if M is of type* III*, then a Spin-structure on $TM \oplus 2L$ gives a* Pin^+*-structure on TP, where L is the nontrivial line bundle over M; if M is of type* I*, then a Spin-structure on $TM \oplus \gamma$ gives a* Pin^c*-structure on TP, where γ is a complex line bundle over M such that $c_1(\gamma) \equiv w_2(M) \pmod 2$.*

Proof. Let $i\colon P \subset M$ be the inclusion and ν be the normal bundle of this inclusion, then $TP \oplus \nu = i^*TM$. If M is of type II, a Spin-structure on TM induces a Spin-structure on $TP \oplus \nu = TP \oplus \det TP$, therefore by Lemma 2.2, gives a Pin^--structure on TP.

If M is of type III, then $TM \oplus 2L$ admits Spin-structures and such a structure induces a Spin-structure on $TP \oplus 3\det TP$, henceforth a Pin^+-structure on TP.

If M is of type I, then TP has neither Pin^- nor Pin^+-structures. Now $TM \oplus \gamma$ has Spin-structures. Such a structure induces a Spin-structure on $TP \oplus \det TP \oplus i^*\gamma$, and hence a Pin^--structure on $TP \oplus i^*\gamma$. Since $c_1(i^*\gamma) \equiv i^*w_2(M) = w_1(P)^2 + w_2(P) \pmod 2$, we obtain a Pin^c-structure on TP. □

Lemma 2.6. *If M is of type* II *or* III*, then different characteristic submanifolds of M with the* $\mathrm{Pin}^{\pm}$*-structures obtained by* Lemma 2.5 *represent a pair of mutually inverse elements in the corresponding bordism group $\Omega_4^{\mathrm{Pin}^{\pm}}$.*

Proof. If we fix a Spin-structure on TM (or $TM \oplus 2L$), then it's clear that all different characteristic submanifolds with the induced $\mathrm{Pin}^{\pm}$-structure are $\mathrm{Pin}^{\pm}$-bordant, for they are transversal preimages of classifying maps of $\pi_1(M)$ and all such maps are homotopic. Now we fix a characteristic submanifold P, then the two $\mathrm{Pin}^{\pm}$-structures on TP are related by the action of $w_1(P)$, and it's a general fact that P with such two $\mathrm{Pin}^{\pm}$-structures give rise to a pair of mutually inverse elements in the corresponding bordism group [11, p.190]. □

3. Main Results

Now we are ready to state the classification of the manifolds under consideration.

Theorem 3.1. *Two smooth, closed, orientable fibered type 5-manifolds M and M', with fundamental group $\mathbb{Z}/2$ and torsion-free second homology group, are diffeomorphic if and only if they have the same w_2-type,* $\operatorname{rank} H_2(M) = \operatorname{rank} H_2(M')$*, and* $[P] = [P'] \in \Omega_4^{\mathrm{Pin}^{\dagger}}/\pm$*, where P and P' are the characteristic submanifolds and $\dagger = c, -, +$ for types* I, II, III *respectively.*

Remark 3.2. It is known that $\Omega_4^{\mathrm{Pin}^-} = 0$ [11]. Therefore, $\operatorname{rank} H_2(M)$ is the only diffeomorphism invariant for the type II manifolds.

There are topological versions of the central extensions mentioned above and we have groups $\mathrm{TopPin}^{\dagger}(n)$, $\dagger \in \{c, +, -\}$. For the preliminaries on $\mathrm{TopPin}^{\dagger}(n)$ we refer to [11] and [9]. Therefore we have corresponding results in the topological category.

Lemma 3.3. *Let M be a topological, orientable 5-manifold with $\pi_1(M) \cong \mathbb{Z}/2$ and $H_2(M;\mathbb{Z})$ torsion-free. Let $P \subset M$ be a characteristic submanifold (with $\pi_1(P) \cong \pi_1(M)$).*

Then TP admits a $\mathrm{TopPin}^\dagger$*-structure, where*

$$\dagger = \begin{cases} c & \text{if } M \text{ is of type I} \\ - & \text{if } M \text{ is of type II} \\ + & \text{if } M \text{ is of type III} \end{cases}$$

Theorem 3.4. *Two topological, closed, orientable fibered type 5-manifolds M and M', with fundamental group $\mathbb{Z}/2$ and torsion-free second homology group, are homeomorphic if and only if they have the same w_2-type,* $\operatorname{rank} H_2(M) = \operatorname{rank} H_2(M')$ *and $[P] = [P'] \in \Omega_4^{\mathrm{TopPin}^\dagger}/\pm$, where P and P' are characteristic submanifolds and $\dagger = c, -, +$ for type* I, II, III *respectively.*

The groups $\Omega_4^{\mathrm{Pin}^\pm}$ and $\Omega_4^{\mathrm{TopPin}^\pm}$ are computed in [11]. $\Omega_4^{\mathrm{TopPin}^c}$ is computed in [9, p.654]. (Note that the rôle of Pin^+ and Pin^- in [9] are reversed since in that paper the authors consider normal structures whereas here we use the convention in [11], looking at the tangential Gauss-map.) In a similar way we will compute $\Omega_4^{\mathrm{Pin}^c}$ below. We list the values of these groups:

$\dagger$	$\Omega_4^{\mathrm{Pin}^\dagger}$	invariants	generators
c	$\mathbb{Z}/8 \oplus \mathbb{Z}/2$	(arf, w_2^2)	$\mathbb{RP}^4$, $\mathbb{CP}^2$
$+$	$\mathbb{Z}/16$	?	$\mathbb{RP}^4$
$-$	0	–	–
$\dagger$	$\Omega_4^{\mathrm{TopPin}^\dagger}$	invariants	generators
c	$\mathbb{Z}/2 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2$	(KS, arf, w_2^2)	E_8, $\mathbb{RP}^4$, $\mathbb{CP}^2$
$+$	$\mathbb{Z}/2 \oplus \mathbb{Z}/8$	(KS, arf)	E_8, $\mathbb{RP}^4$
$-$	$\mathbb{Z}/2$	KS	E_8

Computation of $\Omega_4^{\mathrm{Pin}^c}$: the extension

$$1 \to \mathrm{Pin}^- \to \mathrm{Pin}^c \to U(1) \to 1$$

induces Gysin-sequence (compare [9, p.654])

$$\cdots \to \Omega_4^{\mathrm{Pin}^-} \to \Omega_4^{\mathrm{Pin}^c} \xrightarrow{\cap c} \Omega_2^{\mathrm{Pin}^-}(BU(1)) \to \Omega_3^{\mathrm{Pin}^-} \to \cdots .$$

Since $\Omega_4^{\mathrm{Pin}^-} = \Omega_3^{\mathrm{Pin}^-} = 0$ (see [11]), we have an isomorphism

$$\Omega_4^{\mathrm{Pin}^c} \xrightarrow{\cap c} \Omega_2^{\mathrm{Pin}^-}(BU(1))$$

and the latter group is the same as $\Omega_2^{\mathrm{TopPin}^-}(BU(1))$, which is computed in [9]. The invariants in Theorem 3.1 are subject to certain relations.

Definition 3.5. Denote $r = \operatorname{rank} H_2(M)$, $q = [P] \in \Omega_4^{\mathrm{Pin}^+}/\pm = \{0, 1, \ldots, 8\}$ and $(q, s) = [P] \in \Omega_4^{\mathrm{Pin}^c}/\pm = \{0, 1, \ldots, 4\} \times \{0, 1\}$.

As an application of the semi-characteristic class [17], we have

Theorem 3.6. *Let M be a smooth, orientable 5-manifold with $\pi_1(M) \cong \mathbb{Z}/2$ and torsion-free $H_2(M)$, having the invariants as above. Then these invariants subject to the following relations*

type	*relation*
I	$q+s+r \equiv 1 \pmod 2$
II	$r \equiv 1 \pmod 2$
III	$q+r \equiv 1 \pmod 2$

Now we give a list of all the manifolds under consideration, realizing the possible invariants. We need some preliminaries.

By a computation of the surgery exact sequence, it is shown in [24] that in the smooth (or PL) category, there are 4 distinct diffeomorphism types of manifolds which are homotopy equivalent to $\mathbb{RP}^5$, these are called fake $\mathbb{RP}^5$. An explicit construction using links of singularities (Brieskorn spheres) can be found in [7]. Following the notations there, we denote these fake $\mathbb{RP}^5$ by $X^5(q)$, $q = 1, 3, 5, 7$, with $X^5(1) = \mathbb{RP}^5$. These manifolds fall into the class of manifolds under consideration. They are of type III and the Pin^+-bordism class of the corresponding characteristic submanifold is $q \in \Omega_4^{\mathrm{Pin}^+}/\pm = \{0, 1, \dots, 8\}$, see [7]. In our list of standard forms these fake projective spaces will serve as building blocks under the operation $\sharp_{S^1}$—"connected-sum along S^1", which we explain now, compare [9].

Connected sum along a circle. Let M_i $(i = 1, 2)$ be oriented 5-manifolds with fundamental group $\mathbb{Z}/2$ or $\mathbb{Z}$, and at least one of the fundamental groups is $\mathbb{Z}/2$. Denote the trivial oriented 4-dimensional real disc bundle over S^1 by E. Choose embeddings of E into M_1 and M_2, representing a generator of $\pi_1(M_i)$, such that the first embedding preserves the orientation and the second reverses it. Then we define

$$M_1 \sharp_{S^1} M_2 := (M_1 - E) \cup_\partial (M_2 - E).$$

Note that if one of the 5-manifolds admits an orientation reversing automorphism, then the construction doesn't depend on the orientations, and this is the case for the building blocks in the list below, namely, $S^2 \times \mathbb{RP}^3$, $S^2 \times S^2 \times S^1$, $X^5(q)$ and $\mathbb{CP}^2 \times S^1$ admit orientation reversing automorphisms. (The fact that $X^5(q)$ admits orientation reversing automorphisms follows from that $\mathbb{RP}^5$ admits orientation reversing automorphisms and that the action of $Aut(\mathbb{RP}^5)$ on the structure set $\mathscr{S}(\mathbb{RP}^5)$ is trivial.)

The Seifert-van Kampen theorem implies that $\pi_1(M_1 \sharp_{S^1} M_2) \cong \mathbb{Z}/2$. The Mayer-Vietoris exact sequence implies that $H_2(M_1 \sharp_{S^1} M_2)$ is torsion-free, and hence $M_1 \sharp_{S^1} M_2$ is of fibered type. The homology $\operatorname{rank} H_2(M_1 \sharp_{S^1} M_2) = \operatorname{rank} H_2(M_1) + \operatorname{rank} H_2(M_2) + 1$ if both fundamental groups are $\mathbb{Z}/2$, and $\operatorname{rank} H_2(M_1 \sharp_{S^1} M_2) = \operatorname{rank} H_2(M_1) + \operatorname{rank} H_2(M_2)$ if one of the fundamental groups is $\mathbb{Z}$.

Since $\pi_1 SO(4) \cong \mathbb{Z}/2$, there are actually two possibilities to form $M_1 \sharp_{S^1} M_2$. However, from the classification result, it turns out that this ambiguity happens only when we construct $X^5(q) \sharp_{S^1} X^5(q')$. This does depend on the framings, and therefore $X^5(q) \sharp_{S^1} X^5(q')$ represents two manifolds. Note that the characteristic submanifold of $M_1 \sharp_{S^1} M_2$ is $P_1 \sharp_{S^1} P_2$ (see [9, p.651] for the definition of $\sharp_{S^1}$ for nonorientable 4-manifolds with fundamental group $\mathbb{Z}/2$). Therefore if we fix Pin^+-structures on each of the characteristic submanifolds, then $X^5(q)\,\sharp_{S^1} X^5(q')$ is well-defined.

This construction allows us to construct manifolds with a given bordism class of characteristic submanifold. Note that $P_1\,\sharp_{S^1} P_2$ corresponds to the addition in the bordism group $\Omega_4^{\mathrm{Pin}^\dagger}$. Now for $q = 0, 2, 4, 6, 8$, choose $l, l' \in \{1, 3, 5, 7\}$ and appropriate Pin^+-structures on the characteristic submanifolds of $X^5(l)$ and $X^5(l')$, we can form a manifold $X^5(l)\,\sharp_{S^1} X^5(l')$ such that the characteristic submanifold $[P] = q \in \Omega_4^{\mathrm{Pin}^+}/\pm$. We denote this manifold also by $X^5(q)$. For example, we can form $X^5(0) = X^5(1)\,\sharp_{S^1} X^5(1)$ and $X^5(2) = X^5(1)\,\sharp_{S^1} X^5(1)$ with different glueing maps.

With these notations, the list of standard forms of the manifolds under consideration is given as follows:

Theorem 3.7. *Every closed smooth orientable fibered type 5-manifold with fundamental group $\mathbb{Z}/2$ and second homology group $\mathbb{Z}^r$ is diffeomorphic to exactly one of the following standard forms:*

$$\textit{type I}: X^5(q)\,\sharp_{S^1}(S^2\times\mathbb{RP}^3)\,\sharp_{S^1}((\sharp_k\, S^2\times S^2)\times S^1),\ r = 2k+(5+(-1)^q)/2,\ q\in\{0,\dots,4\};$$

$$X^5(q)\,\sharp_{S^1}(\mathbb{CP}^2\times S^1)\,\sharp_{S^1}((\sharp_k\, S^2\times S^2)\times S^1),\ r = 2k+(3+(-1)^q)/2,\ q\in\{0,\dots,4\};$$

$$\textit{type II}: (S^2\times\mathbb{RP}^3)\,\sharp_{S^1}((\sharp_k\, S^2\times S^2)\times S^1),\ r = 2k+1;$$

$$\textit{type III}: X^5(q)\,\sharp_{S^1}((\sharp_k\, S^2\times S^2)\times S^1),\ r = 2k+(1+(-1)^q)/2,\ q\in\{0,\dots,8\}.$$

Where $\sharp_k\, S^2 \times S^2$ is the connected sum of k copies of $S^2 \times S^2$.

Remark 3.8. There can be other descriptions of the manifolds in the list. For example, we have a (more symmetric) description of the type II standard forms

$$\underbrace{(S^2\times\mathbb{RP}^3)\,\sharp_{S^1}\cdots\,\sharp_{S^1}(S^2\times\mathbb{RP}^3)}_{k\quad\text{times}}.$$

Remark 3.9. Note that the universal covers of the manifolds under consideration have torsion-free second homology, therefore, according to the results of Smale and Barden, are diffeomorphic to $\sharp_r(S^2\times S^3)$ or $B\sharp_{r-1}(S^2\times S^3)$, where B is the nontrivial S^3-bundle over S^2. From this point of view, Theorem 3.7 gives the classification of orientation preserving free involutions on $\sharp_r(S^2\times S^3)$ and $B\sharp_{r-1}(S^2\times S^3)$, which act trivially on H_2. For example, consider the orientation preserving free involution on $S^2\times S^3$ given by $(x,y)\mapsto(r(x),-y)$, where $r\colon S^2\to S^2$ is the reflection along a line and $-\colon S^3\to S^3$ is the antipodal map. Then the quotient space is actually the sphere bundle of the nontrivial orientable $\mathbb{R}^3$-bundle over $\mathbb{RP}^3$. From Theorem 3.1 it is easy to see that this is just $X^5(0)$.

Remark 3.10. The above list may be of use in the study of geometric structures on these manifolds. Geiges and Thomas [7] show that the type II and type III manifolds admit contact structures. On the other hand, a necessary condition for the existence of contact structures on M^{2n+1} is the reduction of the structure group of TM to $U(n)$, hence the vanishing of integral Stiefel-Whitney classes $W_{2i+1}(M)$. It is easy to see that the type I manifolds satisfy this necessary condition. These manifolds also satisfy the necessary conditions on the cup length and Betti numbers in [2] for the existence of Sasakian structures. Therefore it would be interesting to study these geometric structures on these manifolds.

The proof of Theorem 3.7. By the Van-Kampen theorem and the Mayer-Vietoris sequence it is easy to see that all the manifolds in the list are orientable, with fundamental group $\mathbb{Z}/2$ and torsion-free H_2, and the π_1-action on H_2 is trivial. Therefore we only need to verify that these manifolds have different invariants and realize all the possible invariants.
Type II: $\operatorname{rank} H_2((S^2 \times \mathbb{R}\mathrm{P}^3) \sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1)) = 2k + 1$.
Type III: the characteristic submanifold of $X^5(q) \sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1)$ is just that of $X^5(q)$, which corresponds to $q \in \Omega_4^{\mathrm{Pin}^+}/\pm = \{0, \cdots, 8\}$.
Type I: similarly, the manifold $X^5(q) \sharp_{S^1} (\mathbb{C}\mathrm{P}^2 \times S^1) \sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1)$ has characteristic submanifold invariant $(q, 1) \in \Omega_4^{\mathrm{Pin}^c}/\pm$. □

To give a list of standard forms of the manifolds under consideration in the topological case, we need a topological 5-manifold which is homotopy equivalent to $S^2 \times \mathbb{R}\mathrm{P}^3$ and whose characteristic submanifold represents the nontrivial element in $\Omega_4^{\mathrm{TopPin}^-} = \mathbb{Z}/2$. Note that by Theorem 3.4, if such manifolds exist, then the homeomorphism type is unique. Following the notation in [9], we denote this manifold by $*(S^2 \times \mathbb{R}\mathrm{P}^3)$. We now give the construction of $*(S^2 \times \mathbb{R}\mathrm{P}^3)$.

Let $W = S^2 \times \mathbb{R}\mathrm{P}^3 \sharp_{S^1} E_8 \times S^1$, so that $\pi_1(W) = \mathbb{Z}/2$ and the characteristic submanifold of W is $S^2 \times \mathbb{R}\mathrm{P}^2 \sharp E_8$. Let $h\colon W \to S^2 \times \mathbb{R}\mathrm{P}^3$ be a degree 1 normal map which extends the degree 1 normal map $f\colon S^2 \times \mathbb{R}\mathrm{P}^2 \sharp E_8 \to S^2 \times \mathbb{R}\mathrm{P}^2$. Then by doing codimension 1 surgery on h we obtain a W' with characteristic submanifold $P = *(S^2 \times \mathbb{R}\mathrm{P}^2)$ and a degree 1 normal map $h'\colon W' \to S^2 \times \mathbb{R}\mathrm{P}^3$ extending a homotopy equivalence $f'\colon *(S^2 \times \mathbb{R}\mathrm{P}^2) \to S^2 \times \mathbb{R}\mathrm{P}^2$ (cf. [9] for the construction of $*(S^2 \times \mathbb{R}\mathrm{P}^2)$). The π-π theorem allows us to do further surgeries on the complement of a tubular neighbourhood of P to obtain a homotopy equivalence.

In the topological category there are four fake $\mathbb{R}\mathrm{P}^5$'s. Two of them are smoothable. We denote these manifolds by $X^5(p, q)$ ($p \in \{0, 1\}$, $q \in \{1, 3\}$) such that the characteristic submanifold of $X^5(p, q)$ is $(p, q) \in \Omega_4^{\mathrm{TopPin}^+}/\pm = \{0, 1\} \times \{0, 1, 2, 3, 4\}$. Similar to the smooth case, we can also construct $X^5(p, q)$ ($p \in \{0, 1\}$, $q \in \{0, 2, 4\}$) by circle connected sum of fake $\mathbb{R}\mathrm{P}^5$. (Note that the Kirby-Siebenmann invariant is additive under the connected sum operation [20]).

Theorem 3.11. *Every closed topological orientable fibered type 5-manifold with fundamental group $\mathbb{Z}/2$ and second homology group $\mathbb{Z}^r$ is homeomorphic to exactly one of the following standard forms:*

$$\textit{type I}: X^5(p, q) \sharp_{S^1} (S^2 \times \mathbb{R}\mathrm{P}^3) \sharp_{S^1} ((\sharp k\, S^2 \times S^2) \times S^1),$$
$$r = 2k + (5 + (-1)^q)/2,\ q \in \{0, \ldots, 4\},\ p = 0, 1;$$

$X^5(p,q)\,\sharp_{S^1}(\mathbb{C}\mathrm{P}^2\times S^1)\,\sharp_{S^1}((\sharp k\, S^2\times S^2)\times S^1)$,
$r = 2k + (3+(-1)^q)/2$, $q\in\{0,\dots,4\}$, $p = 0,1$;

type II : $(S^2\times \mathbb{R}\mathrm{P}^3)\,\sharp_{S^1}((\sharp k\, S^2\times S^2)\times S^1)$, $r = 2k+1$;

$*(S^2\times \mathbb{R}\mathrm{P}^3)\,\sharp_{S^1}((\sharp k\, S^2\times S^2)\times S^1)$, $r = 2k+1$;

type III : $X^5(p,q)\,\sharp_{S^1}((\sharp k\, S^2\times S^2)\times S^1)$, $r = 2k+(1+(-1)^q)/2$, $q\in\{0,\dots,4\}$, $p = 0,1$.

From the above list, we can also give a homotopy classification.

Theorem 3.12. *The homotopy type of M^5 is determined by its w_2-type,* $\operatorname{rank} H_2(M)$*, and in the type* I *case the number $\langle w_2(M)^2\cup t + t^5, [M]\rangle \in \mathbb{Z}/2$, where $t\in H^1(M;\mathbb{Z}/2)$ is the nonzero element.*

Proof. Note that $X^5(q)$ and $X^5(p,q)$ are homotopy equivalent to $\mathbb{R}\mathrm{P}^5$ and the operation $\sharp_{S^1}$ preserves homotopy equivalence. This proves the theorem for the type II and III cases. For type I manifolds, the s-component of the characteristic submanifold P is determined by $\langle w_2(P)^2, [P]\rangle$. Since $w_2(P) = i^*(w_2(M)+t^2)$, $\langle w_2(P)^2,[P]\rangle = \langle w_2(M)^2\cup t+t^5,[M]\rangle$, and this is a homotopy invariant. $\square$

4. Bordism and Surgery

§4A. The framework of modified surgery. The main tool used in our solution of the classification problem is the modified surgery developed by Kreck [13], [14]. We first briefly describe how this theory is applied in our situation.

Let $p\colon B\to BO$ be a fibration, and $\bar\nu\colon M^{2m-1}\to B$ be a lift of the normal Gauss map $\nu\colon M\to BO$ classifying the stable normal bundle of M. Such a lift $\bar\nu$ is called a *normal B-structure* of M, and the pair $(M,\bar\nu)$ is called a *normal k-smoothing in B* if the map $\bar\nu$ is a $(k+1)$-equivalence. Manifolds with normal B-structures form a bordism theory $\Omega_*(B,p)$, described in Stong [22, Chap. II].

Suppose $(M_i^{2m-1},\bar\nu_i)$ $(i=1,2)$ are two normal $(m-1)$-smoothings in B, and suppose that $(W^{2m},\bar\nu)$ is a B-bordism between $(M_1^{2m-1},\bar\nu_1)$ and $(M_2^{2m-1},\bar\nu_2)$. Then the surgery obstruction for W^{2m} being B-bordant rel. boundary to an s-cobordism (implying that M_1 and M_2 are diffeomorphic) is a $(-1)^m$-quadratic form over (Λ,S), where $\Lambda=\mathbb{Z}[\pi_1(B)]$ is the group ring and $S\subset\Lambda$ is a certain form-parameter subgroup. The surgery obstruction lies in an abelian group $L_{2m}^{s,\tau}(\pi_1(B),w_1(B),S)$ ([13, Theorem 5.2 b]), where $w_1(B)$ is the orientation character. This group is related to Wall's L-group in the following diagram ([13, p.37])

$$\begin{tikzcd} 0 \arrow[r] & L^s_{2m}(\pi_1,w_1) \arrow[r] & L^{s,\tau}_{2m}(\pi_1,w_1) \arrow[r] \arrow[d] & Wh(\pi_1) \\ & & L^{s,\tau}_{2m}(\pi_1,w_1,S) \arrow[d] & \\ & & 0 & \end{tikzcd}$$

where $Wh(\pi_1)$ is the Whitehead group (see [19]).

In our case, $\pi_1 = \mathbb{Z}/2$, $Wh(\mathbb{Z}/2) = 0$ and $L^s_6(\mathbb{Z}/2) = \mathbb{Z}/2$. Therefore our surgery obstruction group is either 0 or $\mathbb{Z}/2$. In the latter case, it is isomorphic to $L^s_6(\mathbb{Z}/2)$, the non-trivial element is detected by the Kervaire-Arf invariant (see Wall [24, §13A]). Since the closed manifold $S^3 \times S^3$ admits a framing with Arf invariant 1, we may eliminate the surgery obstruction by connected sum in the interior of W. We have the following:

Proposition 4.1. *Two smooth 5-manifolds M_1 and M_2 with fundamental group $\mathbb{Z}/2$ are diffeomorphic if they have bordant normal 2-smoothings in some fibration B.*

The fibration B is called the *normal 2-type of M* if p is 3-coconnected. This is an invariant of M. Because of this proposition, the solution to the classification problem consists of two steps: first, determine the normal 2-types B for the 5-manifolds under consideration, and then determine invariants to detect the corresponding bordism groups $\Omega_5(B, p)$.

§4B. Normal 2-types. Let M^5 be a fibered type 5-maniofold. The universal coefficient theorem implies that $H_2(\widetilde{M}) \otimes_{\mathbb{Z}[\pi_1]} \mathbb{Z} \to H_2(M)$ is an isomorphism. Since the $\pi_1(M)$-action on $H_2(\widetilde{M})$ is trivial, we have $H_2(\widetilde{M}) \otimes_{\mathbb{Z}[\pi_1]} \mathbb{Z} = H_2(\widetilde{M})$, therefore $H_2(\widetilde{M}) \to H_2(M)$ is an isomorphism, also is the second Hurewicz map $\pi_2(M) \to H_2(M)$. Now suppose $\pi_1(M) \cong \mathbb{Z}/2$ and $H_2(M) \cong \mathbb{Z}^r$.

We start with the description of the normal 2-types for type II manifolds. It is the simplest situation and illuminates the ideas.

Type II: consider the fibration

$$p\colon B = \mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r \times B\,\mathrm{Spin} \to BO,$$

where $p\colon B \to BO$ is trivial on the first two factors and on $B\,\mathrm{Spin}$ it is the canonical projection from $B\,\mathrm{Spin}$ onto BO. A lift $\bar{\nu}\colon M \to B$ is given as follows: the map to $\mathbb{RP}^\infty$ is the classifying map of the fundamental group; choose a basis $\{u_1, \ldots, u_r\}$ of the free part of $H^2(M) \cong \mathbb{Z}^r \oplus \mathbb{Z}/2$, by realizing each element u_i by a map to $\mathbb{CP}^\infty$ we get a map to $(\mathbb{CP}^\infty)^r$; a Spin-structure on νM gives rise to a map to $B\,\mathrm{Spin}$. It's easy to see that (B, p) is the normal 2-type of type II manifolds and that $\bar{\nu}$ induces an isomorphism on π_1 and H_2. Since the second Hurewicz maps $\pi_2(M) \to H_2(M)$ and $\pi_2((\mathbb{CP}^\infty)^r) \to H_2((\mathbb{CP}^\infty)^r)$ are isomorphisms, $\bar{\nu}$ is a normal 2-smoothing.

Type III: let η be the canonical real line bundle over $\mathbb{RP}^\infty$, and $2\eta = \eta \oplus \eta$. Consider the fibration

$$p\colon B = \mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r \times B\,\mathrm{Spin} \xrightarrow{f_1 \times f_2} BO \times BO \xrightarrow{\oplus} BO,$$

where $f_1\colon \mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r \to BO$ is the classifying map of $p_1^*(2\eta)$, (where $p_1\colon \mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r \to \mathbb{RP}^\infty$ is the projection map,) $f_2\colon B\,\mathrm{Spin} \to BO$ is the canonical projection and $\oplus\colon BO \times BO \to BO$ is the H-space structure on BO induced by the Whitney sum of vector bundles. A lift $\bar{\nu}\colon M \to B$ is given as follows: the map to $\mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r$ is the same as in type II. Since $w_2(2\eta) = w_1(\eta)^2$ is the nonzero element in $\mathrm{Ext}(H_1(\mathbb{RP}^\infty), \mathbb{Z}/2)$ and $w_2(M)$ is the nonzero element in $\mathrm{Ext}(H_1(M), \mathbb{Z}/2)$, we have $w_2(\bar{\nu}^* 2\eta) = w_2(\nu M)$. This implies that $\nu M - \bar{\nu}^* 2\eta$ admits a Spin-structure. Such a structure induces a map to

$B\,\mathrm{Spin}$. Then $\bar{\nu}$ is a lift of ν. It is easy to see that (B,p) is the normal 2-type of type III manifolds and $\bar{\nu}$ is a normal 2-smoothing.

Type I: let γ be the canonical complex line bundle over $\mathbb{C}\mathrm{P}^\infty$. Consider the fibration

$$p\colon B = \mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r \times B\,\mathrm{Spin} \xrightarrow{f_1\times f_2} BO \times BO \xrightarrow{\oplus} BO,$$

where $f_1\colon \mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r \to BO$ is the classifying map of $p_2^*\gamma$, $p_2\colon \mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r \to \mathbb{C}\mathrm{P}^\infty$ is the projection map to the first $\mathbb{C}\mathrm{P}^\infty$. A lift $\bar{\nu}\colon M \to B$ is given as follows: since the Bockstein homomorphism $\beta\colon H^2(M;\mathbb{Z}/2) \to H^3(M;\mathbb{Z})$ is trivial, $w_2(M)$ is the mod 2 reduction of an integral cohomology class. Since $w_2(M)$ is not contained in $\mathrm{Ext}(H_1(M),\mathbb{Z}/2)$, this integral cohomology class can be taken as a primitive one, say, u_1 and we extend it to a basis $\{u_1,\dots,u_r\}$. Then the map to $\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r$ is the same as above. Now $\nu M - \bar{\nu}^*\gamma$ admits a Spin-structure, this gives rise to a map $M \to B\,\mathrm{Spin}$. Then $\bar{\nu}$ is a lift of ν. It is easy to see that (B,p) is the normal 2-type of type I manifolds and $\bar{\nu}$ is a normal 2-smoothing.

§4C. **Computation of the bordism groups.** In this subsection we calculate the bordism groups $\Omega_5(B,p)$ for our types:

$$\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r),\ \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r; p_1^*2\eta),\ \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r; p_2^*\gamma).$$

The main tools are the Atiyah-Hirzebruch spectral sequence and the Adams spectral sequence. Before doing the calculation, we need to compute $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty)$, $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty;2\eta)$ and $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty\times\mathbb{C}\mathrm{P}^\infty;p_2^*\gamma)$. These groups can be calculated via the Adams spectral sequence. Here we give an alternative argument, emphasizing the role of the characteristic submanifolds.

There are long exact sequences (this is a special case of [6, (3.2)])

$$\cdots \to \Omega_n^{\mathrm{Spin}} \to \Omega_n^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty;k\eta) \xrightarrow{\partial} \Omega_{n-1}^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty;(k+1)\eta) \to \Omega_{n-1}^{\mathrm{Spin}} \to \dots$$

and

$$\cdots \to \Omega_n^{\mathrm{Spin}}(\mathbb{C}\mathrm{P}^\infty;\gamma) \to \Omega_n^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty\times\mathbb{C}\mathrm{P}^\infty;p_2^*\gamma) \xrightarrow{\partial} \Omega_{n-1}^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty\times\mathbb{C}\mathrm{P}^\infty;\eta\times\gamma) \to \dots$$

where the maps ∂ correspond to taking a characteristic submanifold. In particular we have an isomorphism $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty) \xrightarrow{\cong} \Omega_4^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty;\eta)$, together with exact sequences

$$0 \to \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty;2\eta) \to \Omega_4^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty;3\eta)$$

and

$$\Omega_5^{\mathrm{Spin}}(\mathbb{C}\mathrm{P}^\infty;\gamma) \to \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty\times\mathbb{C}\mathrm{P}^\infty;p_2^*\gamma) \to \Omega_4^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty\times\mathbb{C}\mathrm{P}^\infty;\eta\times\gamma) \to \Omega_4^{\mathrm{Spin}}(\mathbb{C}\mathrm{P}^\infty;\gamma)$$

Furthermore, we have

$$\Omega_n^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty;\eta) \cong \Omega_n^{\mathrm{Pin}^-},\ \Omega_n^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty;3\eta) \cong \Omega_n^{\mathrm{Pin}^+},\ \Omega_n^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty\times\mathbb{C}\mathrm{P}^\infty;\eta\times\gamma) \cong \Omega_n^{\mathrm{Pin}^c}.$$

This is seen as follows: first, given $[X^n,f] \in \Omega_n^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty;\eta)$, clearly

$$w_1(f^*\eta) = w_1(X) = w_1(\det TX).$$

Therefore by Lemma 2.2, the Spin-structure on $TX \oplus f^*\eta$ induces a Pin^--structure on TX and we have a well-defined map $\Omega_n^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty;\eta) \to \Omega_n^{\mathrm{Pin}^-}$. Given X^n together with a Pin^--structure, by letting $f\colon X \to \mathbb{R}\mathrm{P}^\infty$ be the classifying map for $w_1(X)$, we obtain $[X, f] \in \Omega_n^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty;\eta)$. These two maps are inverse to each other. The Pin^+ and Pin^c cases are similar.

The $\mathrm{Pin}^{\pm}$-bordism groups in low dimensions were calculated in [11]: we have $\Omega_4^{\mathrm{Pin}^-} = 0$ and $\Omega_4^{\mathrm{Pin}^+} \cong \mathbb{Z}/16$, generated by $\pm\mathbb{R}\mathrm{P}^4$. Also it is clear that under the map

$$\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty;2\eta) \to \Omega_4^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty;3\eta) \cong \Omega_4^{\mathrm{Pin}^+},$$

the element $[\mathbb{R}\mathrm{P}^5, \text{inclusion}]$ goes to $\pm\mathbb{R}\mathrm{P}^4$, therefore the map

$$\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty;2\eta) \to \Omega_4^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty;3\eta)$$

is an isomorphism. An easy Atiyah-Hirzebruch spectral sequence calculation shows that $\Omega_5^{\mathrm{Spin}}(\mathbb{C}\mathrm{P}^\infty;\gamma) \cong \widetilde{\Omega}_7^{\mathrm{Spin}}(\mathbb{C}\mathrm{P}^\infty) = 0$ and $\Omega_4^{\mathrm{Spin}}(\mathbb{C}\mathrm{P}^\infty;\gamma) \cong \widetilde{\Omega}_6^{\mathrm{Spin}}(\mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}\oplus\mathbb{Z}$. Therefore the map $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty\times\mathbb{C}\mathrm{P}^\infty;p_2^*\gamma) \to \Omega_4^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty\times\mathbb{C}\mathrm{P}^\infty;\eta\times\gamma)$ is also an isomorphism. To summarize, we have

Lemma 4.2. *Taking characteristic submanifolds gives isomorphisms*

$$\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty) \cong \Omega_4^{\mathrm{Pin}^-},\ \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty;2\eta) \cong \Omega_4^{\mathrm{Pin}^+},\ \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty\times\mathbb{C}\mathrm{P}^\infty;p_2^*\gamma) \cong \Omega_4^{\mathrm{Pin}^c}.$$

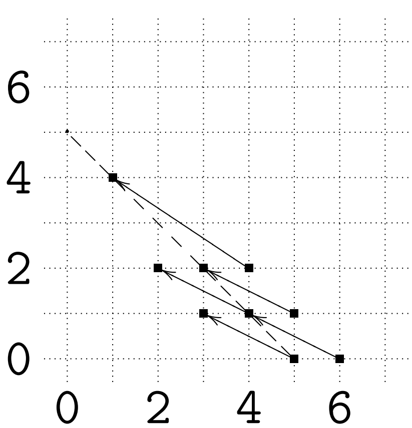
Now we begin the calculation of the bordism groups of interest. As in the last subsection, we start with the type II manifolds, which is the simplest case.

<u>Type II</u>: recall that the normal 2-type is

$$p\colon B = \mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r \times B\,\mathrm{Spin} \to BO,$$

where $p\colon B \to BO$ is trivial on the first two factors and is the canonical projection from $B\,\mathrm{Spin}$ onto BO. Therefore the bordism group $\Omega_5(B,p)$ is the Spin-bordism group $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty\times(\mathbb{C}\mathrm{P}^\infty)^r)$. To compute this bordism group, we apply the Atiyah-Hirzebruch spectral sequence. The E^2-terms are $E^2_{p,q} = H_p(\mathbb{R}\mathrm{P}^\infty\times(\mathbb{C}\mathrm{P}^\infty)^r;\Omega_q^{\mathrm{Spin}})$.

To illuminate the situation, we first consider the group $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty\times\mathbb{C}\mathrm{P}^\infty)$. The relevant terms and differentials in the spectral sequence are depicted as follows:



The E^2-terms are:

- $E^2_{1,4} = H_1(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}/2$,
- $E^2_{2,2} = H_2(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$,
- $E^2_{3,1} = E^2_{3,2} = H_3(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$,
- $E^2_{4,1} = E^2_{4,2} = H_4(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^3$,
- $E^2_{5,0} = H_5(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong (\mathbb{Z}/2)^3$,
- $E^2_{5,1} = H_5(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^3$,
- $E^2_{6,0} = H_6(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}/2$.

The differential $d_2\colon E^2_{p,1} \to E^2_{p-2,2}$ is dual to the Steenrod square

$$Sq^2\colon H^{p-2}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2) \to H^p(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2);$$

the differential $d_2\colon E^2_{p,0} \to E^2_{p-2,1}$ is the mod 2 reduction composed with the dual of the Steenrod square

$$H_p(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}) \to H_p(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2) \xrightarrow{(Sq^2)^*} H_{p-2}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2).$$

With these identifications, the differentials d_2 starting from or ending at the line $p+q=5$ are easily computed. Let $\alpha \in H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2)$, $\beta \in H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2)$ denote the generators, then on the E^3-page, we have three nontrivial terms in the line $p+q=5$: $E^3_{5,0} = \mathbb{Z}/2$, dual to $\alpha^3\beta$; $E^3_{4,1} = \mathbb{Z}/2$, dual to $\alpha^2\beta$; and $E^3_{1,4} = \mathbb{Z}/2$. The terms $E^3_{5,0}$ and $E^3_{4,1}$ must survive to infinity, for there are no non-trivial differentials starting from or ending at these two positions (see the picture above).

There is a possibly non-trivial differential $d_3\colon E^3_{4,2} \to E^3_{1,4}$. To see this differential is indeed non-trivial, we just need to note that the terms $E^3_{1,4} = E^2_{1,4} = H_1(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2)$ come from $\mathbb{R}\mathrm{P}^\infty$ and $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty) = \Omega_4^{\mathrm{Pin}^-} = 0$. Therefore on the E^∞-page, in the line $p+q=5$, the nontrivial terms are

$$E^\infty_{5,0} = H_3(\mathbb{R}\mathrm{P}^\infty) \otimes H_2(\mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}/2, \quad E^\infty_{4,1} = H_2(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2) \otimes H_2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

The calculation is finished once the extension problem is solved. We state the result in the following lemma. Let $\tau\colon \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$ be the involution on $\mathbb{C}\mathrm{P}^\infty$ with $\tau_* = -1$ on $H_2(\mathbb{C}\mathrm{P}^\infty)$, then τ induces an involution τ_* on $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty)$. Let $\alpha \in H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2)$, $\beta \in H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2)$ be the nonzero elements.

Lemma 4.3. *The short exact sequence*

$$0 \to \mathbb{Z}/2 \to \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \to \mathbb{Z}/2 \to 0$$

is nonsplit, thus $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}/4$. The elements ± 1 are represented by $\mathbb{R}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^1 \hookrightarrow \mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty$. A bordism class $[X^5, f]$ equals ± 1 if and only if $\langle \alpha^3 \cup \beta, f_*[X] \rangle = 1 \in \mathbb{Z}/2$. There is a relation $\langle \alpha \cup \beta^2, f_*[X] \rangle = 0$. The action τ_* is the multiplication by -1.

Proof. There is a product map

$$\varphi\colon \Omega_3^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty) \otimes \Omega_2^{\mathrm{Spin}}(\mathbb{C}\mathrm{P}^\infty) \to \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty),$$

induced by the product of manifolds. There is a corresponding product map on the Atiyah-Hirzebruch spectral sequences

$$\Phi\colon E^r_{p,q}(1) \otimes E^r_{s,t}(2) \to E^r_{p+s,q+t}(3),$$

where on the E^∞-page Φ is compatible with the filtrations on the bordism groups and on the E^2-page it is just the cross product map (see [23, p. 352]). It is easy to see that the Atiyah-Hirzebruch spectral sequence of $\Omega_2^{\mathrm{Spin}}(\mathbb{C}\mathrm{P}^\infty)$ collapses on the line $p+q=2$. Also since $\Omega_3^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty) \cong \Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$ (by [11]), we see that the Atiyah-Hirzebruch spectral sequence of $\Omega_3^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty)$ collapses on the line $p+q=3$. From the knowledge of $E^\infty_{5,0}(3)$ and $E^\infty_{4,1}(3)$ discussed above, we see there are surjections

$$\Phi\colon E^\infty_{3,0}(1) \otimes E^\infty_{2,0}(2) \to E^\infty_{5,0}(3),\ \ \Phi\colon E^\infty_{2,1}(1) \otimes E^\infty_{2,0}(2) \to E^\infty_{4,1}(3).$$

Therefore φ is surjective. Now $\Omega_3^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty) \cong \mathbb{Z}/8$ is generated by $[\mathbb{R}\mathrm{P}^3, \text{inclusion}]$ and $\Omega_2^{\mathrm{Spin}}(\mathbb{C}\mathrm{P}^\infty) \cong \Omega_2^{\mathrm{Spin}} \oplus H_2(\mathbb{C}\mathrm{P}^\infty)$. The group Ω_2^{Spin} is generated by T^2 with the Lie group spin structure. The product $\varphi(\mathbb{R}\mathrm{P}^3, T^2) = 0$, since the map $\mathbb{R}\mathrm{P}^3 \times T^2 \to \mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty$ factors through $\mathbb{R}\mathrm{P}^\infty$ and $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty) = 0$. Therefore we have a surjection

$$\mathbb{Z}/8 \otimes \mathbb{Z} \to \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty).$$

This shows that $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}/4$, generated by $\mathbb{R}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^1 \hookrightarrow \mathbb{R}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty$ and $[X, (\mathrm{id}_{\mathbb{R}\mathrm{P}^\infty} \times \tau) \circ f] = -[X, f]$. The fact that a bordism class $[X^5, f]$ equals ± 1 if and only if $\langle \alpha^3 \cup \beta, f_*[X]\rangle = 1 \in \mathbb{Z}/2$ comes from the fact that $E^\infty_{5,0}$ is dual to $\alpha^3\beta$. The relation $\langle \alpha \cup \beta^2, f_*[X]\rangle = 0$ comes from the fact that the dual of d_2 maps $\alpha\beta$ to $\alpha\beta^2$. $\square$

In general, on the E^∞-page of the Atiyah-Hirzebruch spectral sequence for $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r)$, the nontrivial terms in the line $p+q=5$ are

$$\begin{aligned}
E^\infty_{5,0} &= \bigoplus_i H_3(\mathbb{R}\mathrm{P}^\infty) \otimes H_2(\mathbb{C}\mathrm{P}^\infty_i) \oplus \bigoplus_{i \neq j} H_1(\mathbb{R}\mathrm{P}^\infty) \otimes H_2(\mathbb{C}\mathrm{P}^\infty_i) \otimes H_2(\mathbb{C}\mathrm{P}^\infty_j) \\
&\cong (\mathbb{Z}/2)^{r + r(r-1)/2} \\
E^\infty_{4,1} &= \bigoplus_i H_2(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2) \otimes H_2(\mathbb{C}\mathrm{P}^\infty_i; \mathbb{Z}/2) \\
&\cong (\mathbb{Z}/2)^r
\end{aligned}$$

Using the same argument as in Lemma 4.3, we have the following:

Proposition 4.4 (type II). *The bordism group $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r)$ is isomorphic to $(\mathbb{Z}/4)^r \oplus (\mathbb{Z}/2)^{r(r-1)/2}$. Let $\alpha \in H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2)$, $\beta_i \in H^2(\mathbb{C}\mathrm{P}^\infty_i; \mathbb{Z}/2)$ be the nonzero elements, τ_i be the involution on $\mathbb{C}\mathrm{P}^\infty_i$ with $\tau_{i*} = -1$ on H_2, then*

1. *the $\mathbb{Z}/2$-factors are determined by the invariants $\langle \alpha \cup \beta_i \cup \beta_j, f_*[X]\rangle \in \mathbb{Z}/2$, with $i, j = 1, \cdots r$, and $i > j$,*
2. *a bordism class $[X, f]$ has component ± 1 in the i-th $\mathbb{Z}/4$-factor if and only if $\langle \alpha^3 \cup \beta_i, f_*[X]\rangle = 1 \in \mathbb{Z}/2$, $i = 1, \cdots r$,*
3. *there are relations $\langle \alpha \cup \beta_i^2, f_*[X]\rangle = 0$ for all i,*

(4) *the action of τ_i on the bordism group is multiplication by -1 on the i-th $\mathbb{Z}/4$-factor and trivial on other factors.*

Type III: the normal 2-type is

$$p\colon B = \mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r \times B\,\mathrm{Spin} \to BO,$$

where the map on $\mathbb{RP}^\infty$ is the classifying map of the vector bundle 2η. Therefore the bordism group $\Omega_5(B,p)$ is the twisted Spin-bordism group

$$\Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r; p_1^* 2\eta) = \widetilde{\Omega}_7^{\mathrm{Spin}}(\mathrm{Th}(p_1^* 2\eta)).$$

In the Atiyah-Hirzebruch spectral sequence, the E^2-terms are

$$E^2_{p,q} = \widetilde{H}_p(\mathrm{Th}(p_1^* 2\eta); \Omega_q^{\mathrm{Spin}}).$$

Since 2η is orientable, we may apply the Thom isomorphism and after a degree shift $p \mapsto p-2$ we have $E^2_{p,q} = H_p(\mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r; \Omega_q^{\mathrm{Spin}})$. Therefore the E^2-terms are the same as in the type II case, and in the identification of the differentials d_2, we need to replace Sq^2 by $Sq^2 + w_2(2\eta)$.

As before, we first look at the group $\Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty \times \mathbb{CP}^\infty; p_1^* 2\eta)$. Clearly $\Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty; 2\eta) \cong \mathbb{Z}/16$ is a direct summand. Besides this, there are two terms on the E^∞-page at positions $(5,0)$ and $(4,1)$ respectively, each isomorphic to $\mathbb{Z}/2$. The extension problem is solved in the following lemma.

Lemma 4.5. *We have an isomorphism*

$$\Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty \times \mathbb{CP}^\infty; p_1^*(2\eta)) \cong \mathbb{Z}/4 \oplus \Omega_4^{\mathrm{Pin}^+}.$$

A bordism class $[X,f]$ has component ± 1 in the $\mathbb{Z}/4$-factor if and only if $\langle \alpha^3 \cup \beta, f_*[X]\rangle = 1 \in \mathbb{Z}/2$. There is a relation $\langle \alpha^3 \cup \beta, f_*[X]\rangle = \langle \alpha \cup \beta^2, f_*[X]\rangle$. The action τ_* of the involution τ on $\mathbb{CP}^\infty$ is the multiplication by -1 on the $\mathbb{Z}/4$-factor and trivial on the $\Omega_4^{\mathrm{Pin}^+}$-factor.

Proof. From the above discussion we have

$$\Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty \times \mathbb{CP}^\infty; p_1^*(2\eta)) \cong G \oplus \Omega_4^{\mathrm{Pin}^+},$$

where the order of G is 4. To determine G, the geometric argument in Lemma 4.3 doesn't work since now we have $\Omega_3^{\mathrm{Spin}}(\mathbb{RP}^\infty; 2\eta) \cong \Omega_2^{\mathrm{Pin}^+} = 0$. Thus we turn to consider the Adams spectral sequence for $\widetilde{\Omega}^{\mathrm{Spin}}_{t-s-2}(\mathbb{RP}^\infty \times \mathbb{CP}^\infty; p_1^*(2\eta)) = \pi_{t-s}^S(Th(p_1^*(2\eta)) \wedge MSpin))$ at prime 2:

$$Ext_{\mathcal{A}}^{s,t}(H^*(Th(p_1^*(2\eta)) \wedge MSpin; \mathbb{F}_2); \mathbb{F}_2) \Rightarrow \pi_{t-s}^S(Th(p_1^*(2\eta)) \wedge MSpin))/\text{non 2-torsion},$$

where $\mathcal{A}$ is the mod 2 Steenrod algebra. We have

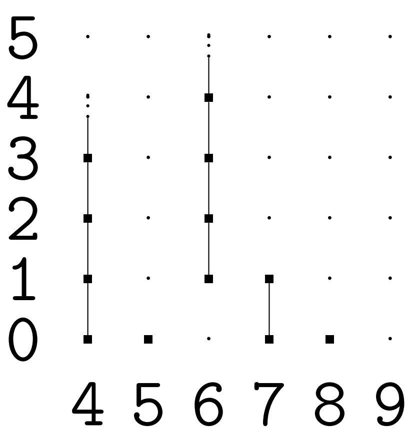
$$H^*(Th(p_1^*(2\eta)) \wedge MSpin; \mathbb{F}_2) \cong \widetilde{H}^*(Th(p_1^*(2\eta); \mathbb{F}_2) \otimes_{\mathbb{F}_2} H^*(MSpin; \mathbb{F}_2),$$

where $\widetilde{H}^*(Th(p_1^*(2\eta); \mathbb{F}_2)$ is a free $\mathbb{F}_2[t,x]$-module on one generator u_2 of degree 2 (the Thom class), where $\deg t = 1$ and $\deg x = 2$, and

$$Sq(u_2) = u_2 + t^2 u_2.$$

From this we may write down the $\mathcal{A}$-module structure of $H^*(Th(p_1^*(2\eta)) \wedge MSpin; \mathbb{F}_2)$ in degree≤ 9, produce a minimal free $\mathcal{A}$-resolution of $H^*(Th(p_1^*(2\eta)) \wedge MSpin; \mathbb{F}_2)$ which corresponds to the E_2-term of the spectral sequence. In practice, we may ignore the pure terms from $\mathbb{RP}^\infty$, since we already know the contribution of $\mathbb{RP}^\infty$ is a $\mathbb{Z}/16$-summand.

In low degrees, the E_2-page of the spectral sequence is depicted as follows (with horizontal index $t-s$ and vertical index s. The calculation is confirmed by Olbermann and Abczynski using a computer program developed by Bruner):



This shows that $G \cong \mathbb{Z}/4$.

The fact that the generators of the $\mathbb{Z}/4$-factor are detected by the invariant $\langle \alpha^3 \cup \beta, f_*[X]\rangle \in \mathbb{Z}/2$ and the relation $\langle \alpha^3 \cup \beta, f_*[X]\rangle = \langle \alpha \cup \beta^2, f_*[X]\rangle$ are seen from the Atiyah-Hirzebruch spectral sequence, as in the type II case. From this, we claim that $[X^5(0), f]$ represents a generator of $\mathbb{Z}/4$, where

$$f\colon X^5(0) \to \mathbb{RP}^\infty \times \mathbb{CP}^\infty$$

is a normal 2-smoothing. To see this, recall that

$$X^5(0) = (\mathbb{RP}^5 - S^1 \times D^4) \cup_\partial (\mathbb{RP}^5 - S^1 \times D^4),$$

and $\mathbb{RP}^5 - S^1 \times D^4$ is the disc bundle $D(2\eta)$ over $\mathbb{RP}^3$. Therefore $X^5(0)$ is actually the sphere bundle $S(2\eta \oplus \mathbb{R})$. The cohomology groups are easily computed and we see that $\langle \alpha^3 \cup \beta, f_*[X]\rangle = 1$.

Now let $r\colon X^5(0) \to X^5(0)$ be the fiberwise antipodal map, we have a commutative diagramm

$$\begin{tikzcd} X^5(0) \arrow[r, "f"] \arrow[d, "r"] & \mathbb{RP}^\infty \times \mathbb{CP}^\infty \arrow[d, "(\mathrm{id},\tau)"] \\ X^5(0) \arrow[r, "f"] & \mathbb{RP}^\infty \times \mathbb{CP}^\infty. \end{tikzcd}$$

Since r is orientation reversing, we conclude that the action of τ on the $\mathbb{Z}/4$ factor is multiplication by -1. It's also clear that the action of τ on the $\Omega_4^{\mathrm{Pin}^+}$ is trivial. $\square$

In the general situation, the calculation is similar, and we have

Proposition 4.6 (type III)**.** *The bordism group $\Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r; p_1^*(2\eta))$ is isomorphic to $(\mathbb{Z}/4)^r \oplus (\mathbb{Z}/2)^{r(r-1)/2} \oplus \Omega_4^{\mathrm{Pin}^+}$. Furthermore,*

1. *the $\mathbb{Z}/2$-factors are determined by the invariants $\langle \alpha \cup \beta_i \cup \beta_j, f_*[X]\rangle \in \mathbb{Z}/2$, with $i, j = 1, \cdots r$, and $i > j$,*
2. *a bordism class $[X, f]$ has component ± 1 in the i-th $\mathbb{Z}/4$-factor if and only if $\langle \alpha^3 \cup \beta_i, f_*[X]\rangle = 1 \in \mathbb{Z}/2$, $i = 1, \cdots r$,*
3. *there are relations $\langle \alpha \cup \beta_i^2, f_*[X]\rangle = \langle \alpha^3 \cup \beta_i, f_*[X]\rangle$ for all i,*
4. *the action τ_i on the bordism group is the multiplication by -1 on the i-th $\mathbb{Z}/4$-factor and trivial on other factors.*

<u>Type I:</u> recall that the normal 2-type is

$$p\colon B = \mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r \times B\,\mathrm{Spin} \to BO,$$

where the map p on the first $\mathbb{CP}^\infty$ is the classifying map of the vector bundle γ. Therefore the bordism group $\Omega_5(B, p)$ is the twisted Spin-bordism group

$$\Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r; p_2^*\gamma) = \widetilde{\Omega}_7^{\mathrm{Spin}}(\mathrm{Th}(p_2^*\gamma)).$$

As before we apply the Thom isomorphism and the E^2-terms in the Atiyah-Hirzebruch spectral sequence are $E^2_{p,q} = \widetilde{H}_p(\mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r; \Omega_q^{\mathrm{Spin}})$, where in the identification of the differentials d_2, we replace Sq^2 by $Sq^2 + w_2(\gamma)$. The calculation is analogous to the type II case.

Proposition 4.7 (type I)**.** *The bordism group $\Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty \times (\mathbb{CP}^\infty)^r; p_2^*\gamma)$ is isomorphic to $(\mathbb{Z}/4)^{r-1} \oplus (\mathbb{Z}/2)^{r(r-1)/2} \oplus \Omega_4^{\mathrm{Pin}^c}$. Furthermore,*

1. *the $\mathbb{Z}/2$-factors are determined by the invariants $\langle \alpha \cup \beta_i \cup \beta_j, f_*[X]\rangle \in \mathbb{Z}/2$, with $i, j = 1, \cdots r$, and $i > j$,*
2. *a bordism class $[X, f]$ has component ± 1 in the i-th $\mathbb{Z}/4$-factor if and only if $\langle \alpha^3 \cup \beta_i, f_*[X]\rangle = 1 \in \mathbb{Z}/2$, $i = 2, \cdots, r$,*
3. *there are relations $\langle \alpha^5 + \alpha^3 \cup \beta_1, f_*[X]\rangle = 0$ and $\langle \alpha \cup \beta_i^2, f_*[X]\rangle = \langle \alpha \cup \beta_1 \cup \beta_i, f_*[X]\rangle$ for all i,*
4. *the action τ_i ($i \geq 2$) on the bordism group is the multiplication by -1 on the i-th $\mathbb{Z}/4$-factor and trivial on other factors.*

5. Proofs of the Main Results

In this section we prove Theorem 3.1 and Theorem 3.6. From the point of view of Propositioni 4.1, the key point to prove Theorem 3.1 is to show that for manifolds having the same invariants stated in the theorem, we can find appropriate normal 2-smoothings in B, such that they are bordant in $\Omega_5(B, p)$. (In some applications, this is done by understanding the action of the group of fiber homotopy equivalences $Aut(B, p)$ on $\Omega_n(B, p)$. But in our situation, we find it more practical to find the smoothings directly.)

Lemma 5.1. *Let M^5 be a fibered type manifold with $\pi_1(M) \cong \mathbb{Z}/2$ and $H_2(M) \cong \mathbb{Z}^r$. Let $t \in H^1(M;\mathbb{Z}/2)$ be the nonzero element, and let $\{t^2, x_1, \cdots, x_r\}$ be a basis of $H^2(M;\mathbb{Z}/2)$. Then $\{t^3, tx_1, \cdots, tx_r\}$ is a basis of $H^3(M;\mathbb{Z}/2)$.*

Proof. Consider the Leray-Serre cohomology spectral sequence for the fibration $\widetilde{M} \to M \to \mathbb{R}\mathrm{P}^\infty$ with $\mathbb{Z}/2$-coefficients. Note that $\dim H^2(M;\mathbb{Z}/2) = r+1$ and $\dim H^2(\widetilde{M};\mathbb{Z}/2) = r$. This implies that the differential

$$d_2\colon E_2^{0,2} = H^2(\widetilde{M};\mathbb{Z}/2) \to E_3^{3,0} = H^3(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}/2)$$

must be trivial. Therefore, the elements $t^3, tx_1, \cdots, tx_r$ all survive to form a basis of $H^3(M;\mathbb{Z}/2)$. $\square$

The proof of Theorem 3.1. First of all, by Lemma 2.6, we see that in the type II and III cases, $[P] \in \Omega_4^{\mathrm{Pin}^\pm}/\{\pm 1\}$ is an invariant for M. Since we don't have a statement for Pin^c, we will give an alternative argument below for the type I case.

Let $f\colon M^5 \to \mathbb{R}\mathrm{P}^\infty$ be the classifying map of π_1, $t = f^*\alpha \in H^1(M;\mathbb{Z}/2)$. Consider the nondegenerate symmetric bilinear form

$$\lambda\colon H^2(M;\mathbb{Z}/2) \times H^2(M;\mathbb{Z}/2) \xrightarrow{\cup} H^4(M;\mathbb{Z}/2) \xrightarrow{\cup t} H^5(M;\mathbb{Z}/2) \cong \mathbb{Z}/2.$$

<u>Type II</u>: note that since $\langle t^5, [M]\rangle = \langle \alpha^5, f_*[M]\rangle$ and $\Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty) = 0$, we have $\lambda(t^2, t^2) = 0$. From this and the relations in Proposition 4.4 we see that $\lambda(x, x) = 0$ for all x. Therefore we may extend t^2 to a symplectic basis of λ, $\{t^2, u_1, \cdots, u_r\}$. Especially we have $\lambda(t^2, u_1) = 1$, $\lambda(u_1, u_1) = 0$ and $\lambda(t^2, u_i) = \lambda(u_1, u_i) = 0$ for $i > 1$. Now let $u_i' = u_i + u_1$ for $i > 1$, then $\lambda(t^2, u_i') = 1$ for all i and $\lambda(u_i', u_j') = \lambda(u_i, u_j)$. We may lift $\{u_1', \cdots, u_r'\}$ to a basis of the free part of $H^2(M)$ and get a map $M \to (\mathbb{C}\mathrm{P}^\infty)^r$. Together with the canonical map $f\colon M \to \mathbb{R}\mathrm{P}^\infty$ and the classifying map of a Spin-structure $M \to B\,\mathrm{Spin}$, we obtain a normal 2-smoothing $\bar{\nu}\colon M \to B = \mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r \times B\,\mathrm{Spin}$.

Now suppose M' is another manifold, with a normal 2-smoothing $\bar{\nu}'$ constructed as above, then by Proposition 4.4, (composing $\bar{\nu}'$ with some τ_i to interchange ± 1 in the $\mathbb{Z}/4$-factors if necessary) $[M, \bar{\nu}] = [M', \bar{\nu}'] \in \Omega_5^{\mathrm{Spin}}(\mathbb{R}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty)^r)$. Proposition 4.1 implies that they are diffeomorphic.

For the other two cases, the procedure of finding an appropriate map to $(\mathbb{C}\mathrm{P}^\infty)^r$ is similar, thus we will omit the details.

<u>Type III</u>: first note that by the relation in Proposition 4.6, for all $x \in H^2(M;\mathbb{Z}/2)$, $\lambda(t^2, x) = \lambda(x, x)$. There are two different cases:

(1) if $\lambda(t^2, t^2) = 0$: then there exists a u_1 such that $\lambda(t^2, u_1) = 1$. On the orthogonal complement of $\mathrm{span}(t^2, u_1)$, we have $\lambda(x, x) = 0$, thus there exists a symplectic basis $\{u_2, \cdots, u_r\}$. Then the argument is the same as in the previous case.

(2) if $\lambda(t^2, t^2) = 1$: let U be the orthogonal complement of $\mathrm{span}(t^2)$, then $\lambda(x, x) = \lambda(t^2, x) = 0$ for all $x \in U$. There exists a symplectic basis of U, $\{u_1, \cdots, u_r\}$. Let $u_i' = u_i + t^2$, then $\lambda(t^2, u_i') = 1$ for all i and $\lambda(u_i', u_j') = \lambda(u_i, u_j) + 1$. The remaining argument is the same as in the previous case.

For M and M' having the same rank $H_2 = r$, like in the type II case, we may use maps to $(\mathbb{CP}^\infty)^r$ constructed above to make the corresponding bordism classes have equal $(\mathbb{Z}/4)^r \oplus (\mathbb{Z}/2)^{r(r-1)/2}$-component. Now if M and M' have $[P] = [P'] \in \Omega_4^{\mathrm{Pin}^+}/\pm$, then by choosing an appropriate Spin-structure on $TM \oplus f^*(2\eta)$, we may make the $\Omega_4^{\mathrm{Pin}^+}$-component equal.

Type I: let $u_1 = w_2(M)$. By the relation in Proposition 4.7, for all $x \in H^2(M;\mathbb{Z}/2)$, $\lambda(u_1, x) = \lambda(x, x)$. To find the map to $(\mathbb{CP}^\infty)^r$, we have four cases:

(1) if $\lambda(t^2, t^2) = 1$ and $\lambda(u_1, u_1) = 0$: then λ is nondegenerate on $\mathrm{span}(t^2, u_1)$. Let U be the orthogonal complement of $\mathrm{span}(t^2, u_1)$, then and for all $x \in U$ $\lambda(x, x) = 0$. There exists a symplectic basis $\{u_2, \cdots, u_r\}$.
(2) if $\lambda(t^2, t^2) = 0$ and $\lambda(u_1, u_1) = 0$: then exists a u_2 such that $\lambda(u_1, u_2) = 1$ and $\lambda(t^2, u_2) = 0$. λ is nondegenerate on $\mathrm{span}(u_1, u_2)$. On the orthogonal complement we have $\lambda(x, x) = 0$. Therefore there is a symplectic basis $\{t^2, u_3, \cdots, u_r\}$.
(3) if $\lambda(t^2, t^2) = 1$ and $\lambda(u_1, u_1) = 1$: let U be the orthogonal complement of $\mathrm{span}(u_1)$, then there exists a symplectic basis $\{u_2, u_3, \cdots, u_r\}$ for U and we may choose $u_2 = t^2 + u_1$.
(4) if $\lambda(t^2, t^2) = 0$ and $\lambda(u_1, u_1) = 1$: then on the orthogonal complement of $\mathrm{span}(x_1)$ there is a symplectic basis $\{t^2, u_2, \cdots, u_r\}$.

Now we need to consider the $\Omega_4^{\mathrm{Pin}^c}$-component. Note that since the manifolds in the list given in Theorem 3.6 exhaust all possible values of $\mathrm{rank} H_2$ and $[P]$, an M of type I must be diffeomorphic to some manifolds in the list. Now we just need to show that the manifolds in the list are not diffeomorphic to each other.

The s-component of $[P] \in \Omega_4^{\mathrm{Pin}^c}$ is determined by $w_2(P)^2$, therefore varying Pin^c-structures on P^4 will not change the s-component. Thus we see that the two subfamilies

$$X^5(q) \,\sharp_{S^1} (S^2 \times \mathbb{RP}^3) \,\sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1)$$

and

$$X^5(q) \,\sharp_{S^1} (\mathbb{CP}^2 \times S^1) \,\sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1)$$

don't have coincidence.

Let Q^4 be a characteristic submanifold of $X^5(q)$, then a characteristic submanifold of $X^5(q) \,\sharp_{S^1} (S^2 \times \mathbb{RP}^3) \,\sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1)$ can be taken as $P = Q \sharp (S^2 \times \mathbb{RP}^2) \sharp (S^2 \times S^2)$. We have $[S^2 \times \mathbb{RP}^2] = [S^2 \times S^2] = 0 \in \Omega_4^{\mathrm{Pin}^c}$. So we see $[P] = q \in \Omega_4^{\mathrm{Pin}^c}/\pm$ and different q's give non-diffeomorphic $X^5(q) \,\sharp_{S^1} (S^2 \times \mathbb{RP}^3) \,\sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1)$. Similar for the manifolds $X^5(q) \,\sharp_{S^1} (\mathbb{CP}^2 \times S^1) \,\sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1)$, since $[\mathbb{CP}^2] = (0, 1) \in \Omega_4^{\mathrm{Pin}^c}$.

□

The relations among the invariants are essentially seen in the previous proof, but there is a more conceptual way to see this.

The proof of Theorem 3.6. We will use the semi-characteristic class defined by R. Lee in [17]. We work with $\mathbb{Q}$-coefficient, in this case, the semi-characteristic class of an odd dimensional manifold with a free $\mathbb{Z}/2$-action is a homomorphism

$$\chi_{1/2} \colon \Omega_5(\mathbb{Z}/2) \to L^5(\mathbb{Q}[\mathbb{Z}/2]) \cong \mathbb{Z}/2,$$

where $\Omega_5(\mathbb{Z}/2)$ is the bordism group of closed smooth oriented manifolds with an orientation-preserving free $\mathbb{Z}/2$-action, and $L^5(\mathbb{Q}[\mathbb{Z}/2])$ is the symmetric L-group of the rational group ring $\mathbb{Q}[\mathbb{Z}/2])$. We refer to [17] and [4] for details.

Let M^5 be an oriented smooth 5-manifold with fundamental group $\mathbb{Z}/2$, then the semi-characteristic class $\chi_{1/2}(\widetilde{M};\mathbb{Q}) \in \mathbb{Z}/2$ is defined. There is a characteristic class formula [4, Theorem C]

$$\chi_{1/2}(\widetilde{M};\mathbb{Q}) = \langle w_4(M) \cup f^*(\alpha), [M]\rangle,$$

where $f\colon M \to \mathbb{RP}^\infty$ is the classifying map of the covering and $\alpha \in H^1(\mathbb{RP}^\infty;\mathbb{Z}/2)$ is the nonzero element. On the other hand, $\chi_{1/2}(\widetilde{M};\mathbb{Q})$ is identified with (see [4, p.57])

$$\begin{aligned}\hat{\chi}_{1/2}(\widetilde{M};\mathbb{Q}) \quad &:= \dim_{\mathbb{Q}} H_0(\widetilde{M};\mathbb{Q}) + \dim_{\mathbb{Q}} H_1(\widetilde{M};\mathbb{Q}) + \dim_{\mathbb{Q}} H_2(\widetilde{M};\mathbb{Q}) \pmod 2 \\ &\equiv 1 + r \pmod 2.\end{aligned}$$

Type II: the Wu classes of M are $v_1 = 0$ and $v_2 = 0$ since $w_1(M) = w_2(M) = 0$. Therefore $w_4(M) = Sq^2 v_2 = 0$. This means r is odd.

Type III: the Wu classes of M are $v_1 = 0$ and $v_2 = w_2(M) = t^2$. Therefore $w_4(M) = Sq^2 v_2 = t^4$ and $\langle w_4(M) \cup f^*(\alpha), [M]\rangle = \langle \alpha^5, \bar{\nu}_*[M]\rangle$. By the Atiyah-Hirzebruch spectral sequence, there is a nonsplit exact sequence

$$0 \to \mathbb{Z}/8 \to \Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty; 2\eta) \to H_5(\mathbb{RP}^\infty) \to 0.$$

Note that the bordism class $[M, \bar{\nu}] \in \Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty; 2\eta)$ corresponds to the Pin^+-bordism class of a characteristic submanifold, which we denote by q. Therefore $\bar{\nu}_*[M] \equiv q \pmod 2$. This implies $r + q$ is odd.

Type I: the Wu classes of M are $v_1 = 0$ and $v_2 = w_2(M) = \bar{\nu}^* w_2(\gamma)$. Therefore $w_4(M) = Sq^2 v_2 = \bar{\nu}^* w_2(\gamma)^2$ and $\langle w_4(M) \cup f^*(\alpha), [M]\rangle = \langle \alpha \cup \beta^2, \bar{\nu}_*[M]\rangle$. Check on the generators of $\Omega_5^{\mathrm{Spin}}(\mathbb{RP}^\infty \times \mathbb{CP}^\infty; \gamma)$, $\mathbb{RP}^5 \sharp_{S^1} (S^2 \times \mathbb{RP}^3)$ with $(q = 1, s = 0)$ and $\mathbb{RP}^5 \sharp_{S^1} (\mathbb{CP}^2 \times S^1)$ with $(q = 1, s = 1)$, it is seen that $\langle \alpha \cup \beta^2, \bar{\nu}_*[M]\rangle \equiv q + s \pmod 2$. This implies the relation $q + s + r \equiv 1 \pmod 2$. □

6. Circle Bundles over 1-connected 4-manifolds

As an application of the main results, in this section we study the classification of certain circle bundles over simply-connected 4-manifolds.

Let X^4 be a simply-connected 4-manifold, smooth or topological, and let ξ be a complex line bundle over X, with first Chern class $c_1(\xi) \in H^2(X;\mathbb{Z})$. Choose a Riemannian metric on ξ, and then the total space of the corresponding circle bundle is a 5-manifold M. The homotopy long exact sequence of the fiber bundle shows that $\pi_1(M) \cong \mathbb{Z}/m$ if $c_1(\xi)$ is an m-multiple of a primitive element.

In [5], a classification of M in terms of the topological invariants of X and $c_1(\xi)$ is obtained for $m = 1$, using the classification theorem of Smale and Barden. It is also known that $H_2(M)$ is torsion-free of rank $H_2(X) - 1$ and that M is of fibered type. In this section, we will apply the classification results to the $m = 2$ case, to give classification of M in terms of the topological invariants of X and $c_1(\xi)$. We will also identify M in the list of standard forms in Theorem 3.7 and Theorem 3.11.

§6A. Invariants of M. In this subsection we collect the basic algebraic-topological invariants of M.

Proposition 6.1. *Let M^5 be a circle bundle over a simply-connected 4-manifold X, with first Chern class $c_1(\xi) = 2 \cdot$ primitive, then*

1. $\pi_1(M) \cong \mathbb{Z}/2$
2. $H_2(M) \cong \mathbb{Z}^r$ *where* $r = \operatorname{rank} H_2(X) - 1$.
3. *the $\pi_1(M)$–action on $H_2(\widetilde{M})$ is trivial.*
4. *the type of M^5 is given by*

type I	*type* II	*type* III
$w_2(X) \neq 0$ $w_2(X) \not\equiv c_1(\widetilde{\xi}) \pmod 2$	$w_2(X) = 0$	$w_2(X) \equiv c_1(\widetilde{\xi}) \pmod 2$

Proof. First of all, the homotopy long exact sequence

$$\pi_1(S^1) \to \pi_1(M) \to \pi_1(X)$$

implies that $\pi_1(M)$ is a cyclic group. The Gysin sequence

$$0 \to H_2(M) \to H_2(X) \xrightarrow{\cap c_1} H_0(X) \to H_1(M) \to 0$$

shows that $H_2(M)$ is torsion-free of rank equal to $\operatorname{rank} H_2(X) - 1$ and $H_1(M) \cong \mathbb{Z}/2$ since $c_1(\xi) = 2 \cdot (\text{primitive})$. Note that the universal cover $\widetilde{M}$ is a circle bundle over X, denoted by $\widetilde{\xi}$, with first Chern class $c_1(\widetilde{\xi}) = \frac{1}{2} c_1(\xi)$. The $\pi_1(M)$-action on $\widetilde{M}$ is the antipodal map on each fiber, and thus the commutative diagram

$$\begin{tikzcd} H_2(\widetilde{M}) \arrow[r, "T_*"] \arrow[dr, "p_*"'] & H_2(\widetilde{M}) \arrow[d, "p_*"] \\ & H_2(X) \end{tikzcd}$$

shows that the action on $H_2(\widetilde{M})$ is trivial. For the Stiefel-Whitney class, if X is smooth, we have $TM \oplus \mathbb{R} = p^*(TX \oplus \xi)$ (where p is the projection map), this implies $w_2(M) = p^* w_2(X)$. In general, $X - pt$ admits a smooth structure, then the same argument holds, see [5, Lemma 3]. $\square$

§6B. Smoothings of M.

Proposition 6.2. *Let $\xi \colon S^1 \hookrightarrow M^5 \to X$ be a nontrivial circle bundle over a closed, simply-connected, topological 4-manifold. If $c_1(\xi)$ is an odd multiple of a primitive element, then M is smoothable; if $c_1(\xi)$ is an even multiple of a primitive element, then M admits a smooth structure if and only if $KS(X) = 0$.*

Proof. Let M^5 be a topological 5-manifold, then by [12], the obstruction for smoothing M lies in $H^4(M; \pi_3(Top/O)) = H^4(M; \pi_3(Top/PL)) = H^4(M; \mathbb{Z}/2) \cong H_1(M; \mathbb{Z}/2)$. The latter group is trivial if $c_1(\xi)$ is an odd multiple of a primitive element. On the other hand, we have $TM \oplus \mathbb{R} = \pi^*(TX \oplus \xi)$, where π is the projection map. Therefore the obstruction for smoothing M is $\pi^* KS(X)$. It is seen from the Gysin sequence that $\pi^*\colon H^4(X;\mathbb{Z}/2) \to H^4(M;\mathbb{Z}/2)$ is injective if $c_1(\xi)$ is an even multiple of a primitive element. Therefore M admits a smooth structure if and only if $KS(X) = 0$. □

Now we give a geometric description of the characteristic submanifold of a circle bundle over simply-connected X^4.

Lemma 6.3. *Let $\xi\colon S^1 \hookrightarrow M^5 \to X$ be a circle bundle, $\pi_1(M) \cong \mathbb{Z}/2$. Let $F \subset X$ be an embedded surface dual to $c_1(\widetilde{\xi})$, $N(F)$ be a tubular neighborhood of F in X, $S^1 \hookrightarrow B \to F$ be the restriction of ξ on F. Then there is a double cover map $\partial N(F) \to B$ and the characteristic submanifold of M is $P^4 = (X - \mathring{N}(F)) \cup_\partial B$.*

In other words, the characteristic submanifold P is obtained by removing a tubular neighborhood of an embedded surface dual to $c_1(\widetilde{\xi})$ and then identifying antipodal points on on each fiber.

Proof. Since $c_1(\xi) = 2\cdot$(primitive), the circle bundle is the pull-back of the circle bundle over $\mathbb{C}\mathrm{P}^2$ with first Chern class $= 2\cdot$(primitive):

$$\begin{array}{ccccc}
S^1 & \xrightarrow{=} & S^1 & & \\
\downarrow & & \downarrow & & \\
M^5 & \xrightarrow{f} & \mathbb{R}\mathrm{P}^5 & \supset & \mathbb{R}\mathrm{P}^4 = \mathbb{R}\mathrm{P}^3 \cup D^4 \\
\downarrow & & \downarrow & & \\
X & \xrightarrow{g} & \mathbb{C}\mathrm{P}^2 & & = \mathbb{C}\mathrm{P}^1 \cup D^4
\end{array}$$

Now $P = f^{-1}(\mathbb{R}\mathrm{P}^4) = f^{-1}(D^4 \cup_{S^3} \mathbb{R}\mathrm{P}^3)$. Let $F = g^{-1}(\mathbb{C}\mathrm{P}^1)$ be the transvere preimage of $\mathbb{C}\mathrm{P}^1$, then the normal bundle ν of F in X is the pullback of the Hopf bundle, and the restriction of ξ on F is $\nu \otimes \nu$, therefore there is a double cover $\partial N(F) \to B$. It is easy to see that $P^4 = (X - \mathring{N}(F) \cup_\partial B$. □

Lemma 6.4. *Let P be as above. Then $KS(P) = KS(X)$.*

Proof. We identify $N(F)$ with the normal 2-disk bundle, let V be the associated $\mathbb{R}\mathrm{P}^2$-bundle obtained by identifying antipodal points on $\partial N(F)$. Then by the construction,

$$P = X \cup_{N(F)\times\{0\}} N(F) \times I \cup_{N(F)\times\{1\}} V.$$

Therefore P is bordant to $X \sqcup V$. It was shown by Hsu [10] and Lashof-Taylor [15] that the Kirby-Siebenmann invariant is a bordism invariant, thus $KS(P) = KS(X) + KS(V) = KS(X)$ since V is smooth. □

§6C. Classification. Now we can give a classification of circle bundles over 1-connected 4-manifolds, and identify them with the standard forms in Theorem 3.7 and Theorem 3.11, in terms of the topology of X and ξ.

For the type II manifolds it is an immediate consequence of Theorem 3.1 and Theorem 3.4.

Theorem 6.5 (type II)**.** *Let X be a closed, simply-connected, topological spin 4-manifold, $\xi\colon S^1 \hookrightarrow M^5 \to X$ be a circle bundle over X with $c_1(\xi) = 2\cdot$(primitive). Then we have*

(1) *if $KS(X) = 0$, then M is smoothable and M is diffeomorphic to*

$$(S^2 \times \mathbb{RP}^3) \sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1);$$

(2) *if $KS(X) = 1$, then M is non-smoothable and M is homeomorphic to*

$$*(S^2 \times \mathbb{RP}^3) \sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1).$$

Where $k = \operatorname{rank} H_2(X)/2 - 1$.

Remark 6.6. Note that for a spin 4-manifold X, $\operatorname{rank} H_2(X)$ is even, and thus k is an integer.

For smooth manifolds of type III, we do not know a good invariant detecting the bordism group $\Omega_4^{\mathrm{Pin}^+}$. Therefore we could only determine the diffeomorphism type up to an ambiguity of order 2. This is based on the following exact sequence (see [11, §5])

$$0 \to \mathbb{Z}/2 \to \Omega_4^{\mathrm{Pin}^+} \xrightarrow{\cap w_1^2} \Omega_2^{\mathrm{Pin}^-} \to 0,$$

where $\cap w_1^2$ is the operation of taking a submanifold dual to w_1^2. The generators of $\Omega_2^{\mathrm{Pin}^-}$ is $\pm\mathbb{RP}^2$ and $\cap w_1^2$ maps $\pm\mathbb{RP}^4$ to $\pm\mathbb{RP}^2$. The image of $[P]$ in $\Omega_2^{\mathrm{Pin}^-}$ can be determined from the data of the circle bundle.

In the topological case, we have an epimorphism (see [11, §9])

$$\Omega_4^{\mathrm{TopPin}^+} \to \Omega_2^{\mathrm{TopPin}^-} \cong \mathbb{Z}/8,$$

which is an isomorphism on the subgroup generated by $\mathbb{RP}^4$. By Lemma 6.4, we have $KS(P) = KS(X)$. Therefore by Theorem 3.4, we have a complete topological classification.

Theorem 6.7 (type III)**.** *Let X be a closed, simply-connected topological 4-manifold, and let $\xi\colon S^1 \hookrightarrow M^5 \to X$ be a circle bundle over X with $c_1(\xi) = 2\cdot$(primitive) and $w_2(X) \equiv c_1(\widetilde{\xi}) \pmod 2$. Then we have*

(1) *if X is smooth, then the diffeomorphism type of M (with the induced smooth structure) is determined up to an ambiguity of order 2 by $\operatorname{rank} H_2(X)$ and $\langle c_1(\widetilde{\xi})^2, [X]\rangle \in (\mathbb{Z}/8)/\pm = \{0, 1, 2, 3, 4\}$.*

(2) *M is homeomorphic to $X^5(p, q) \sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1)$, where $q = \langle c_1(\widetilde{\xi})^2, [X]\rangle \in (\mathbb{Z}/8)/\pm = \{0, 1, 2, 3, 4\}$, $k = (\operatorname{rank} H_2(X) - (3 + (-1)^q)/2)/2$, $p = KS(X)$.*

Proof. We only need to prove (1), since the proof of (2) is similar. We see from the proof of Lemma 6.3 that $P = f^{-1}(\mathbb{RP}^4)$, where $f\colon P \to \mathbb{RP}^4$ induces an isomorphism on π_1. If the mod 2 degree of f is 1, then the submanifold dual to $w_1(P)$ is $f^{-1}(\mathbb{RP}^3)$, and the submanifold V dual to $w_1(P)^2$ is $f^{-1}(\mathbb{RP}^2)$. Now we have the following commutative diagram

$$\begin{array}{ccccc} S^1 & \xrightarrow{=} & S^1 & & \\ \downarrow & & \downarrow & & \\ \partial N(F) & \xrightarrow{f} & \mathbb{R}\mathrm{P}^3 & & \supset \mathbb{R}\mathrm{P}^2 = D^2 \cup S^1 \\ \downarrow & & \downarrow & & \\ F & \xrightarrow{g} & \mathbb{C}\mathrm{P}^1 & & = D^2 \cup pt \end{array}$$

Let $d = \deg g = \langle c_1(\widetilde{\xi})^2, [X]\rangle$ and $D = g^{-1}(pt) = \{p_1, \cdots, p_d\}$, it is seen that $V = f^{-1}(\mathbb{R}\mathrm{P}^2) = (F - D) \cup_\partial d \cdot S^1$ (where the glueing map is of degree 2) and $[V] = d \cdot [\mathbb{R}\mathrm{P}^2] \in \Omega_2^{\mathrm{Pin}^-}$. If the mod 2 degree of f is zero, then we consider the circle bundle over $X \sharp \mathbb{C}\mathrm{P}^2$ with first Chern class $(c_1(\xi), 2)$. The corresponding map has nonzero mod 2 degree, the image of the corresponding characteristic submanifold in $\Omega_2^{\mathrm{Pin}^-}$ equals to that of the original one plus 1. Finally $\langle (c_1(\widetilde{\xi}), 1)^2, [X \sharp \mathbb{C}\mathrm{P}^2]\rangle = \langle c_1(\widetilde{\xi})^2, [X]\rangle + 1$. This proves the theorem. □

For the manifolds of type I, we have

Theorem 6.8 (type I)**.** *Let X be a closed, simply-connected non-spin topological 4-manifold, and let $\xi\colon S^1 \hookrightarrow M^5 \to X$ be a circle bundle over X with $c_1(\xi) = 2\cdot$(primitive) and $w_2(X) \not\equiv c_1(\widetilde{\xi})$* (mod 2)*. We have*

(1) *if $KS(X) = 0$, then M is smoothable and*

- *if $\langle w_2(X)^2, [X]\rangle \equiv \langle c_1(\widetilde{\xi})^2, [X]\rangle$* (mod 2)*, then M is diffeomorphic to*
$$X^5(q) \sharp_{S^1} (S^2 \times \mathbb{R}\mathrm{P}^3) \sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1),$$
where $q = \langle c_1(\widetilde{\xi})^2, [X]\rangle \in (\mathbb{Z}/8)/\pm = \{0, 1, 2, 3, 4\}$ and
$$k = \frac{1}{2}(\operatorname{rank} H_2(X) - \frac{1}{2}(7 + (-1)^q));$$
- *if $\langle w_2(X)^2, [X]\rangle \not\equiv \langle c_1(\widetilde{\xi})^2, [X]\rangle$* (mod 2)*, then M is diffeomorphic to*
$$X^5(q) \sharp_{S^1} (\mathbb{C}\mathrm{P}^2 \times S^1) \sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1),$$
where $q = \langle c_1(\widetilde{\xi})^2, [X]\rangle \in (\mathbb{Z}/8)/\pm = \{0, 1, 2, 3, 4\}$ and
$$k = \frac{1}{2}(\operatorname{rank} H_2(X) - \frac{1}{2}(5 + (-1)^q)).$$

(2) *if $KS(X) = 1$, then M is non-smoothable and*

- *if $\langle w_2(X)^2, [X]\rangle \equiv \langle c_1(\widetilde{\xi})^2, [X]\rangle$* (mod 2)*, then M is homeomorphic to*
$$X^5(1, q) \sharp_{S^1} (S^2 \times \mathbb{R}\mathrm{P}^3) \sharp_{S^1} ((\sharp_k S^2 \times S^2) \times S^1),$$
where $q = \langle c_1(\widetilde{\xi})^2, [X]\rangle \in (\mathbb{Z}/8)/\pm = \{0, 1, 2, 3, 4\}$ and
$$k = \frac{1}{2}(\operatorname{rank} H_2(X) - \frac{1}{2}(7 + (-1)^q));$$

- *if* $\langle w_2(X)^2, [X]\rangle \not\equiv \langle c_1(\widetilde{\xi})^2, [X]\rangle \pmod 2$*, then* M *is homeomorphic to*
$$X^5(1,q)\,\sharp_{S^1}(\mathbb{C}\mathrm{P}^2\times S^1)\,\sharp_{S^1}((\sharp_k\, S^2\times S^2)\times S^1),$$
where $q = \langle c_1(\widetilde{\xi})^2, [X]\rangle \in (\mathbb{Z}/8)/\pm = \{0,1,2,3,4\}$ *and*
$$k = \frac{1}{2}(\operatorname{rank} H_2(X) - \frac{1}{2}(5+(-1)^q)).$$

Remark 6.9. Note that for a 4-manifold X, $\langle w_2(X)^2, [X]\rangle \equiv \operatorname{rank} H_2(X) \pmod 2$. This ensures that k is an integer.

Proof. We only need to prove (1); the proof of (2) is similar. Recall that we have $\Omega_4^{\mathrm{Pin}^c} \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$, with generators $\mathbb{R}\mathrm{P}^4$ and $\mathbb{C}\mathrm{P}^2$. Thus the q-component is determined as in the type III case. The s-component of P is determined by the bordism number $\langle w_2(P)^2, [P]\rangle \in \mathbb{Z}/2$. (Here we use the notations given before Theorem 3.6.) Since $KS(X) = 0$, there exists an integer m such that $X_0 = X\,\sharp\, m(S^2\times S^2)$ is smooth. Note that if we do the same construction on X_0 we get $P_0 = P\,\sharp\, m(S^2\times S^2)$, and $\langle w_2(P_0)^2, [P_0]\rangle = \langle w_2(P)^2, [P]\rangle$. Therefore, to compute the s-component, we may assume that X is smooth. Recall that $P = (X - \mathring{N}(F)) \cup_\partial B$, it is seen that the bordism class of P is determined by the bordism class of the pair (X, F), which can be viewed as a singular manifold $(X, f) \in \Omega_4(BU(1)) \cong \Omega_4 \oplus H_4(BU(1))$. We have two homomorphisms
$$\Omega_4(BU(1)) \to \mathbb{Z}/2, \quad [X,F] \mapsto \langle w_2(P)^2, [P]\rangle$$
and
$$\Omega_4(BU(1)) \to \mathbb{Z}/2, \quad [X, c_1(\widetilde{\xi})] \mapsto \langle w_2(X)^2 + c_1(\widetilde{\xi})^2, [X]\rangle.$$
By a check on the generators $(\mathbb{C}\mathrm{P}^2\,\sharp\,(S^2\times S^2), c_1(\widetilde{\xi}) = (1,0,1))$ and $(\mathbb{C}\mathrm{P}^2\,\sharp\,(S^2\times S^2), c_1(\widetilde{\xi}) = (0,0,1))$, we see that $s = \langle w_2(P)^2, [P]\rangle = \langle w_2(X)^2 + c_1(\widetilde{\xi})^2, [X]\rangle \pmod 2$. The two cases correspond to the values $s = 0$ and $s = 1$. For the proof of (2), the only change is that $\Omega_4^{\mathrm{Top}} \cong \mathbb{Z}\oplus\mathbb{Z}/2$ with generators $\mathbb{C}\mathrm{P}^2$ and $*\mathbb{C}\mathrm{P}^2\,\sharp\,\overline{\mathbb{C}\mathrm{P}^2}$ [10]. $\square$

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