HOMOTOPY REPRESENTATIONS OVER THE ORBIT CATEGORY

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Abstract. Let G be a finite group. The unit sphere in a finite-dimensional orthogonal G-representation motivates the definition of homotopy representations, due to tom Dieck. We introduce an algebraic analogue, and establish its basic properties including the Borel-Smith conditions and realization by finite G-CW-complexes.

1. Introduction

Let G be a finite group. The unit spheres $S(V)$ in finite-dimensional orthogonal representations of G provide the basic examples of smooth G-actions on spheres. Moreover, character theory reveals intricate relations between the dimensions of the fixed sets $S(V)^H$, for subgroups $H \leq G$, and the structure of the isotopy subgroups $\{G_x \,|\, x \in S(V)\}$. Our goal is to better understand the constraints on these basic invariants, in order to construct new smooth *non-linear* finite group actions on spheres (see [7], [8]).

In order to put this problem in a more general setting, tom Dieck [11, II.10.1] introduced geometric *homotopy representations*, as finite G-CW-complexes X with the property that each fixed set X^H is homotopy equivalent to a sphere. In this paper, we study an algebraic version of this notion for R-module chain complexes over the orbit category $\varGamma_G = \mathrm{Or}_{\mathcal{F}}\, G$, with respect to a ring R and a family $\mathcal{F}$ of subgroups of G. We usually work with $R = \mathbb{Z}_{(p)}$, for some prime p, or $R = \mathbb{Z}$. This theory was developed by Lück [9, §9, §17] and tom Dieck [11, §10-11].

The homological dimensions of the various fixed sets are encoded in a (super) class function $\underline{n}\colon \mathcal{F} \to \mathbb{Z}$. We say that a finite projective chain complex $\mathbf{C}$ over $R\varGamma_G$ is an R-homology $\underline{n}$-sphere if the reduced homology of $\mathbf{C}(K)$ is the same as the reduced homology of an $\underline{n}(K)$-sphere (with coefficients in R) for all $K \in \mathcal{F}$. In Section 4 we show that the dimension functions of such complexes satisfy the well-known Borel-Smith relations (see Theorem 4.2).

If $\mathbf{C}$ is an R-homology $\underline{n}$-sphere, which satisfies the internal homological conditions observed for representation spheres (see Definition 2.6), then we say that $\mathbf{C}$ is an *algebraic homotopy representation*. By [11, II.10], these conditions are all necessary for $\mathbf{C}$ to be chain homotopy equivalent to a geometric homotopy representation. In Proposition 2.8, we show more generally that these conditions hold for $\mathbf{C}$ an R-homology $\underline{n}$-sphere, whenever its homology dimension function $\underline{n} = \mathrm{Dim}\, \mathbf{C}$, where Dim denote the chain complex dimension function. In this case, we say that $\mathbf{C}$ is a *tight* complex.

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In general, $\underline{n}(K) \leq \operatorname{Dim} \mathbf{C}(K)$ for each $K \in \mathcal{F}$, and one would expect obstructions to finding a tight complex chain homotopy equivalent to a given R-homology $\underline{n}$-sphere. Our first main result shows the relevance of the internal homological conditions for this question.

Theorem A. *Let $\mathbf{C}$ be a finite free chain complex of $R\varGamma_G$-modules which is a homology $\underline{n}$-sphere. Then $\mathbf{C}$ is chain homotopy equivalent to a finite free chain complex $\mathbf{D}$ which is tight if and only if $\mathbf{C}$ is an algebraic homotopy representation.*

When these conditions hold for $R = \mathbb{Z}$, then we apply [7, Theorem 8.10], [10] to obtain a geometric realization result.

Corollary B. *Let $\mathbf{C}$ be a finite free chain complex of $\mathbb{Z}\varGamma_G$-modules which is a homology $\underline{n}$-sphere. If $\mathbf{C}$ is an algebraic homotopy representation, and in addition, if $\underline{n}(K) \geq 3$ for all $K \in \mathcal{F}$, then there is a finite G-CW-complex X such that $\mathbf{C}(X^?; \mathbb{Z})$ is chain homotopy equivalent to $\mathbf{C}$ as chain complexes of $\mathbb{Z}\varGamma_G$-modules.*

We are interested in constructing finite G-CW-complexes with a given family of isotropy subgroups. In particular, we will use Theorem A and Corollary B to study the following:

Question. Which finite groups G admit a finite G-CW-complex X with rank one isotropy, such that X is homotopy equivalent to a sphere ?

One motivation for this work is that rank one isotropy examples lead to free G-CW-complex actions of finite groups on *products* of spheres (see Adem and Smith [1]).

In [7] we gave the first non-trivial example, by constructing a finite G-CW-complex $X \simeq S^n$ for the symmetric group $G = S_5$, with cyclic 2-group isotropy, but the arguments used special features of the isotropy family. Corollary B now provides an effective general method for the geometric realization of algebraic models. The algebraic homotopy representation conditions are easy to check locally over $R = \mathbb{Z}_{(p)}$ at each prime, and fit well with the local-to-global procedure for constructing chain complexes $\mathbf{C}$ over $\mathbb{Z}\varGamma_G$. In a sequel [8] to this paper, we apply Corollary B to construct infinitely many new examples with rank one isotropy, for certain interesting families of rank two groups.

Here is a brief outline of the paper. In Section 2 we give the precise setting and definitions for the concepts just presented (see Definition 2.6) and prove the "only if" direction of Theorem A. The "if" direction of Theorem A is proved in Section 3. Corollary B is also proved in this section. In Section 4 we discuss the Borel-Smith conditions.

2. Algebraic homotopy representations

Let G be a finite group and $\mathcal{F}$ be a family of subgroups of G which is closed under conjugations and taking subgroups. The orbit category $\operatorname{Or}_{\mathcal{F}} G$ is defined as the category whose objects are orbits of type G/K, with $K \in \mathcal{F}$, and where the morphisms from G/K to G/L are given by G-maps:

$$\operatorname{Mor}_{\operatorname{Or}_{\mathcal{F}} G}(G/K, G/L) = \operatorname{Map}_G(G/K, G/L).$$

The category $\varGamma_G = \operatorname{Or}_{\mathcal{F}} G$ is a small category, and we can consider the module category over $\varGamma_G$. Let R be a commutative ring with 1. A (right) $R\varGamma_G$-module M is a contravariant functor from $\varGamma_G$ to the category of R-modules. We denote the R-module $M(G/K)$ simply by $M(K)$ and write $M(f)\colon M(L) \to M(K)$ for a G-map $f\colon G/K \to G/L$.

The category of $R\varGamma_G$-modules is an abelian category, so the usual concepts of homological algebra, such as kernel, direct sum, exactness, projective module, etc., exist for $R\varGamma_G$-modules. Note that an exact sequence of $R\varGamma_G$-modules $0 \to A \to B \to C \to 0$ is exact if and only if

$$0 \to A(K) \to B(K) \to C(K) \to 0$$

is an exact sequence of R-modules for every $K \in \mathcal{F}$. For an $R\varGamma_G$-module M the R-module $M(K)$ can also be considered as an $RW_G(K)$-module in an obvious way where $W_G(K) = N_G(K)/K$. We will follow the convention in [9] and consider $M(K)$ as a right $RW_G(K)$-module. In particular, we will consider the sequence above as an exact sequence of right $RW_G(K)$-modules. The further details about the properties of modules over the orbit category, such as the definitions of free and projective modules, can be found in [7] (see also Lück [9, §9,§17] and tom Dieck [11, §10-11]).

In this section we consider chain complexes $\mathbf{C}$ of $R\varGamma_G$-modules. When we say a chain complex we always mean a non-negative complex, so $\mathbf{C}_i = 0$ for $i < 0$. We call a chain complex $\mathbf{C}$ *projective* (resp. *free*) if for all $i \geq 0$, the modules $\mathbf{C}_i$ are projective (resp. free). We say that a chain complex $\mathbf{C}$ is *finite* if $\mathbf{C}_i = 0$ for $i > n$, and the chain modules $\mathbf{C}_i$ are all finitely-generated $R\varGamma_G$-modules. We define the isotropy family of a chain complex $\mathbf{C}$ over $R\varGamma_G$ as the family of subgroups

$$\mathrm{Iso}(\mathbf{C}) = \{H \in \mathcal{F} \mid \mathbf{C}(H) \neq 0\}.$$

Given a G-CW-complex X, there is an associated chain complex of $R\varGamma_G$-modules

$$\mathbf{C}(X^?; R): \quad \cdots \to R[X_n{}^?] \xrightarrow{\partial_n} R[X_{n-1}{}^?] \to \cdots \xrightarrow{\partial_1} R[X_0{}^?] \to 0$$

where X_i denotes the set of i-dimensional cells in X and $R[X_i{}^?]$ is the $R\varGamma_G$-module defined by $R[X_i{}^?](H) = R[X_i^H]$ for every $H \in \mathcal{F}$. We denote the homology of this complex by $H_*(X^?; R)$. Note that the chain complex $\mathbf{C}(X^H; R)$ is actually defined for all subgroups $H \leq G$, but for a given family of subgroups $\mathcal{F}$, we restrict its values from $\mathrm{Or}(G)$ to the full sub-category $\mathrm{Or}_{\mathcal{F}} G$. In particular, we have $\mathrm{Iso}(\mathbf{C}(X^?; R)) = \mathcal{F} \cap \{H \leq G \mid X^H \neq \emptyset\}$. If the family $\mathcal{F}$ includes all the isotropy subgroups of X, then the complex $\mathbf{C}(X^?; R)$ is a chain complex of free $R\varGamma_G$-modules, hence projective $R\varGamma_G$-modules, otherwise it may not be a chain complex of projective $R\varGamma_G$-modules.

Given a finite dimensional G-CW-complex X, there is a *dimension function*

$$\mathrm{Dim}\, X\colon \mathscr{S}(G) \to \mathbb{Z},$$

given by $(\mathrm{Dim}\, X)(H) = \dim X^H$ for all $H \in \mathscr{S}(G)$ where $\mathscr{S}(G)$ denote the set of all subgroups of G. In a similar way, we define the following.

Definition 2.1. The *dimension function* of a finite dimensional chain complex $\mathbf{C}$ over $R\varGamma_G$ is defined as the function $\mathrm{Dim}\, \mathbf{C}\colon \mathscr{S}(G) \to \mathbb{Z}$ which has the value

$$(\mathrm{Dim}\, \mathbf{C})(H) = \dim \mathbf{C}(H)$$

for all $H \in \mathrm{Iso}(\mathbf{C})$, where the dimension of a chain complex of R-modules is defined as the largest integer d such $C_d \neq 0$. If $H \notin \mathrm{Iso}(\mathbf{C})$, then we take $(\mathrm{Dim}\, \mathbf{C})(H) = -1$.

Remark 2.2. Recall that a function $\underline{n}\colon \mathscr{S}(G) \to \mathbb{Z}$ is called a *super class function* if it is constant on conjugacy classes of G. We say that a super class function $\underline{n}\colon \mathscr{S}(G) \to \mathbb{Z}$ is *defined on* $\mathcal{F}$, if $\underline{n}(H) = -1$ for all subgroups $H \notin \mathcal{F}$. For such a function, we sometimes use the notation $\underline{n}\colon \mathcal{F} \to \mathbb{Z}$ instead of $\underline{n}\colon \mathscr{S}(G) \to \mathbb{Z}$.

Note that the dimension function $\operatorname{Dim} \mathbf{C}$ of a chain complex $\mathbf{C}$ over $R\varGamma_G$ is a super class function defined on $\mathcal{F}$, in fact, it is defined on the smaller family $\operatorname{Iso}(\mathbf{C})$.

In a similar way, we can define the *homological dimension function* of a chain complex $\mathbf{C}$ of $R\varGamma_G$-modules as the function $\operatorname{HomDim} \mathbf{C}\colon \mathcal{F} \to \mathbb{Z}$ where for each $H \in \mathcal{F}$, the integer

$$(\operatorname{HomDim} \mathbf{C})(H) = \operatorname{hdim} \mathbf{C}(H)$$

is defined as the homological dimension of the complex $\mathbf{C}(H)$.

Let us write $(H) \leq (K)$ whenever $H^g \leq K$ for some $g \in G$. Here (H) denotes the set of subgroups conjugate to H in G. The notation $(H) < (K)$ means that $(H) \leq (K)$ but $(H) \neq (K)$.

Definition 2.3. We call a function $\underline{n}\colon \mathscr{S}(G) \to \mathbb{Z}$ *monotone* if it satisfies the property that $\underline{n}(K) \leqslant \underline{n}(H)$ whenever $(H) \leq (K)$. We say that a monotone function $\underline{n}$ is *strictly monotone* if $\underline{n}(K) < \underline{n}(H)$, whenever $(H) < (K)$. □

We have the following:

Lemma 2.4. *The dimension function of a projective chain complex of $R\varGamma_G$-modules is a monotone function.*

Proof. By the decomposition theorem for projective $R\varGamma_G$-modules [11, Chap. I, Theorem 11.18], every projective $R\varGamma_G$-module P is of the form $P \cong \oplus_H E_H P_H$ where P_H is a projective $N_G(H)/H$-module. If $\underline{n}(K) = n \neq -1$, then $K \in \mathcal{F}$ and $\mathbf{C}_n(K) \neq 0$, so $\mathbf{C}_n$ must have a summand $E_H P_H$ with $(K) \leq (H)$. But then we will have $\mathbf{C}_n(L) \neq 0$, and hence $\underline{n}(K) \leq \underline{n}(L)$, for every $(L) \leq (K)$. If $\underline{n}(K) = -1$, then the inequality $\underline{n}(K) \leq \underline{n}(L)$ holds for every $(L) \leq (K)$. □

We are particularly interested in chain complexes which have the homology of a sphere when evaluated at every $K \in \mathcal{F}$. To specify the restriction maps in dimension zero, we will consider chain complexes which are augmented: chain complexes $\mathbf{C}$ together with a map $\varepsilon\colon \mathbf{C}_0 \to \underline{R}$ such that $\varepsilon \circ \partial_1 = 0$ where $\underline{R}$ denotes the constant functor. We often consider ε as a chain map $\mathbf{C} \to \underline{R}$ by considering $\underline{R}$ as a chain complex over $R\varGamma_G$ which is concentrated at zero. By the *reduced homology* of an augmented complex $\varepsilon\colon \mathbf{C} \to \underline{R}$, we always mean the homology of the chain complex

$$\widetilde{\mathbf{C}} = \{\cdots \to C_n \xrightarrow{\partial_n} \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \underline{R} \to 0\}$$

where $\underline{R}$ is considered to be at dimension -1. Note that the complex $\widetilde{\mathbf{C}}$ is the -1 shift of the mapping cone of the chain map $\varepsilon : \mathbf{C} \to \underline{R}$.

Definition 2.5. Let $\underline{n}$ be a super class function defined on $\mathcal{F}$, and let $\mathbf{C}$ be a chain complex over $R\varGamma_G$ with respect to a family $\mathcal{F}$ of subgroups.

(i) We say that $\mathbf{C}$ an *R-homology $\underline{n}$-sphere* if $\mathbf{C}$ is an augmented complex such that the reduced homology of $\mathbf{C}(K)$ is the same as the reduced homology of an $\underline{n}(K)$-sphere (with coefficients in R) for all $K \in \mathcal{F}$.
(ii) We say that $\mathbf{C}$ is *oriented* if the $W_G(K)$-action on the homology of $\mathbf{C}(K)$ is trivial for all $K \in \mathcal{F}$.

Note that we do not assume that the dimension function is strictly monotone as in Definition II.10.1 in [11].

In transformation group theory, a G-CW-complex X is called a *homotopy representation* if it has the property that X^H is homotopy equivalent to the sphere $S^{n(H)}$ where $n(H) = \dim X^H$ for every $H \leq G$ (see tom Dieck [11, Section II.10]). We now introduce an algebraic analogue of this useful notion for chain complexes over the orbit category.

In [11, II.10], there is a list of properties that are satisfied by homotopy representations. We will use algebraic versions of these properties to define an analogous notion for chain complexes.

Definition 2.6. Let $\mathbf{C}$ be a finite projective chain complex over $R\varGamma_G$, which is an R-homology $\underline{n}$-sphere. We say $\mathbf{C}$ is an *algebraic homotopy representation* (over R) if

(i) The function $\underline{n}$ is a monotone function.
(ii) If $H, K \in \mathcal{F}$ are such that $n = \underline{n}(K) = \underline{n}(H)$, then for every G-map $f\colon G/H \to G/K$ the induced map $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ is an R-homology isomorphism.
(iii) Suppose $H, K, L \in \mathcal{F}$ are such that $H \leq K, L$ and let $M = \langle K, L\rangle$ be the subgroup of G generated by K and L. If $n = \underline{n}(H) = \underline{n}(K) = \underline{n}(L) > -1$, then $M \in \mathcal{F}$ and $n = \underline{n}(M)$.

Under condition (iii) of Definition 2.6, the isotropy family $\mathcal{F}$ has an important maximality property.

Corollary 2.7. *Let $\mathbf{C}$ be a projective chain complex of $R\varGamma_G$-modules, If condition* (iii) *holds, then the set of subgroups $\mathcal{F}_H = \{K \in \mathcal{F} \,|\, (H) \leq (K),\ \underline{n}(K) = \underline{n}(H) > -1\}$ has a unique maximal element, up to conjugation.*

In the remainder of this section we will assume that R is a principal ideal domain. The important examples for us are $R = \mathbb{Z}_{(p)}$ or $R = \mathbb{Z}$. The main result of this section is the following proposition.

Proposition 2.8. *Let $\mathbf{C}$ be a finite projective chain complex over $R\varGamma_G$, which is an R-homology $\underline{n}$-sphere. If the equality $\underline{n} = \operatorname{Dim}\mathbf{C}$ holds, then $\mathbf{C}$ is an algebraic homotopy representation.*

Before we prove Proposition 2.8, we make some observations and give some definitions for projective chain complexes.

Lemma 2.9. *Let $\mathbf{C}$ be a projective chain complex over $R\varGamma_G$. Then, for every G-map $f\colon G/H \to G/K$, the induced map $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ is an injective map with an R-torsion free cokernel.*

Proof. It is enough to show that if P a projective $R\varGamma_G$-module, then for every G-map $f\colon G/H \to G/K$, the induced map $P(f)\colon P(K) \to P(H)$ is an injective map with a torsion free cokernel. Since every projective module is a direct summand of a free module, it is enough to prove this for a free module $P = R[X^?]$. Let $f\colon G/H \to G/K$ be the G-map defined by $f(H) = gK$. Then the induced map $P(f)\colon R[X^K] \to R[X^H]$ is the linearization of the map $X^K \to X^H$ given by $x \mapsto gx$. Since this map is one-to-one, we can conclude that $P(f)$ is injective with torsion free cokernel. □

When $H \leq K$ and $f\colon G/H \to G/K$ is the G-map defined by $f(H) = K$, then we denote the induced map $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ by r_H^K and call it the *restriction* map. When H and K are conjugate, so that $K = H^g$ for some $g \in G$, then the map $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ induced by the G-map $f\colon G/H \to G/K$ defined by $f(H) = gK$ is called the *conjugation* map and usually denoted by c_K^g. Note that every G-map can be written as a composition of two G-maps of the above two types, so every induced map $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ can be written as a composition of restriction and conjugation maps.

Since conjugation maps have inverses, they are always isomorphisms. So, the condition (ii) of Definition 2.6 is actually a statement only about restriction maps. To study the restriction maps more closely, we consider the image of $r_H^K\colon \mathbf{C}(K) \to \mathbf{C}(H)$ for a pair $H \leq K$ and denote it by $\mathbf{C}_H^K$. Note that $\mathbf{C}_H^K$ is a subcomplex of $\mathbf{C}(H)$ as a chain complex of R-modules. Also note that if $\mathbf{C}$ is a projective chain complex, then $\mathbf{C}_H^K$ is isomorphic to $\mathbf{C}(K)$, as a chain complex of R-modules, by Lemma 2.9.

Lemma 2.10. *Let $\mathbf{C}$ be a projective chain complex over $R\varGamma_G$. Suppose that $K, L \in \mathcal{F}$ such that $H \leq K$ and $H \leq L$, and let $M = \langle K, L\rangle$ be the subgroup generated by K and L. If $\mathbf{C}_H^K \cap \mathbf{C}_H^L \neq 0$ then $M \in \mathcal{F}$ and we have $\mathbf{C}_H^K \cap \mathbf{C}_H^L = \mathbf{C}_H^M$.*

Proof. As before it is enough to prove this for a free $R\varGamma_G$-module $P = R[X^?]$ where X is a G-set whose isotropy subgroups lie in $\mathcal{F}$. Note that the restriction maps r_H^K and r_H^L are linearizations of the maps $X^K \to X^H$ and $X^L \to X^H$, respectively, which are defined by inclusion of subsets. Then it is clear that the intersection of images of r_H^K and r_H^L would be $R[X^K \cap X^L]$ considered as an R-submodule of $R[X^H]$. There is a well known equality $X^K \cap X^L = X^M$ for fixed point sets. Therefore, if $\mathbf{C}_H^K \cap \mathbf{C}_H^L \neq 0$, then we must have $X^M \neq \emptyset$. This implies that $M \in \mathcal{F}$ and that $\operatorname{im} r_H^K \cap \operatorname{im} r_H^L = \operatorname{im} r_H^M$. □

Now, we are ready to prove Proposition 2.8.

Proof of Proposition 2.8. The first condition in Definition 2.6 follows from Lemma 2.4. For (ii) and (iii), we use the arguments similar to the arguments given in II.10.12 and II.10.13 in [11].

To prove (ii), let $f\colon G/H \to G/K$ be a G-map. By Lemma 2.9, the induced map $\mathbf{C}(f) : \mathbf{C}(K) \to \mathbf{C}(H)$ is injective with torsion free cokernel. Let $\mathbf{D}$ denote the cokernel of $\mathbf{C}(f)$. Then we have a short exact sequence of R-modules

$$0 \to \mathbf{C}(K) \to \mathbf{C}(H) \to \mathbf{D} \to 0$$

where both $\mathbf{C}(K)$ and $\mathbf{C}(H)$ have dimension n. Now consider the long exact *reduced* homology sequence (with coefficients in R) associated to this short exact sequence:

$$\cdots \to 0 \to H_{n+1}(\mathbf{D}) \to H_n(\mathbf{C}(K)) \xrightarrow{f^*} H_n(\mathbf{C}(H)) \to H_n(\mathbf{D}) \to \cdots$$

Note that $\mathbf{D}$ has dimension less than or equal to n, so $H_{n+1}(\mathbf{D}) = 0$ and $H_n(\mathbf{D})$ is torsion free. Since $H_n(\mathbf{C}(K)) = H_n(\mathbf{C}(H)) = R$, we obtain that f^* is an isomorphism. Since both $\mathbf{C}(K)$ and $\mathbf{C}(H)$ have no other reduced homology, we conclude that $\mathbf{C}(f)$ induces an R-homology isomorphism between associated augmented complexes. Since the induced map $\underline{R}(f) : \underline{R}(K) \to \underline{R}(H)$ is the identity map id : $R \to R$, the chain map $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ is an R-homology isomorphism.

To prove (iii), observe that there is a Mayer-Vietoris type exact sequence associated to the pair of complexes $\mathbf{C}_H^K$ and $\mathbf{C}_H^L$ which gives an exact sequence of the form

$$0 \to H_n(\mathbf{C}_H^K \cap \mathbf{C}_H^L) \to H_n(\mathbf{C}_H^K) \oplus H_n(\mathbf{C}_H^L) \to H_n(\mathbf{C}_H^K + \mathbf{C}_H^L) \to H_{n-1}(\mathbf{C}_H^K \cap \mathbf{C}_H^L) \to 0.$$

Here we again take the homology sequence as the reduced homology sequence.

Let $i^K\colon \mathbf{C}_H^K \to \mathbf{C}(H)$, $i_H^L\colon \mathbf{C}_H^L \to \mathbf{C}(H)$, and $j\colon \mathbf{C}_H^K + \mathbf{C}_H^L \to \mathbf{C}(H)$ denote the inclusion maps. We have zero on the left-most term since $\mathbf{C}_H^K + \mathbf{C}_H^L$ is an n-dimensional complex. To see the zero on the right-most term, note that by Lemma 2.9, $\mathbf{C}_H^K \cong \mathbf{C}(K)$ and $\mathbf{C}_H^L \cong \mathbf{C}(L)$ as chain complexes of R-modules, so they have the same homology. This gives that $H_i(\mathbf{C}_H^K) = H_i(\mathbf{C}_H^L) = 0$ for $i \leq n-1$.

Also note that by part (ii), the composition

$$H_n(\mathbf{C}(K)) \cong H_n(\mathbf{C}_H^K) \xrightarrow{i_*^K} H_n(\mathbf{C}_H^K + \mathbf{C}_H^L) \xrightarrow{j_*} H_n(\mathbf{C}(H))$$

is an isomorphism. So, j_* is surjective. Since $H_{n+1}(\mathbf{C}(H)/(\mathbf{C}_H^K + \mathbf{C}_H^L)) = 0$, we see that j_* is also injective. Therefore, j_* is an isomorphism. This implies that i_*^K is an isomorphism. Similarly one can show that $i_*^L\colon H_n(\mathbf{C}_H^L) \to H_n(\mathbf{C}_H^K + \mathbf{C}_H^L)$ is also an isomorphism. Using these isomorphisms and looking at the exact sequence above, we conclude that $H_n(\mathbf{C}_H^K \cap \mathbf{C}_H^L) \cong R$ and $H_i(\mathbf{C}_H^K \cap \mathbf{C}_H^L) = 0$ for $i \leq n-1$. So, $\mathbf{C}_H^K \cap \mathbf{C}_H^L$ is an R-homology n-sphere.

Since $n > -1$, this implies that $\mathbf{C}_H^K \cap \mathbf{C}_H^L \neq 0$, and hence $M = \langle K, L \rangle \in \mathcal{F}$ by Lemma 2.10. Moreover, $\mathbf{C}_H^K \cap \mathbf{C}_H^L = \mathbf{C}_H^M$. This proves that $\underline{n}(M) = n$ as desired. □

3. The Proof of Theorem A

In this section we will again assume that R is a principal ideal domain. The main examples for us are $R = \mathbb{Z}_{(p)}$ or $R = \mathbb{Z}$, as before.

Definition 3.1. We say a chain complex $\mathbf{C}$ of $R\varGamma_G$-modules is *tight at* $H \in \mathcal{F}$ if

$$\operatorname{Dim} \mathbf{C}(H) = \operatorname{HomDim} \mathbf{C}(H).$$

We call a chain complex of $R\varGamma_G$-modules *tight* if it is tight at every $H \in \mathcal{F}$.

Suppose that $\mathbf{C}$ is a finite projective complex over $R\varGamma_G$ which is an R-homology $\underline{n}$-sphere. If $\mathbf{C}$ is chain homotopy equivalent to a tight complex, then Proposition 2.8 shows that $\mathbf{C}$ is an algebraic homotopy representation. This establishes one direction of Theorem A. The other direction uses the assumption that the chain modules of $\mathbf{C}$ are free over $R\varGamma_G$.

Theorem 3.2. *Let* $\mathbf{C}$ *be a finite free chain complex of* $R\Gamma_G$*-modules which is a homology* $\underline{n}$*-sphere. If* $\mathbf{C}$ *is an algebraic homotopy representation over* R*, then* $\mathbf{C}$ *is chain homotopy equivalent to a finite free chain complex* $\mathbf{D}$ *which is tight.*

We need to show that the complex $\mathbf{C}$ can be made *tight* at each $H \in \mathcal{F}$ by replacing it with a chain complex homotopic to $\mathbf{C}$. The proof is given in several steps.

3A. Tightness at maximal isotropy subgroups. Let H be a *maximal element* in $\mathcal{F}$. Consider the subcomplex $\mathbf{C}^{(H)}$ of $\mathbf{C}$ formed by free summands of $\mathbf{C}$ isomorphic to $R[G/H^{?}]$. The complex $\mathbf{C}^{(H)}$ is a complex of isotypic modules of type $R[G/H^{?}]$. Recall that free $R\Gamma_G$-module F is called *isotypic* of type G/H if it is isomorphic to a direct sum of copies of a free module $R[G/H^{?}]$, for some $H \in \mathcal{F}$. For extensions involving isotypic modules we have the following:

Lemma 3.3. *Let*

$$\mathcal{E}\colon 0 \to F \to F' \to M \to 0$$

be a short exact sequence of $R\Gamma_G$*-modules such that both* F *and* F' *are isotypic free modules of the same type* G/H*. If* $M(H)$ *is* R*-torsion free, then* $\mathcal{E}$ *splits and* M *is stably free.*

Proof. This is Lemma 8.6 of [7]. The assumption that R is a principal ideal domain ensures that finitely-generated R-torsion free modules are free. □

Note that $\mathbf{C}^{(H)}(H) = \mathbf{C}(H)$, since H is maximal in $\mathcal{F}$. This means that $\mathbf{C}^{(H)}$ is a finite free chain complex over $R\Gamma_G$ of the form

$$\mathbf{C}^{(H)}\colon 0 \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to 0$$

which is a R-homology $\underline{n}(H)$-sphere, with $\underline{n}(H) \leq d$.

Lemma 3.4. *Let* $\mathbf{C}$ *be a finite free chain complex of* $R\Gamma_G$*-modules. Then* $\mathbf{C}$ *is chain homotopy equivalent to a finite free chain complex* $\mathbf{D}$ *which is tight at every maximal element* $H \in \mathcal{F}$*.*

Proof. We apply [7, Proposition 8.7] to the subcomplex $\mathbf{C}^{(H)}$, for each maximal element $H \in \mathcal{F}$. The key step is provided by Lemma 3.3. □

3B. The inductive step. To make the complex $\mathbf{C}$ tight at every $H \in \mathcal{F}$ we use a downward induction, but the situation at an intermediate step is more complicated than the first step considered above.

Suppose that $H \in \mathcal{F}$ is such that $\mathbf{C}$ tight at every $K \in \mathcal{F}$ such that $(K) > (H)$. Let $\mathbf{C}^H$ denote the subcomplex of $\mathbf{C}$ with free summands of type $R[G/K^{?}]$ satisfying $(H) \leq (K)$. In a similar way, we can define the subcomplex $\mathbf{C}^{>H}$ of $\mathbf{C}$ whose free summands are of type $R[G/K^{?}]$ with $(H) < (K)$. The complex $\mathbf{C}^{>H}$ is a subcomplex of $\mathbf{C}^H$. Let us denote the quotient complex $\mathbf{C}^H/\mathbf{C}^{>H}$ by $\mathbf{C}^{(H)}$. As before the complex $\mathbf{C}^{(H)}$ is isotypic with isotropy type $R[G/H^{?}]$. We have a short exact sequence of chain complexes of free $R\Gamma_G$-modules

$$0 \to \mathbf{C}^{>H} \to \mathbf{C}^H \to \mathbf{C}^{(H)} \to 0.$$

By evaluating at H, we obtain an exact sequence of chain complexes

$$0 \to \mathbf{C}^{>H}(H) \to \mathbf{C}^{H}(H) \to \mathbf{C}^{(H)}(H) \to 0$$

which is just the sequence

$$0 \to \mathbf{C}^{>H}(H) \to \mathbf{C}(H) \to S_H\mathbf{C} \to 0$$

defining the splitting functor S_H (see [9, Lemma 9.26]). Note that we also have a sequence

$$0 \to \mathbf{C}^{H} \to \mathbf{C} \to \mathbf{C}/\mathbf{C}^{H} \to 0.$$

If we can show that $\mathbf{C}^H$ is homotopy equivalent to a complex $\mathbf{D}'$ which is tight at H, then by push-out of $\mathbf{D}'$ along the injective map $\mathbf{C}^H \to \mathbf{C}$, we can find a complex $\mathbf{D}$ homotopy equivalent to $\mathbf{C}$ which is tight at every $K \in \mathcal{F}$ with $(H) \leq (K)$. So it is enough to show that $\mathbf{C}^H$ is homotopy equivalent to a complex $\mathbf{D}'$ which is tight at H.

Lemma 3.5. *Let $\mathbf{C}$ be a finite free chain complex of $R\Gamma_G$-modules, such that $\mathbf{C}$ is tight at every $K \in \mathcal{F}$ with $(H) < (K)$, for some $H \in \mathcal{F}$. Suppose*

(i) $n = \operatorname{hdim} \mathbf{C}(H) \geq \dim \mathbf{C}(K)$, *for all $(H) < (K)$, and that*

(ii) $H_{n+1}(S_H\mathbf{C}) = 0$.

Then $\mathbf{C}^H$ is homotopy equivalent to a finite free chain complex $\mathbf{D}'$ which is tight at every $K \in \mathcal{F}$ with $(H) \leq (K)$.

Proof. We first observe that $\mathbf{C}^{>H}$ has dimension $\leq n$, since $\mathbf{C}^{>H}(K) = \mathbf{C}(K)$ for $(H) < (K)$, and $\dim \mathbf{C}(K) \leq n$. Let $d = \dim \mathbf{C}(H)$. If $d = n$, then we are done, so assume that $d > n$. Then $\dim \mathbf{C}^{(H)} = d$, and $\mathbf{C}^{(H)}$ is a complex of the form

$$\mathbf{C}^{(H)}\colon 0 \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to 0.$$

We claim that the first map in the above chain complex is injective. Note that since $\mathbf{C}^{(H)}$ is isotypic of type (H), it is enough to show that this map is injective when it is calculated at H. In other words we claim that $H_d(\mathbf{C}^{(H)}(H)) = H_d(S_H\mathbf{C}) = 0$ when $d > n$. To show this consider the short exact sequence $0 \to \mathbf{C}^{>H}(H) \to \mathbf{C}(H) \to S_H\mathbf{C} \to 0$. Since the complex $\mathbf{C}^{>H}$ has dimension $\leq n$, the corresponding long exact sequence gives that $H_d(S_H\mathbf{C}) \cong H_d(\mathbf{C}(H)) = 0$ when $d > n+1$. If $d = n+1$, then this is true by assumption (ii) in the lemma. Now we apply [7, Proposition 8.7] to $\mathbf{C}^{(H)}$ to obtain a tight complex $\mathbf{D}'' \simeq \mathbf{C}^{(H)}$, and then let $\mathbf{D}' \simeq \mathbf{C}^H$ denote the pullback of $\mathbf{D}''$ along the surjection $\mathbf{C}^H \to \mathbf{C}^{(H)}$. □

3C. **Verifying the inductive step.** To complete the proof of Theorem 3.2, we need to show that the assumptions in Lemma 3.5 hold at an intermediate step of the downward induction. We will make detailed use of the internal homological conditions (i), (ii), and (iii) in Definition 2.6, satisfied by an algebraic homotopy representation $\mathbf{C}$. We proceed as follows:

(1) The dimension assumptions in Lemma 3.5 follow from the condition (i), since when $\underline{n}$ is monotone, we have

$$\operatorname{HomDim} \mathbf{C}(H) = \underline{n}(H) \geq \underline{n}(K) = \operatorname{HomDim} \mathbf{C}(K) = \operatorname{Dim}(K)$$

for all $K \in \mathcal{F}$ with $(H) < (K)$.

(2) The assumption that $H_{n+1}(S_H\mathbf{C}) = 0$ is established in Corollary 3.8.It follows from the conditions (ii) and (iii) and the Mayer-Vietoris argument given below.

In the rest of the section, we assume that $\mathbf{C}$ is a finite projective chain complex of $R\varGamma_G$-modules, which is an R-homology $\underline{n}$-sphere, and satisfies the conditions (i), (ii), and (iii) in Definition 2.6. Assume also that $\mathbf{C}$ is tight for all $K \in \mathcal{F}$ with $(H) < (K)$ for some fixed subgroup $H \in \mathcal{F}$. Let $\mathcal{K}_H$ denote the set of all subgroups $K \in \mathcal{F}$ such that $H < K$ and $\underline{n}(H) = \underline{n}(K)$. By condition (iii) of Definition 2.6, this collection has a unique maximal element M. Let $\mathbf{C}_H^K$ denote the image of the restriction map

$$r_H^K \colon \mathbf{C}(K) \to \mathbf{C}(H),$$

for every $K \in \mathcal{F}$ with $H \le K$. Note that $\mathbf{C}_H^K$ is a subcomplex of $\mathbf{C}(H)$ and by Lemma 2.9, it is isomorphic to $\mathbf{C}(K)$. Moreover, if $K \in \mathcal{K}_H$, then by condition (ii), the subcomplex $\mathbf{C}_H^K$ is an R-homology n-sphere and the map

$$H_n(\mathbf{C}_H^M) \to H_n(\mathbf{C}_H^K)$$

induced by the inclusion map $\mathbf{C}_H^M \hookrightarrow \mathbf{C}_H^K$ is an isomorphism. More generally, the following also holds.

Lemma 3.6. *Let $\mathbf{C}$ and $H \in \mathcal{F}$ be as above, and let $K_1, \dots, K_m$ be a set of subgroups in $\mathcal{K}_H$. Then the subcomplex $\sum_{i=1}^m \mathbf{C}_H^{K_i}$ is an R-homology n-sphere and the map*

$$H_n(\mathbf{C}_H^M) \to H_n(\sum_{i=1}^m \mathbf{C}_H^{K_i}) \tag{3.7}$$

induced by the inclusion maps is an isomorphism.

Proof. The case $m = 1$ follows from the remarks above. For $m > 1$, we have the following Mayer-Vietoris type long exact sequence

$$0 \to H_n(\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m}) \to H_n(\mathbf{D}_{m-1}) \oplus H_n(\mathbf{C}_H^{K_m}) \to H_n(\mathbf{D}_m) \to H_{n-1}(\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m}) \to$$

where $\mathbf{D}_j = \sum_{i=1}^j \mathbf{C}_H^{K_i}$ for $j = m-1, m$. By the inductive assumption, we know that $\mathbf{D}_{m-1}$ is an R-homology n-sphere and the map $H_n(\mathbf{C}_H^M) \to H_n(\mathbf{D}_{m-1})$ is an isomorphism. Note that

$$\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m} = (\sum_{i=1}^{m-1} \mathbf{C}_H^{K_i}) \cap \mathbf{C}_H^{K_m} = \sum_{i=1}^{m-1} (\mathbf{C}_H^{K_i} \cap \mathbf{C}_H^{K_m}) = \sum_{i=1}^{m-1} \mathbf{C}_H^{\langle K_i, K_m \rangle}$$

where the last equality follows from Lemma 2.10. We can apply Lemma 2.10 here because $\mathbf{C}_H^M \subseteq \mathbf{C}_H^K$ for all $K \in \mathcal{K}_H$ gives that $\mathbf{C}_H^{K_i} \cap \mathbf{C}_H^{K_m} \neq 0$ for every $i = 1, \dots, m-1$. Note that we also obtain $\langle K_i, K_m \rangle \in \mathcal{K}_H$ for all i. Applying our inductive assumption again to these subgroups, we obtain that $\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m}$ is an R-homology n-sphere and that the induced map

$$H_n(\mathbf{C}_H^M) \to H_n(\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m})$$

is an isomorphism. This gives that $H_i(\mathbf{D}_m) = 0$ for $i \leq n-1$. We also obtain a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbf{C}_H^M) \arrow[r] \arrow[d] & H_n(\mathbf{C}_H^M) \oplus H_n(\mathbf{C}_H^M) \arrow[r] \arrow[d] & H_n(\mathbf{C}_H^M) \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & H_n(\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m}) \arrow[r] & H_n(\mathbf{D}_{m-1}) \oplus H_n(\mathbf{C}_H^{K_m}) \arrow[r] & H_n(\mathbf{D}_m) \arrow[r] & 0
\end{tikzcd}$$

Since all the vertical maps except the map φ are known to be isomorphisms, we obtain that φ is also an isomorphism by the five lemma. This completes the proof. □

Corollary 3.8. *Let* $\mathbf{C}$ *and* $H \in \mathcal{F}$ *are as above. Then* $H_{n+1}(S_H\mathbf{C}) = 0$.

Proof. Let $\mathcal{K}_H = \{K_1, \dots, K_m\}$. By condition (ii), we know that the composition

$$H_n(\mathbf{C}(M)) \xrightarrow{\cong} H_n(\mathbf{C}_H^M) \to H_n(\sum_{i=1}^{m} \mathbf{C}_H^{K_i}) \to H_n(\mathbf{C}(H))$$

is an isomorphism. However, we have just proved that the middle map is an isomorphism, and that all the modules involved in the composition are isomorphic to R. Therefore, the map induced by inclusion

$$H_n(\sum_{i=1}^{m} \mathbf{C}_H^{K_i}) \to H_n(\mathbf{C}(H))$$

is an isomorphism. Note that if $(H) \leq (K)$ and $\underline{n}(K) < n$, for some $K \in \mathcal{F}$, then $\dim \mathbf{C}(K) < n$. This means that

$$H_n(\mathbf{C}^{>H}(H)) \cong H_n(\sum_{i=1}^{m} \mathbf{C}_H^{K_i}) \cong H_n(\mathbf{C}(H)).$$

From the exact sequence $0 \to \mathbf{C}^{>H}(H) \to \mathbf{C}(H) \to S_H\mathbf{C} \to 0$, and the fact that $\operatorname{HomDim} \mathbf{C}(H) = n$, we conclude that $H_{n+1}(S_H\mathbf{C}) = 0$, as required. □

This completes the proof of Theorem A. In [7], we proved the following realization theorem for free $\mathbb{Z}\Gamma_G$-module chain complexes, with respect to any family $\mathcal{F}$, which are $\mathbb{Z}$-homology $\underline{n}$-spheres satisfying certain extra conditions.

Theorem 3.9 ([7, Theorem 8.10], [10])**.** *Let* $\mathbf{C}$ *be a finite free chain complex of* $\mathbb{Z}\Gamma_G$*-modules which is a* $\mathbb{Z}$*-homology* $\underline{n}$*-sphere. Suppose that* $\underline{n}(K) \geq 3$ *for all* $K \in \mathcal{F}$*. If* $\mathbf{C}_i(H) = 0$ *for all* $i > \underline{n}(H) + 1$*, and all* $H \in \mathcal{F}$*, then there is a finite G-CW-complex X such that* $\mathbf{C}(X^{?}; \mathbb{Z})$ *is chain homotopy equivalent to* $\mathbf{C}$ *as chain complexes of* $\mathbb{Z}\Gamma_G$*-modules.*

Note that a homology $\underline{n}$-sphere $\mathbf{C}$ with $\operatorname{Dim} \mathbf{C} = \underline{n}$, and $\underline{n}(K) \geq 3$ for all $K \in \mathcal{F}$, will automatically satisfy these conditions. So Corollary B follows immediately from Theorem A and Theorem 3.9.

Remark 3.10. The construction actually produces a finite G-CW-complex X with the additional property that all the non-empty fixed sets X^H are simply-connected. Moreover, by construction, $W_G(H) = N_G(H)/H$ will act trivially on the homology of X^H. Therefore

X will be an *oriented* geometric homotopy representation (in the sense of tom Dieck). From the perspective of Theorem A, since we don't specify any dimension function, a G-CW-complex X with all fixed sets X^H integral homology spheres will lead (by three-fold join) to a homotopy representation. The same necessary and sufficient conditions for existence apply.

4. Borel-Smith conditions for chain complexes

If G is a finite group and X is a finite G-CW-complex which is a mod-p homology sphere, then by Smith theory for every p-subgroup $H \leq G$, the fixed point space X^H is also a mod-p homology sphere if it is non-empty. So if we take $R = \mathbb{Z}/p$ and $\varGamma_G$ as the orbit category over the family $\mathcal{F}_p$ of all p-subgroups of G, then the chain complex $\mathbf{C}(X^?; \mathbb{Z})$ over $R\varGamma_G$ is a finite free chain complex which is an R-homology $\underline{n}$-sphere. Here, as before, we take $\underline{n}(H) = -1$ when $X^H = \emptyset$. In this case, it is known that the super class function $\underline{n}$ satisfies certain conditions called the Borel-Smith conditions (see [3, Thm. 2.3 in Chapter XIII] or [11, Section 5]). These conditions are given as follows:

Definition 4.1. Let G be a finite group and let $f \colon \mathscr{S}(G) \to \mathbb{Z}$ be super class function, where $\mathscr{S}(G)$ denote the family of all subgroups of G. We say the function f satisfies the Borel-Smith conditions, if it has the following properties:

(i) If $L \lhd K \leq G$ are such that $K/L \cong \mathbb{Z}/p$, and p is odd, then $f(L) - f(K)$ is even.

(ii) If $L \lhd K \leq G$ are such that $K/L \cong \mathbb{Z}/p \times \mathbb{Z}/p$, and if L_i/L denote the subgroups of order p in K/L, then

$$f(L) - f(K) = \sum_{i=0}^{p} (f(L_i) - f(K)).$$

(iii) If $L \lhd K \lhd N \leq G$ are such that $L \lhd N$, and $K/L \cong \mathbb{Z}/2$, then $f(L) - f(K)$ is even if $N/L \cong \mathbb{Z}/4$, and $f(L) - f(K)$ is divisible by 4 if N/L is the quaternion group of order 8.

We show that these conditions are satisfied by the homological dimension function $\underline{n}$ of a finite projective complex $\mathbf{C}$ over $R\varGamma_G$ which is an R-homology $\underline{n}$-sphere, where $R = \mathbb{Z}/p$ and $\varGamma_G$ is taken over a given family $\mathcal{F}$ of subgroups of G.

Theorem 4.2. *Let G be a finite group, $R = \mathbb{Z}/p$, and $\mathcal{F}$ be a given family of subgroups of G. If $\mathbf{C}$ is a finite projective chain complex over $R\varGamma_G$, which is an R-homology $\underline{n}$-sphere, then the function $\underline{n}$ satisfies the Borel-Smith conditions.*

In the case of a mod-p homology sphere X, these conditions are proved using a reduction argument to the corresponding subquotients. For a subquotient K/L, the reduction comes from considering the fixed point space X^L as a K-space. To do a similar reduction for chain complexes over $R\varGamma_G$, we first introduce a new functor for $R\varGamma_G$-modules, called the *deflation* functor. We will introduce this functor as a restriction functor between corresponding module categories. For this discussion R can be taken as any commutative ring with 1 and $\mathcal{F}_G$ is any family subject to the extra conditions we assume during the construction.

Let N be a normal subgroup of G. We define a functor

$$F\colon \Gamma_{G/N} \to \Gamma_G$$

by considering a G/N-set (or G/N-map) as a G-set (or G-map) via composition with the quotient map $G \to G/N$. For this definition to make sense, the families $\mathcal{F}_{G/N}$ and $\mathcal{F}_G$ should satisfy the property that if $(K/N) \in \mathcal{F}_{G/N}$, then $N \leq K \in \mathcal{F}_G$. Since we always assume the families are nonempty, the above assumption also implies that $N \in \mathcal{F}_G$. For notational simplicity from now on let us denote K/N by $\overline{K}$ for every $N \leq K$.

If a family $\mathcal{F}_G$ is already given, we will always take $\mathcal{F}_{G/N} = \{\overline{K} \mid N \leq K \text{ and } K \in \mathcal{F}_G\}$ and the condition above will be automatically satisfied. We also assume that $N \in \mathcal{F}_G$ to have a nonempty family for $\mathcal{F}_{G/N}$.

The functor F gives rise to two functors (see [9, 9.15]):

$$\mathrm{Res}_F\colon \text{Mod-}R\Gamma_G \to \text{Mod-}R\Gamma_{G/N}$$

and

$$\mathrm{Ind}_F\colon \text{Mod-}R\Gamma_{G/N} \to \text{Mod-}R\Gamma_G \ .$$

The first functor Res_F takes a $R\Gamma_G$-module M to the $R\Gamma_{G/N}$-module

$$\mathrm{Def}^G_{G/N}(M) := M \circ F\colon \Gamma_{G/N} \to R\text{-Mod}.$$

We call this functor the *deflation functor*. Note that

$$(\mathrm{Def}^G_{G/N} M)(\overline{K}) = M(K).$$

The induction functor $\mathrm{Inf}^G_{G/N} := \mathrm{Ind}_F$ associated to F is called the *inflation functor*. It takes a $R\Gamma_{G/N}$-module M to the module

$$\mathrm{Inf}^G_{G/N}(M) = M \otimes_{R\Gamma_{G/N}} R\,\mathrm{Mor}(??, F(?)).$$

Note that by general properties of restriction and induction functors associated to a functor F, the functor $\mathrm{Def}^G_{G/N}$ is exact and $\mathrm{Inf}^G_{G/N}$ respects projectives (see [9, 9.24]). Further properties can be obtained by computing the affects of these functors on free modules.

Lemma 4.3. *Let X be a G-set. Then, we have*

$$\mathrm{Def}^G_{G/N} R[X^{?}] = R[(X^N)^{?}]$$

In particular, if $H \in \mathcal{F}_G$ implies $HN \in \mathcal{F}_G$, then the functor $\mathrm{Def}^G_{G/N}$ respects projectives.

Proof. For every $K \in \mathcal{F}_G$ such that $N \leq K$, we have

$$(\mathrm{Def}^G_{G/N} R[X^{?}])(\overline{K}) = R[X^{?}](K) = R[X^K] = R[(X^N)^{K/N}] = R[(X^N)^{?}](\overline{K}).$$

Note that $(G/H)^N = G/HN$ as an G/N-set. If $H \in \mathcal{F}_G$ implies $HN \in \mathcal{F}_G$, then by assumption $\overline{HN} \in \mathcal{F}_{G/N}$. Hence $R[((G/H)^N)^{?}]$ is free as an $R\Gamma_{G/N}$-module and $\mathrm{Def}^G_{G/N}$ respects projectives. $\square$

The rest of the section is devoted to the proof of Theorem 4.2. As a first step of the proof we extend the given family $\mathcal{F}$ to the family $\mathscr{S}(G)$ of all subgroups of G by taking $\mathbf{C}(H) = 0$ for every $H \notin \mathcal{F}$. Note that over the extended family, $\mathbf{C}$ is still a finite projective chain complex over $R\Gamma_G$.

The conditions (i) and (iii) comes from the period of the cohomology of quotient groups. For (i), let $L \lhd K \leq G$ be such that $K/L \cong \mathbb{Z}/p$ with p is odd. Consider the complex $\mathrm{Def}^K_{K/L} \mathrm{Res}^G_K \mathbf{C}$. This is a finite projective complex over $R\varGamma_{K/L}$ because both restriction and deflation functors preserve projectives (note that the condition in Lemma 4.3 is satisfied because we extended our family $\mathcal{F}$ to the family of all subgroups of G).

So, for (i), the problem reduces to showing that if $G = \mathbb{Z}/p$, then given a finite projective $R\varGamma_G$-complex $\mathbf{C}$ which is an R-homology $\underline{n}$-sphere, the difference $\underline{n}(1) - \underline{n}(G)$ is even. In this case, $H_0 = \underline{R}$ is projective, so we can add $\mathbf{C}_{-1} = \underline{R}$ and consider the homology of the reduced complex $\widetilde{\mathbf{C}}$. The complex $\widetilde{\mathbf{C}}$ has nontrivial cohomology only at two dimensions, say m and k with $m \geq k$, so we get an extension of the form

$$0 \to H_m(\mathbf{C}) \to \mathbf{C}_m / \operatorname{im} \partial_{m+1} \to \cdots \to \mathbf{C}_{k+1} \to \ker \partial_k \to H_k(\mathbf{C}) \to 0.$$

where the homology modules are $I_1 R$ and $I_G R$. Since $\ker \partial_k$ is projective and $\operatorname{im} \partial_{m+1}$ has finite projective resolution, the above extension must be a non-split extension. However, $\mathrm{Ext}^i_{R\varGamma_G}(I_1 R, I_G R) = 0$ for all $i \geq 0$, so we must have $H_k(\mathbf{C}) = I_G R$ and $H_m(\mathbf{C}) = I_1 R$. In particular, we have $\underline{n}(1) \geq \underline{n}(G)$, meaning that the dimension function $\underline{n}$ is monotone in the sense defined in [11, page 211].

Now to prove $\underline{n}(1) - \underline{n}(G)$ is even, let $\widetilde{\mathbf{C}}^G$ denote the subcomplex of $\widetilde{\mathbf{C}}$ consists of all projectives of type $R[G/G^?]$, and let $\mathbf{D} = \widetilde{\mathbf{C}}/\widetilde{\mathbf{C}}^G$ denote the quotient complex. The complex $\mathbf{D}$ has nontrivial homology only at dimensions m and $k+1$. Evaluating at subgroup 1, we obtain a chain complex of free RG-modules

$$0 \to Q_d \to \cdots \to Q_{m+1} \xrightarrow{\partial_{m+1}} Q_m \to \cdots \to Q_{k+1} \xrightarrow{\partial_{k+1}} Q_k \to \cdots \to Q_0 \to 0.$$

whose homology is R at dimensions m and $k+1$. This gives an exact sequence of the form

$$0 \to R \to Q_m / \operatorname{im} \partial_{m+1} \to \cdots \to Q_{k+2} \to \ker \partial_{k+1} \to R \to 0.$$

Using the fact that free RG-modules are both projective and injective, we conclude that all the modules in the above sequence, except the two R's on the both ends, are projective as RG-modules, so it is a periodic resolution. Since the group G has cohomology with period 2, we obtain that $m - k$ is even. The proof of condition (iii) is similar.

For condition (ii), we may assume that $G = K/L = \mathbb{Z}/p \times \mathbb{Z}/p$. Since the complex $\mathbf{C}$ is a finite complex of projective modules, for any $R\varGamma$-module M, we have

$$H^n(\mathrm{Hom}_{R\varGamma_G}(\mathbf{C}, M)) = 0$$

for $n > d$, where d is the dimension of the chain complex $\mathbf{C}$. Consider the hyper-cohomology spectral sequence for the complex $\mathbf{C}$. This is a spectral-sequence with E_2-term given by

$$E_2^{s,t} = \mathrm{Ext}^s_{R\varGamma_G}(H_t(\mathbf{C}), M) \tag{4.3}$$

which converges to $H^{s+t}(\mathrm{Hom}_{R\Gamma_G}(\mathbf{C}, M))$. Since $\underline{R}$ is a projective $R\Gamma_G$-module (note that $\mathcal{F}_G$ is the family of all subgroups of G after the subquotient reduction), we can replace $H_t(\mathbf{C})$ with the reduced homology $\widetilde{H}_t(\mathbf{C})$. So, we have nonzero terms for $E_2^{s,t}$ only when t is equal to $n_1 = \underline{n}(1)$, $n_G = \underline{n}(G)$, or $n_i = \underline{n}(H_i)$ where H_i are the subgroups of G of order p. Since $\underline{n}$ is monotone, we have $n_1 \geq n_i \geq n_G$ for all $i \in \{1, \dots, p+1\}$. In the proof below we assume $n_1 > n_i > n_G$ for all i. For the remaining cases, a proof can be given in a similar way. The required formula is

$$n_1 - n_G = \sum_{i=1}^{p+1} (n_i - n_G).$$

Note that by adding free summands to the complex $\mathbf{C}$, we can assume that all the cohomology between dimensions n_1 and n_G is concentrated at the dimension $n_M = \max_i\{n_i\}$. Then the homology at this dimension will be an $R\Gamma_G$-module which is filtered by Heller shifts of homology groups $H_t(\mathbf{C})$ at dimensions $t = n_i$ for $i = 1, \dots, p+1$. Note that homology of the complex $\mathbf{C}$ at dimension n_i is $I_{H_i}R$, where $I_{H_i}R$ denotes the $R\Gamma_G$ module with value R at H_i and zero at all the other subgroups. We have the following lemma.

Lemma 4.4. *If $i, j \in \{1, \dots, p+1\}$ are such that $i \neq j$, then*

$$\mathrm{Ext}^m_{R\Gamma_G}(I_{H_i}R, I_{H_j}R) = 0$$

for every $m \geq 0$.

Proof. The projective resolution of I_{H_i} is formed by projective modules of type $E_H P$ with $H = 1$ or H_i. Since

$$\mathrm{Hom}_{R\Gamma_G}(E_H P, I_{H_j}R) \cong \mathrm{Hom}_{RW_G(H)}(P, I_{H_j}(H)) = 0$$

when $i \neq j$, we obtain the desired result. □

As a consequence of Lemma 4.4, we conclude that all the extensions in this filtration of $H_{n_M}(\mathbf{C})$ are split extensions. So, the homology module $H_{n_M}(\mathbf{C})$ is isomorphic to a direct sum of Heller shifts of modules $I_{H_i}R$. In particular, we obtain that, for any $R\Gamma_G$-module M,

$$\mathrm{Ext}^s_{R\Gamma_G}(H_{n_M}(\mathbf{C}), M) \cong \oplus_i \mathrm{Ext}^{s+n_M-n_i}_{R\Gamma_G}(I_{H_i}R, M)$$

for every $s \geq 0$.

The spectral sequence given at (4.3) converges to zero at large dimensions and it has only three non-zero horizontal lines, so it gives a long exact sequence of the form

$$\cdots \to \mathrm{Ext}^{k+n_1-n_G+1}_{R\Gamma_G}(I_G R, M) \xrightarrow{\delta} \mathrm{Ext}^k_{R\Gamma_G}(I_1 R, M) \xrightarrow{\gamma} \oplus_{i=1}^{p+1} \mathrm{Ext}^{k+n_1-n_i+1}_{R\Gamma_G}(I_{H_i}R, M)$$

$$\to \mathrm{Ext}^{k+n_1-n_G+2}_{R\Gamma_G}(I_G R, M) \xrightarrow{\delta} \mathrm{Ext}^{k+1}_{R\Gamma_G}(I_1 R, M) \to \cdots$$

where k is an integer such that $k > d - n_1$ and M is any $R\Gamma_G$-module. If we take $M = I_1 R$, then $\mathrm{Ext}^k_{R\Gamma_G}(I_1 R, M) \cong H^k(G, R)$, so it is isomorphic to the tensor product of an exterior algebra with a polynomial algebra

$$\wedge_R(a_1, a_2) \otimes R[x_1, x_2]$$

where $\deg a_i = 1$ and $\deg x_i = 2$. When $M = I_1 R$, the other Ext groups in the above exact sequence also reduce to the cohomology of the group G but with some dimension shifts.

Lemma 4.5. *For every $i \in \{1, \dots, p+1\}$, we have*

$$\mathrm{Ext}^m_{R\varGamma_G}(I_{H_i}R, I_1R) \cong \mathrm{Ext}^{m-1}_{R\varGamma_G}(I_1R, I_1R) \cong H^{m-1}(G; R)$$

for every $m \geq 1$. We also have

$$\mathrm{Ext}^m_{R\varGamma_G}(I_GR, I_1R) \cong \oplus_p \mathrm{Ext}^{m-2}_{R\varGamma_G}(I_1R, I_1R) \cong \oplus_p H^{m-2}(G; R)$$

for every $m \geq 2$. Here $\oplus_p$ denotes the direct sum of p-copies of the same R-module.

Proof. Since we already observed that $\mathrm{Ext}^k_{R\varGamma_G}(I_1R, I_1R) \cong H^k(G, R)$ for every $k \geq 0$, it is enough to show the first isomorphisms. Let $i \in \{1, \dots, p+1\}$ and $S_{H_i}R$ denote the $R\varGamma_G$ module with value R at subgroups 1 and H_i and zero at every other subgroup. We assume that the restriction map is an isomorphism. So we have a non-split exact sequence of $R\varGamma_G$-modules of the form

$$0 \to I_1R \to S_{H_i}R \to I_{H_i}R \to 0.$$

Note that the projective resolution of $S_{H_i}R$ will only include projective modules of the form $E_{H_i}P$, so we have $\mathrm{Ext}^m_{R\varGamma_G}(S_{H_i}R, I_1R) = 0$ for all $m \geq 0$. The long exact Ext-group sequence associated to the above short exact sequence will give the desired isomorphism for the module $I_{H_i}R$.

For the second statement in the lemma, we again only need to show that the isomorphism

$$\mathrm{Ext}^m_{R\varGamma_G}(I_GR, I_1R) \cong \oplus_p \mathrm{Ext}^{m-2}_{R\varGamma_G}(I_1R, I_1R)$$

holds for all $m \geq 2$. Let N denote the $R\varGamma_G$-module defined as the kernel of the map $\underline{R} \to I_GR$ which induces the identity homomorphism at G. Since the constant module $\underline{R}$ is projective as a $R\varGamma_G$-module, we have

$$\mathrm{Ext}^m_{R\varGamma}(I_GR, I_1R) \cong \mathrm{Ext}^{m-1}_{R\varGamma}(N, I_1R)$$

for $m \geq 2$. Note that there is an exact sequence of the form

$$0 \to \oplus_p I_1R \to \oplus_{i=1}^{p+1} S_{H_i}R \to N \to 0.$$

Since $\mathrm{Ext}^m_{R\varGamma_G}(S_{H_i}R, I_1R) = 0$ for all $m \geq 0$, we obtain

$$\mathrm{Ext}^m_{R\varGamma}(I_GR, I_1R) \cong \mathrm{Ext}^{m-1}_{R\varGamma}(N, I_1R) \cong \oplus_p \mathrm{Ext}^{m-2}_{R\varGamma}(I_1R, I_1R) \cong \oplus_p H^{m-2}(G; R)$$

for every $m \geq 2$. This completes the proof of the lemma. $\square$

The proof of Theorem 4.2. Using the Ext-group calculations given in Lemma 4.5, we obtain a long exact sequence of the form

$$\cdots \to \oplus_p H^{k+n_1-n_G-1}(G; R) \xrightarrow{\delta} H^k(G; R) \xrightarrow{\gamma} \oplus_{i=1}^{p+1} H^{k+n_1-n_i}(G; R)$$

$$\to \oplus_p H^{k+n_1-n_G}(G; R) \xrightarrow{\delta} H^{k+1}(G; R) \to \cdots$$

where $k > d - n_1$. We claim that the map γ is injective. Observe that if $\gamma = \oplus \gamma_i$, then for each i, the map γ_i can be defined as multiplication with some cohomology class u_i. To see this observe that γ is the map induced by the differential

$$d_{n_1-n_M+1}\colon \operatorname{Ext}^k_{R\Gamma_G}(H_{n_1}(\mathbf{C}), I_1R) \to \operatorname{Ext}^{k+n_1-n_M+1}_{R\Gamma_G}(H_{n_M}(\mathbf{C}), I_1R)$$

on the hypercohomology spectral sequence given at (4.3). This spectral sequence has an $\operatorname{Ext}^*_{R\Gamma}(I_1R, I_1R)$ module structure where the multiplication is given by the Yoneda product, i.e., by splicing the corresponding extensions (see [2, Section 4]).

Under the isomorphisms given in Lemma 4.5, the differential $d_{n_1-n_M+1}$ becomes a map $H^k(G,R) \to \oplus_i H^{k+n_1-n_i}(G,R)$ and the Yoneda product of Ext-groups is the same as the usual cup product multiplication in group cohomology under the canonical isomorphism $\operatorname{Ext}^m_{R\Gamma_G}(I_1R, I_1R) \cong H^m(G,R)$ (for comparison of different products on group cohomology see [4, Proposition 4.3.5]). So we can conclude that γ_i is the map defined by multiplication with the cohomology $u_i \in H^{n_1-n_i+1}(G,R)$ where the multiplication is the usual cup product multiplication in group cohomology.

Suppose now that γ is not injective. Then for each i, the class u_i must be a multiple of a_1a_2 (note that by the first part the class u_i is an even dimensional class). But, then the restriction of the entire spectral sequence to some H_i will result with a spectral sequence with zero differentials. This is because $\operatorname{Res}^G_{H_i} I_GR = 0$ and $\operatorname{Res}^G_{H_i} I_{H_j}R = 0$ if $i \neq j$. So, if γ is not injective, the restriction of the spectral sequence to a subgroup H_i gives a spectral sequence which collapses. But, the restriction of $\mathbf{C}$ to a proper subgroup is still a finite projective chain complex, so this gives a contradiction. Hence, we can conclude that γ is injective.

The fact that γ is injective gives that for every $k \geq 0$, we have a short exact sequence of the form

$$0 \to H^k(G;R) \xrightarrow{\gamma} \oplus_{i=1}^{p+1} H^{k+n_1-n_i}(G;R) \to \oplus_p H^{k+n_1-n_G}(G;R) \to 0.$$

Since $\dim_R H^m(G;R) = m+1$, we obtain

$$(k+1) + p(k+n_1-n_G+1) = \sum_{i=1}^{p+1}(k+n_1-n_i+1).$$

Cancelling the $(k+1)$'s and grouping the terms in a different way gives the desired equality. The proof of Theorem 4.2 is complete. □

Remark 4.6. The fact that the dimension function of an algebraic $\underline{n}$-homology sphere satisfies the Borel-Smith conditions suggests that more of the classical results on finite group actions on spheres might hold for finite projective chain complexes over a suitable orbit category. For example, one could ask for an algebraic version of the results of Dotzel-Hamrick [5] on p-groups. Other potential applications of algebraic models to finite group actions are outlined in [6].

Example 4.7. An important test case for groups acting on spheres, or on products of spheres [1], is the rank two group $\mathrm{Qd}(p) = (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes SL_2(p)$. At present, it is not known whether $\mathrm{Qd}(p)$ can act freely on a product of two spheres, but Ünlü [12] showed that $\mathrm{Qd}(p)$ does not act on a finite complex homotopy equivalent to a sphere with rank one isotropy.

We apply the Borel-Smith conditions prove an algebraic version of this result.

Proposition 4.8. *Let p be an odd prime, $G = \mathrm{Qd}(p)$, $R = \mathbb{Z}/p$, and $\mathcal{F}$ be the family of all subgroups $H \leq G$ such that $\mathrm{rank}_p(H) \leq 1$. Let $\underline{n}$ be a super class function with $\underline{n}(1) \geq 0$. Then, there exists no finite projective chain complex $\mathbf{C}$ over $R\Gamma_G$ which is an R-homology $\underline{n}$-sphere.*

Proof. First observe that we can extend the family $\mathcal{F}$ to the family $\mathscr{S}(G)$ of all subgroups of G by taking $\mathbf{C}(H) = 0$ for all subgroups such that $H \notin \mathcal{F}$. Note that for these subgroups, we take $\underline{n}(H) = -1$. Observe that by Theorem 4.2, the dimension function $\underline{n}$ defined on $\mathscr{S}(G)$ satisfies the Borel-Smith conditions.

Now the rest of the argument follows as in Ünlü [12, Theorem 3.3]. Let P be a Sylow p-subgroup of $\mathrm{Qd}(p)$. The group P is isomorphic to the extra-special p-group of order p^3 and exponent p. If $Z(P)$ is the center of P, then the quotient group $P/Z(P)$ is isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$. Applying the Borel-Smith condition (ii) for this quotient, we get $\underline{n}(Z(P)) = -1$. In G, it is possible to find two Sylow p-subgroups P_1 and P_2 such that $E = P_1 \cap P_2 \cong \mathbb{Z}/p \times \mathbb{Z}/p$ and $Z(P_1)$ and $Z(P_2)$ are distinct subgroups of order p in E. Two such Sylow p-subgroups can be given as $P_i = (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \langle A_i \rangle$ for $i = 1, 2$ where

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Note that by the above argument $\underline{n}(Z(P_i)) = -1$, and non-central p-subgroups in E are conjugate to each other. So, we obtain that $\underline{n}(K) = -1$ for every subgroup K of order p in E. By the Borel-Smith conditions applied to E, we get $\underline{n}(1) = -1$, contradicting our assumption on $\underline{n}$. $\square$

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