QUOTIENTS OF $S^2 \times S^2$

I. HAMBLETON AND J. A. HILLMAN

ABSTRACT. We consider closed topological 4-manifolds M with universal cover $S^2 \times S^2$ and Euler characteristic $\chi(M) = 1$. All such manifolds with $\pi = \pi_1(M) \cong \mathbb{Z}/4$ are homotopy equivalent. In this case, we show that there are four homeomorphism types, and propose a candidate for a smooth example which is not homeomorphic to the geometric quotient. If $\pi \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, we show that there are three homotopy types (and between 6 and 24 homeomorphism types).

1. INTRODUCTION

The goal of this paper is to characterize 4-manifolds with universal cover $S^2 \times S^2$ up to homeomorphism in terms of standard invariants, continuing the program of [8, Chapter 12]. A 4-manifold M has universal covering space $\widetilde{M} \cong S^2 \times S^2$ if and only if $\pi = \pi_1(M)$ is finite, $\chi(M)|\pi| = 4$ and its Wu class $v_2(M)$ is in the image of $H^2(\pi; \mathbb{F}_2)$. There are eight such manifolds which are geometric quotients, with π acting through a subgroup of $\mathrm{Isom}(\mathbb{S}^2 \times \mathbb{S}^2) = (O(3) \times O(3)) \rtimes \mathbb{Z}/2$.

We first recall that closed topological manifolds with $\pi_1(M) = 1$ or $\pi_1(M) = \mathbb{Z}/2$ have already been classified (without assumption on the universal convering):

(i) If $|\pi| \leq 2$, and M is orientable, then M is classified up to homeomorphism by its intersection form on $H_2(M; \mathbb{Z})/Tors$, $w_2(M)$ and the Kirby-Siebenmann invariant (see Freedman [2] for $\pi = 1$, and [5, Theorem C] for $\pi = \mathbb{Z}/2$).

(ii) If $\pi = \mathbb{Z}/2$, and M is non-orientable, then M is classified up to homeomorphism by explicit invariants (see [7, Theorem 2]), and a complete list of such manifolds is given in [7, Theorem 3].

If we further impose the condition that $\widetilde{M} = S^2 \times S^2$, then there are two orientable geometric $\mathbb{Z}/2$-quotients, namely the 2-sphere bundles $S(\eta \oplus 2\epsilon)$ and $S(3\eta)$ over RP^2, where η is the canonical line bundle over RP^2. The second manifold is non-spin and has a non-smoothable homotopy equivalent "twin" $*M$ with KS $\neq 0$.

In the non-orientable case, there are two geometric quotients: $S^2 \times RP^2$ and $S^2 \tilde{\times} RP^2 = S(2\eta \oplus \epsilon)$, and one further smooth manifold $RP^4 \sharp_{S^1} RP^4$ obtained by removing a tubular neighbourhood of $RP^1 \subset RP^4$, and gluing the complements together along the boundary. Each of these has a homotopy equivalent twin $*M$ with KS $\neq 0$, so there are six such non-orientable manifolds.

Date: December 12, 2017.
2010 *Mathematics Subject Classification.* 57N13.
This research was partially supported by an NSERC Discovery Grant.

We now assume that $|\pi| = 4$, which implies that any quotient M of $S^2 \times S^2$ is non-orientable and $\chi(M) = 1$. If $\pi = \mathbb{Z}/4$, there is just one geometric quotient $\mathbb{M}$ obtained by the free action generated by $(u, v) \mapsto (-v, u)$, for $(u, v) \in S^2 \times S^2$.

Theorem A. *Let N be a closed topological 4-manifold with $\widetilde{N} = S^2 \times S^2$.*

(i) If $\pi_1(N) = \mathbb{Z}/4$, then N is homotopy equivalent to the unique geometric quotient $\mathbb{M}$.
(ii) Every self-homotopy equivalence of $\mathbb{M}$ is homotopic to a self-homeomorphism.
(iii) There are four such manifolds up to homeomorphism, of which exactly two have non-trivial Kirby-Siebenmann invariant.

Remark 1.1. An analysis of one construction of the geometric example $\mathbb{M}$ leads to the construction of another smooth 4-manifold in this homotopy type, which may not be homeomorphic to the geometric manifold.

Remark 1.2. If π has order 2 or 4 then $Wh(\pi) = 0$ and the natural homomorphism from $L_4(1)$ to $L_4(\pi, -)$ is trivial (see Wall [17]). Thus if M is non-orientable we may surger the normal map $M\#E_8 \to M\#S^4 = M$ to obtain a twin: there is a homotopy equivalent 4-manifold $*M$ with the opposite Kirby-Siebenmann invariant. On the other hand, when $|\pi| = 4$ the mod 2 Hurewicz homomorphism is trivial. Hence pinch maps have trivial normal invariant, so do not provide "fake" self homotopy equivalences (see [1, p. 420]).

In the remaining cases, where $\pi = \mathbb{Z}/2 \times \mathbb{Z}/2$, we classify the homotopy types.

Theorem B. *There are two quadratic 2-types of PD_4-complexes X with $\chi(X) = 1$ and $\pi_1(X) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, and seven homotopy types in all.*

(i) All such complexes have universal cover homotopy equivalent to $S^2 \times S^2$.
(ii) The two quadratic 2-types are represented by the total spaces of the two RP^2-bundles over RP^2.
(iii) A third homotopy type includes a smooth manifold N with $RP^4\#_{S^1}RP^4$ as a double cover.
(iv) The remaining four homotopy types do not include closed manifolds.

The statement about the quadratic 2-types was proved in [8, Chapter 12, §6]. The homeomorphism classification appears difficult: all we can say at this stage is that in each case the TOP structure set has 8 members, so that there are between 6 and 24 homeomorphism types of such manifolds, of which half are not stably smoothable. To resolve this ambiguity, more information is needed about self-homotopy equivalences.

Here is an outline of the paper. After some preliminary homotopy theoretic remarks in §2, we review the constructions of the non-orientable smoothable quotients of $S^2 \times S^2$ with $\pi = \mathbb{Z}/2$ (see §§3 - 5). In §6 we show that there are four homeomorphism types with $\pi \cong \mathbb{Z}/4$, and in §7 we construct a smooth example which may not be homeomorphic to the geometric quotient.

In §8 we construct a new smooth 4-manifold N in the quadratic homotopy type of the bundle space $RP^2\tilde{\times}RP^2$, but distinguished from it by its non-orientable double covers (see Definition 8.1). In particular, N is not a geometric quotient. In §9 and §10 we show that there are no other homotopy types of 4-manifolds with $\pi \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ and $\chi = 1$.

In §11 we estimate the size of the group of homotopy classes of based self-equivalences of $RP^2 \times RP^2$, and show in §12 that the strategy of §7 does not extend easily to provide a candidate for a smooth fake $RP^2 \times RP^2$. A final §13 provides an alternate approach to Theorem A via a stable homeomorphism classification result.

Acknowledgement. JAH would like to thank the Department of Mathematics and Statistics at McMaster University and the Fields Institute for their hospitality and support in September and October 2009, and in October 2017, respectively.

2. some general results

Let X be a connected cell complex with fundamental group π, and let $G_\#(X)$ be the group of based self homotopy equivalences of X which induce the identity on all homotopy groups. Let $P_n(X)$ be the nth stage of the Postnikov tower for X. This may be constructed by adjoining cells of dimension $\geq n+2$ to X. Then $G_\#(P_2(X)) \cong H^2(\pi; \pi_2(X))$, and there are exact sequences

$$H^n(P_{n-1}(X); \pi_n(X)) \to G_\#(P_n(X)) \to G_\#(P_{n-1}(X)),$$

for $n > 2$, by Tsukiyama [16, Theorem 2.2 and Proposition 1.5], respectively. The image on the right is the subgroup which stabilizes

$$k_n(X) \in H^{n+1}(P_{n-1}(X), \pi_n(X)).$$

In particular, if $H^k(P_{k-1}(X); \pi_k(X)) = 0$ for $2 \leq k \leq n$ then self homotopy equivalences of $P_n(X)$ are detected by their actions on the homotopy groups.

If X is a PD_4-complex such that π is finite then any based self-homotopy equivalence of X lifts to a based self-homotopy equivalence of the 1-connected PD_4-complex $\widetilde{X}$. Hence if also $\pi_2(X) \neq 0$ then that any based self-homotopy equivalence of X which induces the identity on π and $\pi_2(X)$ is in $G_\#(X)$, by [1, Theorem 3.1]. (Compare [6, Theorem A]).

Lemma 2.1. *Each homotopy type within the quadratic 2-type of a PD_4-complex X with π finite may be obtained by varying the attaching map of the top cell to the 2-skeleton $X^{(3)}$. The torsion subgroup of $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(\pi_2(X))$ acts transitively on the set of PD_4-polarizations of the quadratic 2-type.*

Proof. In the statement, Γ_W denotes the quadratic functor of Whitehead. These results are contained in [14, Chap. 2] (see [4, Theorem 1.1] for the oriented case). $\square$

Thus the cardinality of this subgroup is an upper bound for the number of homotopy types within the quadratic 2-type. The ring homomorphism $H^*(X; \mathbb{F}_2) \to H^*(X^{(3)}; \mathbb{F}_2)$ induced by the inclusion of the 3-skeleton is an isomorphism in degrees ≤ 3.

Since $\pi_i(G/TOP) = 0$ in all odd dimensions and the first significant k-invariant of G/TOP is 0, there is a 6-connected map $G/TOP \to K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 4)$ (see [11, §2]). Hence if X is a closed 4-manifold then

$$[X, G/TOP] \cong H^2(X; \mathbb{Z}/2) \oplus H^4(X; \mathbb{Z}).$$

In these low dimensions, Poincaré duality with $\mathbb{L}$-theory coefficients

$$[X, G/TOP] = H^0(X; \mathbb{L}_0) \cong H_4(X; \mathbb{L}_0)$$

on the left-hand side agrees with ordinary Poincaré duality on the right-hand side. This gives $[X, G/TOP] \cong H_2(X; \mathbb{Z}/2) \oplus H_0(X; \mathbb{Z}^w)$, where $w = w_1(X)$.

Kim, Kojima and Raymond [10] defined a $\mathbb{Z}/4$-valued quadratic function $q_{KKR}(M)$ on $\pi_2(M) \otimes \mathbb{Z}/2$, for M a closed non-orientable 4-manifold, by the rule

$$q_{KKR}(M)(x) = e(\nu(S_x)) + 2|\operatorname{Self}(S_x)|,$$

where $S_x \colon S^2 \to M$ is a self-transverse immersion representing x, $e(\nu(S_x))$ is the Euler number of the normal bundle and $\operatorname{Self}(S_x)$ is the set of double points of the image of S_x. This is an enhancement of the mod 2 equivariant intersection pairing on $\widetilde{M}$, and is a homotopy invariant for M.

We introduce some notation for later use. Let A be the antipodal involution of S^2, and let $\eta \colon S^3 \to S^2$ be the Hopf fibration. Let $\bar{\eta} \colon S^3 \to RP^2$ be the composite of η with the projection $S^2 \to RP^2 = S^2/x \sim A(x)$.

3. Non-orientable quotients of $S^2 \times S^2$ with $\pi = \mathbb{Z}/2$

There are two quadratic 2-types of non-orientable PD_4-complexes X with $\pi = \mathbb{Z}/2$ and $\chi(X) = 2$. All such quotients of $S^2 \times S^2$ have the quadratic 2-type of $S^2 \times RP^2$ (see [8, Chapter 12, §5]). Let $K = \overline{S^2 \times RP^2 \setminus D^4}$ be the 3-skeleton of $S^2 \times RP^2$, let $I_1, I_2 \colon S^2 \to \widetilde{K} = \overline{S^2 \times S^2 \setminus 2D^4}$ be the inclusions of the factors, and let $[J]$ be the homotopy class of a fixed lift $\widetilde{J} \colon S^3 \to \widetilde{K}$ of the natural inclusion $J \colon S^3 = \partial D^4 \to K$. The torsion subgroup of $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(\pi_2(S^2 \times RP^2))$ is isomorphic to $(\mathbb{Z}/2)^2$, and is generated by the images of $\eta_i = I_i \circ \eta$, for $i = 1, 2$.

The four homotopy types represented by the PD_4-complexes $W_\alpha = K \cup_{[J]+\alpha} e^4$ corresponding to $\alpha = 0$, η_1, η_2 and $\eta_1 + \eta_2$ are in fact distinct, as we shall see. Clearly $W_0 = K \cup_{[J]} D^4 \simeq S^2 \times RP^2$. There are two other closed 4-manifolds, namely the bundle space $S^2 \tilde{\times} RP^2$ and the manifold $RP^4 \#_{S^1} RP^4$, which are described in the next section, and shown to have distinct homotopy types in §5. In [3] it is shown that the PD_4-complex $P_{HM} = W_{\eta_2}$ is not homotopy equivalent to a closed 4-manifold, but note that [3] writes the factors in the opposite order. The remaining two possibilities W_{η_1} and $W_{\eta_1+\eta_2}$ must give $S^2 \tilde{\times} RP^2$ and $RP^4 \#_{S^1} RP^4$. According to [10, p. 80], $W_{\eta_1} \simeq S^2 \tilde{\times} RP^2$ and $W_{\eta_1+\eta_2} \simeq RP^4 \#_{S^1} RP^4$. Is this easily seen?

Since the homomorphisms $H^*(W_\alpha; \mathbb{F}_2) \to H^*(K; \mathbb{F}_2)$ are isomorphisms in degrees ≤ 3, $H^1(W_\alpha; \mathbb{F}_2) = \langle x \rangle$, where $x^3 = 0$ in all cases. The group $H^2(K; \mathbb{F}_2)$ is generated by x^2 and the class u pulled back by the projection to S^2. The latter map extends to a map from P_{HM} to S^2, and so $u^2 = 0$ in $H^4(P_{HM}; \mathbb{F}_2)$. Since $x^4 = 0$ also, it follows that $v_2(P_{HM}) = 0$. On the other hand, this projection does not extend in this way when $\alpha = \eta_1$ or $\eta_1 + \eta_2$.

The only other quadratic 2-type with $\pi = \mathbb{Z}/2$, $w_1 \neq 1$ and $\chi = 2$ is that of $RP^4 \# CP^2$ (the nontrivial RP^2-bundle over S^2), which contains two homotopy types. One of these is not homotopy equivalent to a closed 4-manifold, by [14, §3.3.1]. These PD_4-complexes have universal cover $\simeq S^2\tilde{\times}S^2$, and do not cover PD_4-complexes with $\chi = 1$ (see [8, Lemma 12.3].)

4. $S^2\tilde{\times}RP^2$ and $RP^4\#_{S^1}RP^4$

Let E be a regular neighbourhood of $RP^2 = \{[x : y : z : 0 : 0] \mid x^2 + y^2 + z^2 = 1\}$ in RP^4. Then $\nu = \overline{RP^4 \setminus E}$ is a regular neighbourhood of $RP^1 = \{[0 : 0 : 0 : u : v] \mid u^2 + v^2 = 1\}$, and $\partial E = \partial\nu$ is both the total space on an S^1-bundle over RP^2 and the mapping torus $S^2\tilde{\times}S^1 = S^2 \times [0, 1]/(s, 0) \sim (A(s), 1)$. In particular, $\pi_1(\partial E) \cong \mathbb{Z}$, and so E is not the product $RP^2 \times D^2$. On passing to the universal cover we see that $S^4 = \widetilde{E}\cup\widetilde{\nu}$. We may assume that

$$\widetilde{E} = \{(u, v, x, y, z) \in S^4 \mid u^2 + v^2 \leq \frac{1}{4}\}.$$

Let $h\colon \widetilde{E} \to S^2 \times D^2$ be the homeomorphism given by $h(\tilde{e}) = (x/r, y/r, z/r, 2u, 2v)$, where $r = \sqrt{x^2 + y^2 + z^2}$, for all $\tilde{e} = (u, v, x, y, z) \in \widetilde{E}$. It follows that we may write $E = S^2 \times D^2/(s, d) \sim (A(s), -d)$, and the projection $p\colon E \to RP^2$ is then given by $p([s, d]) = [s] \in RP^2$. The space E is also an orbifold bundle with general fibre S^2 over the marked disc $D(2)$, via the projection $p'([s, d]) = d^2$. Here we view D^2 as the unit disc in the complex plane.

Doubling E along its boundary gives the total space of an S^2-bundle over RP^2. This space DE is non-orientable and $v_2(DE) \neq 0$, since the core RP^2 in E has self-intersection 1 (mod 2). Thus DE is the nontrivial, non-orientable S^2-bundle space

$$S^2\tilde{\times}RP^2 = S^2 \times S^2/(s, t) \sim (A(s), R_\pi t).$$

where R_π is rotation of S^2 through π radians. We shall view S^2 as the purely imaginary quaternions of length 1. The antipodal map is then multiplication by -1, and we may identify R_π with conjugation by $\pm\mathbf{k}$, i.e., rotation about the $\mathbf{k}$ axis. Composition of the double covering of RP^2 with the projection of $S^2 \times S^2$ onto its first factor induces the S^2-bundle projection $DE \to RP^2$. The space DE is also the total space of an orbifold bundle with general fibre S^2 over the orbifold $S(2, 2)$ (the double of $D(2)$).

We may construct a different 4-manifold by identifying two copies of E via a diffeomorphism of their boundaries which does not extend across E. The action of conjugation by $e^{\pi\mathbf{i}t}$ on S^2 is rotation through $2\pi t$ radians about the $\mathbf{i}$-axis.

Definition 4.1. Let E_1 and E_2 be two copies of E, and let $\xi\colon \partial E_1 \to \partial E_2$ be the map given by

$$\xi([s, x]_1) = [e^{\pi\mathbf{i}t}se^{-\pi\mathbf{i}t}, x]_2,\ \forall\ s \in S^2,\ \forall\ x = e^{2\pi\mathbf{i}t} \in S^1.$$

We define $RP^4\#_{S^1}RP^4 = E_1 \cup_\xi E_2$.

Note that $e^{\pi\mathbf{i}t}$ is a square root for $x = e^{2\pi\mathbf{i}t}$. This "twist map" ξ does not extend to a homeomorphism from E_1 to E_2 (see [9, Corollary 2.2]).

This manifold is the total space of an orbifold bundle with regular fibre S^2 over $S(2, 2)$. The exceptional fibres are the cores RP^2 of the copies of E, and each has self-intersection 1. Hence $v_2(RP^4\#_{S^1}RP^4) \neq 0$. We shall show in the next section that $RP^4\#_{S^1}RP^4$ is not homotopy equivalent to a bundle space [10], and hence it is not geometric.

The universal cover of $RP^4\#_{S^1}RP^4$ is the union $\widetilde{E}_1 \cup_{\tilde{\xi}} \widetilde{E}_2$, where $\tilde{\xi}$ is the lift of ξ given by $\tilde{\xi}((s,x)_1) = (xsx^{-1}, x)_2$, for all $(s,x) \in S^2 \times S^1 = \partial\widetilde{E}_1$. This lift is isotopic to the identity, and so $\widetilde{E}_1 \cup_{\tilde{\xi}} \widetilde{E}_2 \cong S^2 \times S^2$.

We may make this explicit as follows. Let $P(r,x) = \sin(\frac{\pi}{2}r)x + \cos(\frac{\pi}{2}r)\mathbf{j}$, for $0 \le r \le 1$ and $x \in S^1$. Then $P(0,x) = \mathbf{j}$ and $P(1,x) = x$, for all $x = e^{2\pi \mathbf{i} t} \in S^1$. Write the second factor of $S^2 \times S^2$ as the union of two hemispheres $S^2 = D_- \cup D_+$, with boundary in the $(\mathbf{i},\mathbf{j})$-plane. Let $V\colon D_+ \to S^3$ be the function defined by $V(d) = P(r, e^{2\pi\mathbf{i}t})$ for all $d = re^{2\pi\mathbf{i}t} \in D_2$. Here we identify D_+ with D^2, and then R_π corresponds to multiplication by -1. Then the function $H\colon S^2 \times S^2 \to \widetilde{E}_1 \cup \widetilde{E}_2$, defined by

$$H(s,d) = (s,d)_1 \in \widetilde{E}_2,\ \forall\ (s,d) \in S^2 \times D_-$$

and

$$H(s,d) = (V(d)sV(d)^{-1}, d)_2 \in \widetilde{E}_2,\ \forall\ (s,d) \in S^2 \times D_+,$$

is a homeomorphism. Hence $RP^4\#_{S^1}RP^4 \cong S^2 \times S^2/\langle\psi\rangle$, where ψ is the free involution given by

$$\psi(s,d) = (A(s), R_\pi(d)),\ \forall\ (s,d) \in S^2 \times D_-,$$

and

$$\psi(s,d) = (V(R_\pi(d))^{-1}V(d)A(s)V(d)^{-1}V(R_\pi(d)), R_\pi(d)),\ \forall\ (s,d) \in S^2 \times D_+.$$

The factor $V(R_\pi(d))^{-1}V(d)$ may be written more explicitly as

$$V(R_\pi(d))^{-1}V(d) = \cos(\pi r)\mathbf{1} - \sin(\pi r)\cos(2\pi t)\mathbf{j} + \sin(\pi r)\sin(2\pi t)\mathbf{k}.$$

Thus $V(R_\pi(d))^{-1}V(d) = \mathbf{1}$ when $r = 0$ and $V(R_\pi(d))^{-1}V(d) = -\mathbf{1}$ when $r = 1$.

5. Distinguishing the homotopy types

We shall follow [10] in using the mod 2 intersection pairing (in the guise of v_2) and the invariant q_{KKR} to show that $RP^4\#_{S^1}RP^4$ is not homotopy equivalent to either of the S^2-bundle spaces. As our construction of $RP^4\#_{S^1}RP^4$ differs slightly from that of [10], we shall give details of the geometric computation of q_{KKR} for these manifolds.

Let $M = S^2 \times RP^2$, $S^2 \tilde{\times} RP^2$ or $RP^4\#_{S^1}RP^4$, and let $x, y \in \pi_2(M)$ be the classes corresponding to the first and second factors of $S^2 \times S^2$. Then $x + y$ corresponds to the diagonal. In each case x is represented by the (general) fibres of the (orbifold) bundle projections to RP^2, $S(2,2)$ and $S(2,2)$, respectively, which are embedded with trivial normal bundle, and so $q_{KKR}(M)(x) = 0$, while the normal Euler number of the diagonal is ± 2.

Let $f\colon S^2 \to S^2$ be the map which folds one hemisphere onto another, and let $g\colon S^2 \to RP^2$ be the 2-fold cover. The 2-sphere $\{(f(s), s) | s \in S^2\}$ represents y, and has trivial normal bundle, since f is null homotopic. Its image in $S^2 \times RP^2$ has a single double point, and so $q_{KKR}(S^2 \times RP^2)(y) \equiv 2 \pmod 4$. The graph $\Gamma_g \subset S^2 \times RP^2$ is an embedded 2-sphere which lifts to the diagonal embedding in $S^2 \times S^2$. Since there are no self intersections, $q_{KKR}(S^2 \times RP^2)(x + y) \equiv 2 \pmod 4$ also. Hence $q_{KKR}(S^2 \times RP^2)$ is nontrivial for $S^2 \times RP^2$.

In $S^2\tilde{\times}RP^2$ the fibre of the bundle projection to RP^2 represents y. Hence

$$q_{KKR}(S^2\tilde{\times}RP^2)(x) = q_{KKR}(S^2\tilde{\times}RP^2)(y) = 0.$$

The image of the diagonal has a circle of self-intersections. However id_{S^2} is isotopic to a self-homeomorphism of S^2 which is the identity on one hemisphere and moves the equator off itself in the other hemisphere. Hence the diagonal embedding is isotopic to an embedding whose image has just one self-intersection. Hence $q_{KKR}(S^2\tilde{\times}RP^2)(x+y) = 0$ also, and so $q_{KKR}(S^2\tilde{\times}RP^2)$ is identically 0 for $S^2\tilde{\times}RP^2$.

In $RP^4\#_{S^1}RP^4$ the class y is represented by the image of $\{\mathbf{j}\} \times S^2$. Double points in the image correspond to pairs $\{s, s'\} \subset S^2$ such that $\psi(\mathbf{j}, s) = (\mathbf{j}, s')$. If $\{s, s'\}$ is such a pair then $s, s' \in D_+$, $s' = R_\pi(s)$ and

$$\mathbf{j}V(R_\pi(s))^{-1}V(s) = -V(R_\pi(s))^{-1}V(s)\mathbf{j}.$$

On using the explicit formula for $V(R_\pi(d))^{-1}V(d)$ given at the end of §3, we see that we must have $\cos(\pi r) = 0$ and $\cos(2\pi t) = 0$. Thus there are just two possibilities for s, differing by the rotation R_π. We may check that the double point is transverse. Hence $|\,\mathrm{Self}(S_y)| = 1$. Since $\{\mathbf{j}\} \times S^2$ has trivial normal bundle in $S^2 \times S^2$, $q_{KKR}(RP^4\#_{S^1}RP^4)(y) \equiv 2 \pmod 4$, and so $RP^4\#_{S^1}RP^4$ is not homotopy equivalent to $S^2\tilde{\times}RP^2$. It is not homotopy equivalent to $S^2 \times RP^2$ either, since $v_2(RP^4\#_{S^1}RP^4) \neq 0$. Thus these three manifolds may be distinguished by the invariants v_2 and q_{KKR}.

6. 4-manifolds with $\pi \cong \mathbb{Z}/4$ and $\chi = 1$

Let $M = S^2 \times S^2/\langle\sigma\rangle$, where $\sigma(s,t) = (t, A(s))$ for all $s, t \in S^2$. Let $[s,t]$ be the image in M of $(s,t) \in S^2 \times S^2$. Let $s \mapsto \bar{s}$ be reflection across the equator $S^1 \subset S^2$, and fix a basepoint $e = \bar{e}$ on the equator. We shall take (e,e) and $[e,e]$ as basepoints for $S^2 \times S^2$ and M, respectively. Let $\pi = \pi_1(M) = \mathbb{Z}/4$ and $\Lambda = \mathbb{Z}[\pi] = \mathbb{Z}[t]/(t^4-1)$. Since $\widetilde{M} = S^2 \times S^2$, $\pi_k(M) = \pi_k(S^2) \oplus \pi_k(S^2)$, for all $k \geq 2$. Let $\Pi = \pi_2(M)$, considered as a $\mathbb{Z}[\pi]$-module.

Every PD_4-complex X with $\pi_1(X) \cong \mathbb{Z}/4$ and $\chi(X) = 1$ is homotopy equivalent to M (see [8, Chapter 12, §5]).

Theorem 6.1. *Let $M = S^2 \times S^2/\langle\sigma\rangle$, as above. The natural map from π to $Aut_\pi(\Pi)$ is an isomorphism, while $G_\#(M)$ has order ≤ 2.*

Proof. The group $\pi = \mathbb{Z}/4$ acts on $\Pi = \mathbb{Z}^2$ via $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and so $\Pi \cong \Lambda/(t^2+1) = \mathbb{Z}[i]$. Hence the natural map from π to $Aut_\pi(\Pi) \cong \mathbb{Z}[i]^\times = \langle i \rangle$ is an isomorphism. Similarly, π acts on $\pi_4(M) = (\mathbb{Z}/2)^2$ via swapping the summands.

The Hopf maps corresponding to the factors of $\widetilde{M}$ generate $\pi_3(M) \cong \mathbb{Z}^2$. Hence $\pi = \mathbb{Z}/4$ acts on $\pi_3(M)$ via $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Simple calculations using the standard periodic resolution of the augmentation module $\mathbb{Z}$ give $H^0(\pi;\Pi) \cong H^0(\pi;\pi_3(M)) \cong \mathbb{Z}$, while

$$H^{2k-1}(\pi;\Pi) = H^{2k}(\pi;\pi_3(M)) = \mathbb{Z}/2$$

and $H^{2k}(\pi;\Pi) = H^{2k-1}(\pi;\pi_3(M)) = 0$, for all $k > 0$.

The homotopy fibre of the classifying map from $P_2(M)$ to $K(\pi,1)$ is $K(\Pi,2)$. Since $K(\Pi,2)$ has no cohomology in odd degrees, while $H^2(K(\Pi,2);\pi_3(M)) \cong \pi_3(M)^2$, all the terms with $p+q$ odd in the Leray-Serre spectral sequence

$$H^p(\pi; H^q(K(\Pi,2);\pi_3(M)) \Rightarrow H^{p+q}(P_2(M);\pi_3(M))$$

are 0. Hence the spectral sequence collapses, so $H^3(P_2(M);\pi_3(M)) = 0$ and

$$H^4(P_2(M);\pi_3(M)) \cong \mathbb{Z} \oplus T.$$

where T has order 4. Therefore $G_\#(P_2(M)) = G_\#(P_3(M)) = 0$.

We may assume that $P_3(M)$ has a single 5-cell, attached along a map which factors through one of the S^2 factors of $\widetilde{M}$. Therefore the connecting homomorphism

$$H^4(M;\pi_4(M)) \to H^5(P_3(M), M;\pi_4(M))$$

is zero, and restriction from $H^4(P_3(M);\pi_4(M))$ to $H^4(M;\pi_4(M))$ is an isomorphism. Since

$$H^4(M;\pi_4(M)) \cong \overline{H_0(M;\pi_4(M))} = \mathbb{Z}/2,$$

by Poincaré duality, we see that $G_\#(P_4(M))$ has order at most 2.

Self homotopy equivalences of M extend to self-homotopy equivalences of $P_j(M)$, for all $j > 0$. Conversely, if $j \geq 3$ then every self-map f of $P_j(M)$ restricts to a self-map of M, by cellular approximation, and if f is a self-homotopy equivalence then so is the restriction, by duality in the universal cover $\widetilde{M} = S^2 \times S^2$ and the Whitehead theorems. Moreover, if $j \geq 4$ then homotopies of self maps of $P_j(M)$ restrict to homotopies of self-maps of M. Thus $G_\#(M) = G_\#(P_4(M))$, and so has order ≤ 2. □

We do not know if $G_\#(M)$ is trivial, but can avoid this issue by a different argument. Let $M_o = \overline{M \setminus D^4}$, and let $j_o \colon M_o \to M$ be the natural inclusion.

Lemma 6.2. *Let h be a based self homotopy equivalence of M which induces the identity on π and Π. Then h is based homotopic to a self-homeomorphism of M.*

Proof. The map h induces the identity on all homotopy groups, since $\pi_k(M) = \pi_k(S^2) \oplus \pi_k(S^2)$, for all $k \geq 2$. Let $P_j(h)$ be the extension of h to a self-homotopy equivalence of $P_j(M)$. Since $P_3(h)$ induces the identity on π, Π and $\pi_3(M)$, there is a homotopy H_t from $P_3(h)$ to the identity, by the exact sequences and main result of Tsukiyama [16, Theorem 1].

We may assume that $H(M_o \times [0,1])$ has image in M. Thus $h|_{M_o} \sim j_o$, and so we may assume that $h|_{M_o} = j_o$, by the homotopy extension property. We then see that h is a pinch map: $h \sim id_M \vee \gamma$ for some $\gamma \in \pi_4(M)$. Now $\gamma = \mu \circ \eta \circ S\eta$ for some $\mu \in \Pi$, since $\pi_4(S^2) = \mathbb{Z}/2$ is generated by $\eta \circ S\eta$. Since $H_2(M;\mathbb{F}_2) = H_2(\pi;\mathbb{F}_2)$, the Hurewicz homomorphism from Π to $H_2(M;\mathbb{F}_2)$ is 0. Therefore h has trivial normal invariant, by the arguments in Cochran and Habegger [1, Theorem 5.1]. Hence h is homotopic to a self-homeomorphism of M. □

Remark 6.3. As pointed out by Kirby and Taylor [11, Theorem 18], the argument for [1, Theorem 5.1] does not require simple connectivity, but for the general statement w_2 must be replaced by the Wu class $v_2(M)$. For any closed 4-manifold, the second Stiefel-Whitney class $w_2(\nu_X)$ of the stable normal bundle is the Wu class $v_2(X) = w_2(X) + w_1(X)^2$. In the present case $v_2(M) = w_2(M)$, since $w_1^2(M) = 0$.

Theorem 6.4. *Every based self-homotopy equivalence of M is based homotopy equivalent to a homeomorphism.*

Proof. Interchange of factors and reflection across the equator of S^2 may be used to define basepoint preserving homeomorphisms r and s of M, with $r([x,y]) = [y,x]$ and $s([x,y]) = [\bar{x},\bar{y}]$, for all $[x,y] \in M$. Let c be the equatorial arc from $c(0) = e$ to $c(1) = -e$ in S^2. Then $\gamma(t) = [c(t), e]$ represents a generator of π, while $r\gamma(t) = \gamma(1-t)$ and $s\gamma(t) = \gamma(t)$, for all $0 \le t \le 1$. Therefore $\pi_1(r) = -1$ and $\pi_1(s) = 1$. Clearly $\pi_2(r)$ and $\pi_2(s)$ have matrices $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $-I$, respectively.

Let h be the homeomorphism which drags the basepoint $\star$ around a loop representing a generator g of π. Then $\pi_1(h) = id_\pi$, since π is abelian, while $\pi_2(h)$ acts through g, and so has matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ or its inverse. It is now clear that if f is a based self-homotopy equivalence of M there is a based self-homeomorphism F such that $\pi_i(f) = \pi_i(F)$, for $i = 1$ and 2. Since $F^{-1}f$ induces the identity on π and Π, the theorem follows from Lemma 6.2. $\square$

Corollary 6.5. *There are four homeomorphism types of manifolds homotopy equivalent to M.*

Proof. The normal invariant map in the surgery exact sequence

$$S_{TOP}(M) \to [M, G/TOP] = H^2(X; \mathbb{Z}/2) \oplus H^4(X; \mathbb{Z})$$

is a bijection, since the groups $L_5(\mathbb{Z}/4, -)$ and $L_4(\mathbb{Z}/4, -)$ are both trivial (see Wall [17, Theorem 3.4.5]). Since every self-homotopy equivalence of M is homotopic to a self-homeomorphism, by Theorem 6.4, there are four homeomorphism types of manifolds homotopy equivalent to M. $\square$

As observed in the Introduction, every such manifold has a fake twin. In particular, if $h \colon M' \to M$ is a homotopy equivalence with nontrivial normal invariant $\eta(h) \in H^2(M; \mathbb{Z}/2)$, then every closed 4-manifold with $\pi \cong \mathbb{Z}/4$ and $\chi = 1$ is homeomorphic to one of M, M', $*M$ or $*M'$. The normal invariant of $M \sharp E_8 \to M$ is non-trivial in $H^4(M; \mathbb{Z}) \cong \mathbb{Z}/2$. After surgery, this produces the twin manifold $*M$.

Similarly, we have the manifold $*M'$ whose normal invariant is non-trivial in both summands of $[M, G/TOP]$, and $KS(*M') = 0$ by the formula on [11, p. 398]. In contrast, both M' and $*M$ have non-trivial Kirby-Siebenmann invariant. We do not know whether $*M'$ admits a smooth structure (see the next section for a candidate).

In general, the normal invariant is an invariant of a map. However, in this case the normal invariant and the Kirby-Siebenmann invariant distinguish homeomorphism types completely.

7. A smooth fake version of M ?

In this section we construct another smooth manifold M'' with $\pi_1(M'') = \mathbb{Z}/4$, which is homotopy equivalent to the geometric quotient M. At present we are not able to determine whether M'' is homeomorphic to M or to $*M'$.

Let $M^+ = S^2 \times S^2/\langle \sigma^2 \rangle = S^2 \times S^2/(s, s') \sim (A(s), A(s'))$ be the orientable double cover of $M = S^2 \times S^2/\langle \sigma \rangle$. Let $\Delta = \{(s, s) \mid s \in S^2\}$ be the diagonal in $S^2 \times S^2$. We may isotope Δ to a nearby sphere which meets Δ transversely in two points, by rotating the first factor, and so Δ has self-intersection ± 2. The diagonal is invariant under σ^2, and so $\delta = \Delta/\langle\sigma^2\rangle \cong RP^2$ embeds in M^+ with an orientable regular neighbourhood. Since $\sigma(\Delta) \cap \Delta = \emptyset$ this also embeds in M. We shall see that the complementary region also has a simple description.

We shall identify S^3 with the unit quaternions $\mathbb{H}_1$, and view S^2 as the unit sphere in the space of purely imaginary quaternions. The standard inner product on the latter space is given by $v \bullet w = \mathfrak{Re}(v\bar{w})$, for v, w purely imaginary quaternions. Let

$$C_x = \{(s,t) \in S^2 \times S^2 \mid s \bullet t = x\},\ \forall\, x \in [-1,1].$$

Then $C_1 = \Delta$ and $C_{-1} = \sigma(\Delta)$, while $C_x \cong C_0$ for all $|x| < 1$. The map $f \colon S^3 \to C_0$ given by $f(q) = (q\mathbf{i}q^{-1}, q\mathbf{j}q^{-1})$ for all $q \in S^3$ is a 2-fold covering projection, and so $C_0 \cong RP^3$.

It is easily seen that $N = \cup_{x \geq \varepsilon} C_x$ and $\sigma(N)$ are regular neighbourhoods of Δ and $\sigma(\Delta)$, respectively, while $C = \cup_{x \in [-\varepsilon,\varepsilon]} C_x \cong C_0 \times [-\varepsilon, \varepsilon]$. In particular, N and $\sigma(N)$ are each homeomorphic to the total space of the unit disc bundle in T_{S^2}, and $\partial N \cong C_0 \cong RP^3$. The subsets C_x are invariant under σ^2. Hence $N(\delta) = N/\langle\sigma^2\rangle$ is the total space of the tangent disc bundle of RP^2. In particular, $\partial N(\delta) \cong L(4,1)$ and δ represents the nonzero element of $H_2(M;\mathbb{F}_2)$, since it has self-intersection 1 in $\mathbb{F}_2$.

Remark 7.1. It is not hard to show that any embedded surface representing the nonzero element of $H_2(M;\mathbb{F}_2)$ is non-orientable but lifts to M^+, and so has an orientable regular neighbourhood.

We also see that $C/\langle\sigma^2\rangle \cong L(4,1) \times [-\varepsilon,\varepsilon]$. Since $f(q.\frac{1}{\sqrt{2}}(\mathbf{1}+\mathbf{k})) = \sigma(f(q))$, the map $\tilde{\sigma} \colon S^3 \to S^3$ defined by right multiplication by $\frac{1}{\sqrt{2}}(\mathbf{1}+\mathbf{k})$ lifts σ. Hence $C_0/\langle\sigma\rangle = S^3/\langle\tilde{\sigma}\rangle = L(8,1)$, and so $MC = C/\langle\sigma\rangle$ is the mapping cylinder of the double cover $L(4,1) \to L(8,1)$. Since $S^2 \times S^2 = N \cup C \cup \sigma(N)$ it follows that $M = N(\delta) \cup MC$.

This construction suggests a candidate for another smooth 4-manifold in the same (simple) homotopy type. Let $M'' = N(\delta) \cup MC'$, where MC' is the mapping cylinder of the double cover $L(4,1) \to L(8,5)$. Then $\pi_1(M'') \cong \mathbb{Z}/4$ and $\chi(M'') = 1$, and so there is a homotopy equivalence $h \colon M'' \simeq M$.

Some questions for further investigation:

(i) Is there an easily analyzed explicit choice for $h \colon M'' \to M$, with computable codimension 2 Kervaire invariant?
(ii) Are M and M'' homeomorphic? diffeomorphic?
(iii) Is there a computable homeomorphism (or diffeomorphism) invariant that can be applied here?

We remark that most readily computable invariants are invariants of homotopy type.

8. PD_4-complexes with $\pi \cong (\mathbb{Z}/2)^2$ and $\chi = 1$

We now consider the remaining cases, where $\pi \cong (\mathbb{Z}/2)^2$. As mentioned in the Introduction, there are two geometric quotients, namely $RP^2 \times RP^2$ and the non-trivial bundle $RP^2 \tilde{\times} RP^2$. In this section, we will construct a third smooth manifold N homotopy equivalent to $RP^2 \tilde{\times} RP^2$, but which is not a geometric quotient. By [8, Chapter 12,

§6], there are two equivalence classes of quadratic 2-types realized by PD_4-complexes X with universal cover $X \simeq S^2 \times S^2$ and

$$\pi_1(X) \cong \pi = \langle t, u \mid t^2 = u^2 = (tu)^2 = 1\rangle.$$

Let $\{t^*, u^*\}$ be the dual basis for $H^1(\pi; \mathbb{F}_2)$. If X is a PD_4-complex with $\pi_1(X) \cong \pi$ and $\chi(X) = 1$, then we may assume that $v_1(X) = t^* + u^*$ and $v_2(X)$ is either t^*u^* or $t^*u^* + (u^*)^2$. This is an easy consequence of Poincaré duality with coefficients $\mathbb{F}_2$ and the Wu formulas.

Let X^+ denote the orientation double cover of X. If $v_2(X) = t^*u^*$ then $v_2(X^+) = t^{*2} \neq 0$ and both non-orientable double covers have $v_2 = 0$, while if $v_2(X) = t^*u^* + (u^*)^2$ then $v_2(X^+) = 0$ and just one of the non-orientable double covers has $v_2 = 0$. The two possibilities for v_2 are realized respectively by $RP^2 \times RP^2$ and the nontrivial bundle space $RP^2 \tilde{\times} RP^2 = S^2 \times S^2/\pi$, where π acts by $t(s, s') = (-s, s')$ and $u(s, s') = (R_\pi(s), -s')$, for all $s, s' \in S^2$.

We now construct a third smooth manifold N with universal cover $S^2 \times S^2$ and fundamental group π as follows. The map $f([s, x]) = [-s, x]$ defines a free involution of $S^2 \tilde{\times} S^1$, with quotient $RP^2 \times S^1$. By Definition 4.1, the manifold $RP^4 \#_{S^1} RP^4 = E_1 \cup_\xi E_2$, with glueing map $\xi \colon \partial E_1 \to \partial E_2$. The maps f and ξ commute.

Definition 8.1. Let N denote the quotient space of $RP^4 \#_{S^1} RP^4$ by the free involution F given by the formula $F([s, d]_i) = [-s, d]_{3-i}$ for all $[s, d]_i \in E_i$ and $i = 1, 2$.

We shall see below that N is in the quadratic 2-type of $RP^2 \tilde{\times} RP^2$. Let F^+ be the lift of F to the universal cover of $RP^4 \#_{S^1} RP^4$. Then the orientable double cover of N is $S^2 \times S^2/\langle H^{-1}F^+H\rangle$. Now $H^{-1}F^+H(s, s') = (-s, R(s'))$, where R is reflection across the equator, and so this double cover is the S^2-bundle space over RP^2 which is a spin manifold. A similar argument shows that the other non-orientable double cover is $RP^2 \times S^2$.

Thus N is not homotopy equivalent to a bundle space. Nor is it homotopy equivalent to the total space of an orbifold bundle over a 2-orbifold: indeed, any such bundle would have general fibre RP^2, and so the total space would be foliated by copies of RP^2. Hence the base orbifold would have no singular points.

9. THE QUADRATIC 2-TYPE OF $RP^2 \times RP^2$

Let $K = \overline{RP^2 \times RP^2 \setminus D^4}$, let $I_1, I_2 \colon S^2 \to \widetilde{K} = \overline{S^2 \times S^2 \setminus 4D^4}$ be the inclusions of the factors, and let $[J]$ be the homotopy class of a fixed lift $\widetilde{J} \colon S^3 \to \widetilde{K}$ of the natural inclusion $J \colon S^3 = \partial D^4 \to K$. Then $\Pi = \pi_2(K) \cong \mathbb{Z}^2$ is generated by I_1 and I_2, and

$$\Pi \cong \Lambda/(t + 1, u - 1) \oplus \Lambda/(t - 1, u + 1)$$

as a Λ-module, $\Lambda = \mathbb{Z}[\pi]$ denotes the integral group ring. The Hurewicz homomorphism $h \colon \pi_3(K) \to H_3(\widetilde{K}; \mathbb{Z}) \cong \mathbb{Z}^3$ is surjective, with kernel the image of $\Gamma_W(\Pi)$, generated by Whitehead products and composites with η. Then $h([J])$ generates $H_3(\widetilde{K}; \mathbb{Z})$ as a Λ-module, and $H_3(\widetilde{K}; \mathbb{Z}) \cong \Lambda/(1 - t)(1 - u)\Lambda$.

The elements $\eta_1 = I_1 \circ \eta$, $\eta_2 = I_2 \circ \eta$ and $\zeta = [I_1, I_2]$ are a basis for $\Gamma_W(\Pi) \cong \mathbb{Z}^3$. Since $\Gamma_W(\Pi)$ is torsion free and $2\eta_i = [I_i, I_i]$, we see that $t\eta_i = u\eta_i = \eta_i$ for $i = 1, 2$, while $t\zeta = u\zeta = -\zeta$. Hence $\mathbb{Z}^w \otimes_\Lambda \Gamma_W(\Pi) \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2$, and the torsion subgroup is generated by the images of η_1 and η_2. Since the k-invariant is symmetric under the involution which interchanges the summands of Π, there are three homotopy types of PD_4-complexes $X_\alpha = K \cup_{[J]+\alpha} e^4$ in this quadratic 2-type, represented by $\alpha = 0$, η_1 and $\eta_1 + \eta_2$.

As above, let $\{t^*, u^*\}$ be the basis of $H^1(\pi; \mathbb{F}_2)$ dual to $\{t, u\}$. Let X^t_α and X^u_α be the covering spaces associated to the subgroups $\langle t\rangle = \mathrm{Ker}(u^*)$ and $\langle u\rangle = \mathrm{Ker}(t^*)$ of π, respectively. Since the homomorphisms $H^*(X_\alpha; \mathbb{F}_2) \to H^*(K; \mathbb{F}_2)$ are isomorphisms in degrees ≤ 3 and $(t^*)^3 = (u^*)^3 = 0$ in $H^3(RP^2 \times RP^2; \mathbb{F}_2)$, we see that $(t^*)^3 = (u^*)^3 = 0$ in $H^3(X_\alpha; \mathbb{F}_2)$, for all α. It follows easily from the nonsingularity of Poincaré duality that the rings $H^*(X_\alpha; \mathbb{F}_2)$ are all isomorphic. In particular, $w_1(X_\alpha) = t^* + u^*$, $v_2(X_\alpha) = t^*u^*$ and $x^4 = 0$, for all $x \in H^1(X_\alpha; \mathbb{F}_2)$, in each case. Hence $X^+_\alpha \simeq S^2 \times S^2/\langle\sigma^2\rangle$, while the non-orientable double covers X^t_α and X^u_α each have $v_2 = 0$. Thus the manifold N defined in (8.1) is not in this quadratic 2-type.

We shall adapt the argument of [3, §3] to show that if $\alpha \neq 0$ then X_α is not homotopy equivalent to a closed 4-manifold.

Theorem 9.1. *Let M be a closed 4-manifold with $\pi = \pi_1(M) \cong (\mathbb{Z}/2)^2$ and $\chi(M) = 1$, and such that $x^4 = 0$ for all $x \in H^1(M; \mathbb{F}_2)$. Then M is homotopy equivalent to $RP^2 \times RP^2$.*

Proof. Our hypotheses imply that M is in the quadratic 2-type of $RP^2 \times RP^2$, and so $M \simeq X_\alpha = K \cup_{[J]+\alpha} e^4$, for some $\alpha = 0$, η_1 or $\eta_1 + \eta_2$. If M in the quadratic 2-type of $RP^2 \tilde{\times} RP^2$ then there is a class $x \in H^1(M; \mathbb{F}_2)$ such that $x^3 \neq 0$. Poincaré duality considerations then imply that $x^4 \neq 0$ (see [8, Chapter 12, §§4-6]).

Suppose that $\alpha = \eta_1$ or $\eta_1 + \eta_2$. Then the image of α in $\pi_3(RP^2)$ under composition with the projection pr_1 to the first factor is $\bar{\eta}$. Hence the composite of the inclusion $K \subset RP^2 \times RP^2$ with pr_1 extends to a map $p\colon X_\alpha \to L = RP^2 \cup_{\bar{\eta}} e^4$. (Note that $\mathrm{Ker}(\pi_1(p) = \langle u\rangle$.) Let $\tilde{p}\colon X^u_\alpha \to \widetilde{L}$ be the induced map of double covers, and let $f\colon X_\alpha \to RP^{k+1}$ (for k large) be the classifying map for the double cover $X^u_\alpha \to X_\alpha$.

Let $a = \tilde{p}^*(c)$ be the image of the generator of $H^2(\widetilde{L}; \mathbb{F}_2) = \mathbb{F}_2$, let $\bar{b} = (u^*)^2 \in H^2(X_\alpha; \mathbb{F}_2)$, and let b be the image of $\bar{b}$ in $H^2(X^u_\alpha; \mathbb{F}_2)$. The 3-skeleton of X^u_α is K^u, and so the covering transformation t acts on $H^2(X^u_\alpha; \mathbb{F}_2)$ via the identity. Hence the quadratic form q used in [3] in computing the Arf invariant $A(X_\alpha, f)$ of the covering $X^u_\alpha \to X_\alpha$ is an enhancement of the ordinary cup product. The pair $\{a, b\}$ is a symplectic basis with respect to the cup product, and $q(a) = 1$, by the argument of [3]. Since $(u^*)^3 = (u^*)^4 = 0$ in $H^*(X_\alpha; \mathbb{F}_2)$, $Sq_i\bar{b} = Sq^{2-i}\bar{b} = 0$ for $i = 0$ or 1. Hence we also have $q(b) = 1$, by [3, Proposition 1.5], and so $A(X_\alpha, f)$ is nonzero. But this contradicts the assumption that X_α is homotopy equivalent to a closed manifold, by [3, Proposition 2.2]. Hence $\alpha = 0$ and so M is homotopy equivalent to $RP^2 \times RP^2$. □

Corollary 9.2. *There is exactly one homotopy type for a closed manifold in the quadratic 2-type of $RP^2 \times RP^2$.*

The inclusion $RP^2 \to L = RP^2 \cup_{\bar{\eta}} e^4$ induces isomorphisms on π_i for $i \leq 2$. Since L is covered by $S^2 \cup_\eta e^4 \cup_{A\eta} e^4 \simeq S^2 \cup_\eta e^4 \vee S^4 = CP^2 \vee S^4$, $\pi_3(L) = 0$. Hence we may view L as the 4-skeleton of $P_2(RP^2)$. (See [13].)

10. THE QUADRATIC 2-TYPE OF $RP^2\tilde{\times}RP^2$

Now let $K' = \overline{RP^2\tilde{\times}RP^2 \setminus D^4}$, and let $\Pi' = \pi_2(K')$. Let $J' \colon S^3 = \partial D^4 \to K'$ be the natural inclusion, and let $\eta_i' \colon S^3 \to K'$ (with $i = 1, 2$) be "Hopf maps" factoring through inclusions of the factors of the universal cover $S^2 \times S^2$. We again find that $\mathbb{Z}^w \otimes_\Lambda \Gamma_W(\Pi') \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2$, and the torsion subgroup is generated by the images of η_1' and η_2'. However there are four homotopy types of PD_4-complexes $Y_\alpha = K' \cup_{[J']+\alpha} e^4$ in this quadratic 2-type, represented by $\alpha = 0$, η_1', η_2' and $\eta_1' + \eta_2'$.

Let $\{t^*, u^*\}$ be the basis of $H^1(\pi; \mathbb{F}_2)$ dual to $\{t, u\}$, and let Y_α^t and Y_α^u be the covering spaces associated to the subgroups $\langle t\rangle = \mathrm{Ker}(u^*)$ and $\langle u\rangle = \mathrm{Ker}(t^*)$ of π, respectively. We may assume that u^* is induced from the base RP^2, so $(u^*)^3 = 0$, and then $(t^*)^3 \neq 0$, since $v_2(RP^2\tilde{\times}RP^2) \neq 0$. We again find that the $\mathbb{F}_2$-cohomology rings of the Y_α are all isomorphic. In particular, $v_2(Y_\alpha) = t^*u^* + (u^*)^2$ in each case. Hence in each case Y_α^+ is homotopy equivalent to the S^2-bundle space over RP^2 which is a spin manifold. The covering space Y_α^t is homotopy equivalent to $S^2 \times RP^2$, since $v_2(Y_\alpha^t) = 0$, while Y_α^u is homotopy equivalent to one of either $RP^4\#_{S^1}RP^4$ or $S^2\tilde{\times}RP^2$, since $v_2(Y_\alpha^u) \neq 0$.

Theorem 10.1. *Let M be a closed 4-manifold with $\pi = \pi_1(M) \cong (\mathbb{Z}/2)^2$ and $\chi(M) = 1$, and such that $x^4 \neq 0$ for some $x \in H^1(M; \mathbb{F}_2)$. Then M is homotopy equivalent to $RP^2\tilde{\times}RP^2$ or N.*

Proof. We shall adapt the proof of Theorem 9.1, again based on the arguments of [3]. In this case M must be in the quadratic 2-type of $RP^2\tilde{\times}RP^2$, and so $M \simeq Y_\alpha = K' \cup_{[J']+\alpha} e^4$ for some $\alpha = 0$, η_1', η_2' or $\eta_1' + \eta_2'$. The double covering space M^t is homotopy equivalent to $RP^2 \times S^2$. As in Theorem 9.1, the covering automorphism induces the identity on $H^2(M^u; \mathbb{F}_2)$.

Suppose that $\alpha = \eta_2'$ or $\eta_1' + \eta_2'$. The composite of the inclusion $K' \subset RP^2\tilde{\times}RP^2$ with the bundle projection extends to a map $p \colon Y_\alpha \to L$. Let $\tilde{p} \colon Y_\alpha^u \to \widetilde{L}$ be the induced map of double covers, and let $a = \tilde{p}^*(c)$ be the image of the generator of $H^2(\widetilde{L}; \mathbb{F}_2)$. Let $\bar{b} = (t^*)^2 \in H^2(Y_\alpha; \mathbb{F}_2)$, and let b be the image of $\bar{b}$ in $H^2(Y_\alpha^t; \mathbb{F}_2)$. Then $\{a, b\}$ is a symplectic basis for the cup product pairing. We again find that $q(a) = q(b) = 1$, so the Arf invariant associated to the 2-fold covering $Y_\alpha^u \to Y_\alpha$ is nonzero, contradicting the hypothesis that M is a closed manifold. Therefore either $\alpha = 0$ or $\alpha = \eta_1'$. Since $Y_0 = RP^2\tilde{\times}RP^2$ and N are manifolds in this quadratic 2-type, and are not homotopy equivalent, we must have $Y_{\eta_1'} \simeq N$ and M must be one of these two manifolds. □

The manifolds $RP^2\tilde{\times}RP^2$ and N may be distinguished by their (non-orientable) double covers. However, in general we do not know which of the PD_4-complexes W_α of §3 are double covers of the PD_4-complexes X_β or Y_γ of §8 or §9.

11. SELF HOMOTOPY EQUIVALENCES

The standard cellular decomposition of RP^2 has three cells, with basepoint $* = [1 : 0 : 0]$ and 1-skeleton $RP^1 = \{[x : y : 0]\}$. Reflection across the (x, y)-plane in $\mathbb{R}^3$ induces an involution h of RP^2 given by $h([x : y : z]) = [x : y : -z]$, for all $[x : y : z] \in RP^2$. Then h

fixes $RP^1 \cup \{[0:0:1]\}$. The image of h generates $E_*(RP^2) = \mathbb{Z}/2$, while $\pi_2(h) = -1$, so $G_\#(RP^2) = 1$. Since h is freely isotopic to the identity, $E(RP^2) = 1$.

The product cell structure for M has basepoint $(*,*)$ and intermediate skeleta

$$RP^1_1 \vee RP^1_2 \subset RP^2_1 \cup T \cup RP^2_2 \subset RP^2 \times RP^1 \cup_T RP^1 \times RP^2,$$

where $RP^k_1 = RP^k \times \{*\}$ and $RP^k_2 = \{*\} \times RP^k$, for $k = 1, 2$, and T is the torus $RP^1 \times RP^1 \subset RP^2 \times RP^2$. Let $j_i \colon RP^2 \to M$ be the inclusions of RP^2_i into M, for $i = 1, 2$, and let $p_1, p_2 \colon M \to RP^2$ be the projections to the factors.

Let f be a based self homotopy equivalence of $M = RP^2 \times RP^2$. Then $f^*w = w$, and so either $\pi_1(f)$ interchanges the generators corresponding to the factors or $\pi_1(f) = id_\pi$. In the latter case $\pi_2(f)$ has diagonal matrix with respect to the basis for $\Pi = \pi_2(M)$ corresponding to the factors of M. Since there are based homeomorphisms switching the factors and inducing the diagonal actions on Π, we may assume that $\pi_1(f) = id_\pi$ and $\pi_2(f) = id_\Pi$, if we seek exotic self-homotopy equivalences. Such maps induce the identity on all homotopy groups. Let $f \in G_\#(M)$, and let $f_i = p_i f$, for $i = 1, 2$. Then $f_i j_i \sim id$, for $i = 1, 2$, so we may assume that f restricts to the identity on $RP^2_1 \vee RP^2_2$.

The results of Tsukiyama [16] cited in §1 imply that $G_\#(P_2(M)) \cong H^2(\pi; \Pi)$ and that the natural homomorphism from $G_\#(P_3(M))$ to $G_\#(P_2(M))$ has image the stabilizer of $k_3(M)$, and kernel a quotient of $H^3(P_2(M); \pi_3(M))$. It is easily seen that $H^2(\pi; \Pi) \cong (\mathbb{Z}/2)^2$. Since π acts diagonally on Π, it fixes the Hopf maps η_1 and η_2, and so acts trivially on $\pi_3(M) \cong \mathbb{Z}^2$. As noted at the end of §8, we may assume that $P_2(RP^2)$ has 4-skeleton $L = RP^2 \cup_{\bar\eta} e^4$. Hence

$$H^3(P_2(M); \mathbb{Z}) \cong H^3(L \times L; \mathbb{Z}) = Tor(H^2(L; \mathbb{Z}), H^2(L; \mathbb{Z})) = \mathbb{Z}/2,$$

and so $H^3(P_2(M); \pi_3(M)) \cong (\mathbb{Z}/2)^2$. However, the action of $H^3(P_2(M); \pi_3(M))$ on the homotopy type of $P_3(M)$ may not be effective. Thus $G_\#(P_3(M))$ has order at most 16.

If f is homotopic to id_M as a map into $P_3(M)$ then we may assume that $f = id_M$ on the 3-skeleton $M_o = \overline{M \setminus D^4}$. The map f is then a pinch map. Since the mod 2 Hurewicz homomorphism is trivial in this case, such maps have trivial normal invariant, and so are homotopic to homeomorphisms.

If $M' = RP^2 \tilde{\times} RP^2$ is the nontrivial bundle space and $w = w_1(M')$ then

$$H^*(M'; \mathbb{F}_2) \cong \mathbb{F}_2[w, x]/(x^3, w^2(w + x)),$$

where $x \in H^1(M'; \mathbb{F}_2)$ is induced from the base RP^2. Every self homotopy equivalence of M' must induce the identity on this cohomology ring, and so on π. It must also act diagonally on $\pi_2(M')$, with respect to the standard basis. Reflections of the factors of $\widetilde{M'} = S^2 \times S^2$ across their equators commute with the action of π, and cover homeomorphisms of M' which preserve a basepoint and induce the diagonal automorphisms of $\pi_2(M')$. As before, it would be enough to understand the subgroup $G_\#(M'_o)$, where $M'_o = \overline{M' \setminus D^4})$.

The manifold N constructed in §7 has the same 3-skeleton as $RP^2 \tilde{\times} RP^2$, and $H^*(N; \mathbb{F}_2) \cong H^*(M'; \mathbb{F}_2)$. (See [8, Chapter 12, §4].) Hence self homotopy equivalences of N must induce the identity on π, and similar estimates apply for $G_\#(N_o)$, where $N_o = \overline{N \setminus D^4})$. Are the nontrivial diagonal automorphisms of $\pi_2(N)$ realized by homeomorphisms?

In each case, $S_{TOP}(M)$ has 8 elements, half of which have domains with nontrivial Kirby-Siebenmann invariant, and so the image of Homeo(M) in the group of (free homotopy classes of) self homotopy equivalences of M has index at most 4. However, whether every self homotopy equivalence of M is homotopic to a homeomorphism remains open. To make further progress we need explicit representatives for the self homotopy equivalences.

12. RECONSTRUCTING $RP^2 \times RP^2$?

Arguments similar to those of §7 show that $RP^2 \times RP^2 \cong N(\delta) \cup MC''$, where MC'' is the mapping cylinder of the double cover of $S^3/Q(8)$. Since $S^3/Q(8)$ is the only 3-manifold with fundamental group non-cyclic and of order 8, we cannot vary the construction by replacing MC'' by another such mapping cylinder. It is tempting to consider instead replacing the unit tangent disc bundle $N(\delta)$ by another disc bundle over RP^2, with Euler class k times the generator of $H^2(RP^2;\mathbb{Z}^w) \cong \mathbb{Z}$ and total space N_k. Then ∂N_k is a prism manifold with fundamental group of order $2k$. This double covers another prism manifold, and so we may adjoin a mapping cylinder, to get a closed nonorientable 4-manifold M_k with $\chi(M_k) = 1$ and $\pi_1(M_k) \cong (\mathbb{Z}/2)^2$.

Unfortunately this construction gives nothing new, for $RP^2 \times RP^2$ is the union of such a disc bundle and the corresponding mapping cylinder, for any $k \geq 1$! Let $f_k(z) = z^k$, for $z \in \widehat{\mathbb{C}} = S^2$, and let $f_k'(z) = A(f_k(z)) = (-1)\bar{z}^{-k}$ be the image of $f_k(z)$ under the antipodal map of S^2. The graphs of f_k and f_k' are disjoint, and are interchanged by the canonical generators of $(\mathbb{Z}/2)^2$, acting on $S^2 \times S^2$ in the standard way. Therefore their images in $RP^2 \times RP^2$ coincide, and are diffeomorphic to RP^2. Since the graph of f_k has self-intersection number $2k$, tubular neighbourhoods of this copy of RP^2 are diffeomorphic to N_k. It is straightforward to check that the homomorphism induced by the double cover maps $H^2(RP^2;\mathbb{Z}^w)$ onto $2H^2(S^2;\mathbb{Z})$. As in §4, the complementary region is a mapping cylinder of the type proposed above.

13. Stable classification

Let $\xi\colon B \to BTop$ denote the normal 1-type of the geometric quotient M of $S^2 \times S^2$ with fundamental group $\pi = \mathbb{Z}/4$. We may assume that $B = BTopSpin \times K(\pi,1)$, since $w_2(\widetilde{M}) = 0$ (see [14, Theorem 5.2.1 and §8.1]). Let $\gamma\colon B \to K(\pi,1)$ be the projection onto the second factor.

In order to compute the bordism group $\Omega_4(B,\xi)$ we use the Atiyah-Hirzebruch spectral sequence with $E^2_{p,q} = H_p(\pi;\Omega_q^{TopSpin})$ where the coefficients

$$\Omega_q^{TopSpin} = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}/2, \qquad \text{for } 0 \leq 1 \leq 4,$$

are twisted by w_1. We have $E^2_{p,0} = H_p(\pi;\mathbb{Z}_-) = \mathbb{Z}/2$, for p even, and $E^2_{p,0} = 0$ for p odd. Similarly, $E^2_{0,4} = \mathbb{Z}/2$. The first differential

$$d_2\colon E^2_{p,q} \to E^2_{p-2,q+1}$$

is dual to the map

$$\hat{d}\colon H^{p-2}(\pi;\mathbb{Z}/2) \to H^p(\pi;\mathbb{Z}/2)$$

for the cases $(4,2)$ and $(3,1)$. Note that the cohomology ring $H^*(\pi;\mathbb{Z}/2) = P(u)\otimes E(x)$, where $|u| = 2$ and $|x| = 1$, with $Sq^1 u = 0$ and $x^2 = 0$. The classes $w_1(\nu_M) = x$ and $w_2(\nu_M) = u$.

According to Teichner [15, §2] the map $\hat{d}$ is given by the formula

$$\hat{d}(\alpha) = Sq^2\alpha + (Sq^1\alpha)w_1 + \alpha w_2.$$

We compute using this formula and obtain:

$$\begin{aligned}
&\hat{d}\colon H^1(\pi;\mathbb{Z}/2) \to H^3(\pi;\mathbb{Z}/2), \qquad && \hat{d}(x) = xu \neq 0\\
&\hat{d}\colon H^2(\pi;\mathbb{Z}/2) \to H^4(\pi;\mathbb{Z}/2), \qquad && \hat{d}(u) = 0\\
&\hat{d}\colon H^3(\pi;\mathbb{Z}/2) \to H^5(\pi;\mathbb{Z}/2), \qquad && \hat{d}(xu) = 0\\
&\hat{d}\colon H^4(\pi;\mathbb{Z}/2) \to H^6(\pi;\mathbb{Z}/2), \qquad && \hat{d}(u^2) = u^3 \neq 0.
\end{aligned}$$

After dualizing, we get $E^3_{0,4} = \mathbb{Z}/2$, $E^3_{3,1} = 0$, $E^3_{2,2} = H_2(\pi;\mathbb{Z}/2) = \mathbb{Z}/2$, and $E^3_{4,0} = \mathbb{Z}/2$. Moreover, the only nonzero entry on the line $p+q = 5$ of the E^3 page is $E_{3,2} = E^2_{2,2} = \mathbb{Z}/2$.

We remark that the non-zero element in $E^3_{4,0} = \mathbb{Z}/2$ is represented by the image of the E_8-manifold under the inclusion map

$$\Omega_4^{TopSpin}(*) \to \Omega_4(B,\xi).$$

However. we have a factorization:

$$\Omega_4^{TopSpin}(*) \to \Omega_4(B,\xi) \to \Omega_4^{TopSpin^c}(*),$$

and the E_8-manifold represents a non-trivial element in $\Omega_4^{TopSpin^c}(*)$, as noted in [7, p. 654]. Hence the $E^3_{4,0}$-term survives to $E^\infty_{4,0}$. The conclusion is that

$$\Omega_4(B,\xi) = \mathbb{Z}/2 \oplus H_2(\pi;\mathbb{Z}/2) \oplus \mathbb{Z}/2.$$

Let $c\colon M \to B$ denote the classifying map of the ξ-structure on M. To detect elements in this bordism group, we can define

$$\Omega_4(B,\xi)_M = \{(M',c'))\colon \gamma_* c'_*[M'] = \gamma_* c_*[M] \in H_4(\pi;\mathbb{Z}/2)\}$$

If $f\colon M' \to M$ is a homotopy equivalence, then we can cover f by a bundle map. The projection of the difference $[M',c'] - [M,c]$ into $H_2(\pi;\mathbb{Z}/2)$ is detected by the first component of the normal invariant $\eta(f) \in [M, G/TOP]$, with respect to the identification

$$[M, G/TOP] = H^2(M;\mathbb{Z}/2) \oplus H^4(M;\mathbb{Z}) \cong H_2(\pi;\mathbb{Z}/2) \oplus \mathbb{Z}/2$$

given by Poincaré duality. We will call this the *reduced normal invariant of* M, and denote by $\overline{\eta}(M') \in H_2(\pi;\mathbb{Z}/2)$ the equivalence class of $\eta(f)$ modulo the action on normal invariants by homotopy self equivalences of M. If this is zero, then the difference $[M',c'] - [M,c]$ is detected by the KS invariant.

Lemma 13.1. *Suppose that $f\colon M \to M$ is a self homotopy equivalence. Then the elements $(M, c\circ f)$ and (M,c) are ξ-bordant.*

Proof. By functoriality, the homotopy equivalence $f\colon M \to M$ induces a self homotopy equivalence $\phi\colon B \to B$, such that $c \circ f \simeq \phi \circ c$. However, since $B = BTopSpin \times K(\pi, 1)$ has the homotopy type of $K(\mathbb{Z}, 4) \times K(\pi, 1)$ through dimensions ≤ 5, the composition $\phi \circ c$ is determined by the map $\phi^*\colon H^4(B; \mathbb{Z}) \to H^4(B; \mathbb{Z})$. Either $\phi \circ c \simeq c$ or $\phi \circ c$ differs from c by a non-trivial map $K(\pi, 1) \to K(\mathbb{Z}, 4)$. In this case, the normal invariant of f would have non-zero component in $H^2(\pi; \mathbb{Z}/2) \subset [M, G/TOP]$. But this would imply a change in the Kirby-Siebenmann invariant from domain to range of f, by the formula in [11, p. 398], which is impossible for a self homotopy equivalence. □

Corollary 13.2. *Stably homeomorphic manifolds homotopy equivalent to M are homeomorphic. Such manifolds are distinguished by their reduced normal invariant and the KS invariant.*

Proof. The calculations above, and the theory of [12, §4], show that there are 4 distinct stable homeomorphism classes. However the structure set $S_{TOP}(M)$ has 4 elements, so there can be no non-trivial self-homotopy equivalences. If follows that the choice of a homotopy equivalence $f\colon M' \to M$ is unique up to homotopy. Hence the reduced normal invariant $\bar{\eta}(M') \in H_2(\pi; \mathbb{Z}/2)$ is a well-defined invariant of M'. □

References

[1] T. D. Cochran and N. Habegger, *On the homotopy theory of simply connected four manifolds*, Topology **29** (1990), 419–440.

[2] M. H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), 357–453.

[3] I. Hambleton and R. J. Milgram, *Poincaré transversality for double covers*, Canad. J. Math. **30** (1978), 1319–1330.

[4] I. Hambleton and M. Kreck, *On the classification of topological* 4*-manifolds with finite fundamental group*, Math. Ann. **280** (1988), 85–104.

[5] ———, *Cancellation, elliptic surfaces and the topology of certain four-manifolds*, J. Reine Angew. Math. **444** (1993), 79–100.

[6] ———, *Homotopy self-equivalences of 4-manifolds*, Math. Z. **248** (2004), 147–172.

[7] I. Hambleton, M. Kreck, and P. Teichner, *Nonorientable* 4*-manifolds with fundamental group of order* 2, Trans. Amer. Math. Soc. **344** (1994), 649–665.

[8] J. A. Hillman, *Four-manifolds, geometries and knots*, Geometry & Topology Monographs, vol. 5, Geometry & Topology Publications, Coventry, 2002 (revision 2007).

[9] M. H. Kim and F. Raymond, *The diffeotopy group of the twisted* 2*-sphere bundle over the circle*, Trans. Amer. Math. Soc. **322** (1990), 159–168.

[10] M. H. Kim, S. Kojima, and F. Raymond, *Homotopy invariants of nonorientable* 4*-manifolds*, Trans. Amer. Math. Soc. **333** (1992), 71–81.

[11] R. C. Kirby and L. R. Taylor, *A survey of 4-manifolds through the eyes of surgery*, Surveys on surgery theory, Vol. 2, Ann. of Math. Stud., vol. 149, Princeton Univ. Press, Princeton, NJ, 2001, pp. 387–421.

[12] M. Kreck, *Surgery and duality*, Ann. of Math. (2) **149** (1999), 707–754.

[13] J. Siegel, *Higher order cohomology operations in local coefficient theory*, Amer. J. Math. **89** (1967), 909–931.

[14] P. Teichner, *Topological four manifolds with finite fundamental group*, Ph.D. thesis, University of Mainz, Shaker Verlag, ISBN 3-86111-182-9; http://math.berkeley.edu/ teichner/Papers/phd.pdf, 1992.

[15] ———, *On the star-construction for topological 4-manifolds*, Geometric topology (Athens, GA, 1993), AMS/IP Stud. Adv. Math., vol. 2, Amer. Math. Soc., Providence, RI, 1997, pp. 300–312.
[16] K. Tsukiyama, *Self-homotopy-equivalences of a space with two nonvanishing homotopy groups*, Proc. Amer. Math. Soc. **79** (1980), 134–138.
[17] C. T. C. Wall, *Classification of Hermitian Forms. VI. Group rings*, Ann. of Math. (2) **103** (1976), 1–80.

Department of Mathematics & Statistics
McMaster University
Hamilton, ON L8S 4K1, Canada
E-mail address: hambleton@mcmaster.ca

School of Mathematics and Statistics
University of Sydney
Sydney, NSW 2006, Australia
E-mail address: jonathan.hillman@sydney.edu.au