FREE INVOLUTIONS ON 6-MANIFOLDS

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INTRODUCTION

In this paper, we give the diffeomorphism classification of smooth, closed, orientable manifolds \( M \) of dimension six with \( \pi_1 M = Z_2 \) and \( \pi_2 M = 0 \). This is equivalent to the classification of free differentiable orientation-preserving involutions on a connected sum of finitely many copies of \( S^3 \times S^3 \). In this case, it is therefore possible to carry out the program proposed in [5] for the study of involutions on \((n - 1)\)-connected \(2n\)-manifolds \((n \geq 3)\).

The paper is organized as follows. Section 1 contains an explanation of the notation and an exposition of the results needed from [1] and [5]. In Section 2, we state the classification results, Theorems 2 and 3, and give an example. The remaining sections contain the proofs.

1. BILINEAR FORMS

Let \( K \) be a finite orientable Poincaré complex of dimension six [8] with \( \pi_1 K = Z_2 \) and \( \pi_2 K = 0 \). The generator of \( \pi_1 K \) will be denoted by \( T \). Then the integral homology and cohomology groups of the universal covering space \( \tilde{K} \) are modules over the integral group ring \( \Lambda \) of \( Z_2 \) via the action of \( T \). In particular, \( H_3(\tilde{K}) \cong r\Lambda \oplus Z_+ \oplus Z_+ \) for some integer \( r \), where \( Z_+ \) is the group of integers with trivial action of \( Z_2 \). This can easily be shown, if it is recalled that since \( H_3(\tilde{K}) \) is a free abelian group it has the form \( r_0 Z_+ \oplus r_1 Z_- \oplus r_2 \Lambda \) as a \( \Lambda \)-module. From the spectral sequence of the covering \( \tilde{K} \to K \), we deduce the values \( r_0 = 2 \) and \( r_1 = 0 \).

Let us write \( H = H_3(\tilde{K}) \) and consider the effect of the involution on the intersection pairing \( \lambda: H \times H \to Z \). This is a unimodular, skew-symmetric bilinear form with the further properties

(1) \( \lambda(Tx, Ty) = \lambda(x, y) \) for all \( x, y \) in \( H \), and
(2) \( \lambda(x, x) = \lambda(x, Tx) = 0 \) for all \( x \) in \( H \).

Associated with \( \lambda \), there is the Browder-Livesay self-intersection map \( \phi: H \otimes Z_2 \to Z_2 \) (see [1] and Sections 5 and 6 below). This is related to \( \lambda \) by the equation

\[ \phi(x + y) - \phi(x) - \phi(y) = \lambda(x, Ty) \pmod{2}, \]

valid for all \( x, y \) in \( H \). Although \( \phi \) is actually defined on \( H \otimes Z_2 \), it will cause no confusion to write \( \phi(x) \) for \( x \) in \( H \), instead of \( \phi(x \otimes 1) \). The geometry of \( K \) therefore gives the algebraic data \((\lambda, \phi, H)\). Any such triple, satisfying the relations listed above, will be called a \( Z_2 \)-form.

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In fact, the $Z_2$-forms that come from Poincaré complexes have an additional structure. From obstruction theory, there is a 2-connected map $f: \mathbb{R}P^3 \to K$, covered by $\tilde{f}: S^3 \to \tilde{K}$. In [5], it is shown that we can choose $f$ so that $e_0 = \tilde{f}_* [S^3]$ generates a $Z_+^t$ direct summand of $H$. Let $(e_1, \ldots, e_r, e_0, e_\infty)$ be a set of $\Lambda$-generators of $H$ containing $e_0$ such that $(e_1, \ldots, e_r)$ is a $\Lambda$-base for an $r\Lambda$ complementary summand to $Z_+^t \oplus Z_+^t$ generated by $(e_0, e_\infty)$. Such a set will be referred to as a *basis* of $H$. It is not difficult to see that the only basis changes $B$ of $H$ that come from homotopy equivalences of $K$ have the property

$$(\ast) \quad Be_0 - e_0 = (1 + T)z, \quad \text{for some } z \in H.$$ 

This condition ensures that $e_0' = Be_0$ can be represented by a mapping of $\mathbb{R}P^3 \to K$ if $e_0$ has such a representation. The following definitions are motivated by this geometric fact. Suppose $H = r\Lambda \oplus Z_+^t \oplus Z_+^t$.

**Definition 1.** A *based* $Z_2$-form on $H$ is a $Z_2$-form $(\lambda, \phi, H)$ together with a generator $e_0$ of a $Z_+$ direct summand of $H$.

**Definition 2.** Two based forms $(\lambda, \phi, e_0)$ and $(\lambda', \phi', e_0')$ on $H$ are *base-equivalent* if there exists a $\Lambda$-isomorphism $B: H \to H$ such that

1. $\lambda'(Bx, By) = \lambda(x, y)$,
2. $\phi'(Bx) = \phi(x)$, $e_0' = Be_0$, and
3. $Be_0 - e_0 = (1 + T)z$ for some $z$ in $H$.

The discussion of the preceding paragraph can be summed up: *With each Poincaré complex of our type, there is associated a based $Z_2$-form whose base-equivalence class is a homotopy invariant.*

It will be useful to observe that, given a based $Z_2$-form $(\lambda, \phi, e_0, H)$, we can, by a purely algebraic argument, find a splitting basis $B$ of $H$ for $\lambda$. More precisely, there exists a basis change $B: H \to H$ with property (*), such that the direct-sum splitting of $H$ into $H_1 = r\Lambda$ and $H_0 = Z_+^t \oplus Z_+^t$, given by the new basis, is an orthogonal splitting with respect to $\lambda$. This implies that, in the new basis $(e_0', e_\infty')$ for $H_0$,

$$\lambda(e_0', e_\infty') = 1 \quad \text{and} \quad \lambda(e_0', e_0') = \lambda(e_\infty', e_\infty') = 0.$$ 

The proof of this fact is an immediate consequence of the following result of [4]. In the statement, we denote $G/2G$ by $\overline{G}$, for an abelian group $G$. Given $\lambda$, a $Z_2$-form on a $\Lambda$-module $N$, we construct a form $\overline{\lambda}$ on $\overline{N}$ by reducing the values of $\lambda$ modulo 2.

**Lemma 1.** Let $\overline{\lambda}$ be the reduction of a nonsingular $Z_2$-form on a $\Lambda$-module $N \cong r\Lambda \oplus M$, where $M$ has no $\Lambda$-free direct summand. Then $\lambda$ restricted to $M$ is nonsingular.

**Proof.** We set $Q = (1 + T)r\Lambda$ and let $P$ be the subgroup of $N$ generated by a $\Lambda$-base for the $r\Lambda$ summand, so that as a free abelian group $N = P \oplus Q \oplus M$.

Then, if

$$\text{Ann}(Q) = \{ x \in \overline{N} \mid \overline{\lambda}(x, y) = 0 \text{ for all } y \in \overline{Q} \},$$

It is clear that $\overline{M} \oplus \overline{Q} \subseteq \text{Ann}(Q)$. Suppose $\overline{\lambda} \mid \overline{M} \times \overline{M}$ is singular. This implies that there exists a nonzero $z \in \overline{M} \cap \text{Ann}(\overline{M})$. Since $\overline{\lambda}$ is nonsingular on $\overline{N}$, there is an
\[ x \in \overline{P} \text{ with } \overline{\lambda}(x, z) = 1. \] By adding suitable multiples of \( z \) to basis elements of \( \overline{Q} \), we obtain \( \overline{Q}' \) of the same rank (as a \( \mathbb{Z}_2 \)-vector space) with

\[ \text{Ann}(\overline{Q}') \supseteq \overline{Q}' \oplus \overline{M} \oplus \langle x \rangle. \]

Since \( \overline{Q}' \) is also a direct summand of \( \overline{N} \), there is a subgroup \( \overline{T} \) of \( \overline{N} \) such that \( \overline{N} \cong \text{Ann}(\overline{Q}') \oplus \overline{T} \). Clearly, rank \( \overline{T} = \text{rank} \overline{Q}' \). Now there is a contradiction: \( \text{rank} \overline{N} = 2(\text{rank} \overline{Q}) + \text{rank} \overline{M} \geq 2(\text{rank} \overline{Q}) + \text{rank} \overline{M} + 1 \).

We conclude this section by describing a condition the map \( \phi \) must satisfy for \( K \) to be smoothable. Choose an embedding of \( H_0 = \mathbb{Z}_+ \oplus \mathbb{Z}_+ \) so that \( H \cong H_0 \oplus H_1 \). Then \( \phi \mid H_0 \) is an associated quadratic map to \( \lambda \mid H_0 \) (in the usual sense); for if \( x \) is in \( H_0 \), then \( Tx = x \). Denote by \( A(\phi, H_0) \) the Arf invariant of \( \phi \mid H_0 \). The following calculation shows that \( A(\phi, H_0) \) is in fact independent of the choice of embedding of \( H_0 \).

**Lemma 2.** Let \( B: H \rightarrow H \) be a basis change, and let \( H' \) = BH. Then \( A(\phi, H'_0) = A(\phi, H_0) \).

**Proof.** Pick a basis \( (e_0, e_\infty) \) of \( H_0 \) containing \( e_0 \), and set \( e'_0 = Be_0 \) and \( e'_\infty = Be_\infty \). Then

\[ Be_0 = ae_0 + be_\infty + (1 + T)x \quad \text{for some } x \text{ in } H_1, \]

and

\[ Be_\infty = ce_0 + de_\infty + (1 + T)y \quad \text{for some } y \text{ in } H_1. \]

Using the fact that \( \lambda(e'_0, e'_\infty) = 1 \mod 2 \) from Lemma 1, we deduce that \( ad + bc \equiv 1 \mod 2 \). This clearly implies that \( \phi(e'_0) \phi(e'_\infty) = \phi(e_0) \phi(e_\infty) \).

Now suppose we are given a Poincaré complex \( K \) as above, with its map \( \phi \) defined on \( H_3(K) \otimes \mathbb{Z}_2 \). Set \( A(K) = A(\phi, H_3(K)) \), where in view of Lemma 2, the notation for the Arf invariant has been simplified. The following restriction on \( \phi \) was obtained in [5].

**Theorem 1.** Let \( M \) be a closed, smooth, oriented 6-manifold with \( \pi_1 M = \mathbb{Z}_2 \) and \( \pi_2 M = 0 \). Then \( A(M) = 0 \).

2. THE CLASSIFICATION

Our classification is contained in the next two results. All manifolds mentioned are smooth, closed, and oriented, and they have dimension six.

**Theorem 2.** Suppose \( K \) is a finite, oriented Poincaré complex that is the homotopy type of a manifold \( M^6 \), with \( \pi_1 M = \mathbb{Z}_2 \) and \( \pi_2 M = 0 \). Then \( K \) has exactly two smoothings.

**Theorem 3.** Homotopy types of 6-manifolds \( M \) with \( \pi_1 M = \mathbb{Z}_2 \) and \( \pi_2 M = 0 \) are in bijective correspondence with the sets of invariants

1. a \( \Lambda \)-module \( H = r\Lambda \oplus \mathbb{Z}_+ \oplus \mathbb{Z}_+ \), for some even integer \( r \geq 0 \),

2. a based \( \mathbb{Z}_2 \)-form \( (\lambda, \phi, e_0, H) \) on \( H \) with \( A(\phi, H) = 0 \), modulo the equivalence relation generated by base-equivalence of \( \mathbb{Z}_2 \)-forms.

In a special case we have computed the classification also for Poincaré complexes.
PROPOSITION 1. There are exactly ten homotopically distinct, finite, oriented Poincaré complexes $K$ of dimension six, with $\pi_1 K = \mathbb{Z}_2$ and $\tilde{K} \cong S^3 \times S^3$. Only two are smoothable.

This is discussed in Section 6. We remark that because of the existence of a splitting basis in each base-equivalence class, and the fact that $L_6(\mathbb{Z}_2, +) \cong \mathbb{Z}_2$, the classification of Theorem 3 is computable.

3. PROOF OF THEOREM 2

Suppose $M$ is a manifold of the kind considered above. According to surgery theory, the proof of Theorem 2 amounts to computing $\mathcal{F}_{PL}(M)$ and the action of $L_7(\mathbb{Z}_2, +)$ on it [9]. In dimension six, it is clearly enough to work in the PL category.

LEMMA 3. $[M, G/PL] \cong [M, G/PL_{(2)}] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Proof. It is known that $G/PL_{(odd)} = BO \otimes \mathbb{Z} [1/2]$, and that $[M, BO \otimes \mathbb{Z} [1/2]]$ can be computed by means of a spectral sequence with

$$E_2^{p,q} = H^p(M; KO^q(\ast)) \otimes \mathbb{Z} [1/2].$$

However, $\tilde{h}^p(M, \mathbb{Z} [1/2]) = 0$ unless $p = 3$ or $p = 6$, and $KO^q(\ast) \otimes \mathbb{Z} [1/2] = 0$ unless $q \equiv 0 \pmod{4}$. Therefore $E_2^{p,-p} = 0$ for all $p$, and $[M, G/PL_{(odd)}] = 0$.

Now it is clear that

$$[M, G/PL_{(2)}] \cong [M, Y] \oplus [M, K(Z_2, 6)],$$

where $Y$ is the 2-stage Postnikov system occurring as a factor of $G/PL_{(2)}$ [7]. From the exact sequence

$$[M, K(Z_2, 4)] \rightarrow [M, Y] \rightarrow [M, K(Z_2, 2)] \rightarrow 0$$

and the fact that $H^4(M; \mathbb{Z}) = 0$, we see that $[M, Y] \cong \mathbb{Z}_2$.

We shall now prove Theorem 2 by calculating the surgery obstruction $\gamma: [M, G/PL] \rightarrow L_6(\mathbb{Z}_2, +) = \mathbb{Z}_2$, using the formula of [9, p. 178]. The map $L_7(\mathbb{Z}_2, +) \rightarrow \mathcal{F}_{PL}(M)$ is trivial [6, p. 48]. In fact, if $g: M \rightarrow G/PL$ corresponds to the essential map $M \rightarrow K(Z_2, 6)$, then

$$\gamma(M, g) = \{1 + w_2(M)\} g^*(1 + Sq^2 + Sq^4 Sq^2)k[M] = g^*k[M] = 1,$$

where $k = k_2 + k_6$, and $k_i$ is in $H^i(G/PL; \mathbb{Z}_2)$. Similarly, if $g$ corresponds to the essential map $M \rightarrow K(Z_2, 2)$, then $\gamma(M, g) = 0$, since $Sq^2 = 0$ on $H^2(M; \mathbb{Z}_2)$.

4. CONSTRUCTION OF 6-MANIFOLDS

In this section, we prove half of Theorem 3 by constructing a one-to-one map from the equivalence classes of invariants to the homotopy classes of manifolds. This is a special case of a construction in [5].

Suppose we are given a based form $(\lambda, \phi, e_0, H)$, and set $w_2 = 1 + \phi(e_0)$ in $\mathbb{Z}_2$. Let $\xi$ be the orientable 3-plane bundle over $RP^3$, with second Stiefel-Whitney class
w_2. Then ξ is either 3ε or ε ⊕ 2η, where ε (respectively, η) is the trivial (respectively, nontrivial) line bundle over RP^3. After changing the Z_2-form, if necessary, within its base-equivalence class, we may assume that H ≅ H_0 ⊕ H_1 is a split decomposition of H for λ with e_0 in H_0. Let (e_1, ..., e_r, e_0, e_∞) be the splitting basis, and denote by D(ξ) and S(ξ) the disk and sphere bundles, respectively, associated with ξ.

We begin the construction by forming W_0 = D(ξ) ∪ f D^3 × D^3 and using an embedding f: S^2 × D^3 → S(ξ), obtained from trivializing S(ξ) over a disk D^3 ⊂ RP^3. Observe that ∂W_0 = S(ξ_0), where ξ_0 is i*(ξ ⊕ η), the bundle ξ ⊕ η pulled back over i: RP^2 ⊂ RP^3.

If r = 0, we finish by attaching D(ξ_0) to W_0 along ∂W_0 with some diffeomorphism. If r > 0, we attach r disjoint handles D^3 × D^3 to W_0 along ∂W_0 to obtain W. We construct the required embeddings f_i: S^2 × D^3 → ∂W_0 by first picking r unknotted and unlinked embeddings f_i^0: S^2 × D^3 → ∂W_0 (i = 1, ..., r) inside disjoint embedded disks D^3 ⊂ ∂W_0. These embeddings are then moved by regular homotopies η_i: S^2 × D^3 × I → ∂W_0 (i = 1, ..., r) (with both ends embedded) whose intersections and self-intersection numbers are prescribed by λ and φ as in [9, p. 53]. For f_i we take η_i | S^2 × D^3 × 1.

An easy surgery argument (see [5]) now shows that ∂W ≅ S(ξ_0) also, and we finish as before by attaching D(ξ_0) with a diffeomorphism h: ∂W → S(ξ_0). Denote M = W ∪_h D(ξ_0) by Γ(θ), where θ = (λ, φ, e_0, H). The following lemma shows that we may omit the map h from our notation.

**Lemma 4.** Any two choices of the diffeomorphism h: ∂W → S(ξ_0) result in homotopy-equivalent manifolds M.

This is the main step in showing that Γ is surjective. It will be carried out in the next two sections. First we apply the result to conclude that Γ is well-defined on equivalence classes and is one-to-one.

**Lemma 5.** Let θ = (λ, φ, e_0, H) and θ' = (λ', φ', e_0', H). Then M = Γ(θ) and M' = Γ(θ') are homotopy-equivalent if and only if the forms θ and θ' are base-equivalent.

**Proof.** Since any homotopy equivalence induces a base-equivalence of the forms and w_2 is a homotopy invariant, the necessity is clear.

Now suppose that e_0 and e_0' are contained in splitting bases and that B: H → H gives a base-equivalence of θ and θ'. Then φ(e_0) = φ'(e_0'). Because B is based, there is a map RP^3 → M' representing Be_0 that is a 2-connected and can be taken to be an embedding, by Haefliger's theorem [3]. In fact, by general position, we can assume that this embedding lies in W'; therefore we let N ⊂ W' be a small tubular neighborhood. One can use the basis Be_1 of H to attach handles to N inside W' and thus to produce an embedding of W ⊂ interior W'. It is easy to see that W' - W is an h-cobordism between ∂W' and ∂W, so that W ≅ W'. From Lemma 4, we conclude that M ≅ M'.

### 5. Reduction to involutions on S^3 × S^3

We shall prove Lemma 4 by listing the possible homotopy types of oriented Poincaré complexes K^n, then proving that the smoothable homotopy types can be specified by our invariants.
The first step is to reduce the problem to the case where \( r = 0 \). Consider a normal cell decomposition [8] of \( K \) induced by a splitting basis of \( H_3(\tilde{K}) \) with respect to \( \lambda \). It will be necessary to have a notation for the skeleta \( K^i \):

\[
K^3 = \mathbb{R}P^3 \vee S_\infty^3 \vee L_r, \quad \text{where} \quad L_r = \bigvee_{k=1}^{r} S_k^3,
\]

\[
K^{i+1} = K^i \cup D^{i+1} \quad \text{for} \quad 3 \leq i \leq 5.
\]

One can show [5] that for \( i = 4 \) and \( i = 5 \),

\[
\tilde{K}^i \simeq S_t^1 \vee N_r, \quad \text{where} \quad N_r = S_0^3 \vee S_\infty^3 \vee L_r \vee L_r^*.
\]

As the notation indicates, the obvious inclusion \( j: S_\infty^3 \vee L_r \subset \tilde{K}^4 \subset \tilde{K} \) has the property that \( j_*[S_3^3] = \epsilon_i \) for \( i = 1, \cdots, r \) and \( i = \infty \), while the inclusion \( \mathbb{R}P^3 \subset \tilde{K}^4 \subset \tilde{K} \) is covered by \( S_0^3 \subset \tilde{K}^4 \subset \tilde{K} \) and represents \( e_0 \). Finally, \( L_r^* \) is another copy of \( L_r \), the image of \( L_r \) under the covering transformation \( T \) in \( \tilde{K} \).

We ask what complexes \( K^i \) have the same homology and cup-product as \( K \). Clearly, \( K^4 \) is determined by homology. However, the attaching map of the 5-cell has homotopy class in \( \pi_4 \tilde{K} \simeq \pi_4 S^4 \oplus \pi_4 N_r \). Let \( \alpha \) be the summand from \( \pi_4 N_r \) (which is a direct sum of copies of \( Z_2 \equiv \pi_4 S^3 \)). This element \( \alpha \) must have the property that \( (1 - T)\alpha = 0 \). It is not detected by homology or cup-product. Similarly, the summand of the homotopy class of the attaching map for the 6-cell that is not detected by this means is \( \beta \), in

\[
(Z_2)^2 \oplus \bigoplus_{k=1}^{r} \pi_5(S^3 \vee S^3)_{(k)}.
\]

We shall use \( e_k \) for the inclusions \( S_k^3 \subset \tilde{K} \) as well as for the homology classes they represent.

**Lemma 6.** Let \((e_1, \cdots, e_r, e_0, e_\infty)\) be a splitting basis for \( K \). There exists a normal cell decomposition of \( K \) induced by the basis, as above, such that

1. \( \alpha = e_0 \circ \alpha_0 + e_\infty \circ \alpha_\infty \), where \( \alpha_0 \) and \( \alpha_\infty \) are in \( \pi_4 S^3 \), and

2. \( \beta = \sum_{k=1}^{r} e_k \circ \beta_k + e_0 \circ \beta_0 + e_\infty \circ \beta_\infty + \sum_{k=1}^{r} m_k [e_k, T e_k] \),

where the element \( \beta_k \) is in \( \pi_5 S^3 \), the coefficient \( m_k \) is in \( Z_2 \), and \([e_k, T e_k]\) generates \( \pi_5(S^3 \vee S^3)_{(k)} \).

**Lemma 7.** The covering space \( \tilde{K} \) is smoothable if and only if \( \beta_k = 0 \) for \( 1 \leq k \leq r \).

**Proof.** In the notation established at the beginning of the section, \( \tilde{K} \simeq N_r \cup_{(1 + T)\beta} D^6 \). Clearly, \( \tilde{K} \) is smoothable if and only if it is the homotopy type of a connected sum of copies of \( S^3 \times S^3 \). In that case, there exists for each \( k = 1, \cdots, r \) a projection \( p_k: \tilde{K} \rightarrow \tilde{K}^k \) such that \( p_k \circ e_k \) is the identity. Hence \( \beta_k \), the obstruction to the existence of \( p_k \), is zero.
Next we identify the coefficients $m_k$ occurring in the expression for $\beta$ of Lemma 6. Recall that in $[1] \phi$ is defined by means of a cohomology operation $\psi: H^3(\tilde{K}, Z_2) \to H^0(K, Z_2) \cong Z_2$ (see also Section 6 below). In fact, if $\tilde{x}$ is the Poincaré dual to $x$ in $H_3(\tilde{K}, Z_2)$, then

$$\phi(x) = \psi(\tilde{x})[K], \quad \text{where } [K] \text{ generates } H_0(K; Z_2).$$

An easy calculation using the definition of $\psi$ yields the following result.

**Lemma 8.** Let $e_k^*$ be reduction modulo 2 of the class in $H^3(\tilde{K})$ dual to $e_k$. Then $m_k = \psi(e_k^*)$ for $k = 1, \ldots, r$.

By the construction of Section 4, each value of $\phi(e_k)$ (and therefore each value of $m_k$) is possible for $k = 1, \ldots, r$ in smoothable complexes. Note that $K_0 = K/L_k$ has the homotopy type of a Poincaré complex with $\pi_1 K_0 = Z_2$ and $\tilde{K}_0 \cong S^3 \times S^3$. The following result is a consequence of Lemmas 6 to 8.

**Lemma 9.** $K$ is smoothable if and only if $K_0$ and $\tilde{K}$ are smoothable.

6. INVOLUTIONS ON $S^3 \times S^3$

First we shall prove part of Proposition 1.

**Lemma 10.** There are ten distinct complexes $K$ of our type with $\tilde{K} \cong S^3 \times S^3$.

**Proof.** The possibilities for $(\alpha_0, \alpha_\infty)$ may be written $(00), (10), (01)$, and $(11)$. Similarly for $(\beta_0, \beta_\infty)$. For constructing $K^5$, given $K^4$, we have three choices: $K^5(00), K^5(10)$, and $K^5(01)$, since $K^5(11)$ is homotopy-equivalent to $K^5(01)$. For constructing $K^6$, given $K^5$, we have three complexes based on $K^5(00)$ or $K^5(10)$, but four on $K^5(01)$. These must all be shown to be distinct.

Suppose that $K$ and $K_1$ are two of the complexes above and that $f: K \to K_1$ is a cellular homotopy equivalence. Since both complexes are orientable, $f$ maps the top cell with degree $\pm 1$ and induces a homotopy equivalence $\tilde{f}: K^5 \to K_1^5$. This proves that $f^* \beta = \beta$. Moreover, because $K^5$ is nonorientable, $\tilde{f}$ maps the 5-cell with degree $\ell$ (odd). Therefore $f^* \alpha = \ell \alpha' = \alpha'$. The only variation is therefore caused by basis changes in $H_2(\tilde{K})$. The only one that affects the attaching maps, namely setting $e_\infty = e_0 + e_\infty$ and $e_0' = e_0$, is allowed for in our list. (Recall here that the isomorphism defined by $e_0' = e_0 + e_\infty$ and $e_\infty' = e_\infty$ is not a base-equivalence.)

**Remark.** The complex corresponding to the choice $\alpha = \beta = 0$ is $\mathbb{R}P^3 \times S^3$; the choice $\alpha = (01)$ and $\beta = 0$ gives $K = S(2 \varepsilon \oplus 2 \eta)$. These are distinguished by $w_2(K)$ or $Sq^2$.

The remainder of the proof of Proposition 1 is contained in the three lemmas below. These show how the homotopy description of Section 5 can be given in terms of the $Z_2$-form, at least for smoothable complexes, and enable us to identify the other invariants $w_2(K)$ and $Sq^2$.

In the statement that follows, recall that $e_0^*, e_\infty^*$ is the (cohomology) dual basis to $e_0, e_\infty$. By $\tilde{e}_\infty^*$ we mean the class in $H^3(K)$ dual to that represented by $S_\infty^3 < K$, reduced modulo 2. It should also be noted that the Poincaré duals of $e_0$ and $e_\infty$ are $e_\infty^*$ and $e_0^*$, respectively.

**Lemma 11.** $\alpha_0 \neq 0$ if and only if $\psi(e_0^*) = \phi(e_\infty) \neq 0$. 

LEMMA 12. (1) $\alpha_\infty \neq 0$ if and only if $\psi(e_0^\infty) = \phi(e_0) = 0$.

(2) $\alpha_\infty \neq 0$ if and only if $\text{Sq}^2 e_0^\infty \neq 0$ (or $w_2 \neq 0$).

LEMMA 13. If $K$ is smoothable, then $\beta_0 = \beta_\infty = 0$.

Application of Lemmas 11 to 13, together with Theorem 1, which eliminates the case $\alpha = (10)$, reduces the list of ten complexes to two possible smooth ones. These are precisely $\text{RP}^3 \times S^3$ and $S(2c \oplus 2\eta)$, and clearly they are smoothable. This proves Proposition 1.

Combined with Lemma 9, the result evidently establishes that the homotopy type of a smoothable complex is completely determined by the base-equivalence class of its based $Z_2$-form, and Lemma 4 follows. As we observed earlier, we can now conclude that the map $\Gamma$ is one-to-one and surjective. This proves Theorem 3.

Proof of Lemma 11. We recall the definition of $\psi$ in [1]. Let $T: \tilde{K} \to \tilde{K}$ be a simplicial, free involution and $z$ a cocycle in $Z^3(\tilde{K}, Z_2)$. Then there exist cochains $v^{3+i}$ for $0 \leq i \leq 3$ in $C^{3+i}(\tilde{K}, Z_2)$ such that

$$z \cup_{3-i} Tz + \delta v^{3+i-1} = (1 + T)v^{3+i} \quad (0 \leq i \leq 3),$$

where $v^2 = 0$. It turns out that cocycle $(1 + T)v^6$ represents a class in $H^6_T(\tilde{K}, Z_2) \cong H^6(K, Z_2)$, which depends only on the cohomology class of $z$. Set $\psi(z) = \text{cls}((1 + T)v^6)$.

This operation can be evaluated on a complex $L$ obtained from $K$ by forming $K/S^3 = (\text{RP}^3 \vee S^4) \cup D^5 \cup D^6$, and then collapsing the resulting $S^4$. Let $j: K \to L$ be the quotient map, and $u$ the generator of $H^3(L, Z_2)$. Evidently,

$$L \simeq (\text{RP}^3 \cup \alpha_0 D^5) \cup D^6,$$

and $j^*u = e_0^\infty$. Using the fact that $\text{Sq}^2$ detects the generator of $\pi_4 S^3$, we see that $\text{Sq}^2 u \neq 0$ as a cochain if and only if $\alpha_0 \neq 0$;

therefore $\psi(u) \neq 0$ if and only if $\alpha_0 \neq 0$. The result now follows by naturality.

Proof of Lemma 12. For this argument, let $L = K/\text{RP}^3$, let $j: K \to L$ be the quotient map, and let $u$ be the generator of $H^3(L, Z_2)$. Clearly,

$$L \simeq (S^3 \cup_{\alpha_\infty} D^5 \cup_{\beta_\infty} D^6) \vee S^4 \quad \text{and} \quad j^*u = e_0^\infty.$$

Part (2) now follows by naturality of $\text{Sq}^2$. Since $e_0^\infty = (1 + T)\sigma^3$, where $\sigma^3$ is in $C^3(\tilde{K}, Z_2)$, it is easy to compute $\psi(e_0^\infty)$ and obtain (1).

Proof of Lemma 13. Suppose that $M$ is a smoothing of $K$ and that $\xi$ is the normal bundle of an embedded $\text{RP}^3$ in $M$. It follows from the decomposition

$$M \simeq D(\xi) \cup D(\xi)$$

that $K/\text{RP}^3 \simeq M/\text{RP}^3 \simeq (\text{Thom space of } \xi)$. But if $\beta_\infty \neq 0$, then $T(\xi)$ carries the nonzero secondary cohomology operation on the Thom class $U$ described in [2]. This implies that the Gitler-Stasheff characteristic class of $\xi$ is nonzero, so that $\xi$ is not a vector bundle.
For the other part, consider the decomposition

\[ M \cong W \cup_h D(\xi_0) \]

of Section 4, where \( \xi_0 \) is the normal bundle to a 2-connected embedding of \( \mathbb{R}P^2 \) in \( M \). By Theorem 1 and Lemma 11, there is a basis of \( H_3(R) \) in which \( \phi(e_\infty) = 0 \).

Therefore \( \beta_0 \) is the only obstruction to a map \( p: M \to \mathbb{R}P^3 \) with the property that \( p|\mathbb{R}P^3 \) is the identity. However, \( W \cong S^3_\infty \vee \mathbb{R}P^3 \); hence we can try to extend the projection \( W \to S^3_\infty \vee \mathbb{R}P^3 \to \mathbb{R}P^3 \) to all of \( M \). Since the homotopy class of the attachment of a cell in a cell decomposition of \( M \) modulo \( W \) factors through the map

\[ h_*: \pi_i(S(\xi_0)) \to \pi_i(\partial W), \]

and since the composition

\[ \pi_i(\partial W) \to \pi_i(W) \to \pi_i(\mathbb{R}P^3) \quad (i > 1) \]

is zero, the extension is possible.

REFERENCES


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