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Splitting of Hermitian Forms over Group Rings

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Let Π be a cyclic group of prime order p. According to a theorem of Reiner [16], a finitely generated $\mathbb{Z}\Pi$-module M which is torsion free over $\mathbb{Z}$ has a decomposition into $\mathbb{Z}\Pi$-submodules

$$M = M_{(0)} \oplus M_{(1)} \oplus M_{(2)} \tag{1}$$

where $M_{(2)}$ is projective over $\mathbb{Z}\Pi$, Π operates trivially on $M_{(0)}$ and through p^{th} roots of 1 on $M_{(1)}$. We call such a splitting a *Reiner splitting*.

Let $h\colon M \times M \to \mathbb{Z}\Pi$ be a non-singular hermitian or skew hermitian form with respect to the involution on $\mathbb{Z}\Pi$ which inverts the elements of Π. Our main concern is with conditions for the existence of an *orthogonal* Reiner splitting

$$M = M_{(0)} \perp M_{(1)} \perp M_{(2)}. \tag{2}$$

Such splittings do not always exist (see Example 8) and are of interest in topology (see § 5).

It is well-known [19] that M is the pull-back of a $\mathbb{Z}$-module M_0 and a Λ_1-module M_1 where $\Lambda_1 = \mathbb{Z}[\tau]$, τ a primitive p^{th} root of 1. We show that h is the pull-back of "almost unimodular" forms $h_0\colon M_0 \times M_0 \to \mathbb{Z}$ and $h_1\colon M_1 \times M_1 \to \Lambda_1$ (Ths. 3 and 6), and further that h has an orthogonal Reiner splitting if and only if h_0 and h_1 have "Jordan splittings" (Th. 7). In §§ 3 and 4 we give conditions under which h_0 and h_1 have Jordan splittings, principally under the assumption of indefiniteness which allows the very effective spinor genus theory of quadratic and hermitian forms to be used.

In § 5 we deal with the topological case. A smooth p-fold covering $\tilde{X}^{2l} \to X^{2l}$ of closed oriented manifolds gives rise to a non-singular ε-hermitian form h on $M = H^l(X;\mathbb{Z})/\text{Torsion}$ (we refer to such forms as "geometric"). Conditions on h implied by the geometry are determined (the most important coming from the Π-signature theorem of [1]) and, when combined with earlier results, show that a geometric h always has an orthogonal Reiner splitting if it is skew hermitian (Th. 30). Necessary and sufficient conditions involving the signature $\sigma(h_0)$ are

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given in Theorems 31 and 32 when h is hermitian and h_0 and h_1 are indefinite. It is also shown (Th. 33) that a geometric hermitian form has an orthogonal Reiner splitting if $M_{(0)}=0$. These theorems yield information about the construction of $\tilde{X}$ as an equivariant handlebody. This approach was used in [13] (p odd) and [9] ($p=2$) and the results of § 5 can be used to extend them.

In a final section, § 6, analogous results for the non-orientable case when $p=2$ and the involution is $a+bT \mapsto a-bT$ ($a, b \in \mathbb{Z}$, $\Pi=\{1, T\}$) are given.

§ 1. Modules over $\mathbb{Z}\Pi$

The results in this section are either contained in [19] or easily derived therefrom. M is the pull-back of a diagram $M_0 \to M_p \leftarrow M_1$ where both maps are epimorphisms, M_i is a projective Λ_i-module ($\Lambda_0=\mathbb{Z}$) and M_p is an $\mathbb{F}_p$-module. The given maps of M, M_0 and M_1 onto M_p are all denoted $x \mapsto x_p$, so that

$$M=\{(x_0, x_1) \in M_0 \oplus M_1 \colon \; x_{0p}=x_{1p}\}. \tag{3}$$

Similarly the maps $M \to M_i$ are denoted by $x \mapsto x_i$ ($i=0, 1$). We often consider M_0 and M_1 as submodules of $M_0 \oplus M_1$. All of this applies when $M=\Lambda := \mathbb{Z}\Pi$, in which case $\Lambda_p=\mathbb{F}_p$. We put $\Gamma=\mathbb{Z} \oplus \Lambda_1$.

A Reiner splitting (1) is not unique but M is characterized by r_0, r_1, r_2 and cls M where r_i = number of summands in a direct sum decomposition of $M_{(i)}$ into indecomposables, and cls $M := \operatorname{cls} M_1$ = the ideal class of Λ_1 belonging to M_1. We have $r_i+r_2=\operatorname{rank}_{\Lambda_i} M_i$ ($i=0, 1$), $r_2=\operatorname{rank}_{\mathbb{F}_p} M_p$. If two of the summands in (1) are zero, say $M=M_{(i)}$ for $i=0, 1$ or 2, we say *M is of type i*.

Proposition 1. *For $i=0$ or 1, let N_i be a direct summand of M_i.*

(a) *N_i is a direct summand of M (of type i) if and only if the image N_{ip} of N_i in M_p is 0.*

(b) *There is a submodule N_{i+1} (indices* mod 2) *of M_{i+1} such that the pull-back N of $N_i \to N_{ip} \leftarrow N_{i+1}$ is a direct summand of M of type 2 if and only if* $\operatorname{rank}_{\mathbb{F}_p} N_{ip} = \operatorname{rank}_{\Lambda_i} N_i$.

Proof. The proof of (a) and the necessity of (b) are contained in the argument on page 79, [19]. For the sufficiency in (b) write $M_i=N_i \oplus P_1 \oplus \cdots \oplus P_k$ where rank P_j is 1 for all j. Then N_{ip} and the P_{jp} span the vector space M_p. By renumbering if necessary we may suppose that $M_p=N_{ip} \oplus P_{1p} \oplus \cdots \oplus P_{lp}$ where all the summands are non-zero. Let $Q_i=N_i \oplus P_1 \oplus \cdots \oplus P_l$ and write $M_i=Q_i \oplus Q_i'$ where $Q_i'=P_{l+1} \oplus \cdots \oplus P_k$. There is a commutative diagram

$$\begin{array}{ccc} Q_i' & \xrightarrow{\;\theta\;} & Q_i \\ \downarrow & & \downarrow \\ Q_{ip}' & \xrightarrow{\text{inclusion}} & M_p \end{array}$$

since Q_i' is a projective Λ_i-module. Then $M_i = Q_i \oplus Q_i''$ where $Q_i'' = (1-\Theta) Q_i'$ and $Q_{ip}'' = 0$. It now follows by Swan's proof that there is a submodule N_{i+1} of M_{i+1} of the required kind. $\square$

Proposition 2. (a) *Suppose* $\chi_p \in \mathrm{Hom}(M_p, \mathbb{F}_p)$ *and* $i=0$, *or* 1. *Then one can find* $\chi_i \in \mathrm{Hom}_{\Lambda_i}(M_i, \Lambda_i)$ *such that*

$$\begin{array}{ccc} M_i & \xrightarrow{\chi_i} & \Lambda_i \\ \downarrow & & \downarrow \\ M_p & \xrightarrow[\chi_p]{} & \mathbb{F}_p \end{array}$$

commutes.

(b) *If* $\chi_i \in \mathrm{Hom}_{\Lambda_i}(M_i, \Lambda_i)$ *has the property that its composite with* $\Lambda_i \to \mathbb{F}_p$ *factors through* $M_i \to M_p$, *then there exists* $\chi_{i+1} \in \mathrm{Hom}_{\Lambda_{i+1}}(M_{i+1}, \Lambda_{i+1})$ *such that* $\chi_0 \oplus \chi_1 \in \mathrm{Hom}(M, \Lambda)$.

§ 2. Hermitian Forms

Let $^-$ denote the usual involution on Λ, the identity on $\mathbb{Z}$ and "complex conjugation" on Λ_1. Let $\varepsilon = \pm 1$ and let $h\colon M \times M \to \Lambda$ be an ε-hermitian form ($^-$-linear in the second variable). It has a unique extension to an ε-hermitian form $\tilde{h}\colon V \times V \to \mathbb{Q}\Pi$ where $V = M \otimes_{\mathbb{Z}} \mathbb{Q}$. Fix a generator T of Π and a primitive p^{th} root of 1, τ, in Λ_1. Identify $\mathbb{Q}\Pi = \mathbb{Q} \oplus F_1$ so that $T = (1, \tau)$, where F_1 = field of quotients of Λ_1. Then $V = V_0 \oplus V_1$ where V_0 (resp. V_1) is a $\mathbb{Q}$-space (resp. F_1-space) and this leads to an orthogonal splitting $\tilde{h} = \tilde{h}_0 \oplus \tilde{h}_1$ where $\tilde{h}_0$ is an ε-symmetric form on V_0 and $\tilde{h}_1$ is an ε-hermitian form on V_1.

If $x \in M_0 \oplus M_1$, $px \in M$ and so the map $x \mapsto \frac{1}{p}(px)$ is a Γ-isomorphism $M_0 \oplus M_1 \simeq \Gamma M \subseteq V$ where $\Gamma = \mathbb{Z} \oplus \Lambda_1$. We identify via this map, so the M_i are lattices in V_i. Define h_i to be the restriction of $\tilde{h}_i$ to M_i $(i=0,1)$. Thus

$$h(x,y) = (h_0(x_0, y_0), h_1(x_1, y_1)) \in \Lambda \tag{4}$$

so there is an ε-symmetric bilinear form $h_p\colon M_p \times M_p \to \mathbb{F}_p$ satisfying

$$h_p(x_p, y_p) = h_0(x_0, y_0)_p = h_1(x_1, y_1)_p = h(x,y)_p. \tag{5}$$

Now suppose h is *non-degenerate*, i.e. the adjoint map $M \to M^*$ given by $y \mapsto h(\ ,y)$ is injective. Then for $i=0$ and 1, $\tilde{h}_i$ is non-degenerate and

$$M_i^\# = \{y \in V_i\colon\ h_i(M_i, y) \subseteq \Lambda_i\}$$

is a lattice containing M_i and isomorphic to M_i^* via h_i. We define the *Jordan invariants* of h_i to be the invariant factors of M_i in $M_i^\#$. If $M_i \to M_i^*$ is bijective, i.e. all Jordan invariants are $= \Lambda_i$, h_i is *non-singular* or *unimodular.*

Define $\pi_0=p$, and $\pi_1=\tau-\tau^{-1}$ if p is odd, $\pi_1=2$ if $p=2$. When p is odd, $(\pi_1)=\pi_1\Lambda_1$ is the only ramified prime in Λ_1. We call h_i *almost unimodular* if its Jordan invariants are all Λ_i or (π_i), i.e. $\pi_i M_i^{\#}\subseteq M_i$.

Theorem 3. *Let h be an ε-hermitian form on M. Then there are unique ε-hermitian forms h_i: $M_i\times M_i\to\Lambda_i$ $(i=0,1)$ satisfying* (4) *and there is an ε-symmetric form h_p: $M_p\times M_p\to\mathbb{F}_p$ satisfying* (5).

Moreover if h is non-singular, h_p is non-singular and h_0 and h_1 are almost unimodular; in fact

$$M_0^{\#}/M_0\simeq(\mathbb{Z}/(p))^{r_0},\qquad M_1^{\#}/M_1\simeq(\Lambda_1/(\pi_1))^{r_1}. \tag{6}$$

We note that it follows from § 1 that the number of Jordan invariants of h_i which $=\Lambda_i$ is r_2 for both $i=0$ and 1.

Proof. By what has already been proved we may assume h non-singular. Now $M_i=M_{(i)}\oplus M_{(2)i}$ for $i=0,1$ so $M_i^{\#}=M'_{(i)}\oplus M'_{(2)i}$ where e.g., $M'_{(i)}$ is the annihilator in $M_i^{\#}$ of $M_{(2)i}$. It suffices to show that

$$M_i=\pi_i M'_{(i)}\oplus M'_{(2)i}. \tag{7}$$

Now $h_i(M_i,M_{(i)})_p=h_p(M_p,M_{(i)p})=0$ (Prop. 1(a)), so the left side$\subseteq$right side. But by Proposition 1, $\ker(M_i\to M_p)=M_{(i)}\oplus\pi_i M_{(2)i}$ whose inner product with the right side of (7) is $=(\pi_i)$, so if x_i is in the right side and $\chi_i=h_i(\ ,x_i)\in M_i^*$, it follows from Proposition 2(b) and the non-singularity of h that $h_i(\ ,y_i)=h_i(\ ,x_i)\in M_i^*$ for some $y_i\in M_i$, so $x_i=y_i\in M_i$ and (7) follows. The non-singularity of h_p is a consequence of $M_i^{(\pi_i)}=M_{(i)}\oplus\pi_i M_{(2)i}$ and

Proposition 4. *h_p is non-singular iff* $\ker(M_i\to M_p)=M_i^{(\pi_i)}$ *for* $i=0$ *or* 1. □

Here, for any ideal A of Λ_i,

$$M_i^A=\{x\in M_i:\ h_i(x,M_i)\subseteq A\}.$$

For the next two results, we do not assume that a form h is given on M, but only that M is the pull-back of (epimorphisms) $M_0\to M_p\leftarrow M_1$. Lemma 5 follows easily from the definitions.

Lemma 5. *Suppose an almost unimodular form h_i is given on the Λ_i-module M_i. Then there are modules P' and Q' such that*

$$M_i^{\#}=P'\oplus Q',\qquad M_i=P'\oplus\pi_i Q'.$$

Let P (resp. Q) be the annihilator of Q' (resp. P') in M_i. Then

$$M_i=P\oplus Q,\qquad M_i^{(\pi_i)}=\pi_i P\oplus Q,\qquad M_i^{\#}=P\oplus\pi_i^{-1}Q.$$

Theorem 6. *Let $\varepsilon=\pm1$ and suppose ε-hermitian forms h_0, h_1 and h_p are given on M_0, M_1 and M_p satisfying for $i=0$ and* 1

$$h_i(x_i,y_i)_p=h_p(x_{ip},y_{ip}) \tag{8}$$

for all x_i and y_i in M_i. Then there is a unique ε-hermitian form $h\colon M\times M\to\Lambda$ satisfying (4). *h_0 and h_1 are the component forms of h as defined in Theorem* 3. *The form h is non-singular iff h_0 and h_1 are almost unimodular and h_p is non-singular.*

Proof. Only the sufficiency of the last statement will be verified. Let $\eta\in M^*$. As in the definition of h_0 and h_1 from h, one can show that $\eta=\eta_0\oplus\eta_1$ where $\eta_i\in M_i^* = \mathrm{Hom}_{\Lambda_i}(M_i,\Lambda_i)$ for $i=0,1$. Then $\eta_i=\tilde{h}_i(\ ,y_i)$ for some y_i in $M_i^{\#}$. Since $M_i^{(\pi_i)}\subseteq\ker(M_i\to M_p)$ by (8), $M_i^{(\pi_i)}\subseteq M$ and so $\eta_i(M_i^{(\pi_i)})_p=\eta_{i+1}(0)_p=0$ (indices mod 2). It follows easily from Lemma 5 that $y_i\in M_i$. Moreover $\eta\in M^*$ implies $h_0(x_0,y_0)_p=h_1(x_1,y_1)_p$ for all x in M, whence $y_{0p}=y_{1p}$ by (8) and the non-degeneracy of h_p. Thus $y\in M$ and since $h(\ ,y)=\eta$, h is non-singular. □

Let h_i be a non-degenerate form on M_i. Then h_i (or M_i) is called *A-modular* (A an ideal in Λ_i) if the Jordan invariants of M_i are all $=A$. A splitting $M_i = N_1\perp N_2\perp\ldots\perp N_t$ is called a *Jordan splitting* if for each μ, N_μ is A_μ-modular with $A_\mu\neq A_\nu$ when $\mu\neq\nu$. A Jordan splitting for an almost unimodular lattice is of the form $N_1\perp N_2$ where N_1 is unimodular or 0, N_2 is (π_i)-modular or 0.

Theorem 7. *If $h\colon M\times M\to\Lambda$ is a non-singular ε-hermitian form, M has an orthogonal Reiner splitting if and only if M_0 and M_1 have Jordan splittings with respect to h_0 and h_1 resp.*

Proof. Since M_0 and M_1 are orthogonal with respect to $\tilde{h}$, their submodules $M_{(0)}$ and $M_{(1)}$ (from any Jordan splitting) are orthogonal with respect to h. If (2) is an orthogonal Reiner splitting, it follows easily from (7) that $M_i=M_{(2)i}\perp M_{(i)}$ is a Jordan splitting. Conversely if $M_i=J_i\perp K_i$ is a Jordan splitting $(i=0,1)$, $M_i^{(\pi_i)} =\pi_i J_i\perp K_i$, so by Propositions 4 and 1, $M=K_0\perp K_1\perp J$ is an orthogonal Reiner splitting where J is the pull-back of $J_0\to M_p\leftarrow J_1$. □

Example 8. Let $p=5$. Then $\rho=\tau+\tau^{-1}$ is a root of $X^2+X-1=0$, so $\rho=\frac{1}{2}(-1+\sqrt{5})$ (choosing a suitable embedding $\Lambda_1\to\mathbb{C}$) and is a unit. Define $M=\Lambda x\oplus\Lambda y$ where $\Lambda x=\mathbb{Z}x$ is of type 0 and Λy is of type 2, and let h be the hermitian form on M with matrix $\begin{pmatrix}3\Sigma & \Sigma\\ \Sigma & T+T^{-1}\end{pmatrix}$ with respect to the generators x, y, where $\Sigma=1+T+\cdots+T^{p-1}$. By projecting the matrix entries into Λ_0 and Λ_1, we see that the matrices of h_0 and h_1 are resp. $\begin{pmatrix}15 & 5\\ 5 & 2\end{pmatrix}$ and (ρ). Thus h_1 is unimodular and h_0 is almost unimodular (since its discriminant is 5 and so its invariant factors are 1 and 5), so h is non-singular by Theorem 6. But h_0 does not have a Jordan splitting since otherwise $2=\pm(a^2+5b^2)$ would be solvable in $\mathbb{Z}$ and 2 would be a quadratic residue (mod 5). Thus h does not have an orthogonal Reiner splitting by Theorem 7. □

Proposition 9. (a) *Suppose h_i is an almost unimodular ε-hermitian form whose Jordan invariants Λ_i and (π_i) have multiplicity r_2 and r_i resp. Then for $i=0,1$, the form $h_i^{\#}:=\pi_i\tilde{h}_i$ on $M_i^{\#}$ is almost unimodular, in fact has Jordan invariants Λ_i (r_i times) and (π_i) (r_2 times).*

(b) *If $M_i=J\perp K$ is a Jordan splitting with respect to h_i, then $M_i^{\#}=\pi_i^{-1}K\perp J$ is a Jordan splitting with respect to $h_i^{\#}$, and conversely.*

(c) *The map* $x \mapsto \pi_i x$: $\pi_i^{-1} M_i \to M_i$ *gives an isometry* $h_i^{\#\#} \simeq \eta h_i$ *where* $\eta = \pm 1$. *If* p *is odd and* $i=1$, $\eta=-1$ *and* $h_i^{\#}$ *is* $(-\varepsilon)$*-hermitian; otherwise* $\eta=1$ *and* $h_i^{\#}$ *is* ε*-hermitian.*

Proposition 10. *Assume* h *is a non-singular* ε*-hermitian form.*
(a) *When* $p=2$ *and* $\varepsilon=-1$, r_0, r_1 *and* r_2 *are all even integers.*
(b) *When* p *is odd and* $\varepsilon=-1$, r_0 *and* r_2 *are even.*
(c) *When* p *is odd and* $\varepsilon=1$, r_1 *is even.*

Proof. Proposition 9 follows from the definitions and Lemma 5. The forms h_{i_p} on $M_{i_p}=M_i/M_i^{(\pi_i)}$ and $(h_i^{\#})_p$ on $(M_i^{\#})_p=M_i^{\#}/(M_i^{\#})^{(\pi_i)}$ are non-degenerate and Proposition 10 follows easily from Proposition 9 and the fact that an alternating form has even rank. □

Proposition 11. *Suppose that* h_i *is almost unimodular and that* N *is an isotropic direct summand of* M_i *of rank* 1. *Then there is a submodule* P *of rank* 1 *such that* $N \oplus P$ *is an orthogonal direct summand of* M_i, *and* $h_i(N, M_i)=h_i(P, M_i)=\Lambda_i$ *or* (π_i).

Proof. Since $\pi_i M_i^{\#} \subseteq M_i$, $h_i(N, M_i)=\Lambda_i$ or (π_i). Suppose first that it is Λ_i. Define P to be a direct complement in M_i of the orthogonal complement of N. Then $L=N+P$ is non-singular and so splits M_i orthogonally since the composite of the canonical homomorphisms $L \to M_i \to M_i^* \to L^*$ is an isomorphism. If $h_i(N, M_i)=(\pi_i)$, apply the first case to $(M_i^{\#}, h_i^{\#})$ and $\pi_i^{-1}N$ (the condition $h_i^{\#}(\pi_i^{-1}N, M_i^{\#})=\Lambda_i$ follows from Lemma 5) and then use $M_i=(M_i^{\#}, h_i^{\#})^{(\pi_i)}$. □

§ 3. Jordan Splittings over Λ_1

Throughout this section p is odd and h_1 is an almost unimodular ε-hermitian form on M_1 with Jordan invariants Λ_1 and (π_1) of multiplicity r_2 and r_1 resp. If f is any form, we set $f(x,x)=f(x)$.

Proposition 12. *Suppose* h_1 *is isotropic and unimodular,* $\varepsilon=1$, *and* M_1 *is of* rank 2. *Then* $h_1(M_1)=\Lambda_1^0$: = *subring of elements of* Λ_1 *fixed by* $^-$.

Proof. Consider M_1 as a lattice in V_1 and write $M_1=Ax_1+Bx_2$ where $h_1(x_1)=0$ and $h_1(x_1,x_2)=1$. Since M_1 is unimodular, $B=\bar{A}^{-1}$, and, since the trace $\operatorname{Tr}\left(\sum_{1}^{\frac{1}{2}(p-1)} \tau^i\right)$ from Λ_1 to Λ_1^0 is -1, $\operatorname{Tr}\Lambda_1=\Lambda_1^0$ and it follows that we may suppose $h_1(x_2)=0$ as well. By (7.2), [2], we may assume that A is an integral ideal such that the conjugate $\bar{\mathfrak{P}}$ of any prime divisor $\mathfrak{P}$ of A is not a prime divisor of A. Consider the lattice $\Lambda_1 y_1 \perp A\bar{A}^{-1} y_2$ on V_1 with $h(y_1)=1$, $h(y_2)=-1$. It is unimodular and is split by the isotropic module $B(y_1+y_2)$ where $B=\Lambda_1 \cap A\bar{A}^{-1}=A$ and so by the previous argument must be isometric to M_1. Thus $1=h_1(ax_1+bx_2)=\operatorname{Tr}(a\bar{b})$ for some $ax_1+bx_2 \in M_1$ so if $c \in \Lambda_1^0$, $c=h_1(cax_1+bx_2) \in h_1(M_1)$. □

The ring Λ_1^0 (or its field of quotients F_1^0) has $\frac{1}{2}(p-1)$ distinct imbeddings into $\mathbb{R}$ and corresponding to each of them h_1 has a signature which we denote $\sigma_i(h_1)$, $1 \leqq i \leqq \frac{1}{2}(p-1)$.

Theorem 13. *h_1 has a Jordan splitting if*

$|\sigma_i(h_1)| \leqq r_2$ *for all i, when* $\varepsilon = 1$,

$|\sigma_i(h_1)| \leqq r_1$ *for all i, when* $\varepsilon = -1$.

Proof. By Proposition 9 it suffices to consider $\varepsilon = 1$, and we may assume $r_2 \geqq 1$ and $r_1 \geqq 2$ by Proposition 10(c). Thus h_1 is isotropic [17] so by Proposition 11, M_1 is split orthogonally by a unimodular or (π_1)-modular isotropic plane H. Suppose H is unimodular. Then we can find $x \in M_1 - \pi_1 M$ which is orthogonal to H and satisfies $h_1(x, M_1) \subseteq (\pi_1)$. Thus $h_1(x) \in (\pi_1) \cap \Lambda_1^0 = \pi_1^2 \Lambda_1^0$ so by Proposition 12 we can find $y \in \pi_1 H$ so that $x + y$ is isotropic. Thus $A(x+y)$ is a direct summand of M_1 for an ideal $A \not\supseteq (\pi_1^{-1})$. Thus $h_1(A(x+y), M_1) = (\pi_1)$ so by Proposition 11 we may suppose that H is (π_1)-modular. The theorem now follows by induction. □

When $\sigma_1(h_1) = \sigma_2(h_1) = \cdots = \sigma_{\frac{1}{2}(p-1)}(h_1)$, we shall say that h_1 *has equal signatures;* this is the case when h arises geometrically (cf. Th. 27).

Lemma 14. *If M_1 is indefinite and if some lattice L in its genus has a Jordan splitting, then M_1 also has a Jordan splitting. Moreover, if L has a Jordan splitting in which each of the two components has equal signatures, then M_1 has a Jordan splitting with the same property.*

Proof. We may assume h_1 is not unimodular or (π_1)-modular. Let $L = N \perp P$ be a Jordan splitting. Denote by J the set of ideals A of Λ_1 with norm (from F_1 to F_1^0) $= \Lambda_1^0$, and by J_0 the set of principal ideals $a\Lambda_1$ with norm $a = 1$. By 5.2(i), [17] there are lattices $N_1 = N, N_2, \ldots, N_s$ in the genus of N such that the s ideals $[N/N_i]$ (= product of the invariant factors of N_i in N) run over a complete set of representatives of J/J_0. Define $L_i = N_i \perp P$ for $i = 1, \ldots, s$. Then $[L/L_i] = [N/N_i]$ and since the L_i are all in the genus of M_1, they represent all classes in that genus. (As remarked on p. 244 of [21], the group $E(\Lambda)/f_\Lambda(E_0)$ in the proof of 5.24(i) on p. 400, [17], is trivial and so by that proof two indefinite lattices R and S in the same genus are in the same class iff $[R/S] = a\Lambda_1$ with norm $a = 1$. See also 5.28, [17].) Thus M_1 is equivalent to one of the L_i and so has a Jordan splitting. The last statement of the lemma follows easily. □

Theorem 15. *If h_1 is indefinite with equal signature, M_1 has a Jordan splitting in which each of the two components has equal signatures.*

Proof. By Proposition 9 we may suppose $\varepsilon = 1$. Since $\pi_1 \notin F_1^0$ but $\pi_1^2 \in F_1^0$, $\pi_1^2 < 0$ at each real place of F_1^0 and it is easy to see that there is an hermitian space W_1 of dimension r_1 with diagonal form $\langle \pm 1, \ldots, \pm 1, \pm \pi_1^{r_1} \rangle$ of determinant $\pi_1^{r_1}$ such that the number of positive (resp. negative) entries is $\leqq$ the number of positive (resp. negative) entries in a diagonalization of $V_1 = F_1 M_1$. We may therefore suppose that $V_1 = W_0 \perp W_1$ by a theorem of Landherr (5.8, [17]). Since $\det M_1$ is a unit at all finite primes $\neq (\pi_1^2)$, W_0 supports a unimodular lattice J by Proposition 6, [20], and local class field theory. Similarly by considering $\pi_1^{-1} h_1$ on W_1, we see that W_1 supports a (π_1)-modular lattice K. Since $J \perp K$ is in the genus of M_1 by Theorems 7.1 and 8.2 of [11] and Proposition 3.2 of [17], the theorem follows from Lemma 15. □

§ 4. Jordan Splittings over $\mathbb{Z}$

Throughout this section we consider the almost unimodular lattice M_0 with ε-symmetric bilinear form h_0, with Jordan invariants $\mathbb{Z}$ and (p) of multiplicity r_2 and r_0 resp. The results apply also to M_1 when $p=2$.

If $\varepsilon=-1$, M_0 has a Jordan splitting by a theorem of Frobenius (Th. 1, § 5, [3]) so we may assume h_0 is symmetric. If the rank of M_0 is 2, the existence of a Jordan splitting is easily determined by reduction theory (see e.g. [6]) so we may assume rank $M_0 \geqq 3$.

If L is any lattice (with a bilinear form), we shall denote its p-adic completion $\mathbb{Z}_p \otimes_{\mathbb{Z}} L$ by L_p in this section only. A lattice L with form f over $\mathbb{Z}$ or $\mathbb{Z}_2$ is called *even* if $f(x) \in 2\mathbb{Z}_2$ for all x, otherwise it is called *odd*; the lattice L with form af, where a is a scalar, is denoted by $a \circ L$. We record several useful results:

Theorem 16 (see 93:29, [15]). *Let $L \perp K$ and $L' = J' \perp K'$ be Jordan splittings of almost unimodular $\mathbb{Z}_2$-lattices. Then L and L' are equivalent if and only if*

(a) *they are equivalent over $\mathbb{Q}_2$,*

(b) *J and J' have the same parity, and $\frac{1}{2} \circ K$ and $\frac{1}{2} \circ K'$ have the same parity,*

(c) $\det J \equiv \det J' \bmod 2^s \mathbb{Z}_2$ *where $s=1$ when J and $\frac{1}{2} \circ K$ are odd, $s=2$ when one is odd and the other is even, $s=3$ when both are even,*

(d) *when J is odd and K is even, $J \perp \langle \det J . \det J' \rangle$ and $J' \perp \langle 1 \rangle$ are equivalent over $\mathbb{Q}_2$.*

Theorem 17 (see Satz 5, [12] and Th. 4.2, [7]). *Let L and L' be almost unimodular indefinite $\mathbb{Z}$-lattices of* rank $\geqq 3$. *If L and L' are in the same genus, then they are equivalent.*

Theorem 18 (see 93:18, [15]). *An even unimodular lattice over $\mathbb{Z}_2$ is an orthogonal direct sum of planes all of which have matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ except possibly for one with matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.*

σ denotes the signature and [ ] is the greatest integer function.

Theorem 19. *If p is odd, an indefinite even lattice M_0 has a Jordan splitting if and only if r_0 is even and*

(a) $\sigma(M_0) \equiv 0 \pmod 8$,

(b) $|\sigma(M_0)| \leqq 8[r_0/8] + 8[r_2/8]$.

Remark. The conditions r_0 even and (a) are equivalent to the p-modular Jordan component of L_p being hyperbolic or, equally well, to $(h_0^{\#})_p$ being hyperbolic. See Proposition 20.

Proof. We note first that M_{02} even and unimodular implies rank $M_0 = r_2 + r_0$ is even. The necessity follows from the fact that the signatures of the Jordan components of M_0 are $\equiv 0 (\bmod 8)$ since they are even and modular ([18]).

Conversely we may assume $\sigma(M_0) \geqq 0$ by scaling by -1 if necessary. Write $\sigma(M_0) = 8m_2 + 8m_0$ with $m_i \in \mathbb{Z}$, $0 \leqq 8m_i \leqq r_i$ for $i=0,2$. Define $s_i = \frac{1}{2}(r_i - 8m_i)$, $L_{(i)}$

$=(s_i H)\perp(m_i \Gamma_8)$ where H is a hyperbolic plane $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and Γ_8 is the even positive definite unimodular lattice of rank 8. Put $L=L_{(2)}\perp(p\circ L_{(0)})$. Then $\sigma(L)=\sigma(M_0)$ so $L_\infty\simeq M_{0\infty}$, so $\det L=\det M_0$ since both are $\pm p^{r_0}$, so $L_q\simeq M_{0q}$ for all $q\neq 2,p$. If $M_{0p}=J_p\perp K_p$ is a Jordan splitting, $\det K_p=\det(p\circ L_{(0)})$ by Proposition 20, so $\det J_p=\det L_{(2)}$ and $L_p\simeq M_{0p}$. By Hilbert reciprocity, L and M_0 are equivalent over $\mathbb{Q}_2$, hence over $\mathbb{Z}_2$ since they are even and unimodular. Thus $L\simeq M_0$ by Theorem 17, so M_0 has a Jordan splitting. □

Proposition 20. *Suppose that both h_0 and $h_0^\#$ are even. Then the p-modular Jordan component of M_0 is hyperbolic if and only if r_0 is even and $\sigma(M_0)\equiv 0 \pmod 8$.*

Proof. Define

$$a(M_0)=\sum e^{2\pi i\psi(u)}\in\mathbb{C}$$

where the sum is over all $u\in M_0^\#/M_0$ and, if $u=x+M_0$, $\psi(u)=\frac{1}{2}\tilde{h}_0(x)+\mathbb{Z}\in\mathbb{Q}/\mathbb{Z}$. Put $A(M_0)=\operatorname{Arg} a(M_0)\in\mathbb{R}/\mathbb{Z}$. Then, [5], $A(M_0)\in\frac{1}{8}\mathbb{Z}/\mathbb{Z}$ and, if we consider $A(M_0)$ to be in $\mathbb{Z}/8\mathbb{Z}$ by multiplying it by 8,

$$\sigma(M_0)\equiv A(M_0) \pmod 8.$$

If $M_{0p}=J\perp K$ is a Jordan splitting at p, $M_0^\#/M_0=p^{-1}K/K=:k$ and $a(M_0)$ is equal to

$$a(k):=\sum_{u\in k} e^{2\pi i g(u)/p}$$

where $g\colon k\to\mathbb{F}_p$ is the quadratic form induced by $\frac{1}{2}h_0^\#$. Note that $A(k)=A(k')+A(k'')$ if $k=k'\perp k''$. If $p=2$, $h_0^\#$ is even by hypothesis; a direct computation shows that $A\begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix}=0$ and $A\begin{pmatrix}4 & 2\\ 2 & 4\end{pmatrix}=4$ so the proposition follows by Th. 18.

Suppose p is odd. If $\alpha\in\mathbb{F}_p^\times$ then $a(\langle\alpha\rangle)$ is a quadratic Gauss sum and can be evaluated using pages 85–87, [14]. The result is $A(\langle\alpha,\beta\rangle)=0$ if $\langle\alpha,\beta\rangle$ is a hyperbolic plane, otherwise $A(\langle\alpha,\beta\rangle)=4$. The proposition follows since K is hyperbolic iff k is hyperbolic. □

Theorem 21. *Let p be odd and let M_0 be odd, indefinite and of* rank $\geqq 3$. *Then M_0 has a Jordan splitting if and only if, when $p\equiv 1 \pmod 4$, $(h_0^\#)_p$ has determinant* 1.

Proof. If $M_0=J\perp K$ is a Jordan splitting, $\det K=\pm p^{r_0}$, whence the necessity. Conversely it is easy to see that one can choose $J=\langle\pm 1,\dots,\pm 1\rangle$ and $K=\langle\pm p,\dots,\pm p\rangle$ so that $J\perp K\simeq M_0$ using Theorems 16 and 17. □

Theorem 22. *Let $p=2$ and suppose that h_0 is indefinite and even and that $h_0^\#$ is even. Then r_0 and r_2 are even. Moreover M_0 has a Jordan splitting if and only if*

(a) $\sigma(M_0)\equiv 0 \pmod 8$.

(b) $|\sigma(M_0)|\leqq 8[r_0/8]+8[r_2/8]$.

The proof is very similar to Theorem 19 and is omitted.

Theorem 23. *Let $p=2$ and suppose that h_0 is indefinite and even and that $h_0^{\#}$ is odd. Then r_2 is even. Moreover M_0 has a Jordan splitting if and only if*

(a) $\sigma(M_0) \equiv s \pmod 8$ *for some integer s satisfying* $|s| \leqq r_0$,

(b) $|\sigma(M_0)| \leqq r_0 + 8[r_2/8]$.

Remark. Condition (a) is obviously vacuous when $r_0 \geqq 4$.

Proof. The unimodular Jordan component of M_{02} is even, hence r_2 is even. If $M_0 = J \perp K$ is a Jordan splitting, $\sigma(J) \equiv 0 \pmod 8$ since J is even and unimodular; thus (a) and (b) follow from $\sigma(M_0) = \sigma(J) + \sigma(K)$.

Conversely we may assume $\sigma(M_0) \geqq 0$ and rank $M_0 \geqq 3$. Let s be the largest integer $\leqq \sigma(M_0)$ satisfying (a). If $\sigma(M_0) = s$, define

$$J = \tfrac{1}{2} r_2 H, \qquad K = s\langle 1 \rangle \perp \tfrac{1}{2}(r_0 - s)H.$$

If $\sigma(M_0) > s$ then $s > r_0 - 8$ so

$$\sigma(M_0) - s < r_0 + 8[r_2/8] - r_0 + 8 = 8[r_2/8] + 8$$

so $\sigma(M_0) - s \leqq 8[r_2/8]$. Define $t = (\sigma(M_0) - s)/8$ and

$$J = t\Gamma_8 \perp \tfrac{1}{2}(r_2 - 8t)H, \qquad K = s\langle 1 \rangle \perp \tfrac{1}{2}(r_0 - |s|)H$$

where $s\langle 1 \rangle$ is interpreted as $(-s)\langle -1 \rangle$ if $s<0$ and as 0 if $s=0$. Put $L = J \perp (2 \circ K)$. Then one can check that $M_0 \simeq L$ using Theorems 16 and 17 (note that a unimodular Jordan component of M_{02}, being even, has determinant $\equiv (-1)^{\frac{1}{2} r_2} \pmod 4$ by Theorem 18). □

Theorem 24. *Let $p=2$ and suppose that h_0 is indefinite and odd and that $h_0^{\#}$ is even. Then r_0 is even. Moreover M_0 has a Jordan splitting if and only if*

(a) $\sigma(M_0) \equiv s \pmod 8$ *for some integer s satisfying* $|s| \leqq r_2$,

(b) $|\sigma(M_0)| \leqq r_2 + 8[r_0/8]$.

Proof. Interchange the roles of h_0 and $h_0^{\#}$ and apply Theorem 23 and Proposition 9. □

Theorem 25. *Let $p=2$ and suppose that h_0 is indefinite of rank* $\leqq 3$ and odd and that $h_0^{\#}$ *is odd. Then M_0 has a Jordan splitting.*

Proof. One shows in the usual way that

$$M_0 \simeq \langle \pm 1, \ldots, \pm 1 \rangle \perp \langle \pm 2, \ldots, \pm 2 \rangle). \quad \square$$

As a supplementary result we have

Theorem 26. *If p is odd, M_0 has a Jordan splitting if either* (a) *or* (b) holds:

(a) $|\sigma(h_0)| \leqq r_2$ *and $(h_0^{\#})_p$ is hyperbolic.*

(b) $|\sigma(h_0)| \leqq r_0$ *and h_{0p} is hyperbolic.*

Proof. Assume (a). Then r_0 is even and we may assume that it and r_2 are >0. Then h_0 is indefinite and by Theorem 21 we may assume it is even as well. By Proposition 20, $\sigma(M_0) \equiv 0 \pmod 8$, so $|\sigma(M_0)| \leqq 8[r_2/8]$ and the theorem follows from Theorem 19. Under assumption (b), the result follows by using the first part and Proposition 9. □

§ 5. Geometric Forms

Let X be a smooth, closed, oriented manifold of dimension $2l$ and Π a finite group that acts differentiably on X, preserving the orientation. The integral bilinear form $B(x,y)=(xy)[X]$ on $M=H^l(X;\mathbb{Z})/\text{Torsion}$ is Π-invariant, unimodular and ε-symmetric where $\varepsilon=(-1)^l$. If we set

$$h(x,y)=\sum_{g\in\Pi} B(g^{-1}x,y)g$$

then $h\colon M\times M\to\mathbb{Z}\Pi$ is a non-singular ε-hermitian form and $B=\varepsilon_1 h$ where $\varepsilon_1\colon \Lambda\to\mathbb{Z}$ is the augmentation, $\varepsilon_1(\sum m_g g)=m_1$.

Extend B and h to $W=\mathbb{R}\otimes_{\mathbb{Z}}M$ and choose on W a positive definite inner product $\langle\,,\rangle$ invariant under Π. Define $A\in\operatorname{End}_{\mathbb{R}}W$ by $B(x,y)=\langle x,Ay\rangle$. Then A commutes with Π, and its adjoint $A^*=\varepsilon A$.

Suppose now that l is even. Then the positive and negative eigen-spaces of A give a decomposition $W=W^+\perp W^-$ invariant under Π. The two real representations ρ^+ and ρ^- of Π thus defined are independent of the choice of $\langle\,,\rangle$. The Π-signature of X is defined as

$$\operatorname{Sign}(\Pi,X)=\rho^+-\rho^-\in RO(\Pi)\subset R(\Pi)$$

and the value of its character on $g\in\Pi$ is $\operatorname{Sign}(g,X)$.

Suppose l is odd. Then A is skew adjoint so $J=A/(AA^*)^{\frac{1}{2}}$ satisfies $J^2=-1$. Thus W yields a complex representation ρ of Π and the Π-signature in this case is

$$\operatorname{Sign}(\Pi,X)=\rho-\rho^*\in R(\Pi)$$

where ρ^* is the contragredient representation.

Π-Signature Theorem (p. 582, [1]). *If Π acts freely on X then* $\operatorname{Sign}(g,X)=0$ *for all $g\neq 1$ in Π.*

We now specialize to the case Π cyclic of prime order p and we refer to the ε-hermitian forms that arise from manifolds with free Π-action as *geometric.*

Theorem 27 (C.T.C. Wall). *If h is a geometric ε-hermitian form and p is odd, then* h_1 has equal signatures.

Proof. The argument is similar to that on page 175, [22]. $\Omega_{2l}(B\Pi)=\Omega_{2l}\oplus\tilde{\Omega}_{2l}(B\Pi)$ and each $\sigma_i(h_1)$ is a bordism invariant, defining a homomorphism $\sigma_i\colon\Omega_{2l}(B\Pi)\to\mathbb{Z}$. Since $\Omega_{2\Pi}(B\Pi)$ is a p-torsion group it suffices to compute on the summand Ω_{2l} (corresponding to trivial p-fold covers) where the result is clear. □

We say that h *has equal signatures* if $\sigma_i(h_1)=\sigma(h_0)$ for all i. We define the signature of a non-degenerate alternating form to be 0.

Theorem 28. *If h is a geometric hermitian form, r_0 and r_1 are even and h has equal signatures. In addition when $p=2$, $h_0^\#$ and $h_1^\#$ are even while h_0 and h_1 are both even or both odd.*

Proof. Suppose $p=2$. If $x\in M$, $h_0(x_0,x_0)\equiv h_1(x_1,x_1)(\operatorname{mod}2)$ and so h_0 is odd iff h_1 is odd (this is obviously independent of h being geometric). Now by Theorem 7.4, [4], $\varepsilon_1 h(x,Tx)\equiv 0(\operatorname{mod}2)$ for all x in M, which is equivalent to $h_0(x_0,x_0)\equiv h_1(x_1,x_1)(\operatorname{mod}4)$. If $z\in M_0^{\#}$ then $(2z,0)\in M$ whence $h_0^{\#}(z,z)=2\tilde{h}_0(z,z)$ is even. Thus $h_0^{\#}$ is even and $h_1^{\#}$ is similarly even.

Return to p arbitrary. Now $W^+=\mathbb{R}^{d_0^+}\perp\mathbb{R}[\tau]^{d_1^+}$ as an $\mathbb{R}\Pi$-module. Define d_0^- and d_1^- similarly. It is easy to see that the Π-signature theorem is equivalent to

$$d_0^+-d_1^+=d_0^--d_1^-.$$

Since $r_i+r_2=d_i^++d_i^-$ for $i=0,1$, we deduce $r_0\equiv r_1(\operatorname{mod}2)$. Thus r_0 and r_1 are even by Proposition 10(c) when p is odd, by Corollary 9, [8] when $p=2$.

Now ε_1 is $1/p$ times the $\mathbb{Z}$-algebra trace of Λ and the latter is the direct sum of the algebra traces of $\mathbb{Z}$ and Λ_1. It follows that $\sigma(\varepsilon_1 h)=\sigma(h_0)+\sigma(\operatorname{Tr}h_1)$. But since $\dim_{\mathbb{R}}W^+=d_0^++(p-1)d_1^+$ with a similar formula for $\dim_{\mathbb{R}}W^-$, $\sigma(\varepsilon_1 h)=\dim W^+-\dim W^-=p\,\sigma(h_0)$ and so $\sigma(\operatorname{Tr}h_1)=(p-1)\,\sigma(h_0)$. This finishes the proof for $p=2$, and for p odd it follows from Theorem 27 and the following lemma.

Lemma 29. *If p is odd and h_1 is hermitian,*

$$\sigma(\operatorname{Tr}h_1)=2\sum_{i=1}^{\frac{1}{2}(p-1)}\sigma_i(h_1).$$

Proof. By taking an orthogonal decomposition of h_1 (over F_1) one reduces to the case of rank 1, say h_1 is the form $xa\bar{y}$ on $F_1\times F_1$. Now extend by $\mathbb{R}$ to an $\mathbb{R}[\tau]$ form. Since $\mathbb{R}[\tau]=\mathbb{C}^{\frac{1}{2}(p-1)}$, $\operatorname{Tr}(xa\bar{y})=\sum\operatorname{Tr}_{\mathbb{C}/\mathbb{R}}(x_ia_i\bar{y}_i)$ where, e.g., $a_1,\ldots,a_{\frac{1}{2}(p-1)}$ are the conjugates of $a\in F_1^0$. But $\sigma(\operatorname{Tr}(x_ia_i\bar{y}_i))=2\operatorname{sign}a_i=2\sigma_i(h_1)$. □

Theorem 30. *If p is odd and h is a geometric skew hermitian form, r_0 and r_1 are even and h has equal signatures.*

Proof. Consider $U=\mathbb{C}u$ as a real space with basis $\{u,iu\}$, $(i=\sqrt{-1})$. Then i has matrix $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. If g is a non-zero skew hermitian form on U, say $g(u,u)=ai$, $a\in\mathbb{R}$, then if G is the matrix of g,

$$J:=G/\sqrt{G^t\cdot G}=\begin{pmatrix}0&\operatorname{sign}a\\-\operatorname{sign}a&0\end{pmatrix}.$$

If we put a complex structure on U by making i act as J, we get the original one if $a<0$ or its conjugate if $a>0$.

Now consider $h\colon W\times W\to\mathbb{R}\Pi$. Choose a decomposition $R\Pi=\mathbb{R}\oplus\mathbb{C}^{\frac{1}{2}(p-1)}$ in which $T=(1,\zeta,\zeta^2,\ldots,\zeta^{\frac{1}{2}(p-1)})$ with $\zeta=\exp\left(\frac{2\pi i}{p}\right)$. Then $W=W_0\perp W_1\perp\ldots\perp W_{\frac{1}{2}(p-1)}$ where W_0 is a real vector space W_j, $j\geqq 1$, is a complex space on which T acts as ζ^j. Let $h_0,h_{1,1},\ldots,h_{1,\frac{1}{2}(p-1)}$ be the component forms. If $1\leqq j\leqq\frac{1}{2}(p-1)$, decompose W_j orthogonally into (complex) lines and use the procedure above to put a new complex structure on each of them. This yields $W_j = U_j \perp U_{-j}$ where T acts on U_j as ζ^j and on U_{-j} as ζ^{-j}, and $W = \sum_{|j| \leqq \frac{1}{2}(p-1)} U_j$ where $U_0 = W_0$. The Π-signature theorem says that the representation of Π on $W' = \sum_{j \neq 0} U_j$ is real. The characteristic polynomial of T on W' is $f = \prod_{j \neq 0} (X - \zeta^j)^{m_j}$ where $m_j = \dim_{\mathbb{C}} U_j$. Since $f \in \mathbb{R}[X]$, $f = \bar{f}$ so $m_j = m_{-j}$ for all j, i.e. the index $\sigma_j(h_1)$ of the skew hermitian form $h_{1,j}$ is 0. Thus h has equal signatures and the evenness of r_0 and r_2 follows easily from this.

Theorem 31. *If h is a geometric skew hermitian form on M, then M admits an orthogonal Reiner splitting.*

Proof. Since h_0 is skew symmetric it has a Jordan splitting by a theorem of Frobenius (§ 5, [3]) and so has h_1 if $p = 2$. If p is odd, h_1 has a Jordan splitting by Theorems 15 and 30, so the theorem follows by Theorem 7. □

Theorem 32. *Let p be odd. If h is a geometric hermitian form with indefinite component forms h_0 and h_1, M has an orthogonal Reiner splitting if and only if, when h_0 is even, $|\sigma(h_0)| \leqq 8[r_0/8] + 8[r_2/8]$.*

Proof. The necessity follows from Theorems 7 and 19. Conversely h_1 has a Jordan splitting by Theorems 15 and 28. By Theorems 19 and 21, and Proposition 20, it suffices to show that $(h_0^\#)_p$ is hyperbolic.

Suppose that h arises from the p-fold covering $\gamma\colon \tilde{X}^{4k} \to X^{4k}$. Then $M = H^{2k}(\tilde{X}, \mathbb{Z})/\text{Torsion}$ and we put $N = H^{2k}(X, \mathbb{Z})/\text{Torsion}$. The map $\gamma^*\colon N \to M$ has image $\subseteq M_0$ and is a monomorphism since it induces an isomorphism $\mathbb{Q} \otimes_{\mathbb{Z}} N \to \mathbb{Q} \otimes_{\mathbb{Z}} M$ (Ch. 3, [4]). Let $g\colon N \times N \to \mathbb{Z}$ be the cup-product pairing; it is unimodular by Poincaré duality. Then for all $x, y \in N$,

$$p^2 g(x, y) = h_0(\gamma^* x, \gamma^* y). \tag{9}$$

If $t^*\colon M \to N$ is induced by the cohomology transfer map, $t^* \gamma^*(x) = px$ for all x (ibid) so $\gamma^* N \supseteq p M_0$. Also $\gamma^* N \subseteq M \cap M_0 = M_0^{(p)}$ by Proposition 4. By Lemma 5 and (9), the map $\frac{1}{p}\gamma^*\colon (N, g) \to (M_0^\#, \tilde{h}_0)$ is an isometry onto a unimodular submodule $N' \supseteq M_0 = (M_0^\#)^{(p)}$. Since the discriminant of $(M_0^\#, \tilde{h}_0)$ is $\pm p^{-r_0}$, the index of N' in $M_0^\#$ is $p^{\frac{1}{2} r_0}$, hence its image in $(M_0^\#)_p = M_0^\# / M_0$ has dimension $\frac{1}{2} r_0$. But this image is a totally isotropic subspace since $h_0^\# = p\tilde{h}_0$ and so $(h_0^\#)_p$ is hyperbolic. □

Theorem 33. *Let $p = 2$. If h is a geometric hermitian form with indefinite component forms h_0 and h_1, M has an orthogonal Reiner splitting if and only if*

$$|\sigma(h_0)| \leqq r_2 + 8 \min\{[r_0/8], [r_1/8]\}$$

and

$$\sigma(h_0) \equiv s (\operatorname{mod} 8)$$

where $s = 0$ if h_0 is even, otherwise $|s| \leqq r_2$.

This follows easily from Theorems 28, 22, 24.

Theorem 34. *If h is a geometric hermitian form and if $r_0=0$, then M has an orthogonal Reiner splitting.*

Remark. Such forms arise when l is even and $\tilde{X}^{2l}$ is $(l-1)$-connected. Thus Theorem 34 can be used to generalize results of [13].

Proof. Since h has equal signatures and $|\sigma(h_0)|\leqq r_2$, h_1 is indefinite (under the assumption $r_1 \neq 0$) and the theorem follows easily from Theorems 28, 22, 24. □

Summary. *The following conditions are necessary in order that the non-singular ε-hermitian form h be geometric. (The conditions are* not *independent.)*

(i) *h has equal signatures* (Ths. 28 and 30).

(ii) *When $\varepsilon=1$, $h_0^\#$ and $h_1^\#$ are even when $p=2$, and h_0 and h_1 are both even or both odd* (Th. 28).

(iii) *r_0 and r_1 are even* (Ths. 28 and 30).

(iv) *r_2 is even unless $\varepsilon=1$ and h_0 is odd* (Prop. 10, Th. 18).

(v) *If p is odd, $(h_0^\#)_p$ is hyperbolic* (proof of Th. 32); if in addition h_0 is even, $\sigma(h_0)\equiv 0 (\text{mod } 8)$ (Prop. 20).

(vi) *If $p=2$ and $x\in M$, $h_0(x_0,x_0)\equiv h_1(x_1,x_1) (\text{mod } 4)$* (proof of Th. 28).

§ 6. The Non-Orientable Case

We now consider the forms which arise in geometry from 2-fold covers $\tilde{X}^{2l}\to X^{2l}$ of closed manifolds where $\tilde{X}$ is orientable and X is non-orientable and prove that an orthogonal Reiner splitting always exists.

Theorem. *Let $p=2$ and suppose h is a non-singular hermitian or skew hermitian form on M with respect to the involution $a+bT\mapsto a-bT$ on Λ. In any Reiner splitting* (1), *$M_{(0)}$ and $M_{(1)}$ are totally isotropic, and there is a Reiner splitting in which $M_{(0)}\oplus M_{(1)}$ is orthogonal to $M_{(2)}$.*

Proof. If h is skew hermitian, Th is hermitian so we need only consider the hermitian case. The proof is similar to (and much easier than) those in § 2 and so we merely sketch it.

Extend h to $\Gamma M\times\Gamma M\to\Gamma=\mathbb{Z}\oplus\mathbb{Z}$; since $\overline{(a,b)}=(b,a)$, M_0 and M_1 are totally isotropic and there is a non-degenerate pairing η: $M_0\times M_1\to\mathbb{Z}$ such that

$$h(x,y)=(\eta(x_0,y_1),\eta(y_0,x_1)).$$

If η': $M_0\to M_1^*$ is the associated monomorphism, one can show that

$$\eta'(M_0)=2M_{(1)}^*\oplus M_{(2)1}^* \tag{10}$$

for any Reiner splitting (1) where $M_{(1)}^*$, e.g., is the annihilator in M_1^* of $M_{(2)1}$ (cf. proof of Th. 3). We let $M_0=M'_{(0)}\oplus M'_{(2)0}$ be the inverse image of (10). It follows that $M=M'_{(0)}\oplus M_{(1)}\oplus M'_{(2)}$ is the desired splitting where $M'_{(2)}$ is the (type 2) pull-back of $M'_{(2)0}\to M_p\leftarrow M_{(2)1}$. □

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