The surgery group $L_3^h(Z(G))$ for G a finite 2-group

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In [C-M,2] a theorem is proved which expresses $L_3^p(Z(G))$ as a simple functor of the rational representation ring $R_{\mathbb{Q}}(G)$ when G is a finite 2-group. In the appendix to [C-M,2] one of us shows that the 2-primary part of $\tilde{K}_0(Z(G))$ is the quotient of a finite group depending only on $R_{\mathbb{Q}}(G)$ and the order of G.

Here we determine the structure of $L_0^p(Z(G))$, and provide a complete determination of a factorization of the map d in the Ranicki-Rothenberg sequence

$$* \quad \cdots \to L_{2i}^p(Z(G)) \xrightarrow{d} H_{od}(Z/2, \tilde{K}_0(Z(G))) \xrightarrow{\partial} L_{2i-1}^h(Z(G)) \to L_{2i-1}^p(Z(G)) \to H_{ev} \to .$$

through the group alluded to above. In particular we apply our results to obtain $L_3^h(Z(G))$, the surgery obstruction group, when G is a generalized quaternion 2-group. This in turn leads to examples of the existence of semi-free group actions on homotopy spheres which do not admit balanced splittings, (see [A-H] for definitions, and the reduction to properties of * in particular pp. 8-9).

In detail we have

<u>Theorem A</u>: <u>Let</u> G <u>be a finite 2-group, then</u> $L_0^p(Z(G)) = Z^{\ell(G)}$ <u>where</u> $\ell(G)$ <u>is the number of irreducible real representations of</u> G.

<u>Theorem B</u>: <u>For</u> G <u>a finite 2-group the kernel</u> K <u>in the map</u>

$$0 \to K \to L_3^h(Z(G)) \to L_3^p(Z(G))$$

is known once the map $\varphi : W_\ell(G) \to D(G)$ _is known, where_ $W_\ell(G)$ _is given in_ [C-M,2 _Appendix, especially_ A.7, A.8], _for_ ℓ _sufficiently large_.

Indeed in §2,3, we give all the information needed to determine K explicitly. Also, note that $W_\ell(G)$ depends only on the rational representation ring of G, while the ℓ is determined by $|G|$. We remark that even the extension is determined from the information in φ, though we don't explain this here. Finally, we point out that the map d in

$*$ $L^p_{od}(Z(G)) \to H_{ev}(Z/2,\widetilde{K}_o(Z(G)))$ is already implicitly determined in [C-M,2], our techniques here can also be used to determine the map

$$L^p_2(Z(G)) \to H_{ev}(Z/2,\widetilde{K}_o(Z(G)))$$

and in each case a theorem similar to B holds.

In §4, we apply these results to the generalized quaternion groups.

Theorem C: _Let_ $Q_{2^i,2}$ _be the generalized quaternion group_ $\{x,y \mid x^{2^i} = y^2 = (xy)^2\}$ _then_ d _is surjective in_ $*$ _for_ $i = 0$ _and_ $L^h_3(Z(Q_{2^i,2})) = (Z/2)^{i+1}$ _injects into_ $L^p_3(Z(Q_{2^i,2}))$.

The application to balanced splittings results since [F-K-W], [M] show that the Swan homomorphism T is onto the 2-torsion in $\widetilde{K}_o(Z(Q_{2^i,2}))$. See also, §4.1, 4.2.

§1. The proof of theorem A.

Consider the diagrams of long exact sequences

1.1
$$\begin{array}{ccccccccc}
\cdots \to H_{od}(Z/2,\tilde{K}_o(\mathbb{C}(G))) & \to & L_1^h(\mathbb{Q}(G)) & \to & L_1^p(\mathbb{Q}(G)) & \to & H_{ev}(Z/2,\tilde{K}_o)\cdots & \to \\
\downarrow s & & \downarrow \bar{s} & & \downarrow & & \downarrow & \\
\cdots \to H_{od}(Z/2,\tilde{K}_o(\hat{\mathbb{C}}_2(G))) & \to & L_1^h(\hat{\mathbb{C}}_2(G)) & \to & L_1^p(\hat{\mathbb{C}}_2(G)) & \to & H_{ev}(Z/2,\tilde{K}_o)\cdots & \to
\end{array}$$

1.2
$$\begin{array}{ccccccccc}
\cdots \to & L_1^h(\mathbb{Q}(G)) & \to & L_1^{p,tor}(Z(G)) & \to & L_o^p(Z(G)) & \to & L_o^h(\mathbb{Q}(G)) & \to \cdots \\
 & \downarrow \bar{s} & & \downarrow r & & \downarrow & & \downarrow & \\
\cdots \to & L_1^h(\hat{\mathbb{C}}_2(G)) & \xrightarrow{\partial} & L_1^{h,tor}(\hat{Z}_2(G)) & \to & L_o^h(\hat{Z}_2(G)) & \xrightarrow{i} & L_o^h(\hat{\mathbb{C}}_2(Q)) & \to \cdots
\end{array}$$

From [C-M,2 p. 33-35] or [R] we have that

1.3
$$L_1^p(\mathbb{Q}(G)) = L_1^p(\hat{\mathbb{C}}_2(G)) = 0 .$$

Since G is a finite group $K_o(\mathbb{C}(G)) = R_{\mathbb{C}}(G)$, $= Z^{\ell}$ where ℓ is the number of irreducible $\mathbb{C}$ representations of G . Also, since G is a 2-group we have that

1.4
$$K_o(\hat{\mathbb{C}}_2(G)) \cong K_o(\mathbb{C}(G))$$

under the natural inclusion [s] . Hence in 1.1 s is an isomorphism and $\bar{s}$ is a surjection of $L_1^h(\mathbb{C}G)) \to L_1^h(\hat{\mathbb{C}}_2(G))$.

Now consider 1.2. In [C-M,2, p. 31] we have shown that $L_o^h(\hat{Z}_2(G)) = Z/2$ injects into $L_o^h(\hat{\mathbb{C}}_2(G))$. So

1.5 $$\partial : L_1^h(\hat{\mathbb{C}}_2(G)) \to L_1^{h,tor}(\hat{Z}_2(G))$$

is onto. But from [C-M,1 §2] and [C-M,2, p. 10] (or arguments totally analogous to those) we have that

$$r : L_1^{p,tor}(Z(G)) \to L_1^{h,tor}(\hat{Z}_2(G))$$

is an isomorphism. Hence from the surjectivity of ∂ and $\bar{s}$ it follows that the map

1.6 $$L_o^p(Z(G)) \to L_o^h(\mathbb{C}(G))$$

is an injection.

At this point, consider the diagram of exact sequences

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \\
 & & L_o^p(Z(G)) & \longrightarrow & L_o^p(\mathcal{M}(G)) & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_o^h(\mathbb{C}(G)) & \longrightarrow & L_o^p(\mathbb{C}(G)) & \longrightarrow & H_{od}(Z/2,\tilde{K}_o)
\end{array}$$

where $\mathcal{M}$ is a Z-maximal order containing $Z(G)$ in $\mathbb{C}(G)$, which shows that

$$L_o^p(Z(G)) \hookrightarrow \mathrm{im} : L_o^p(\mathcal{M}(G)) \hookrightarrow L_o^p(\mathbb{C}(G))$$

Now, $L_o^p(\mathbb{C}(G)) = \coprod_i L_o^p(R_i(G))$, where R_i is the ith irreducible representation algebra. These are classified as to type in [C-M,2, p. 26]. Using Morita equivalence, the results of [M-H,pp. 117-118] for the type 4.3(ii) and 4.3(iv) representations (in the notation of [C-M,2, p. 26]), [M-H, p. 95] for the type 4.3(ii) representations and a direct calculation in the 4.3(i) case we see that im $L_o^p(\mathcal{M}(G))$ in $L_o^p(\mathbb{C}(G))$ is a direct sum of Z's and the proof of theorem A is complete.

Remark 1.8: Similar techniques can be applied to calculate $L_i^p(Z(G))$ for G a finite 2-group when i = 1,2, as well. These results will be written down in their entirety in [C-M-P] where the general case of G a 2-hyperelementary group will also be studied.

Remark 1.9: It is not true for finite 2 groups that $L_o^h(Z(G)) = Z^\ell$, as $K_o(Z(G))_{(2)}$ tends to grow very large and $L_1^p(Z(G))$ is zero except for some Z/2's coming from the type 4.3(i) representations of [C-M,2, p. 26]. So $L_o^h(Z(G)) = Z^\ell \oplus (Z/2)^s$. The Z's may be detected via the Atiyah-Singer G-signature theorem [P], but we have no idea of what occurs with the Z/2's.

§2. Factoring the map d .

Throughout this section we assume that the reader is familiar with the appendix in [C-M,2].

Begin with the local-global pull-back diagram

2.1
$$\begin{array}{ccc} Z(G) & \longrightarrow & \mathcal{M}(G) \\ \downarrow & & \downarrow i \\ \hat{Z}_2(G) & \xrightarrow{j} & \mathcal{M}(G) \otimes_Z \hat{Z}_2 \end{array}$$

where $\mathcal{M}(G)$ is a maximal Z-order for $Z(G)$ in $\mathbb{Q}(G)$ and $\mathcal{M}(G) \otimes_Z \hat{Z}_2$ is a maximal $\hat{Z}_2$ order.

2.1 allows us to construct projective $Z(G)$ modules together with non-singular forms by mixing forms over $\hat{Z}_2(G)$ with forms over $\mathcal{M}(G)$ on $\mathcal{M}(G) \otimes_Z \hat{Z}_2$. Specifically, let $(\mathcal{M}(G)^n, A_n)$, $(\hat{Z}_2(G)^n, B_n)$ be suitable forms and assume there is a C_n in $GL_n(\mathcal{M}(G) \otimes_Z \hat{Z}_2)$ so that

2.2
$$C_n \cdot i(A_n)C_n^* = j(B_n)$$

Then on the projective module W defined by C_n ,

2.3
$$\begin{array}{ccc} W & \longrightarrow & \mathcal{M}(G_T)^n \\ & & \downarrow i^n \\ \downarrow & & \mathcal{M}(G) \otimes_Z \hat{Z}_2 \\ & & \downarrow C_n \\ \hat{Z}_2(G)^n & \xrightarrow{j^n} & \mathcal{M}(G) \otimes_Z \hat{Z}_2 \end{array}$$

2.2 gives a form which becomes A_n when tensoring W with $\mathcal{M}(G)$, and B_n on tensoring with $\hat{Z}_2(G)$. We denote the form on W by

$$[W,\ A_n,\ B_n,\ C_n]\ .$$

In the appendix to [C-M,2], the group $D(G) \subset \tilde{K}_0(ZG)$ is described on page A.2, see in particular Theorem 1.4, as a quotient of $K_1(\mathcal{M}(G) \otimes_Z \hat{Z}_2)$.

Then the following lemma is clear.

Lemma 2.4: The image of $[C_n]$ in $D(G) \subset \tilde{K}_0(Z(G))$ represents

$$d([W,\ A_n,\ B_n,\ C_n])\ .$$

Throughout the remainder of this section we assume $B_n = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ so that C_n makes A_2 2-locally equivalent to a hyperbolic form. (Actually, this assumption holds for every element of $L_0^p(Z(G))$.)

Lemma 2.5: Let A_n be itself hyperbolic except at a single representation $M_n(F)$ where F is a formally real field, then

$$d(W,\ A_n,\ B_n,\ C_n) = 1\ .$$

Proof: At $M_n(F \otimes_{\mathbb{Q}} \hat{\mathbb{Q}}_2)$ we have

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = C_n A_n C_n^*$$

and taking determinants $\pm 1 = (\det C_n)^2 \det(A_n)$ but $\det(A_n)$ is a unit in $Z(\rho_{2^i} + \rho_{2^i}^{-1})$, the ring of algebraic integers in F. Now, use the unit calculations of §4 of [C-M,2], in particular 4.6, 4.7 to see that det (C_n) is likewise a unit in $Z(\rho_{2^i} + \rho_{2^i}^{-1})$, hence in the kernel of d.

Lemma 2.6: Let A_n be hyperbolic except at a representation $M_n(\mathcal{Q}(F))$ where $\mathcal{Q}(F)$ is the type 4.3(i)(a) simple algebra of [C-M,2], then the class of A in the Witt ring is determined by its multisignature at the various real places of F, and if A_n has signature 0 except at the i^{th} place ∞_i, where it has signature ± 2, then

$$d[W, A_n, B_n, C_n] = (\varepsilon_i^{-1})$$

where ε_i is any unit of F positive at all ∞_j, $j \neq i$ and negative at ∞_i.

Proof: The maximal order $\mathcal{M}(G)$ can be chosen to be $M_n(\mathcal{O}_{\mathcal{Q}(F)}) \oplus \mathcal{M}^1$ where $\mathcal{O}_{\mathcal{Q}(F)}$ is a maximal order in $\mathcal{Q}(F)$. Indeed we can take

2.7 $$\mathcal{O}_{\mathcal{Q}(F)} = Z(\tfrac{1+i+j+k}{2}, i, j) \otimes_Z Z(\rho_{2^\ell} + \rho_{2^\ell}^{-1}) .$$

In this $\mathcal{O}_{\mathcal{Q}(F)}$, $1 = \frac{1+i+j+k}{2} + \frac{1-i-j-k}{2}$ and so all elements of the center are even. Now consider the form $A_n = \left[\begin{pmatrix}\varepsilon_i & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]$.

As $F \neq \mathbb{C}$, $\mathcal{O}_{\mathcal{Q}(F)} \otimes_Z \hat{Z}_2 = M_2(\hat{Z}_2(\rho_{2^\ell} + \rho_{2^\ell}^{-1}))$ [C-M,2, Theorem 4.3.(i)], and the involution is given up to equivalence by

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} \to \begin{pmatrix}d & -b\\ -c & a\end{pmatrix}, \quad \begin{pmatrix}\varepsilon_i & \\ & -1\end{pmatrix} \longmapsto \begin{pmatrix}\varepsilon_i & 0 & & \\ 0 & \varepsilon_i & 0 & \\ & 0 & -1 & 0\\ & & 0 & -1\end{pmatrix}$$

Now, we remark that it is sufficient to study C_n in $\mathcal{O}_{\mathcal{Q}(F)} \otimes_Z \hat{\mathbb{Q}}_2$, since, by [C-M,2,A-4] no information is lost by using $K' = \text{im}\, K_1(\mathcal{O} \otimes \hat{Z}_2)$ in $K_1(\mathcal{O} \otimes \hat{\mathbb{Q}}_2)$. Here we may choose

$$C_2 = C^1 \begin{pmatrix} \varepsilon_i^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

where C^1 effects the isomorphism $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = C^1 \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & -1 & 0 \\ & & 0 & -1 \end{pmatrix}$

which is valid over $\mathbb{Q}$, and clearly $\det C^1 = +1$. Thus, $\det C_2 = (\varepsilon_i^{-1})$ and 2.6 follows.

The situation is slightly different at the ordinary quaternion algebra $\mathcal{Q}(\mathbb{Q})$.

Lemma 2.8: Let A_n be hyperbolic except at $M_n(\mathcal{Q}(\mathbb{Q}))$, then A_n has signature $2i$ and

$$d([W, A_n, B_n, C_n]) = (-1)^i_{\mathcal{Q}} .$$

Proof: We may assume $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A_n$. Now there is a $v \in \hat{Z}_2 \otimes_Z Z(i,j,\frac{i+j+k+1}{2})$ with norm $v\bar{v} = -1$. Set

$$C_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}$$

and $N(C_2) = -1$.

The remaining cases are all type II algebras of the form $M_n(Z(\rho_{2^i}))$ or $M_n(Z(\rho_{2^i} - \rho_{2^i}^{-1}))$.

§3. The type II algebras and theorem B.

We begin by obtaining the structure of the units over complex conjugation t in the rings $\hat{Z}_2(\rho_{2^i})$, $\hat{Z}_2(\tau_i)$ where $\tau_i = \rho_{2^i}-\rho_{2^i}^{-1}$. For the next 2 results we assume $i \geqslant 3$.

Theorem 3.1: Let $\varepsilon_1 = 1 + \rho_{2^i} + \rho_{2^i}^{-1} = 1 + \lambda_i$. There is a unit $\nu(i)$ such that $\nu(i)t(\nu(i))=-1$ and as a module over t we have $\hat{Z}_2(\rho_{2^i})^{\cdot} = \hat{Z}_{2+} \times \bar{M}_- \times \Pi\, \hat{Z}_2(t)/t^2 =$ Moreover $\hat{Z}_{2+}$ is generated by ε_1 , and M_- is the module $\hat{Z}_2 \times Z/2^i$ with t action $t(a,b) = (-a,-b+2^{i-1})$. The generators of M_- are ν and ρ_{2^i} .

Proof: Using Artin reciprocity the norms in $\hat{Z}_2(\lambda_i)$ of $\hat{Z}_2(\rho_{2^i})$ have index 2 and ε_1 is not a norm since its norm in $\hat{Z}_2$ is -1 . Now $(\hat{Z}_2(\rho_{2^i}))^{\cdot} = Z/2^i \times (\hat{Z}_2)^{2^{i-1}}$ and $\hat{Z}_2(\lambda_i)^{\cdot} = Z/2 \times (\hat{Z}_2)^{2^{i-2}}$. Write the generators of this latter group -1, ε_1, $n_2 \ldots n_{2^{i-2}}$ where the n_i are all norms, say $n_i = w_i \cdot t(w_i)$. Clearly, the w_i, ε_1, ν and ρ_{2^i} generate $\hat{Z}_2(\rho_{2^i})^{\cdot}$ and 3.1 follows directly.

Similarly, we have

Theorem 3.2: In $\hat{Z}_2(\tau_{i+1})$ there is a unit $\nu(i+1)$ with $\nu(i+1)t(\nu(i+1)) = -1$ and as a module over t we have

$$\hat{Z}_2(\tau_{i+1})^{\cdot} = \hat{Z}_2^{+} \times M_- \times \Pi\, \hat{Z}_2(t)/t^2 = 1$$

Moreover, $\hat{Z}_2^{+}$, M_- are given as in 3.1.

Remark 3.3: The only differences in these 2 discriptions comes on comparing the images of global units, which give all but 1 of the η_i and are the same. However, in $\hat{Z}_2(\tau_{i+1}) - \hat{Z}_2(\lambda_i)$, $\varepsilon_1 5$ is the remaining η_i while in $\hat{Z}_2(\rho_{2^i}) - \hat{Z}_2(\lambda_i)$ the remaining η_i can be taken to be 5.

Remark 3.4: The cases not covered in the above are $\hat{Z}_2(i) - \hat{Z}_2$ where $\hat{Z}_2(i)^{\cdot} = Z/4 \times \hat{Z}_2(t)/t^2{=}1$ with generators i, $i + 2i$, and $\hat{Z}_2(\sqrt{-2}) - \hat{Z}_2$ where $\hat{Z}_2(\sqrt{-2})^{\cdot} = Z/2 \times \hat{Z}_2(t)/t^2{=}1$ with generators -1, $1 + \sqrt{-2}$.

Hence, as in [C-M,2, A.8, A.9] on factoring out global units (and typos) we have

3.5 $$W_\ell(\hat{Z}_2(\rho_{2^i}),t) = (Z/2^\ell)_-^{2^{i-2}} \times Z/2^\ell(t)/t^2{=}1$$

with generators ν, w_j, $1{+}2i$, where $(w_j)t(w_j) = \eta_j$ a global unit, for $i \geqslant 3$

3.6 $$W_\ell(\hat{Z}_2(i),t) = Z/2^\ell(t)/t^2{=}1 \quad .$$

Also,

3.7 $$W_\ell(\hat{Z}_2(\tau_{i+1}),t) = (Z/2^\ell)^{2^{i-2}} \times Z/2^\ell\ (t)/t^2{=}1$$

with generators ν, w_j, w, $wt(w) = 5\varepsilon_1, i \geqslant 3$ and $w_j t(w_j) = \eta_j$ a global unit, while

3.8 $$W_\ell(\hat{Z}_2(\sqrt{-2}),t) = Z/2^\ell(t)/t^2{=}1$$

with generator $1 + \sqrt{-2}$.

Now we have

<u>Theorem</u> 3.9: <u>The image of</u> d <u>in the</u> W_ℓ <u>above is precisely the</u> w_j <u>with norm a global unit</u>.

<u>Proof</u>: From [M-H, p. 118, example 2] we have that ker(rank homomorphism) $r : W(\hat{\mathbb{Q}}_2(\rho_{2^i})) \to Z/2$ is $\mathbb{Z}/2$ generated by $\langle\varepsilon_1\rangle - \langle 1\rangle$, $r : W(\hat{\mathbb{Q}}_2(\tau_{i+1})) \to Z/2$ is $\langle\varepsilon_1\rangle - \langle 1\rangle$ for $i \geq 3$ and $\langle -1\rangle - \langle 1\rangle$ in the remaining cases. In particular, the forms $\langle\eta_1\rangle - \langle 1\rangle$ for η_i a global unit are all trivial. But rationally

$$\begin{pmatrix} w_i^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \eta_i & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} t(w_i^{-1}) & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So 3.9 follows, on checking from §1 that the $\langle\eta_i\rangle - \langle 1\rangle$, $\langle -1\rangle - \langle 1\rangle$, $2(\langle\varepsilon_1\rangle - \langle 1\rangle)$ generate the piece of $L_o^P(Z(G))$ coming from this representation.

Finally, putting 3.9 together with 2.5-2.8, and checking, again using §1, that every element in $L_o^P(Z(G))$ goes to 0 in $L_o(\mathcal{M}(G) \underset{Z}{\otimes} \hat{Z}_2)$, we see that we have completely determined d so theorem B follows.

§4. The proof of theorem C.

We use the notation of [C-M,2, Appendix], and begin by observing that according to [F-K-W], $D(Q_{2^i,2}) = \tilde{K}_o(Z(Q_{2^i,2}))_{(2)} = Z/2$, and in particular, for $Q_{2,2}$, the element $(1_{\mathcal{Q}}, 1_{--}, 1_{-+}, 1_{+-}, \langle 3\rangle_{++})$ (where $\mathcal{Q}$ is the quaternion representation and $(\pm,\pm)$ are the 1 dimensional representations) represents the generator.

But the unit $1 + x + y$ in $\hat{Z}_2(Q_{2,2})$ has image $\langle 3,1,1,1,3 \rangle$, and $1 + 2xy + 2y \mapsto \langle 1,1,1,3,3 \rangle$ $1 + 2x \mapsto \langle -3,1,1,3,3 \rangle$ so the product $(1 + x + y)(1 + 2x)(1 + 2xy + 2y)^{-1} \mapsto \langle -9,1,1,1,3 \rangle$ and, on factoring out squares, we have that $(-1,1,1,1,1)$ also generates $D(Q_{2,2})$. By 2.8 this last element is in the image of d . On the other hand, by definition the elements $(1,\ldots 1,\theta_{++})$ are the image of the Swan homomorphism T . Hence we have

Theorem 4.1: For $G = Q_{2,2}, K_o(G) = D(G) = Z/2$ is in the image of both T and d .

More generally

Theorem 4.2: a) The Swan homomorphism

$$T : (Z/2^{i+2})^{\cdot} \longrightarrow D(Q_{2^i,2}) = Z/2$$

is surjective.

b) For $Q_{2^i,2}$, $i > 1$ the non-trivial element in $D(Q_{2^i,2})$ is represented by ε_1 at the quaternion algebra and ones at the remaining representations.

Proof: We display the representations as $\mathcal{Q}$, $M_2(Q(\lambda_i))$, $M_2(Q(\lambda_{i-1})),\ldots,$ $M_2(Q)$, $Q_{--}, Q_{-+}, Q_{+-}, Q_{++}$ where $\lambda_i = \rho_{2^i} + \rho_{2^i}^{-1}$ then in $W_\ell(Q_{2^i,2})$ we have for $i \geqslant 3$

$$1 + x^{2^{i-2}} + y \mapsto (3,1,3,3,\ldots,3,1\_\_3,1,3)$$
$$1 + 2y \mapsto (-3\ ,\ 3,3,\ldots,3,1,\ 3,1,3)$$
$$1 + 2x^{2^{i-2}} \mapsto (-3,1,9\ \ldots\ \ 9,1,\ 3,1,3)$$
$$1 + 2x^{2^{i-3}} \mapsto (-3,3,1,9\ \ldots\ 9,1,\ 3,1,3)$$

etc. provided $2^{i-j} > 1$.

Comparing successive terms and factoring out squares we have

$$(1,3,1,\ldots\qquad\qquad ,1)=$$
$$(1,1,3,1,\ldots\qquad\qquad ,1)=$$
$$(1,1,1,3,1\ldots\qquad\qquad ,1)=$$
$$(1,1,\ldots,\qquad 3,1,1,1,1)=$$
$$(-1,1,1,\ldots\ ,\ \ldots\ ,\qquad 1)= 1$$

Next use

$$1 + x + y^2 \mapsto (1,3\ ,\ldots\ ,\qquad 3,1,1,3,3)$$
$$3 \mapsto (9,9\ ,\ldots\ ,\qquad 9,3,3,3,3)$$
$$1 + x^{2^{i-1}} - x^{-2^{i-1}} \mapsto (5,1,1,\ldots\ ,\ \ldots\qquad 1)$$

These imply $(3,1,\ldots,\ \ldots\ ,\qquad\qquad 1) =$

$$(1\ \ldots,\ \ldots\ ,\qquad 1,3,1,3,1) =$$
$$(1\ \ldots,\ \ldots\ ,\qquad 1,1,3,3,1) =$$
$$(1\ \ldots,\ \ldots\ ,\qquad 1,1,1,3,3) = 1$$

Finally, note

$$1 + x^s + y \longrightarrow \begin{array}{ll} (3 + \lambda(s),\ 1, \ldots,\ \ldots\ 1,3) & \delta \text{ odd} \\ (3 + \lambda(s), 1, \ldots,\ \ldots) & \delta \text{ even} \end{array}$$

where $\lambda(s) = \rho^{s}_{2^i} + \rho^{-s}_{2^i}$.

These relations show

$$(1, \ldots,\ \ldots,\ 1,3)$$

generates $D(Q_{2^i,2})$ and prove (a).

To prove (b) we do arithmetic in $\hat{Z}_2(\lambda_i)$ but factor out squares. doing this we can factor out by the ideal $4(\lambda_i)$. Thus

$$(1 + \lambda(s)) \sim (1 + \lambda(s))^3 = (1 + 3\lambda(s))(1 + 3\lambda(s)^2)$$

but

$$\lambda(s)^2 = \lambda(2s) + 2$$

so

$$\begin{aligned} (1 + 3\lambda(s)^2) &= (-1 + 3\lambda(2s)) \\ &\sim -1(1 + 5\lambda(2s)) \\ &\sim -1(1 + \lambda(2s)) \end{aligned}$$

Whence,

$$(1 + 3\lambda(s)) \sim (-1)\ (1 + \lambda(s))\ (1 + \lambda(2s))\ .$$

and we obtain

$$(-1)^{i}(1 + 3\lambda(s))(1 + 3\lambda(2s)) \cdots (1 + 3\lambda(2^{i}s)) \sim (1 + \lambda(s))$$

Using the already determined relations we now have

$$(1 + \lambda(s), 1, \ldots 1, 1) \sim (1, 1, \ldots 1, 3)$$

for s odd. On the other hand, $\delta_s = 1 + \lambda(s)$ is a global unit, as we see from [C-M,2, p. 27, lemma 4.5], with norm -1, and -1 together with the δ_s generate the global units mod squares. Hence, ε_i is an odd product of the δ_s, -1 and squares, which completes the proof.

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November 13, 1978