BALANCED SPLITTINGS OF SEMI-FREE ACTIONS ON HOMOTOPY SPHERES

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Let Σ^{n+k} be a homotopy (n+k)-sphere and $\rho : G \times \Sigma \to \Sigma$ a smooth semi-free action of a finite group G on Σ with fixed-point set a manifold F^n of dimension n. A decomposition of Σ into two G-invariant disks will be called a splitting of the action and the induced splitting of Σ^G denoted $F = F_1 \cup F_2$. We ask whether every such action has a splitting with $H_i(F_1) \cong H_i(F_2)$ for $i \geq 0$ (these are called balanced splittings).

One class of actions for which balanced splittings exist is obtained by the "twisted double" construction. Namely, let $\rho : G \times D^{n+k} \to D^{n+k}$ be a semi-free action of G on an (n+k)-disk. Let $\Sigma = D \cup_{\varphi} D$ where $\varphi : \partial D \to \partial D$ is an equivariant diffeomorphism. Our interest in the problem considered here arose from trying to understand the conditions under which a given semi-free action is a twisted double. An action that admits a balanced splitting resembles a twisted double at least homologically and thus exhibits some symmetry. On the other hand, an action with no balanced splitting is rather strongly asymmetrical.

In this paper we introduce a semi-characteristic invariant of the action to detect the existence of balanced splittings and construct some examples of actions whose semi-characteristic invariant is nonzero. Such actions have no balanced splitting. For most of our results, the arguments are outlined here so that the reader who is familiar with work in this area (e.g., by L. Jones [1] and R. Oliver [2]) can follow them. Full details will appear elsewhere.

Before beginning a precise description of the results, we remark that the fixed-point set F will be assumed nonempty and connected throughout to avoid

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trivial cases. In addition, although the structure of the groups G is known (see [7]), since they all admit free linear representations, this classification is not used in the present situation.

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1. Statement of Results

In order to provide an algebraic setting for the invariant, two categories of finitely-generated $\mathbb{Z}G$-modules will be useful. Let $\mathcal{B}(G)$ denote the category of finite Abelian groups of order prime to $|G|$. If we regard the groups in $\mathcal{B}(G)$ as trivial G-modules, then there is an inclusion $\mathcal{B}(G) \to \mathcal{C}(G)$ (from a result of Rim [4]) where $\mathcal{C}(G)$ is the category of cohomologically trivial modules. The Grothendieck groups of these categories are $G_0(\mathcal{B}(G))$ and $G_0(\mathcal{C}(G))$ and a further result of Rim [4] allows the identification,

$$\widetilde{G}_0(\mathcal{C}(G)) \cong \widetilde{K}_0(\mathbb{Z}G).$$

Finally, let $A(G) = \mathrm{Im}(G_0(\mathcal{B}(G)) \to \widetilde{K}_0(\mathbb{Z}G))$ and note that $A(G)$ is just the image of $\partial : K_1(\mathbb{Z}/|G|) \to \widetilde{K}_0(\mathbb{Z}G)$ considered by Swan [5].

Now let F^n be the fixed-point set of a semi-free action of G on Σ^{n+k}. It follows from Smith theory that $\widetilde{H}_i(F) \in \mathcal{B}(G)$ for $i < n$. Similarly, if the decomposition $F = F_1 \cup F_2$ is induced by a splitting of the action, then $\widetilde{H}_i(F_j) \in \mathcal{B}(G)$ $(j = 1, 2)$ for all i, and $\widetilde{H}_i(F_0) \in \mathcal{B}(G)$ for $i < n-1$ where $F_0 = F_1 \cap F_2$. Any decomposition of F satisfying these necessary conditions will be called a <u>splitting</u> of F.

<u>Definition 1.</u> <u>Let</u> X <u>be a finite</u> CW <u>complex with</u> $\widetilde{H}_i(X) \in \mathcal{B}(G)$ <u>for</u> $i \geq 0$. <u>Then</u>

$$\chi_G(X) = \sum_{i \geq 0} (-1)^i [\widetilde{H}_i(X)] \quad \underline{\text{in}} \quad \widetilde{K}_0(\mathbb{Z}G).$$

Theorem A. *Let* (Σ,ρ) *be a smooth semi-free action of a finite group* G *on a homotopy* $(n+k)$*-sphere* Σ *with fixed-point set* F^n. *If* $1 \leq n \leq k-2$, *then a splitting* $F = F_1 \cup F_2$ *is induced by a splitting of the action if and only if* $\chi_G(F_1) = 0$.

The sufficiency part of Theorem A is proved by an equivariant handle attaching argument similar to the argument given by Jones [1]. The necessity is obtained by observing that $\chi_G(F_1) = -\sum_{i\geq 0}(-1)^i[H_i(D_1,F_1)] = \sum_{i\geq 0}(-1)^i[C_i(D_1,F_1)] = 0$ where D_1 is a G-invariant disk such that $F_1 = D_1 \cap F$. Note that $\chi_G(F_1) = \pm\chi_G(F_2)$, so that statement does not depend on the ordering of F_1 and F_2.

If we now assume that $F = F_1 \cup F_2$ is a *balanced splitting* (that is, $H_i(F_1) \cong H_i(F_2)$ for $i \geq 0$ in addition to the conditions above), then $\chi_G(F_1)$ determines a semi-characteristic invariant of the action.

Definition 2. *Let* X *be a finite* CW *complex of dimension* n *with* $\tilde{H}_i(X) \in \mathfrak{D}(G)$ *for* $i < n$ *and* $|H_m(X)| = q^2$ *when* $n = 2m+1$. *In* $G_0(\mathfrak{D}(G))$ *set*

$$\tilde{\chi}_{\frac{1}{2}}(X) = \begin{cases} \sum_{i=1}^{m-1}(-1)^i[H_i(X)] & \text{if } n = 2m \\ \sum_{i=1}^{m-1}(-1)^i[H_i(X)] + (-1)^m[\mathbb{Z}/q] & \text{if } n = 2m+1 . \end{cases}$$

Now let $x \mapsto \bar{x}$ be the involution on $\tilde{K}_0(\mathbb{Z}G)$ induced by sending $[P]$ to $-[P^*]$. Then we wish to define $\chi_{\frac{1}{2}}(X)$ to be the cohomology class in $H^n(\mathbb{Z}/2;A(G))$ represented by the image of $\tilde{\chi}_{\frac{1}{2}}(X)$. This will make sense if we assume that when $n = 2m+1$, $2\tilde{\chi}_{\frac{1}{2}}(X) = 0$ in $A(G)$ (because the subgroup $A(G)$ is actually fixed by the involution $-$). However, we assert that $\chi_{\frac{1}{2}}(F^n)$ is well defined for $F = \Sigma^G$ provided $|H_m(F)|$ is a square when $n = 2m+1$. Since the resulting class is an invariant of the action, we denote it $\chi_{\frac{1}{2}}(\Sigma,\rho)$.

Theorem B. *Let* (Σ,ρ) *be a smooth semi-free* G*-action on a homotopy* $(n+k)$-

sphere with fixed-point set F^n and $1 \leq n \leq k-2$ $(n \neq 3, 4)$.

i) If $n = 2m$, the action has a balanced splitting.

ii) If $n = 2m+1$, the action has a balanced splitting if and only if $|H_m(F)|$ is a square and $\chi_{\frac{1}{2}}(\Sigma, \rho) = 0$.

Among the actions obtained by the twisted double construction are those for which $\Sigma = D \cup_{\varphi} D$ with φ = identity. These are called strong doubles. If the homological conditions on the splitting of F^n are strengthened to include

$$\ker(H_{m-1}(F_0) \to H_{m-1}(F_1)) = \ker(H_{m-1}(F_0) \to H_{n-1}(F_2))$$

for $m = [\frac{n}{2}]$, we call it a strong balanced splitting. Such a condition is satisfied for all $m \geq 0$ if the splitting arises from a splitting of (Σ, ρ) as a strong double.

Theorem C. Under the same hypotheses as Theorem B, (Σ, ρ) has a strong balanced splitting if and only if $\chi_{\frac{1}{2}}(\Sigma, \rho) = 0$ and $|H_m(F^n)|$ is a square (when $n = 2m+1$).

Remark. A (strong) balanced splitting of the fixed-point set F^n exists always for $n = 2m$ and for $n = 2m+1$ if and only if $|H_m(F)|$ is a square. There are examples of actions with $n = 2m+1$ and $|H_m(F)|$ a nonsquare (e.g., a Brieskorn example of an involution on S^5 with fixed-point set a lens space).

Finally, we have some examples to show that the semi-characteristic invariant can be nonzero. Since the exponent of $A(G)$ divides the Artin exponent of G [6], if $|G|$ is odd $\chi_{\frac{1}{2}}(\Sigma, \rho)$ is always zero. Otherwise the Sylow 2-subgroup $Syl_2(G)$ is cyclic or generalized quaternion $Q2^{\ell}$ of order 2^{ℓ} [7], and our examples concern this second case. The fixed-point sets of these examples are actually strong doubles.

Theorem D. Let G be a finite group with $Syl_2(G) = Q2^{\ell}$ admitting a free linear representation of dimension d and n, k integers such that $5 \leq n \leq k-2$. Then there exists a semi-free G-action ρ on a homotopy (n+k)-sphere Σ with $\dim(\Sigma^G) = n$ and $\chi_{\frac{1}{2}}(\Sigma, \rho) \neq 0$ provided $k \equiv 0 \pmod d$ and $n \not\equiv 1 \pmod 4$ (when $\ell = 3$) or $n \not\equiv 0, 1 \pmod 4$ (when $\ell \geq 4$).

Remarks.

1) If $n \equiv 2 \pmod 4$, we have the more complete result (valid for any G admitting a free representation) that each element of $H^n(Z/2; A(G))$ can arise as the $\chi_{\frac{1}{2}}$-obstruction to the existence of a strong balanced splitting.

2) If (Σ, ρ) and (Σ, ρ') are concordant actions, then $\chi_{\frac{1}{2}}(\Sigma, \rho) = \chi_{\frac{1}{2}}(\Sigma, \rho')$. This means that the examples above are not concordant to linear actions.

2. Proof of Theorem B.

In this section we outline the proof of Theorem B.

Lemma 1. If F^n ($n \neq 3, 4$) is a closed orientable manifold with $\tilde{H}_i(F) \in \mathfrak{B}(G)$ for $i < n$, then F has a (strong) balanced splitting if and only if $|H_m(F)|$ is a square when $n = 2m+1$.

If $n = 2m$, we start with a handlebody $N_0 \subset F$ that carries all handles of F of index $\leq m-1$ in a given handle decomposition. Only enough handles of index m are added to make $H_{m-1}(N_0) \cong H_{m-1}(F)$ without creating any m-dimensional homology. If $n = 2m+1$ and $|H_m(F)|$ is a square, there exists a short exact sequence of the form

$$0 \to T \to H_m(F) \to T \to 0.$$

To $N_0 \subset F$, as before consisting of handles of index $\leq m-1$, can be attached handles of index m and m+1 to get $N_1 \subset F$ such that $H_i(N_1) \cong H_i(F)$ for $i < m-1$, $H_m(N_1) \cong T$ and $H_i(N_1) = 0$ for $i > m$.

For the necessity when $n = 2m+1$, let $F = F_1 \cup F_2$ be a balanced splitting and factor the exact sequence of the pair (F, F_1) into

$$0 \to A \to H_{m-1}(F_1) \to H_{m-1}(F) \to \dots \to H_1(F, F_1) \to 0$$

$$0 \to B \to H_m(F_1) \to H_m(F) \to H_m(F, F_1) \to A \to 0$$

$$0 \to H_{2m}(F_1) \to H_{2m}(F) \to \dots \to H_{m+1}(F, F_1) \to B \to 0$$

and use formal manipulations and Poincaré duality to show $[A] = [B]$ in $G_0(\mathfrak{O}(G))$. The middle sequence then shows

$$[H_m(F)] = 2[H_m(F_1)] - 2[A] \quad \text{in} \quad G_0(\mathfrak{O}(G)).$$

Since $G_0(\mathfrak{O}(G))$ is the free Abelian group on generators $[\mathbb{Z}/p]$ where p is a prime not dividing $|G|$, $|H_m(F)|$ is a square.

The next step is to determine the relationship between $\chi_G(F_1)$ and $\chi_{\frac{1}{2}}(\Sigma, \rho)$ when $F = \Sigma^G$ has a balanced splitting $F = F_1 \cup F_2$.

__Lemma__ 2. __Suppose__ $F^n = F_1 \cup F_2$ __is a balanced splitting.__

i) __If__ $n = 2m+1$, $\tilde{\chi}_{\frac{1}{2}}(F) = \chi_G(F_1)$ __and__ $2\tilde{\chi}_{\frac{1}{2}}(F) = 0$.

ii) __If__ $n = 2m$,

$$\tilde{\chi}_{\frac{1}{2}}(F) = \chi_G(F_1) - 2\left(\sum_{i=m}^{2m-1} (-1)^i [H_i(F_1)]\right) + (-1)^m [A]$$

__where__

$$A = \ker(H_{m-1}(F_1) \to H_{m-1}(F)).$$

From the sketch proof for Lemma 1,

$$\tilde{\chi}_{\frac{1}{2}}(F) = \sum_{i=1}^{m-1} (-1)^i [H_i(F)] + (-1)^m [\mathbb{Z}/q]$$

(where $|H_m(F)| = q^2$ and $n = 2m+1$)

$$= (-1)^m [A] + \sum_{i \neq m} (-1)^i [H_i(F_1)] + (-1)^m ([H_m(F_1)] - [A])$$

$$= \chi_G(F_1).$$

By taking the full sequence of the pair $(F\text{-pt}, F_1)$, we obtain $2\chi_G(F_1) = \chi_G(F\text{-pt}) = 0$. The argument for i) is similar.

Recall now that $\chi_{\frac{1}{2}}(\Sigma, \rho)$ lies in

$$H^n(\mathbb{Z}/2; A(G)) = \begin{cases} A(G)/2A(G) & , \quad n = 2m \\ \{x \in A(G) \mid 2x = 0\}, & n = 2m+1. \end{cases}$$

Clearly, Lemma 2 shows that $\chi_{\frac{1}{2}}(\Sigma, \rho)$ is well defined. The proof of Theorem B when $n = 2m+1$ is now an immediate consequence of Lemmas 1, 2 and Theorem A.

For $n = 2m$ there is one more ingredient. This is a simple method for changing one balanced splitting to a new one. Let $M_n(\ell, p)$ denote a regular neighborhood of the complex $S^\ell \cup_p e^{\ell+1}$ embedded in S^n (where $1 \leq \ell \leq n-4$ and $(p, |G|) = 1$). Then S^n has a splitting into two thickened Moore spaces

$$S^n(\ell, p) = M_n(\ell, p) \cup (S^n - \mathring{M}_n(\ell, p)).$$

If $F^n = F_1 \cup F_2$ is a splitting, there is another splitting $F = F_1' \cup F_2'$ obtained by connected sum along F_0 and ∂M:

$$F^n \approx F^n \# S^n(\ell, p) = (F_1 \# M_n(\ell, p)) \cup (F_2 \# (S^n - \mathring{M}_n(\ell, p))).$$

If a splitting of F is understood, $F \# S^n(\ell, p)$ means this new splitting.

Lemma 3. *Suppose* $F^n = F_1 \cup F_2$ *is a (strong) balanced splitting* $(p, |G|) = 1$ *and* $1 \leq \ell \leq n-4$.

i) $F^n \# S^n(\ell, p) \# S^n(n-\ell-2, p)$ is a balanced splitting (a strong balanced splitting unless $n = 2m$ and $\ell = m-1$).

ii) $F^{2m} \# S^{2m}(m-1, p)$ is a balanced splitting.

iii) If $F^n = F_1' \cup F_2'$ is the new splitting in i)

$$\chi_G(F_1') = \begin{cases} \chi_G(F_1) & \text{if } n = 2m+1 \\ \chi_G(F_1) + (-1)^{\ell} 2[\mathbb{Z}/p] & \text{if } n = 2m . \end{cases}$$

iv) For the splitting in ii),

$$\chi_G(F_1') = \chi_G(F_1) + (-1)^{m-1}[\mathbb{Z}/p].$$

If $n = 2m$, a balanced splitting of Σ^G can now be obtained with $\chi_G(\Sigma^G) = 0$ using Lemma 1 and this construction. Essentially the same argument gives the proof of Theorem C also.

3. Construction of Examples

If X is any finite CW complex with $\tilde{H}_i(X) \in \mathcal{P}(G)$ for $i \geq 0$, a G any group with a free linear representation, let N_1 be a regular neighborhood of X in $\mathbb{R}^n$ and $M^m = \mathbb{R}^n \times \mathbb{V}^k$ where $5 \leq n \leq k-2$ and $\mathbb{V}^k$ is a free representation space for G of real dimension k. By the same method as that used for Theorem A, there exists a $([\frac{m}{2}]-1)$-connected G-invariant compact manifold M_1^m (M_1 is a submanifold of M except possibly when $m = 2s$) such that $M_1^G = N_1$, $\tilde{H}_i(M_1) = 0$ for $i \neq [\frac{m}{2}]$ and $(-1)^s[H_s(M_1)] = \chi_G(N_1) = \chi_G(X)$ for $s = [\frac{m}{2}]$. If $m = 2s$, the final handle attaching to produce M_1 must be done so that the cycles representing $H_s(M_1)$ have zero equivariant self-intersection. Using this manifold M_1, we will try to construct an action on some $(n+k)$-sphere with fixed-point set F^n the double of N_1. Since $A(G)$ is also the image of the Swan homomorphism, $(-1)^{[\frac{m}{2}]} \chi_G(X) = [\mathbb{Z}/r]$ ($= \partial r$) for some integer r prime to $|G|$. If $m = 2s+1$, after a further surgery on M_1 (using the bundle map $\nu_{M_1} \to \nu_M$), we obtain M_1' with $\tilde{H}_i(M_1') = 0$, $i \neq s$ and $H_s(M_1') = \mathbb{Z}/r$. Let W^m be the double of M_1 $(m = 2s)$ or M_1' $(m = 2s+1)$. This is a smooth, semi-free, $([\frac{m}{2}]-1)$-connected G-manifold with W^G the double of N_1 and

$$H_s(W) = \begin{cases} P \oplus P^* & , \quad m = 2s \\ \mathbb{Z}/r \oplus \mathbb{Z}/r , & m = 2s+1. \end{cases}$$

where P is a projective $\mathbb{Z}G$-module. Moreover, the geometric self-intersection (self-linking) is trivial on P and P^* (both copies of $\mathbb{Z}/r$), and the intersection form (linking form) is hyperbolic. This means that the obstruction doing surgery on W to obtain a homotopy sphere can be formulated in terms of the "hyperbolic map" in the Ranicki-Rothenberg sequence [3]:

$$\cdots \to L^p_{m+1}(\mathbb{Z}G) \to H^m(\mathbb{Z}/2;\tilde{K}_0(\mathbb{Z}G)) \xrightarrow{\mathbb{H}} L^h_m(\mathbb{Z}G) \to L^p_m(\mathbb{Z}G) .$$

<u>Proposition</u> 1. W^m <u>is equivariantly cobordant to a semi-free action</u> (Σ, ρ) <u>on a homotopy</u> m-<u>sphere with</u> $\Sigma^G = F$ <u>if</u> $\mathbb{H}(\chi_G(X)) = 0$.

To obtain the examples referred to in the Remarks following Theorem D, we note that $\mathbb{H}(x) = 0$ for any $x \in A(G)$ when $\mathbb{H} : H^0(\mathbb{Z}/2, \tilde{K}_0(\mathbb{Z}G)) \to L^h_2(\mathbb{Z}G)$. Clearly, any element of $A(G)$ arises as $\chi_G(X)$ for some suitable X (e.g., a Moore space). For Theorem D itself, we note that when $H \leq G$ is a subgroup, the restriction $A(G) \to A(H)$ is onto (Ullom [6]). The existence of the desired examples now follows from:

<u>Proposition</u> 2. <u>Let</u> $\mathbb{H} : H^m(\mathbb{Z}/2;K_0(\mathbb{Z}G)) \to L^h_m(\mathbb{Z}G)$ <u>be the hyperbolic map</u>.

i) <u>If</u> $G = Q8$, $\mathbb{H} = 0$ <u>when</u> $m \not\equiv 1$ (<u>mod</u> 4) <u>and</u> $\mathbb{H}$ <u>is injective when</u> $m \equiv 1$ (<u>mod</u> 4).

ii) <u>If</u> $G = Q2^\ell$ $(\ell \geq 4)$, $\mathbb{H} = 0$ <u>when</u> $m \equiv 2, 3$ (<u>mod</u> 4).

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