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The Surgery Obstruction Groups for Finite 2-Groups

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In this paper we establish an effective method for calculating the oriented surgery obstruction groups $L^h_*(\mathbf{Z}G)$ for G a finite group of 2-primary order. We show that these groups depend explicitly on the rational representations of G and certain facts about the reduced projective class group $\tilde{K}_0(\mathbf{Z}G)$, and prove that most of the relevant structure of $\tilde{K}_0(\mathbf{Z}G)$ in turn depends only on the rational representations of G.

Surgery obstruction groups in various geometric situations were introduced by Wall [20]. He proved that the ones studied here are basic for the classification (up to h-cobordism) of closed, oriented manifolds with finite fundamental group [23].

Our method, an extension of a program first proposed in [3], uses the Ranicki-Rothenberg exact sequence [16, Theorem 4.3]

$$(*)\qquad \ldots \to L^h_{k+1}(\mathbf{Z}G) \to L^p_{k+1}(\mathbf{Z}G) \xrightarrow{d_{k+1}} H^k(\mathbf{Z}/2; \tilde{K}_0(\mathbf{Z}G)) \to L^h_k(\mathbf{Z}G) \to \ldots$$

The calculation of $L^p_*(\mathbf{Z}G)$ for G a finite 2-group now appears relatively easy. In Bak [2], Pardon [15], Carlsson-Milgram [4], Kolster [11], and an earlier version of this paper [25], the answers were first worked out. In Theorem A we summarize these results and point out that $L^p_*(\mathbf{Z}G)$ depends only on the structure of $\mathbf{Q}G$. Since all the calculations are now documented in the literature we omit them and prove only this last statement (see § 3).

Our main concern, however, is with studying the map d_{k+1} in $(*)$. The involution on $\tilde{K}_0(\mathbf{Z}G)$ is given by $[P] \mapsto -[P^*]$ where P^* is the dual module to P, and only the 2-torsion part of $\tilde{K}_0(\mathbf{Z}G)$ matters in $(*)$. We define a finite abelian 2-group with involution $W_\ell(G)$ which depends only on the rational representations of G and a involution preserving map $\varphi\colon W_\ell(G) \to \tilde{K}_0(\mathbf{Z}G)$ which is onto the 2-primary part of $\tilde{K}_0(\mathbf{Z}G)$. Then we prove that d_{k+1} factors, as a composite

$$L^p_{k+1}(\mathbf{Z}G) \xrightarrow{d'_{k+1}} H^k(\mathbf{Z}/2, W_\ell(G)) \xrightarrow{\varphi_*} H^k(\mathbf{Z}/2, \tilde{K}_0(\mathbf{Z}G))$$

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where d'_{k+1} depends only on the rational representations of G. This is Theorem B and complete information on d'_* is given in Lemmas (5.1)–(5.7), so the calculation of the $L^h_*(\mathbf{Z}G)$ groups up to extensions is reduced to determining φ_* which depends on $\mathrm{Im}(K_1(\hat{\mathbf{Z}}_2G) \to K_1(\hat{\mathbf{Q}}_2G))$. Here the recent work of Oliver ("SK_1 for finite groups rings: II", Aarhus University preprint, 1980) may be useful.

To demonstrate the effectiveness of our method we list in § 2 some explicit calculations for special cases (Theorems C, D, E) including cyclic, elementary abelian, generalized quaternion, dihedral, semi-dihedral, and 2-Sylow subgroups of the symmetric groups. (Proofs are given in §§ 6, 7.)

The results for the semi-dihedral, and quaternion groups are new and have been applied in [1] to construct examples of semi-free group actions on homotopy spheres which are not twisted doubles of actions on disks.

Also, these results have played a crucial role in recent work on free actions of finite groups on spheres by Milgram.

In §§ 1, 2 we review some facts about the rational representations of finite 2-groups, and state our main results, Theorems A to E. The remaining sections contain proofs.

Bak [2] has previously considered the groups $L^h_*(\mathbf{Z}G)$ where G is a finite group with normal abelian 2-Sylow subgroup, so our results overlap for abelian 2-groups. Wall [22] has made extensive calculations for arbitrary finite groups of the "intermediate" obstruction groups $L'_*(\mathbf{Z}G)$. The relation of these groups to $L^h_*(\mathbf{Z}G)$ is given by an exact sequence [22, 5.4]:

$$0 \to L'_{2k}(\mathbf{Z}G) \to L^h_{2k}(\mathbf{Z}G) \to Wh'(G) \otimes \mathbf{Z}/2 \to L'_{2k-1}(\mathbf{Z}G) \to L^h_{2k-1}(\mathbf{Z}G) \to 0.$$

Using our results the maps in this sequence can be calculated in many cases to settle extension problems. For example, L'_1 and L^h_1 are both of exponent 2 for G a generalized quaternion group of order $\geqq 16$ (cf. [22, 5.2.4]).

§ 1. Rational Representations of a 2-Group

There are four basic types of 2-groups necessary in studying the rational representations:

(1.1)
(a) $\mathbf{Z}/2^n = \{x \mid x^{2^n} = 1\}$ $n \geqq 1$ (cyclic)
(b) $D2^n = \{x, y \mid x^{2^{n-1}} = y^2 = 1, yxy^{-1} = x^{-1}\}$ $n \geqq 2$ (dihedral)
(c) $SD2^n = \{x, y \mid x^{2^{n-1}} = y^2 = 1, yxy^{-1} = x^{2^{n-2}-1}\}$ $n \geqq 4$ (semi-dihedral)
(d) $Q2^n = \{x, y \mid x^{2^{n-2}} = y^2 = (xy)^2\}$ $n \geqq 3$ (generalized quaternion).

It is well known that the irreducible faithful representations of these groups are given as follows (here ζ_k is a primitive 2^k-th root of 1):

(1.2)
(a) $\mathbf{Q}(\zeta_k)$
(b) $M_2(\mathbf{Q}(\zeta_k + \zeta_k^{-1}))$
(c) $M_2(\mathbf{Q}(\zeta_k - \zeta_k^{-1}))$

(d) $\Gamma_k = \left(\frac{-1,\,-1}{\mathbf{Q}(\zeta_k+\zeta_k^{-1})}\right) = \mathbf{Q}(\zeta_k+\zeta_k^{-1}) \otimes_{\mathbf{Q}} \left(\frac{-1,\,-1}{\mathbf{Q}}\right)$, the "usual" quaternion algebra over $\mathbf{Q}(\zeta_k+\zeta_k^{-1})$. (Here $\left(\frac{-1,\,-1}{\mathbf{Q}}\right)$ is the quaternion algebra $\mathbf{Q}(i,j)$ $\{i^2=j^2=-1,\ ij=-ji\}$.)

The involution $g \mapsto g^{-1}$ on $\mathbf{Q}G$ induces the involution $\zeta_k \mapsto \zeta_k^{-1}$ on the centres of (a), (c), and the identity on the centres for (b), (d). As algebras with involution (a), (c) then have type II in the classical notation or type U in Wall's notation, and (b), (d) are type I, but more exactly, type **O** and Sp respectively in Wall's terminology. (See [22, p. 5] for definitions.) The following is a refinement of earlier results of Roquette and Witt.

Theorem 1.3 (Fontaine [6]). *Let G be a finite 2-group, and M an irreducible $\mathbf{Q}G$-module. Then there exist subgroups $H \lhd K$ of G, and an irreducible $\mathbf{Q}[K/H]$-module N such that*

(a) *K/H is in* (1.1)

(b) *N is a projective module over one of the algebras in* (1.2), *and*

(c) *Viewing N as a $\mathbf{Q}K$ module then $M = N \otimes_{\mathbf{Q}K} \mathbf{Q}G$*

(d) *The irreducible simple subalgebra of $\mathbf{Q}G$ corresponding to M is of the form $M_\ell(A)$, where A is the algebra in* (1.2) *corresponding to N.*

In particular, the rank, type (in Wall's notation) and centre describe a simple summand of $\mathbf{Q}G$ up to isomorphism as an algebra-with-involution.

Another useful property of $\mathbf{Q}G$ for G a finite 2-group is the existence of an involution invariant maximal order $\mathcal{M} \subset \mathbf{Q}G$ containing $\mathbf{Z}G$ [15, §5]. Further, if $\mathbf{Q}G = \prod_\alpha D_\alpha$ is the decomposition into simple algebras, then $\mathcal{M} = \prod_\alpha \mathcal{M}_\alpha$, where $\mathcal{M}_\alpha$ is an involution invariant maximal order in D_α. Indeed $\mathcal{M}_\alpha$ can be chosen to be a full matrix ring over

$$\mathbf{Z}(\zeta_k),\ M_2(\mathbf{Z}(\zeta_k+\zeta_k^{-1})),\ M_2(\mathbf{Z}(\zeta_k-\zeta_k^{-1}))$$

or a maximal order in the quaternion algebra Γ_k for some k, in the four cases of (1.2).

Theorem A. *Let G be a finite 2-group and $\mathbf{Q}G = \prod_\alpha D_\alpha$ where the D_α are simple involuted algebras.* (1) *There are groups $\Lambda_i(D_\alpha)$ depending only on type of D_α, the centre of D_α and i such that*

$$L_i^p(\mathbf{Z}G) \cong \prod_\alpha \Lambda_i(D_\alpha) \quad \textit{for } 0 \leqq i \leqq 3.$$

(2) *Let $\ell(\alpha)$ be the number of simple summands in $D_\alpha \otimes_{\mathbf{Q}} \mathbf{R}$. The non-zero groups $\Lambda_*(D_\alpha)$ are:*

(a) $\Lambda_0(D_\alpha) = (\mathbf{Z})^{\ell(\alpha)}$ *for each D_α.*

(b) $\Lambda_1(D_\alpha) = (\mathbf{Z}/2)^{2^{k-2}+1}$ *if D_α has type Sp and centre $\mathbf{Q}(\zeta_k+\zeta_k^{-1})$ for $k \geqq 2$.*

(c) $\Lambda_2(D_\alpha)=(\mathbf{Z})^{\ell(\alpha)}$ *if* D_α *has type* U; $\Lambda_2(D_\alpha)=\mathbf{Z}/2$ *if* $D_\alpha=\mathbf{Q}$ *with trivial G-action;* $\Lambda_2(D_\alpha)=(\mathbf{Z}/2)^{2^{k-2}-1}$ *if* D_α *has type Sp and centre* $\mathbf{Q}(\zeta_k+\zeta_k^{-1})$ *for* $k\geq 3$.

(d) $\Lambda_3(D_\alpha)=\mathbf{Z}/2$ *if* D_α *has type* **O** *and* $D_\alpha \neq \mathbf{Q}$ *with trivial G-action.*

The L^p groups above were obtained for G abelian in [2], L_3^p in [4], [15] and L_1^p modulo an extension problem in [15]. The remaining ones are now in [11]. We remark that $\Lambda_2(\mathbf{Q})$ is detected by the ordinary Arf invariant. Other explicit generators for these groups are given in §5. It is of interest to know the divisibility of the varous signatures in L_0^p and L_2^p. These are actually determined by the following method of calculation.

Let $\mathcal{M}$ as before be an involution-invariant maximal order in $\mathbf{Q}G$ containing $\mathbf{Z}G$ and consider the diagrams of exact sequences derived from [17, 7.3]:

$$\begin{array}{ccccccc}
\ldots\to L_{k+1}^{\hat{Y}}(\hat{\mathbf{Q}}G) & \longrightarrow & L_k^p(\mathbf{Z}G) & \longrightarrow & L_k^h(\hat{\mathbf{Z}}G)\oplus L_k^p(\mathbf{Q}G) & \longrightarrow & L_k^{\hat{Y}}(\hat{\mathbf{Q}}G)\to\ldots \\
\| & & \downarrow & & \downarrow & & \| \\
\ldots\to L_{k+1}^{\hat{Y}}(\hat{\mathbf{Q}}G) & \longrightarrow & L_k^X(\mathcal{M}) & \longrightarrow & L_k^h(\hat{\mathcal{M}})\oplus L_k^p(\mathbf{Q}G) & \longrightarrow & L_k^{\hat{Y}}(\hat{\mathbf{Q}}G)\to\ldots
\end{array}$$

where $X=\ker(\tilde{K}_0(\mathcal{M})\to\tilde{K}_0(\mathbf{Q}G))$ and $\hat{Y}=\operatorname{Im}(\tilde{K}_0(\mathbf{Q}G)\to\tilde{K}_0(\hat{\mathbf{Q}}G))$. Using the fact that $\hat{\mathbf{Z}}_pG$ is maximal for p odd, we obtain

$$\ldots\to L_{k+1}^h(\hat{\mathcal{M}}_2)\longrightarrow L_k^p(\mathbf{Z}G)\longrightarrow L_k^h(\hat{\mathbf{Z}}_2G)\oplus L_k^X(\mathcal{M})\longrightarrow L_k^h(\hat{\mathcal{M}}_2)\to\ldots.$$

This exact sequence gives the groups in Theorem A. In §3, we will establish the first part of Theorem A.

The next step in computing $L_*^h(\mathbf{Z}G)$ is to show to what extent the rational representations also determine

(1.4) $$d_{k+1}\colon L_{k+1}^p(\mathbf{Z}G)\to H^k(\mathbf{Z}/2;\tilde{K}_0(\mathbf{Z}G)).$$

Let

$$D(G)=\ker(\tilde{\tilde{K}}_0(\mathbf{Z}G)\to\tilde{K}_0(\mathcal{M})$$

and observe that since $Cl(\mathcal{M})=\ker(\tilde{K}_0(\mathcal{M})\to\tilde{K}_0(\mathbf{Q}G))$ has odd order [8] when G is a 2-group,

(1.5) $$H^*(\mathbf{Z}/2;\tilde{K}_0(\mathbf{Z}G))=H^*(\mathbf{Z}/2;D(G)).$$

Although $D(G)$ is not determined by $\mathbf{Q}G$, generators of it are.

Lemma 1.6. *Given* ℓ *such that* $2^\ell\hat{\mathcal{M}}_2\subset\hat{\mathbf{Z}}_2G$, *there is a finite group* $W_\ell(G)$ *with involution such that*

(a) $W_\ell(G)$ *has exponent* 2^ℓ,

(b) *there is an epimorphism of* $\mathbf{Z}/2$*-modules*

$$\varphi\colon W_\ell(G)\to D(G)$$

(c) $W_\ell(G)=\prod_\alpha W_\ell(D_\alpha)$ *as* $\mathbf{Z}/2$*-modules where the* $W_\ell(D_\alpha)$ *are finite* $\mathbf{Z}/2$*-modules depending only on* ℓ, *the type, and the centre of the simple summand* D_α *of* $\mathbf{Q}G$.

Proof. We need the description given in [18] of $D(G)$ in terms of units. Let $K'_1(\mathscr{M})$ denote the image of $K_1(\mathscr{M})$ under reduced norms in the product of the units of the centres of the $\mathscr{M}_\alpha$ where $\mathscr{M}=\prod_\alpha \mathscr{M}_\alpha$. Similarly, we define $K'_1(\hat{\mathscr{M}}_2)$ and have a map $K'_1(\mathscr{M})\to K'_1(\hat{\mathscr{M}}_2)$. Also there is a map

$$K_1(\hat{\mathbf{Z}}_2 G)\to K'_1(\hat{\mathscr{M}}_2)$$

defined by applying the reduced norm at each summand of $\hat{\mathscr{M}}_2$ to the image of a unit from $\hat{\mathbf{Z}}_2 G$. From [18, p. 14] there is an exact sequence

$$K_1(\hat{\mathbf{Z}}_2 G)\oplus K'_1(\mathscr{M})\to K'_1(\hat{\mathscr{M}}_2)\to D(G)\to 0.$$

If $D(G)$ has the involution induced by $[P]\mapsto -[P^*]$ and the K_1 groups have the involution "conjugate transpose", then this is also a sequence of $\mathbf{Z}/2$-modules. Since $2^\ell \hat{\mathscr{M}}_2\subset\hat{\mathbf{Z}}_2 G$ we can define

$$I_\ell=\{1+2^\ell\alpha\,|\,\alpha\in\hat{\mathscr{M}}_2\}\subset\hat{\mathbf{Z}}_2 G^\times$$

and define $W_\ell(G)$ by the exact sequence

(1.7) $$I_\ell\oplus K'_1(\mathscr{M})\to K'_1(\hat{\mathscr{M}}_2)\to W_\ell(G)\to 0.$$

Now both (a) and (b) are clear and (c) follows because I_ℓ splits as $\hat{\mathscr{M}}_2$ does.

Our main general result about d_{k+1} is the following.

Theorem B. *Let G be a finite 2-group.*

(1) *The boundary map d_{k+1} in (1.4) factors as*

$$L^p_{k+1}(\mathbf{Z}G)\xrightarrow{d'_{k+1}} H^k(\mathbf{Z}/2;\, W_\ell(G))\xrightarrow{\varphi_*} H^k(\mathbf{Z}/2;\, D(G))$$

(2) *For each simple summand D_α of $\mathbf{Q}G$ there exists a homomorphism*

$$d'_{k+1}(\alpha)\colon \Lambda_{k+1}(D_\alpha)\to H^k(\mathbf{Z}/2;\, W_\ell(D_\alpha))$$

such that $d'_{k+1}=\prod_\alpha d'_{k+1}(\alpha)$ under the splittings of Theorem A.1 *and* (1.6) c.

In §4 we will calculate the groups $H^k(\mathbf{Z}/2;\, W_\ell(D_\alpha))$ for all possible D_α (it will frequently be convenient to denote these groups by $H^k(W_\ell(D_\alpha))$) and in §5 we will calculate all the maps $d'_{k+1}(\alpha)$. When combined with Theorem B, this reduces the computation of $L^h_k(\mathbf{Z}G)$ up to group extensions to the computation of

(1.8) $$\varphi_*\colon H^k(W_\ell(G))\to H^k(D(G))$$

and this is the information about $\tilde{K}_0(\mathbf{Z}G)$ needed in addition to the structure of $\mathbf{Q}G$ to obtain the surgery obstruction groups.

§2. Applications

To illustrate the method, we give the surgery obstruction groups for several classes of 2-groups. (Previous specific calculations are in [2] and [22, 5.4].) The first result gives the answer up to extensions for elementary abelian 2-groups (note that $\tilde{K}_0(\mathbf{Z}G)$ has been computed in this case [24, Th. 12.9]), wreath products of $\mathbf{Z}/2$ (the 2-Sylow subgroups of the symmetric groups) and products of these, all of which are among the large class of groups satisfying the assumptions of

Theorem C. *Let G be a finite 2-group with only type* **O** *summands in* $\mathbf{Q}G$. *Then*

$$H^k(\mathbf{Z}/2;\tilde{K}_0(\mathbf{Z}G))\to L^h_k(\mathbf{Z}G)$$

is injective for $k\not\equiv 2 \pmod 4$ and zero for $k\equiv 2 \pmod 4$.

The L^h-groups for each of the 2-groups G listed in (1.1) can also be given. Since $D(G)=0$ for the dihedral groups [7], their L^h-groups are covered by Theorem A. Since $G=\mathbf{Z}/2^n$ is abelian its L^h-groups appear in [2]. We will check this result by our methods in §7 and obtain: for $G=\mathbf{Z}/2^n$, $L^h_0=(\mathbf{Z})^{2^{n-1}+1}\oplus H^0(\mathbf{Z}/2;D(G))$; $L^h_1=0$; $L^h_3=\mathbf{Z}/2$; $L^h_2=(\mathbf{Z})^{2^{n-1}-1}\oplus\mathbf{Z}/2\oplus H^0(\mathbf{Z}/2;D(G))$.

Theorem D. *Let $G=Q2^n$ then $L^h_0=(\mathbf{Z})^{2^{n-2}+3}$; L^h_1 is an extension of $\mathbf{Z}/2$ by $(\mathbf{Z}/2)^{2^{n-3}}$; $L^h_2=\mathbf{Z}/2$ if $n=3$ and $L^h_2=\mathbf{Z}/2\oplus(\mathbf{Z}/2)^{2^{n-3}-1}$ if $n\geqq 4$; $L^h_3=(\mathbf{Z}/2)^{n-1}$.*

Since $SD2^n$ has a subgroup $Q8$, the results of Ullom [19, 3.5, 3.9] give an isomorphism induced by restriction:

$$H^*(\tilde{K}_0(SD\,2^n))\to H^*(\tilde{K}_0(Q8))=\mathbf{Z}/2.$$

Theorem E. *Let $G=SD2^n$ then $L^h_0=\mathbf{Z}/2\oplus(\mathbf{Z})^{2^{n-3}+2^{n-4}+3}$; $L^h_1=\mathbf{Z}/2$; $L^h_2=\mathbf{Z}/2\oplus(\mathbf{Z})^{2^{n-4}}$; L^h_3 is an extension of $\mathbf{Z}/2$ by $(\mathbf{Z}/2)^{n-1}$.*

In stating the above results we have not specified the divisibility of the signatures although this is determined in the calculation. This data would be helpful for comparison with [22] particularly in finding maps in the K_1-Rothenberg sequences.

Remark. In Theorem D, the extension for L^h_1 is split for $n\geqq 4$.

§3. Proof of Theorem B

Our procedure for factoring the maps d_* involves standard "mixing" constructions for modules over $\mathbf{Z}G$. The only addition is that we simultaneously mix the forms.

Consider the pull-back diagram

(3.1)
$$\begin{array}{ccc} \mathbf{Z}G & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow{\scriptstyle i} \\ \hat{\mathbf{Z}}_2G & \xrightarrow{\;j\;} & \hat{\mathcal{M}}_2 \end{array}$$

where $\mathcal{M}$ is (as in §1) an involution-invariant maximal order for $\mathbf{Z}G$ in $\mathbf{Q}G$. Suppose P is a projective $\mathbf{Z}G$ module, then tensoring P with (3.1) over $\mathbf{Z}G$ gives a pull-back diagram

$$\begin{array}{ccc} P & \longrightarrow & \mathcal{M} \otimes P \\ \downarrow & & \downarrow i \\ \hat{\mathbf{Z}}_2 G \otimes P & \xrightarrow{j} & \hat{\mathcal{M}}_2 \otimes P \end{array}$$

and by Swan's basic result, $\hat{\mathbf{Z}}_2 G \otimes P$, and $\hat{\mathcal{M}}_2 \otimes P$ are free. Next, suppose P admits a non-singular hermitian form b: $P \times P \to \mathbf{Z}G$. Then there are forms (matrices)

$$B: (\hat{\mathbf{Z}}_2 G \otimes P) \times (\hat{\mathbf{Z}}_2 G \otimes P) \to \hat{\mathbf{Z}}_2 G$$
$$j(B): (\hat{\mathcal{M}}_2 \otimes P) \times (\hat{\mathcal{M}}_2 \otimes P) \to \hat{\mathcal{M}}_2$$

associated to B, and if $\mathcal{M} \otimes P$ is also free (i.e., if $[P] \in D(G)$) there is also a matrix pairing

$$A: (\mathcal{M} \otimes P) \times (\mathcal{M} \otimes P) \to \mathcal{M}.$$

If P is now reconstructed via the diagram

(3.2)
$$\begin{array}{ccc} P & \longrightarrow & \mathcal{M} \otimes P \\ \Big\downarrow & & \downarrow i' \\ & & \hat{\mathcal{M}}_2 \otimes P \\ & & \downarrow C \\ \hat{\mathbf{Z}}_2 G \otimes P & \xrightarrow{j} & \hat{\mathcal{M}}_2 \otimes P \end{array}$$

where C is an isomorphism, and i' is the usual inclusion of free modules then we must have

(3.3)
$$C \cdot i'(A) \cdot C^* = j(B).$$

Conversely, if we are given

$$A_n: \mathcal{M}^n \times \mathcal{M}^n \to \mathcal{M}$$
$$B_n: (\hat{\mathbf{Z}}_2 G)^n \times (\hat{\mathbf{Z}}_2 G)^n \to \hat{\mathbf{Z}}_2 G$$

together with an isomorphism C_n so that (3.2) is satisfied, then on P defined via (3.2) using C_n, we will have a non-singular bilinear form. (The above remarks hold equally, of course, for quadratic forms.)

We shall denote the non-singular ε-quadratic form obtained in this way from A_n, B_n and C_n by

$$\theta = [P, A_n, B_n, C_n].$$

Remark 3.4. From (1.5) it follows that every projective P with a non-singular ε-quadratic form may be obtained in this way for a finite 2-group G.

From (3.3) it follows that C_n represents a class $[C_n]$ in $H^1(W_\ell(G))$.

Similarly, any ε-quadratic formation [17] over **Z**G can be reconstructed from suitable formations over $\mathscr{M}$ and $\hat{\mathbf{Z}}_2G$ using an equivalence over $\hat{\mathscr{M}}_2$. Again from (1.5) it follows that the formations over $\mathscr{M}$ and $\hat{\mathbf{Z}}_2G$ may be assumed to consist of a hyperbolic form on a free module with two free lagrangians. The equivalence of the two formations over $\hat{\mathscr{M}}_2$ yields two projective lagrangians P_1, P_2 for the formation θ over **Z**G constructed as pull-backs of free modules using matrices C_1, C_2 in $GL_n(\hat{\mathscr{M}}_2)$. By definition $[P_2]-[P_1^*]$ is the image of the class of θ in $L^p_{2r+1}(\mathbf{Z}G)$ under d_{2r+1} [17]. With these preliminaries, it follows that $C_n = C_2(C_1^*)$ represents a class $[C_n]$ in $H^0(W_\ell(G))$. From the naturality of these constructions it is easy to check that

$$d_{k+1}[\theta] = \varphi_*[C_n] \tag{3.5}$$

for all r and all $[\theta]$ in $L^p_{k+1}(\mathbf{Z}G)$. Actually the class $[C_n]$ in $H^k(W_\ell(G))$ also depends only on $[\theta]$ in certain circumstances. From (1.6) we have components of $[C_n]$ at the summands of **Q**G. Let $[C_n]_{\mathbf{Q}}$ denote the component at the trivial representation.

Lemma 3.6. *Let θ be an ε-quadratic form or formation representing an element of $L^p_{k+1}(\mathbf{Z}G)$.*

(1) *If $k+1=2r$ so $\theta=[P, A_n, B_n, C_n]$, and B_n is hyperbolic, then $[C_n]$ depends only on the class of θ in $L^p_{2r}(\mathbf{Z}G)$.*

(2) *Let $k+1=2r+1$ and θ be trivial over $\hat{\mathbf{Z}}_2G$. Then if r is odd, $[C_n]$ can be chosen so that $[C_n]_{\mathbf{Q}}=1$.*

If r is even, or r odd and $[C_n]_{\mathbf{Q}}=1$, $[C_n]$ depends only on the class of θ in $L^p_{2r+1}(\mathbf{Z}G)$.

Proof. (1) From the definition of $W_\ell(G)$ in (1.7), the indeterminacy in the choice of C_n which is not factored out is the group of stable automorphisms of hyperbolic forms over $\hat{\mathbf{Z}}_2G$. Since $L^h_{2r+1}(\hat{\mathbf{Z}}_2G)=0$, these are trivial in $H^*(W_\ell(G))$.

(2) The indeterminacy in the formation case can be described as the images of $(-\varepsilon)$-quadratic forms over $\hat{\mathbf{Z}}_2G$ under the map $L^h_{2r+2}(\hat{\mathbf{Z}}_2G)\to H^0(W_\ell(G))$. These are trivial if $\varepsilon=1$ and for $\varepsilon=-1$ give a class Δ in $H^0(W_\ell(G))$ which is non-trivial at each of the 1-dimensional representations. In fact this class Δ is given by the image of the generator of $L^h_0(\hat{\mathbf{Z}}_2G)=\mathbf{Z}/2$ which has determinant -5. Since $H^0(W_\ell(D_\alpha))=\mathbf{Z}/2$ if D_α is type **O** (see §4) it follows that $[C_n]$ can be chosen so that $[C_n]_{\mathbf{Q}}=1$ when $\varepsilon=-1$. Note that the unit -5 in $(\hat{\mathbf{Z}}_2G)^\times$ has non-trivial image at each 1-dimensional representation so this indeterminacy is factored out in the projection φ: $W_\ell(G)\to D(G)$.

To define d'_{k+1} using the above construction we note that the image of $L^p_0(\mathbf{Z}G)$, $\tilde{L}^p_2(\mathbf{Z}G)$ or $L^p_{2r+1}(\mathbf{Z}G)$ in the corresponding L^p-group of $\hat{\mathbf{Z}}_2G$ is trivial.

Definition 3.7. *Let $x\in L^p_{k+1}(\mathbf{Z}G)$ be represented by a form (formation) θ which is trivial over $\mathbf{Z}_2G$ (and if $k=2r+1$ with r odd the associated $[C_n]$ has $[C_n]_{\mathbf{Q}}=1$). Let*

$$d'_{k+1}(x)=[C_n] \quad \text{in } H^k(\mathbf{Z}/2; W_\ell(G)),$$

and set $d'_{k+1}(x)=0$ *when* x *generates the summand*

$$L_2(\mathbf{Z}) \quad \text{of } L^p_2(\mathbf{Z}G).$$

From (3.5), (3.6) and the remark just before (3.7) we see that d'_{k+1} is well-defined on all of $L^p_{k+1}(\mathbf{Z}G)$. Furthermore,

$$d_{k+1}=\varphi_* \cdot d'_{k+1} \tag{3.8}$$

which is the first part of Theorem B.

In order to check the second part of Theorem B we must refer again to the pull-back diagram (3.1). First we note that

$$L^p_*(\mathbf{Z}G)=\prod \Lambda_*(D_\alpha)$$

as claimed in Theorem A. In fact, $L^p_*(\mathbf{Z}G)$ is the internal direct sum of the subgroups $\Lambda_*(D_\alpha)$ defined by all possible pull-backs of forms (respectively formations) over $\mathcal{M}$ mixed with all automorphisms (respectively forms) over $\hat{\mathcal{M}}_2$. To these must be added only the subgroup $L^h_2(\mathbf{Z})$ denoted $\Lambda_2(\mathbf{Q})$ in Theorem A.

From this definition of the splitting of $L^p_*(\mathbf{Z}G)$ it follows that d'_{k+1} also splits:

$$d'_{k+1}=\prod_\alpha d'_{k+1}(\alpha),$$

where

$$d'_{k+1}(\alpha)\colon \Lambda_{k+1}(D_\alpha)\to H^k(\mathbf{Z}/2;\, W_\ell(D_\alpha)). \tag{3.9}$$

and Theorem B is proved.

§ 4. Calculation of $W_\ell(G)$

The main result of this paper is the calculation of d'_{k+1} and hence of $L^h_k(\mathbf{Z}G)$ in any specific case. In order to state the answer it is necessary to calculate the right-hand side of (3.9). For this we need information about the units in the centres of the factors of $\hat{\mathcal{M}}_2$ which are $\hat{\mathbf{Z}}_2(\zeta_n)$, $\hat{\mathbf{Z}}_2(\lambda_n)$ or $\hat{\mathbf{Z}}_2(\tau_n)$ where $\zeta=\zeta_n$ is a 2^n-root of 1, $\lambda_n=\zeta_n+\zeta_n^{-1}$ and $\tau_n=\zeta_n-\zeta_n^{-1}$. According to [14, Th. 5.7],

$$\hat{\mathbf{Z}}_2(\zeta_n)^\times \cong \mathbf{Z}/2^n \oplus (\hat{\mathbf{Z}}_2^+)^{2^{n-1}}, \tag{4.1}$$

$$\hat{\mathbf{Z}}_2(\lambda_n)^\times \cong \mathbf{Z}/2 \oplus (\hat{\mathbf{Z}}_2^+)^{2^{n-2}}, \tag{4.2}$$

$$\hat{\mathbf{Z}}_2(\tau_{n+1})^\times \cong \mathbf{Z}/2 \oplus (\hat{\mathbf{Z}}_2^+)^{2^{n-1}} \tag{4.3}$$

where $\hat{\mathbf{Z}}_2^+$ is the additive group of $\hat{\mathbf{Z}}_2$. Also $\hat{\mathbf{Z}}_2(\zeta_n)/\hat{\mathbf{Z}}_2(\lambda_n)$ and $\hat{\mathbf{Z}}_2(\tau_{n+1})/\hat{\mathbf{Z}}_2(\lambda_n)$ are quadratic extensions (with t denoting a generator for the Galois group $\mathbf{Z}/2$), and we must determine the structure of the units (4.1), (4.3) considered as modules over the Galois group. In each case the norm map is given by $x\mapsto x\cdot tx$ and $\hat{\mathbf{Z}}_2(\lambda_n)^\times/\text{norms}=\mathbf{Z}/2$ (combine [5, Cor., p. 19] with [5, Prop. 4, p. 136]). Since $\varepsilon_n=1+\lambda_n$ has norm -1 using the extension $\hat{\mathbf{Z}}_2(\lambda_n)/\hat{\mathbf{Z}}_2$, it is not a norm from $\hat{\mathbf{Z}}_2(\zeta_n)$ or $\hat{\mathbf{Z}}_2(\tau_{n+1})$. Hence there is a unit ν_n in $\hat{\mathbf{Z}}_2(\zeta_n)$ with $\nu_n\cdot t\nu_n=-1$, and similarly a unit μ_{n+1} in $\hat{\mathbf{Z}}_2(\tau_{n+1})$ with $\mu_{n+1}\cdot t\mu_{n+1}=-1$.

We denote the $\mathbf{Z}_2$ group-ring of $\mathbf{Z}/2$ by $\hat{\mathbf{Z}}_2[\mathbf{Z}/2]$.

Theorem 4.4. *Assume $n \geqq 3$. As a module over $\hat{\mathbf{Z}}_2[\mathbf{Z}/2]$ under the Galois action,*

$$\hat{\mathbf{Z}}_2(\zeta_n)^\times \cong \hat{\mathbf{Z}}_2^+ \oplus M \oplus (\hat{\mathbf{Z}}_2[\mathbf{Z}/2])^{2^{n-2}-1}$$

where $\hat{\mathbf{Z}}_2^+$ is generated by ε_n (fixed under t), $M = \hat{\mathbf{Z}}_2^+ \oplus \mathbf{Z}/2^n$ (with action $t(a,b) = (-a, -b+2^{n-1})$) is generated by v_n and ζ_n.

Theorem 4.5. *Assume $n \geqq 3$. As a module over $\hat{\mathbf{Z}}_2[\mathbf{Z}/2]$ under the Galois action,*

$$\hat{\mathbf{Z}}_2(\tau_{n+1})^\times \cong \hat{\mathbf{Z}}_2^+ \oplus N \oplus (\hat{\mathbf{Z}}_2[\mathbf{Z}/2])^{2^{n-2}-1}$$

where $\hat{\mathbf{Z}}_2^+$ is generated by ε_n and $N = \hat{\mathbf{Z}}_2^+ \oplus \mathbf{Z}/2$ (with action $t(a,b) = (-a, b+1)$) is generated by μ_{n+1} and -1.

Proof of (4.4). We will first find generators for $\hat{\mathbf{Z}}_2(\lambda_n)^\times$ over $\hat{\mathbf{Z}}_2$ and write $\lambda = \lambda_n$ for convenience. For this it is sufficient to find generators of $\hat{\mathbf{Z}}_2(\lambda)^\times$ modulo squares and by Hensel's lemma there is a surjection

$$\hat{\mathbf{Z}}_2(\lambda)^\times/(1+4J)^\times \to \hat{\mathbf{Z}}_2(\lambda)^\times/(\hat{\mathbf{Z}}_2(\lambda)^\times)^2,$$

where J is the maximal ideal generated by λ. Now $\hat{\mathbf{Z}}_2(\lambda)^\times = (1+J)^\times$, and so there is an exact sequence:

$$1 \to \frac{(1+2J)^\times}{(1+4J)^\times} \to \frac{(1+J)^\times}{(1+4J)^\times} \to \frac{(1+J)^\times}{(1+2J)^\times} \to 1.$$

Now the map $2a \mapsto 1+2a\lambda$ induces an isomorphism:

$$\left(\frac{2\hat{\mathbf{Z}}_2(\lambda)}{4\hat{\mathbf{Z}}_2(\lambda)}\right)^+ \to \frac{(1+2J)^\times}{(1+4J)^\times},$$

and this subgroup has generators

$$1+2\lambda,\ 1+2\lambda^2,\ 1+2\lambda^3, \ldots, 1+2\lambda^{2^{n-2}-1},\ 5.$$

Here we have used the relations $(\lambda_3)^2 = 2$, $(\lambda_n)^{2^{n-2}} \equiv 6 \pmod{4J}$ for $n \geqq 4$ which imply $8 \equiv 0 \pmod{4J}$.

For the quotient group we have the isomorphism

$$(1+I)^\times \to \frac{(1+J)^\times}{(1+2J)^\times}$$

where I is the ideal generated by θ in the truncated polynomial algebra $\mathbf{F}_2(\theta)/\{\theta^{2^{n-2}+1} = 0\}$ and the map is defined by:

$$1 + a_1\theta + a_2\theta^2 + \ldots \mapsto 1 + a_1\lambda + a_2\lambda^2 + \ldots$$

($a_i = 0$ or 1 for all $i \geqq 1$).

From this isomorphism it is clear that the elements

$$1+\lambda,\ 1+\lambda^3,\ 1+\lambda^5, \ldots, 1+\lambda^{2^{n-2}-1}$$

project to generators of the quotient group.

Lemma 4.6. *The elements* 5, $1+\lambda^{2k-1}$ *for* $1\leqq k\leqq 2^{n-3}$ *and* $1-2\lambda^{2\ell-1}$ *for* $1\leqq \ell\leqq 2^{n-3}-1$ *generate* $\mathbf{Z}_2(\lambda)^\times/(1+4J)^\times$.

Proof. We leave the case $n=3$ to the reader and assume $n\geqq 4$. Let $m=1+2a+2^{n-3}$ for $0\leqq a<2^{n-4}$ and calculate (mod $4J$):

$$(1+\lambda^m)^2=1+2\lambda^m+2\lambda^{2+4a}=(1+2\lambda^m)(1+2\lambda^{2+4a}).$$

This relation shows how the elements $1+2\lambda^{2+4a}$ are derived from the indicated generators. Similarly, let $m=2+4a+2^{n-3}$ where $0\leqq a<2^{n-5}$ and expand $(1+\lambda^m)^2$ to get the elements $1+2\lambda^{4+8a}$ and so on. This accounts for all units of the form $1+2\lambda^{2k}$ except $1+2\lambda^{2^{n-2}}=5$ (mod $4J$). The remaining units needed are $1+2\lambda^{2^{n-2}-1}$ and -1. These are obtained from the relations:

$$(1+\lambda)^{2^{n-2}}=(-1)(1+2\lambda^{2^{n-3}})$$

and

$$(1+2\lambda^{2^{n-2}-1})(1+2\lambda^{2^{n-2}-2})=(1+\lambda^{2^{n-2}-1})^2.$$

Lemma 4.7. *The elements* 5, $1+\lambda$ *and its Galois conjugates generate* $\hat{\mathbf{Z}}_2(\lambda)^\times$.

Proof. Let σ generate $\mathrm{Gal}(\hat{\mathbf{Q}}_2(\lambda)/\hat{\mathbf{Q}}_2)=\mathbf{Z}/2^{n-2}$.

Then $(1+\lambda)^\sigma=1+\zeta^3+\zeta^{-3}=1+\lambda+\lambda^3$ (mod $4J$). It is sufficient to find elements γ_k spanned by 5 and the conjugates of $1+\lambda$ such that

$$\gamma_k=(1+\lambda^{3^k}) \bmod (1+J^{k+1}) \qquad \text{for } k\geqq 0.$$

These are provided by defining

$$\gamma_1=(1+\lambda)^{-1}(1+\lambda)^\sigma, \qquad \gamma_2=\gamma_1^{-1}\gamma_1^\sigma,\dots,\gamma_{k+1}=\gamma_k^{-1}\cdot\gamma_k^\sigma,\dots.$$

We can now complete the proof of (4.4). From (4.7) and the fact that $5=\mathrm{norm}\,(1+2i)$, we see that there are global units $u_1,\dots,u_{2^{n-2}-2}$ which are norms of units from $\hat{\mathbf{Z}}_2(\zeta)^\times$ and together with -1 and $\varepsilon=1+\lambda$ generate $\hat{\mathbf{Z}}_2(\lambda)^\times$. Now let $u_j=w_j\cdot tw_j$ for some $w_j\in\hat{\mathbf{Z}}_2(\zeta)^\times$. It is easy to check that $\varepsilon_n,\zeta_n,v_n,w_1,w_2,\dots,w_{2^{n-2}-2}$ and $1+2i$ generate $\hat{\mathbf{Z}}_2(\zeta)^\times$ as a $\hat{\mathbf{Z}}_2[\mathbf{Z}/2]$ module.

The argument for (4.5) is very similar and will be left to the reader. In particular, there are elements w_j with norm a global unit in this case also. However, $5\varepsilon_n$ is a norm from $\hat{\mathbf{Z}}_2(\tau_{n+1})$ and generates the cokernel of $\hat{\mathbf{Z}}_2(\lambda_n)^\times$ upon factoring out global units (in fact, norm $(\mu_{n+1}(1+\tau_{n+1}))=5\varepsilon_n$).

The cases not covered in the above are $\hat{\mathbf{Z}}_2(i)^\times=\mathbf{Z}/4\oplus\hat{\mathbf{Z}}_2[\mathbf{Z}/2]$ with generators i, $1+2i$ and $\hat{\mathbf{Z}}_2(\sqrt{-2})^\times=\mathbf{Z}/2\oplus\hat{\mathbf{Z}}_2[\mathbf{Z}/2]$ with generators $-1, 1+\sqrt{-2}$.

Lemma 4.8. (a) *For* $n\geqq 3$, $W_\ell(\hat{\mathbf{Z}}_2(\zeta_n))=(\mathbf{Z}/2^\ell)^{2^{n-2}-1}\oplus\mathbf{Z}/2^\ell[\mathbf{Z}/2]$ *generated by* $v_n, w_1,\dots,w_{2^{n-2}-2}$ *and* $1+2i$;

(b) $W_\ell(\hat{\mathbf{Z}}_2(i))=\mathbf{Z}/2^\ell[\mathbf{Z}/2]$ *generated by* $1+2i$;

(c) *for* $n\geqq 3$, $W_\ell(\hat{\mathbf{Z}}_2(\tau_{n+1}))=(\mathbf{Z}/2^\ell)^{2^{n-2}-1}\oplus\mathbf{Z}/2^\ell[\mathbf{Z}/2]$ *generated by* $\mu_{n+1}, w_1,\dots,w_{2^{n-2}-2}$ *and* $\mu_{n+1}(1+\tau_{n+1})$;

(d) $W_\ell(\hat{\mathbf{Z}}_2(\sqrt{-2}))=\mathbf{Z}/2^\ell[\mathbf{Z}/2]$ *generated by* $1+\sqrt{-2}$.

Proof. This is immediate from the previous discussion. Note that on the $\mathbf{Z}/2^\ell$ factors in (a) and (c), the involution acts as -1.

Using this result we can calculate $H^*(W_\ell(D_\alpha))$ for D_α of type U.

Lemma 4.9. (a) *If* D_α *has type* $\mathbf{O}$, $W_\ell(D_\alpha)=\mathbf{Z}/2^\ell$ *generated by* 5.

(b) *If* D_α *has type Sp with centre* $\mathbf{Q}(\lambda_n)$,

$$W_\ell(D_\alpha)=(\mathbf{Z}/2)^{2^{n-2}+1}$$

generated by 5, ε_n, $u_1,\dots,u_{2^{n-2}-2}$ *and* -1 (*notation from before* (4.8)).

(c) *In each case,* $W_\ell(D_\alpha)$ *has trivial involution.*

Proof. (a) This follows from Lemma 4.7.

(b) Since ε_n has norm -1 (see [4, 4.5]) it is negative at an odd number of real places of $\mathbf{Q}(\lambda_n)$. Therefore ε_n and its Galois conjugates are not factored out in $W_\ell(D_\alpha)$ as D_α is ramified at all real places (cf. [18]).

(c) Since $\mathbf{Q}(\lambda_n)$ is totally real, the involution on $W_\ell(D_\alpha)$ induced by "conjugate transpose" is trivial.

§ 5. Calculation of $d'_{k+1}(\alpha)$

The maps $d'_{k+1}(\alpha)$ are given in Lemmas (5.1)–(5.7).

In this section we will use the notation (and results) of §§ 3, 4. Let $\theta=[P,A_m,B_m,C_m]$ represent an element of $L^p_0(\mathbf{Z}G)$ where B_n is hyperbolic.

Lemma 5.1. *Let* A_m *be hyperbolic except at a single representation of type* $\mathbf{O}$, *then* $d'_0[P,A_m,B_m,C_m]=1$.

Proof. The assumption means that θ is in $A_0(D_\alpha)$ where D_α has type $\mathbf{O}$ (see § 3). From (3.6), (3.7) $d'_0[\theta]=[C_m]$ in $H^1(\mathbf{Z}/2;\, W_\ell(D_\alpha))$. However (3.3) gives a relation of the form:

$$-1=(\det C_m)^2\, b$$

where $b\in\mathbf{Z}(\lambda_n)^\times$. This equation in $\hat{\mathbf{Z}}_2(\lambda_n)^\times$ implies that $\det C_m$ is a global unit (see (4.2)) and so $[C_m]=1$.

Lemma 5.2. *Let* A_m *be hyperbolic except at a type Sp summand* $M_k(\Gamma(F))$ *where* $\Gamma(F)=\left(\frac{-1,-1}{F}\right)$ *and* $F\neq\mathbf{Q}$, *then*

(a) *the class of* A_m *in the Witt ring is determined by its signatures at the real places of F,*

(b) *if* A_m *has signature* 0 *except at the i-th place where it has signature* -2, *then*

$$d'[P,A_m,B_m,C_m]=[\theta_i^{-1}]$$

where θ_i *is any unit of F negative at the i-th place and positive at the remainder.*

Proof. The first part follows from [13, p. 119] and (4.9). For the second part we first recall that the maximal order $\mathcal{M}$ can be chosen to have the form $M_k(\mathcal{N})\oplus\mathcal{M}_0$ where $\mathcal{N}$ is a maximal order in $\Gamma(F)$ invariant under the standard involution. This follows from work of Scharlau on the structure of involution-invariant orders. If $F=\mathbf{Q}(\zeta+\zeta^{-1})\neq\mathbf{Q}$ then

$$\Gamma(F)\otimes\hat{\mathbf{Z}}_2\cong M_2(\hat{\mathbf{Q}}_2(\zeta+\zeta^{-1}))$$

and $\hat{\mathcal{N}}_2=\mathcal{N}\otimes\hat{\mathbf{Z}}_2$ is conjugate to $M_2(\hat{\mathbf{Z}}_2(\zeta+\zeta^{-1}))$ where the involution induced on the matrix ring is

$$\tau\colon X\mapsto\begin{pmatrix}0&1\\-1&0\end{pmatrix}X^t\begin{pmatrix}0&-1\\1&0\end{pmatrix}.$$

It follows that $1=e+\bar{e}$ for some e in $\mathcal{N}$ so all elements of the centre are even. (Equivalently, from Kolster [11] we know that $WQ_0^{-1}(\mathcal{N},\min)=WQ_0^{-1}(\mathcal{N},\max)$ for such orders). Now define the form A_2 to be $\begin{pmatrix}\theta_i&0\\0&-1\end{pmatrix}$ at $\mathcal{N}$ and hyperbolic at the other representations. Then under the isomorphism above

$$\begin{pmatrix}\theta_i&0\\0&-1\end{pmatrix}\mapsto\begin{pmatrix}\theta_i&&&\\&\theta_i&&\\&&-1&\\&&&-1\end{pmatrix}.$$

Hermitian forms over $(M_2(\hat{\mathbf{Z}}_2(\zeta+\zeta^{-1})),\tau)$ are Morita equivalent to skew-symmetric forms over $(\hat{\mathbf{Z}}_2(\zeta+\zeta^{-1}),\,id)$ so after a short calculation we get the Morita equivalent form

$$\begin{pmatrix}&\theta_i&&\\-\theta_i&&&\\&&&-1\\&&1&\end{pmatrix}$$

over $(\hat{\mathbf{Z}}_2(\zeta+\zeta^{-1}),\,id)$. This is equivalent to the hyperbolic form using the matrix

$$C_2=\begin{pmatrix}\theta_i^{-1}&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}$$

hence $d_0'[P,A_2,B_2,C_2]=[\theta_i^{-1}]$.

The situation at the ordinary quaternion algebra $\Gamma(\mathbf{Q})$ is slightly different.

Lemma 5.3. *Let A_m be hyperbolic except at $M_k(\Gamma(\mathbf{Q}))$, then A_m has signature $2r$ and*

$$d_0'[P,A,B,C]=(-1)^r.$$

Proof. Again the class of A_m is determined by its signature, so we may assume $m=2$ and $A_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. Now let

$$v=i+j+k+2+4(i+1)+\dots$$

and note $v\bar{v}=-1$. Then set $e=\frac{1}{2}(1+i+j+k)$,

$$C_2=\begin{pmatrix}1 & 1\\ -\bar{e} & e\end{pmatrix}\begin{pmatrix}v & 0\\ 0 & 1\end{pmatrix}$$

and get

$$C_2\cdot i(A_2)\cdot C_2^*=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

It is easy to check that $[C_2]=-1$.

These results give d_0' for type **O** and *Sp*.

Lemma 5.4. *Let A_m be hyperbolic except at a single representation D_α of type U with centre $\mathbf{Q}(\zeta_n)$ or $\mathbf{Q}(\tau_{n+1})$, then the image of $d_{2r}'(\alpha)$ is generated by the elements $[w_j]$ (cf. Lemma 4.8).*

Proof. From results of Weber [8], the subgroup of $\mathbf{Z}(\lambda_n)^\times$ spanned by ε_n and its conjugates contains units with arbitrarily prescribed signs at the real places. Also from (1.4) and [13], $\Lambda_{2r}(D_\alpha)$ is a subgroup of $L^p_{2r}(\mathbf{Z}(\zeta_n))$ which is contained in $L^p_{2r}(\mathbf{Q}(\zeta_n))$. Hence $\Lambda_0(D_\alpha)$ is detected by signatures and by (4.7) the forms $\langle u_j\rangle\perp\langle -1\rangle$, $\langle -1\rangle\perp\langle -1\rangle$ and $2(\langle\varepsilon_n\rangle\perp\langle -1\rangle)$ are equivalent over $\mathbf{Q}(\zeta_n)$ to generators. To see that these rational lattices contain integral lattices we use the criterion of [21, Prop. 6]; the units u_j are those used in (4.8). To calculate $d_0'(\alpha)$ we can work over $\hat{\mathbf{Q}}_2(\zeta_n)$ and find the images of these generators in $H^1(W_\ell(D_\alpha))$:

$$\begin{pmatrix}1 & \bar{w}_j^{-1}\\ -\frac{1}{2} & \frac{1}{2}\bar{w}_j^{-1}\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & u_j\end{pmatrix}\begin{pmatrix}1 & -\frac{1}{2}\\ w_j^{-1} & \frac{1}{2}w_j^{-1}\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

where $u_j=w_j\cdot tw_j$ and $\bar{w}_j=tw_j$. Therefore if

$$A_2=\begin{pmatrix}-1 & \\ & u_j\end{pmatrix}\quad\text{and}\quad C_2=\begin{pmatrix}1 & \bar{w}_j^{-1}\\ -\frac{1}{2} & \frac{1}{2}\bar{w}_j^{-1}\end{pmatrix}$$

we get $d_0'[P,A_2,B_2,C_2]=[\det C_2]=[\bar{w}_j]$. The same result for d_2' follows from scaling the generators by i to get generators for the appropriate summand of $L^X_2(\mathcal{M})$. From this result and (4.8) we see that each summand of type U contributes at most two $\mathbf{Z}/2$'s to $L^h_{2r-1}(\mathbf{Z}G)$ (see §7 for an example of this calculation).

Lemma 5.5. *The cokernel of $d_2'(\alpha)$ for a summand of type Sp with centre $\mathbf{Q}(\lambda_n)$ is generated by $[5]$ and $[\varepsilon_n]$ for $n\geqq 3$.*

Proof. Let D denote a summand of type Sp with centre $\mathbf{Q}(\lambda_n)$ and $\mathcal{N}\subset D$ a maximal order. By Morita theory we can assume that D is a quaternion algebra (see (1.2) d) and $\mathcal{N}$ a maximal order inside it (5.2). Assume $n\geqq 3$, then from the Rothenberg sequence $L^h_3(\hat{\mathcal{N}}_2)=\mathbf{Z}/2$ and $L^h_3(\mathcal{N})=0$ from [15, 8.16]; while $L^h_2(\mathcal{N})=\mathbf{Z}/2)^{2^{n-2}-2}$ maps trivially to $L^h_2(\hat{\mathcal{N}}_2)$ from [15, 8.10] and the appendix by Springer to [10]. In addition, Springer's description implies that $L^h_2(\mathcal{N})$ is generated by forms $\langle\delta j\rangle \perp \langle j\rangle$ where the unit δ in $\mathbf{Z}(\lambda_n)^\times$ is a norm from $\hat{\mathbf{Z}}_2(\zeta_n)^\times$ but $\pm\delta$ is not a norm at any real place. Therefore we can take $\delta = u_1, u_2, \ldots, u_{2^{n-2}-2}$ in the notation of § 4.

Now let

$$J(D) = \operatorname{Im}(K_1'(\mathcal{N}) \oplus I_\ell' \to K_1'(\hat{\mathcal{N}}_2))$$

as in (1.7) so that

$$0 \to J(D) \to K_1'(\hat{\mathcal{N}}_2) \to W_\ell(D) \to 0$$

is an exact sequence of $\mathbf{Z}[\mathbf{Z}/2]$-modules. The remarks above give an exact sequence

$$0 \to \mathbf{Z}/2 \to \Lambda_2(D) \to (\mathbf{Z}/2)^{2^{n-2}-2} \to 0.$$

Since the generators of $L_3^h(\hat{\mathcal{N}}_2)$ after Morita equivalence are represented by matrices $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, the discriminant map

$$L_3^h(\hat{\mathcal{N}}_2) \to H^1(K_1'(\hat{\mathcal{N}}_2)) = \mathbf{Z}/2$$

is onto (the right-hand side is generated by $[-1]$). Using the pull-back square of § 3, we find the commutative diagram (here $J = \prod_\alpha J(D_\alpha)$):

$$\begin{array}{ccccc}
L_3^h(\hat{\mathcal{M}}_2) & \longrightarrow & L_2^p(\mathbf{Z}G) & \longrightarrow & L_2^h(\hat{\mathbf{Z}}_2 G) \oplus L_2^h(\mathcal{M}) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(K_1'(\hat{\mathcal{M}}_2)) & \longrightarrow & H^1(W_\ell(G)) & \longrightarrow & H^0(J)
\end{array}$$

where the lower row contains the exact sequence

$$0 \to H^1(K_1'(\hat{\mathcal{N}}_2)) \to H^1(W_\ell(D)) \to H^0(J(D)) \to 0.$$

The generators $\langle\delta j\rangle \perp \langle j\rangle$ map to $[\delta]$ in $H^0(J(D))$ by checking as in (5.2) and hence all generators of $H^1(W_\ell(D))$ are hit except $[5]$ and $[\varepsilon_n]$.

Lemma 5.6. *For a summand D_α of type Sp, $d_1'(\alpha)$ is an isomorphism.*

Proof. As above we consider the map

$$d_1'(\alpha)\colon \Lambda_1(D) \to H^0(\mathbf{Z}/2; W_\ell(D))$$

using the pull-back square (3.1) and let $\mathcal{N} \subset D$ be the maximal order. An element x of $\Lambda_1(D)$ is represented by a formation θ over $\mathcal{N}$ together with a skew-symmetric form ψ over $\hat{\mathcal{N}}_2$ with boundary $i_*\theta$ (see [17] for definitions). Then d_1' sends x to the class represented by the discriminant of ψ. To see that d_1' is in fact an isomorphism in this case we note that the indeterminacy of this construction lies in $L_2^h(D)$ so that $\Lambda_1(D)$ may be identified with

$$\operatorname{cokernel}(L_2^h(D) \to L_2^h(\hat{D}_2)).$$

Now

$$L_2^h(\hat{D}_2) \cong IW(\hat{\mathbf{Q}}_2(\lambda_n))$$

where the right-hand side denotes the kernel of the rank homomorphism on the Witt ring [13, p. 66]. From [13, pp. 76, 81] there is a non-split exact sequence (for $n \geqq 3$)

$$0 \to \mathbf{Z}/2 \to IW(\hat{\mathbf{Q}}_2(\lambda_n)) \to \hat{\mathbf{Q}}_2(\lambda_n)^{\cdot}/(\text{squares}) \to 0$$

and a generator of the $\mathbf{Z}/4$ can be taken to be $\langle -1\rangle \perp \langle \varepsilon_n(2-\lambda_n)\rangle$. In this case, both $\langle -1\rangle \perp \langle 2-\lambda_n\rangle$ and $2(\langle -1\rangle \perp \langle \varepsilon_n(2-\lambda_n)\rangle)$ are in the image of $L_2^h(D)$ so that

$$\Lambda_1(D) \cong \hat{\mathbf{Z}}_2(\lambda_n)^{\times}/(\text{squares})$$

by the discriminant map and the lemma follows. If $n=2$, the element $\langle i+j\rangle \perp \langle i\rangle$ comes from $L_2^h(D)$ and $L_2^h(\hat{D}_2) \cong \hat{\mathbf{Q}}_2^{\cdot}/(\text{squares})$ by the discriminant. Hence

$$\Lambda_1(D) = \hat{\mathbf{Z}}_2^{\times}/(\text{squares})$$

as before.

Lemma 5.7. *The image of $d_3'(\alpha)$ for a summand D_α of type* **O** *is $\mathbf{Z}/2$ generated by* [5] *except when $D_\alpha = \mathbf{Q}$, the trivial representation.*

Proof. This follows from (4.9)a and a similar commutative diagram to that used in (5.5). The generator of $L_0^h(\hat{\mathcal{N}}_2)$ has discriminant 5 and

$$\mathbf{Z}/2 = L_0^h(\hat{\mathcal{N}}_2) \to \Lambda_3(D)$$

is an isomorphism when $D \neq \mathbf{Q}$.

These results (5.1)–(5.7) give a complete calculation of d' and lead directly to answers for L^h in many cases. Consider the situation of Theorem C where G is a finite 2-group with only type **O** representations. Then $d_0 = 0$ from (5.1), $d_2 = 0$ since $L_2^p(\mathbf{Z}G) = \mathbf{Z}/2$ (Arf invariant), $d_1 = 0$ since $L_1^p(\mathbf{Z}G) = 0$ and d_3' is onto from (5.7). By (4.9)a, $W_\ell(G)$ has trivial involution so $D(G)$ does also and d_3 is onto.

§ 6. Proof of Theorem D

Let $G = Q2^n$ in the notation of (1.1)a and observe that

$$\mathbf{Q}G = \left(\frac{-1,\,-1}{\mathbf{Q}(\lambda_{n-1})}\right) \oplus M_2(\mathbf{Q}(\lambda_{n-2})) \oplus \ldots \oplus M_2(\mathbf{Q}) \oplus \mathbf{Q}^4.$$

From Theorem A:

$$L_0^p = (\mathbf{Z})^{2^{n-2}+3}; \quad L_1^p = (\mathbf{Z}/2)^{2^{n-3}+1};$$

$$L_2^p = \mathbf{Z}/2 \quad \text{if } n=3 \quad \text{and} \quad L_2^p = \mathbf{Z}/2 \oplus (\mathbf{Z}/2)^{2^{n-3}-1} \quad \text{if } n \geqq 4;$$

$$L_3^p = (\mathbf{Z}/2)^n.$$

Recall from [7], [18, p. 33] that $D(G)=\mathbf{Z}/2$ generated by $(1,1,\ldots,1,5)$ in $W_\ell(G)$. As in (1.7) we will denote elements of $W_\ell(G)$ by an array $(*,*,\ldots,*)$ of 2-adic units corresponding to the summands of $\mathbf{Q}G$ and the cohomology class in $H^*(W_\ell(G))$ by $[*,*,\ldots,*]$. The notation is chosen sothat the left-hand entry corresponds to the quaternion algebra and the last four entries to the one-dimensional representations sending (x,y) to $(-1,-1)$, $(-1,1)$ $(1,-1)$ and $(1,1)$ respectively.

Lemma 6.1. *For* $G=Q8$,

$$d_{k+1}\colon L^p_{k+1}(\mathbf{Z}G)\to H^k(\mathbf{Z}/2;\tilde{K}_0(\mathbf{Z}G))$$

is onto for $k\not\equiv 1(4)$ *and zero for* $k\equiv 1(4)$.

Proof. Since $W_\ell(G)$ has trivial involution, and d'_1, d'_3 hit the generator of $D(G)$, (by (5.6) and (5.7)) both d_1 and d_3 are onto. Since $L^p_2(\mathbf{Z}Q8)=\mathbf{Z}/2$ detected by the Arf invariant, $d_2=0$. From (5.3) the element $[-1,1,1,1,1]$ is in the image of d_0 so it is enough to check that $(-1,1,1,1,1)$ also generates $D(G)$. In fact the 2-adic unit

$$\begin{aligned}&(1+x+y)(1+2x)(1+2xy+2y)^{-1}\\ &\mapsto(3,1,1,1,3)(-3,1,1,3,3)(1,1,1,3,3)=(-1,1,1,1,3),\end{aligned}$$

so $(-1,1,1,1,1)$ also generates $\tilde{K}_0(\mathbf{Z}Q8)$. In this calculation we have factored out squares (since the answer is in $\mathbf{Z}/2$) and multiples of 8. Note that in every place except the quaternion representation we can factor out -1.

Lemma 6.2. *For* $G=Q2^n(n>3)$, *the non-trivial element of* $D(G)$ *is represented by* ε_{n-1} *at the quaternion algebra and ones at the remaining representatons.*

Proof. The first part of the argument will also give another proof that $D(G)$ is generated by $(1,1,\ldots,1,3)$ so the Swan homomorphism is onto. From our unit calculations in §4 it is easy to see that $W_\ell(\hat{\mathbf{Z}}_2(\lambda_j))$ is generated by 5 and $W_\ell(\Gamma(\mathbf{Z}_2(\lambda_{n-1}))$ is generated by the units 5, ε_{n-1} and its Galois conjugates (4.7).

The first step is to check that the four elements with 3 at a single one-dimensional representation and 1 otherwise are all equivalent.

This can be seen by comparing successive quotients of the units:

$$\begin{aligned}1+2x&\mapsto(-3,3,\ldots,3,1,1,3,3)\\ 1+2y&\mapsto(-3,3,\ldots,3,3,1,1,3)\\ 1+2xy&\mapsto(-3,3,\ldots,3,1,3,1,3)\\ 3&\mapsto(1,1,\ldots,1,3,3,3,3).\end{aligned}$$

Now the sequence of units

$$1+2x^{2^{n-2-j}}\mapsto(-3,3,\ldots,3,1,1,\ldots,1)$$

for $1\leqq j\leqq n-2$ (the first 1 occurs in the $(j+1)$-th place from the left) shows that the element with -3 at the quaternion place and ones at the remainder and all the elements with 3 at a single matrix representation and ones at the remainder are trivial in $D(Q2)^n$. Since $1+x^{2^{n-3}}+y \mapsto (3,1,\dots,1)$, it also follows that $(-1,1,\dots,1)$ is trivial. Finally note:

$$(6.3)\qquad 1+x^s+y \mapsto \begin{cases} (3+\lambda(s),1,\dots,1,3), & s \text{ odd} \\ (3+\lambda(s),1,\dots,1), & s \text{ even} \end{cases}$$

where $\lambda(s)=\zeta_{n-1}^s+\zeta_{n-1}^{-s}$. These relations show $(1,1,\dots,1,3)$ generates $D(Q2^n)$.

To complete the proof we must check that $(1+\lambda_{n-1},1,\dots,1)$ is equivalent to the generator. As in §4, we can factor out $4\lambda_{n-1}$ (since squares are trivial) and get:

$$(1+\lambda(s))=(1+\lambda(s))^3=(1+3\lambda(s))(1+3\lambda(s)^2).$$

But $\lambda(s)^2=\lambda(2s)+2$ gives

$$1+3\lambda(s)^2=-1+3\lambda(2s)=(-1)(1+\lambda(2s)).$$

Therefore

$$1+3\lambda(s)=(-1)(1+\lambda(s))(1+\lambda(2s)),$$

and we obtain

$$(6.4)\qquad (-1)^{n-2}(1+3\lambda(s))(1+3\lambda(2s))\dots(1+3\lambda(2^{n-2}s))=3(1+\lambda(s)).$$

The result follows now by combining (6.3) with (6.4) for $s=1$.

This result proves that d_0 is onto for $G=Q2^n\,(n>3)$ as well and (5.6), (5.7) again imply that d_1, d_3 are onto. For d_2 we can use (5.5) or use naturality since the restriction map

$$D(Q2^n)\to D(Q8)$$

is an isomorphism [7]. Theorem D now follows.

We see that L_1^h is a split extension in this case by comparing the Rothenberg sequences for $Q8$ and $Q2^n$ $(n\geqq 4)$ by the restriction map and noting that on the type Sp summands $\Lambda_1(D_\alpha)$ the restriction map is just the norm homomorphism $\hat{\mathbf{Z}}_2(\lambda_{n-1})^\times \to \mathbf{Z}_2^\times$ (mod squares) by (5.6). From (6.2) the restriction map is zero hence the sequence splits.

§7. Proof of Theorem E

When $G=SD2^n\,(n\geqq 4)$,

$$\mathbf{Q}G\cong M_2(\mathbf{Q}(\tau_{n-1}))\oplus M_2(\mathbf{Q}(\lambda_{n-2}))\oplus\dots\oplus M_2(\mathbf{Q})\oplus\mathbf{Q}^4$$

so that

$$L_0^p=(\mathbf{Z})^{2^{n-3}+2^{n-4}+3};\qquad L_1^p=0;\qquad L_2^p=\mathbf{Z}/2\oplus\mathbf{Z}^{2^{n-4}}$$

and $L_3^p=(\mathbf{Z}/2)^n$.

In §2 we mentioned the fact [19] that $D(SD2^n)=\mathbf{Z}/2$, detected by restriction to the subgroup $Q8$ of G. From (6.1) and naturality this gives $d_2=0$. Also (5.7) implies that d_3 is onto since $D(SD2^n)$ is equal to the image of the Swan homomorphism, so has generator $[1,1,\dots,1,5]$. From (5.1), d'_0 is zero on the type $\mathbf{O}$ summands and $d_0=d_2$ is zero on the type U summands. Therefore $d_0=0$ and the values of $L^h_*(\mathbf{Z}G)$ are determined.

As a further example of these calculations we give another proof of Bak's result on $L^h_*(\mathbf{Z}G)$ when $G=\mathbf{Z}/2^n$. (The groups are listed in § 2.) Since

$$\mathbf{Q}G=\mathbf{Q}(\zeta_n)\oplus\mathbf{Q}(\zeta_{n-1})\oplus\dots\oplus\mathbf{Q}(\zeta_2)\oplus\mathbf{Q}^2$$

then

$$L^p_0=(\mathbf{Z})^{2^{n-1}+1},\quad L^p_1=0,\quad L^p_2=\mathbf{Z}/2\oplus(\mathbf{Z})^{2^{n-1}-1},\quad L^p_3=\mathbf{Z}/2.$$

Here the difficulty is that $D(G)$ is quite complicated [9]. However $d_1=0$ clearly and $d_3=0$ also since the element $[1,1,\dots,1,5]=0$ in $D(G)$ (i.e., the Swan subgroup is trivial [19]). From (5.4), the cokernel of d'_0 or d'_2 on $H^1(W_\ell(G))$ is generated by the 2-adic units $v_n, v_{n-1},\dots,v_3$ at the type U summands (except $\mathbf{Q}(\zeta_2)$ from (4.8)). By definition, $v_k\cdot\bar v_k=-1$ so in the cohomology group we can use instead of v_k, any unit with norm -1 (moduluo $4\lambda_k$). Let $v'_k=1+2i+\zeta_3-\zeta_3^{-1}$ for $3\leqq k\leqq n$ and note that $v'_k\cdot\bar v'_k=(1+2i+\zeta_3-\zeta_3^{-1})(1-2i+\zeta_3^{-1}-\zeta_3)\equiv -1$. These elements however are trivial in $H^1(D(G))$. Consider the units (from $\hat{\mathbf{Z}}_2(G)$) $\alpha_k=1+2x^{2^k}+x^{2^{k-1}}-x^{-2^{k-1}}$, $1\leqq k\leqq n-2$ and note that

$$\begin{aligned}
\alpha_{n-2}&\mapsto(v'_n,3,3,\dots,3,3)\\
\alpha_{n-3}&\mapsto(1,v'_{n-1},3,3,\dots,3,3)\\
&\vdots\\
\alpha_2&\mapsto(1,\dots,1,v'_4,3,3,3,3)\\
\alpha_1&\mapsto(1,\dots,1,1,v'_3,3,3,3)
\end{aligned}$$

modulo the image of d'_0 or d'_2 and the images of $1+x^{2^k}-x^{-2^k}$ for $k\geqq 0$.

The sequence of 2-adic units $1+2x^{2^k}$ for $0\leqq k\leqq n-1$ shows that the elements with 3 at one place and trivial elsewhere are all equivalent to $[1,1,\dots,1,3]=1$ in $D(G)$. This proves that d_0, d_2 are onto and finishes the calculation of $L^h_*(\mathbf{Z}G)$.

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