PROJECTIVE SURGERY OBSTRUCTIONS ON CLOSED MANIFOLDS

by

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Let π be a finite group and $f\colon M^n \to N^n$ a surgery problem of closed topological n-manifolds ($n \geq 5$) with $\pi_1 N = \pi$ and $w_1 N = w$. A basic question is: what elements of $L_h^h(\pi, w)$ are the surgery obstructions of such problems? If $C_n^h(\pi, w)$ denotes the subgroup of $L_n^h(\pi, w)$ generated by these surgery obstructions $\sigma(f)$, we can ask for (i) a calculation of C_n^h, (ii) specific invariants of $f\colon M^n \to N^n$ which detect $\sigma(f)$ and (iii) specific examples of surgery problems with arbitrary obstruction in C_n^h.

Wall proved in [W2] that $\sigma(f)$ is detected by restriction to the 2-Sylow subgroup of π so it is natural to assume that π is a 2-group. Furthermore the calculation of $L_n^h(\pi, w)$ is still complicated because of K_0 or K_1 difficulties (see [W3] and [HM] for more details). In this paper we answer the analogous questions (i) - (iii) about the image $\bar{C}_n^h(\pi, w)$ of C_n^h in $L_n^p(\pi, w)$. These groups are the geometric surgery obstruction groups of Maumary [M] or Taylor [T]; algebraically they are L-groups of quadratic forms on projective (instead of free) $Z\pi$ modules [R1]. The appropriate version of (ii) is then to ask for invariants detecting $\sigma(f \times id)$ where $f \times id\colon M \times S^1 \to N \times S^1$ and the answer to (i) is now possible because the groups L^p are easier to calculate than L^h. We give in Section 3 a calculation of $L_n^p(\pi, w)$ for π a finite 2-group with arbitrary orientation character along the

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lines of [HM, ThmA] and define invariants which detect the elements not in $\bar{C}^h_n(\pi,w)$.

It has been known [W1, p 176] for some time that part (ii) can be attacked by factoring σ: $[N,G/Top] \to L^h_n(\pi,w)$ through $\Omega_n(B\pi \underset{w}{\times} G/Top)$ and using bordism calculations to restrict the images of σ. This was carried out and the image of σ evaluated in L^p by Morgan and Pardon (unpublished) for π abelian and by Taylor and Williams [TW] for π an arbitrary 2-group (in the orientable case $w \equiv 1$).

Another approach is based on the LN-groups of Wall [W1, 12C], which are obstructions to codimension 1 splitting problems. These groups can be used to define invariants which vanish on closed manifold surgery problems but still detect a large part of the Wall group and some calculations for dihedral and quaternion groups, based on [W3] were carried out in an earlier version of this work(*). Cappell and Shaneson independently discovered this technique [CS1], [CS2] and exploited it to analyse an interesting surgery problem with obstruction not zero in $C^h_1(Q8)$ detected by a codimension 3 Arf invariant. This example showed that the list of invariants found by Morgan-Pardon (signature, codim. 0,1,2 Arf) was insufficient in L^h for π non-abelian.

Our results show that these invariants are in fact sufficient for all 2-groups in L^p. The higher co-dimension Arf invariants all vanish in L^p so algebraically they are in the image of $H^n(\tilde{K}_o(\pi)) \to L^h_n(\pi)$. It would be interesting to know the complete list of invariants for L^h. This has been named the "oozing problem" by John Morgan.

In Section 1 we describe Wall's LN-groups and develop some of their properties. Theorem 3 answers a question in [W1, p. 242]. In

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Section 2 the sequences of Section 1 are used to define splitting invariants which generalize those of Browder and Livesay [BL] and the A-invariant described there is recognized as a "twisted" transfer homomorphism (Lemma 5). The calculation of $L_n^p(\pi,w)$ for π a finite 2-group is given in Section 3 based on the sequence in [HM, Section 1] which relates the L^p groups to L^h groups for summands of an involution-invariant maximal order in $Q\pi$ containing $Z\pi$. These in turn are computed by referring to [W3] for L^s and applying the results of [HM, Section 4] in the $L^s - L^h$ Rothenberg sequence. These are summarized in Proposition 9, Theorem 10 and Table 1. The LN-groups needed for Section 5 are also calculated in Proposition 11 and Table 2. Our answer to question (iii) on the realization of elements in $\bar{C}^h$ by specific surgery problems is in Section 4. It is a special case of a construction found with W.-C. Hsiang. In Section 5 we apply the L^p and LN results to prove that the cup product on $H^1(\pi; Z/2)$ and the A,B invariants detect all elements of $L_n^p(\pi,w)$ not in $\bar{C}_n^h(\pi,w)$ when π is a special 2-group (i.e. cyclic, dihedral, semidihedral or quaternion). The computation of $\bar{C}_n^h$ for these groups π is in Propositions 12-16. Finally in Section 6 we prove our main result, Theorem 17, answering questions (i)-(iii) in L^p for a general 2-group.

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1. Obstructions to Codimension One Splitting

First we recall the LN-groups of Wall. Let $\rho \subset \pi$ be an inclusion of groups where ρ is of index 2 and $X \overset{p}{\to} Y$ a universal 2-fold cover inducing $\rho \to \pi$. Let Z be a $K(\rho,1)$ meeting the mapping cylinder M_Y of p in X and write $K(\rho \to \pi)$ for the triad $(M_Y \cup Z; Z, X)$. Wall then considers a cobordism group of objects consisting of: a finite Poincaré pair (N^n,M) and a manifold pair (W^{n+1},V), a finite Poincaré embedding $(N,M) \to (W,V)$ and a smoothing of the embedding $M \to V$ together with a map $(W; N, W - N) \to K(\rho \to \pi)$ compatible with $w(M_Y \cup Z)$. These cobordism groups are denoted $LN_n(\rho \to \pi)$ and Wall proves

<u>Theorem 1</u> ([W1, 11.6]).

There is a natural exact sequence

$$(1.1) \quad \ldots L_{n+1}(\pi) \overset{j}{\to} L_{n+2}(\rho \to \pi) \to LN_n(\rho \to \pi) \to L_n(\pi) \to \ldots$$

<u>Remarks</u> (i) For $(\rho \subset \pi) = (1 \subset Z/2)$ the LN-groups were first discovered by Browder-Livesay [BL] and this sequence by Lopez de Medrano [LM],

(ii) In Wall's treatment the L^s groups are understood,

(iii) If $\phi: \pi \to Z/2$ denotes the homomorphism with kernel ρ and $w: \pi \to Z/2$ the orientation character for $M_Y \cup Z$ the groups $L_k(\pi)$ have orientation $w\phi$ while the relative ones $L_k(\rho \to \pi)$ have orientation w,

(iv) Geometrically the first map j is obtained by pulling back the orientation line bundle over the surgery problem.

In [W1, 12C] Wall gives implicitly another cobordism description of these LN-groups along the lines of [BL]. Let (N_1^n,M_1) be a manifold pair with a map to Y compatible with w(Y). Form E, the pull-back of M_Y over N_1 and let $\partial E = \partial_0 E \cup \partial_1 E$ where $\partial_1 E$ is the pull-back over M_1.

The objects in the new cobordism group will be manifold pairs (W^{n+1},V) together with a homotopy equivalence

$$h: (W,V) \to (E,\partial_1 E)$$

such that h is transverse regular on $M_1 \subset \partial_1 E$ and the induced map $\partial_1 h: M = h^{-1}(M_1) \to M_1$ is a homotopy equivalence. The resulting cobordism group is again $LN_n(\rho \to \pi)$. This involves the appropriate version of Wall's $\pi - \pi$ Theorem. In this formulation there are versions for compact smooth, PL on Top manifolds with different assumptions on the torsion of h. Using the methods of [PR] there is a version for paracompact manifolds modelled on $N \times R$. These different versions lead to groups LN^s, LN^h and LN^p.

The main result of [W1, 12C] is the following expression for the LN-groups in terms of ordinary L-groups. Recall from [W3] that if R is a ring with involution α and $u \in R^\times$ such that $u^\alpha = u^{-1}$ and $x^{\alpha\alpha} = uxu^{-1}$ for all $x \in R$, there are Wall groups $L_n(R,\alpha,u)$.

<u>Theorem 2</u> $LN_n\ (\rho \to \pi,w) \cong L_n(Z\rho,\ \alpha,\ -w(t)g_0^{-1})$ where $t \in \pi$ generates π/ρ, $t^2 = g_0 \in \rho$ and $x^\alpha = w(x)t^{-1}x^{-1}t$ for all $x \in \rho$.

<u>Remarks</u>

(i) In [W1] this was proved under the assumption that t is central of order 2. Similar techniques suffice for the general case.

(ii) The result hold for LN^s, LN^h or LN^p (see also [R3]). Our first result is

<u>Theorem 3</u> There is a natural isomorphism of the exact sequence of Theorem 1 with the sequence:

$(1.2)\ldots L_{n+1}(Z\rho \to Z\pi,\alpha,u) \to L_n(Z\rho,\alpha,u) \to L_n(Z\pi,\alpha,u)\ldots$

where $u = (-1)w(t)g_0^{-1}$ as above. The isomorphism for the middle term is that of Th. 2 and for the last term "scaling by t".

Proof (Sketch). One approach is to follow the spectrum method of Quinn [Q] and Ranicki [R2]. Let $\underline{\underline{L}}(Z\pi,\ w\phi)$ denote the simplicial monoid with n-simplices of algebraic Poincaré (n + 2)-ads over $(Z\pi,\ w\phi)$. Similarly let $\underline{\underline{LN}}(\rho \to \pi,w)$ be a simplicial set of algebraic codimension 1 splitting problems. Then Wall's chapter 12C can be interpreted to give the left vertical arrow in a diagram:

$$\begin{array}{ccc} \underline{\underline{LN}}(\rho \to \pi,w) & \to & \underline{\underline{L}}(Z\pi,\ w\phi) \\ \downarrow & & \downarrow \\ \underline{\underline{L}}(Z\rho,\alpha,u) & \to & \underline{\underline{L}}(Z\pi,\alpha,u) \end{array}$$

The right vertical map is scaling and both induce isomorphisms on homotopy groups. The long exact sequences of homotopy groups are the two sequences (1.1) and (1.2).

2. The A,B, Invariants

We define two invariants for splitting problems. First consider the homomorphism(where $\rho = \ker(\phi:\pi\to Z/2)$)

$$A:\quad L_n(\pi,w) \to LN_{n-2}(\rho \to \pi)$$

defined by the composition of $L_n(\pi,w) \to L_n(\rho \to \pi)$ and the map $L_n(\rho \to \pi) \to LN_{n-2}(\rho \to \pi)$ from Theorem 1.

This homomorphism can be given a more geometrical definition by choosing a manifold X^{n-1} with $\pi_1 X = \pi$ and $w_1 X = w$ and considering the action of $x \in L_n(\pi,w)$ on the base point id: $X \to X$ in S(X) via the Wall realization theorem [W1]. This produces a new element f: $M^{n-1} \to X$ in

S(X) and so a splitting problem relative to any $\rho \subset \pi$ of index 2. A(x) is just the cobordism class of this splitting problem in $LN_{n-2}(\rho \to \pi)$.

In the case $n \equiv 0(4)$, $(\rho \subset \pi) = (1 \subset Z/2)$ and $w \equiv 1$, this is the α-invariant of Atiyah-Singer. From the geometrical definition it follows that $A(x) = 0$ if x acts trivially on $S(X^{n-1})$ for some compact Top manifold X as above. The subgroup of $L_n^h(\pi,w)$ generated by all such x is called the <u>inertia subgroup</u> $I_n^h(\pi,w)$ so we have $I_h^h(\pi,w) \subset \ker A(\rho \to \pi)$ for any subgroup $\rho \subset \pi$ of index 2. Since $I_n^h(\pi,w) \subset C_n^h(\pi,w)$, the subgroup of $L_n^h(\pi,w)$ generated by closed manifold surgery problems, and $A(x) = 0$ for $x \in C_n^h(\pi,w)$ also, the A-invariant can be used to estimate the size of $C_n^h(\pi,w)$. Our results in Section 6 will show that the <u>images</u> of $I_n^h(\pi,w)$ and $C_n^h(\pi,w)$ in $L_n^p(\pi,w)$ are equal for π a finite 2-group.

<u>Question</u>: Are $I_n^h(\pi,w)$ and $C_n^h(\pi,w)$ always equal for any finite group π?

To define the next invariant we let $A_n(\rho \to \pi) = \ker A$ and choose a (possibly different) subgroup $\rho' \subset \pi$ of index 2. Define

$$B: \quad A_n(\rho \to \pi) \to \overline{LN}_{n-3}(\rho' \to \pi, w\phi)$$

as follows: if $x \in A_n(\rho \to \pi)$ choose $y \in L_n(\pi, w\phi)$ mapping to x in sequence (1.1) and consider $A(y) \in LN_{n-3}(\rho' \to \pi, w\phi)$. The indeterminacy in A(y) is the image of the composite

$$\begin{array}{ccc} \gamma: \ LN_{n-1}(\rho \to \pi, w) & \to & L_{n-1}(\pi, w\phi) \\ & & \downarrow \\ & & L_{n-1}(\rho' \to \pi,\ w\phi) \to LN_{n-3}(\rho' \to \pi,\ w\phi). \end{array}$$

where the horizontal maps come from two sequences of type (1.1). We define $\overline{LN}_{n-3}(\rho' \to \pi, w\phi)$ to be the quotient by Imγ and let $B(x) = A(y)$. If $x \in I_n^h(\pi, w)$ then $A(x) = 0$ and $B(x) = 0$. We can identify the composite γ algebraically (Lemma 6) when $\rho' = \rho$ by considering a functor Φ: $\underline{\underline{Q}}(Z\rho, \alpha, u) \to \underline{\underline{Q}}(Z\rho, \alpha, u)$ where α, u are as in Theorem 2 and $\underline{\underline{Q}}(Z\rho, \alpha, u)$ is the category of quadratic forms over $(Z\rho, \alpha, u)$ on free (or projective) modules [W3]. If (M, f) represents a quadratic form then $\Phi(M, f)$ is represented by the module M $t((m \otimes t)\cdot x = m(txt^{-1}) \otimes t)$ and form $\bar{f}(m \otimes t,\ n \otimes t) = t^{-1} f(m,n) t$. This induces a homomorphism

$$\Phi : L_n(Z\rho, \alpha, u) \to L_n(Z\rho, \alpha, u)$$

<u>Lemma 4</u>. The composite

$$L_n(Z\rho, \alpha, u) \xrightarrow{i_*} L_n(Z\pi, \alpha, u) \xrightarrow{i^*} L_n(Z\rho, \alpha, u)$$

is $1+\Phi$, where i_* is the inclusion map and i^* the restriction.

The map A can be identifed as just the transfer of the twisted anti-structures.

Lemma 5. The composite

$$L_n(\pi,w) \overset{S_*}{\to} L_n(Z\pi,\alpha,w(t)g_0^{-1}) \overset{i^*}{\to} L_n(Z\rho,\alpha,w(t)g_0^{-1})$$

is the map A where S_* is induced by "scaling by t" under the identification

$$L_n(Z\rho,\alpha,w(t)g_0^{-1}) \cong LN_{n-2}(\rho \to \pi,w) \text{ of Th. 2.}$$

Proof: From Theorem 3 we have a commutative diagrams $(u = w(t)g_0^{-1})$

$$(2.1)\qquad \begin{array}{ccc} L_n(\pi,w) & \overset{S_*}{\underset{\approx}{\longrightarrow}} & L_n(\pi,\alpha',u) \\ \downarrow & & \downarrow j_* \\ L_n(\rho \to \pi,w) & \overset{\approx}{\longrightarrow} & L_{n+1}(\rho \to \pi,\alpha,u) \\ \downarrow & & \downarrow \partial_* \\ LN_{n-2}(\rho \to \pi,w) & \underset{\approx}{\to} & L_n(Z\rho,\alpha,u) \end{array}$$

where $\alpha'(x) = w(x)t^{-1}x^{-1}t$ for $x \in \pi$ differs from $\alpha(x) = \phi(x)w(x)t^{-1}x^{-1}t$ on elements of $\pi - \rho$. The map j_* is analogous to that of (1.1) and the composite $\partial_* j_* = i^*$. We have used the identification $L_i(R,\alpha,u) = L_{i+2}(R,\alpha,-u)$ given in [W3].

As a consequence of (2.1) in the proof of Lemma 5:

Lemma 6.

The diagram

$$\begin{array}{ccc} LN_{n-1}(\rho \to \pi,w) & \overset{\gamma}{\longrightarrow} & LN_{n-3}(\rho \to \pi, w\phi) \\ \| & & \| \\ L_{n-1}(\rho,\alpha,-w(t)g_0^{-1}) & \overset{1+\Phi}{\to} & L_{n-1}(\rho,\alpha,-w(t)g_0^{-1}) \end{array}$$

commutes, where the vertical isomorphisms are from Th. 2.

3. Calculation of $L^p(\pi,\omega)$

In this section we adapt the method described in [HM, Section 1] to compute $L^p_n(\pi,w)$ for π a finite 2-group and $w\colon \pi \to Z/2$ an arbitrary orientation character. The first step is to identify the types of simple involuted algebras in $Q\pi$ corresponding to the absolutely irreducible characters χ of π. If χ is not $Q(\chi)$-primitive then there is a proper subgroup ρ π and a character ξ of ρ such that $\xi^* = \chi$, $Q(\xi) = Q(\chi)$ and ξ is $Q(\chi)$-primitive [F]. If $D(\chi)$, the summand of $Q\pi$ containing χ, is involution invariant then we can distinguish two cases:

(i) when $D(\xi)$ is involution invariant also (in $Q\rho$) or

(ii) when distinct summands $D(\xi)$ and $D(\xi^t)$, $t \notin \rho$ are permuted by the involution. In case (i) $\ker \xi \leq \ker w$ so that it suffices to consider the summands of $Q(\rho/\ker\xi)$, or equivalently to determine the summands of $Q\pi$ for special 2-groups (cyclic, dihedral, semi-dihedral, quaternion) with arbitrary orientation character. Otherwise if case (ii) applies whenever χ is induced by ξ as above we say that χ is w-$Q(\chi)$-primitive.

The following eight types (D,τ) must be distinguished to fully describe the summands in $Q\pi$ where D is a simple involuted algebra, τ the (anti-) involution on D and ζ denotes a primitive 2^k-th root of 1.

(3.1)

Oa:	$Q(\zeta + \bar{\zeta})$, $\zeta^\tau = \bar{\zeta}$		(k 1)
Ob:	$Q(\zeta - \bar{\zeta})$, $\zeta^\tau = -\bar{\zeta}$		$(k \geq 3)$
Oc:	$Q(i)$, $i^\tau = i$		
Ua:	$Q(\zeta)$, $\zeta^\tau = \bar{\zeta}$		
Ub:	$Q(\zeta)$, $\zeta^\tau = -\bar{\zeta}$		$(k \geq 3)$
Uc:	$Q(\zeta + \bar{\zeta})$, $\zeta^\tau = -\bar{\zeta}$		$(k \geq 3)$

Ud: $Q(\zeta - \bar{\zeta})$, $\zeta^{\tau} = -\bar{\zeta}$ $(k \geq 3)$

Ue: $\Gamma_k = \left(\frac{-1,\ -1}{Q(\zeta+\bar{\zeta})}\right)$, $\zeta^{\tau} = -\bar{\zeta}$ $(k \geq 3)$ and $e_1^{\tau} = -e_1$, $e_2^{\tau} = -e_2$ where $\{1, e_1, e_2, e_1 e_2\}$ is the usual basis of Γ_k over its centre $Q(\zeta+\bar{\zeta})$.

Sp: Γ_k, $\zeta^{\tau} = \bar{\zeta}$ $(k \geq 2)$ and $e_1^{\tau} = -e_1$, $e_2^{\tau} = -e_2$.

GL: D is the sum of two simple algebras interchanged by the involution.

Now the result corresponding to [HM,1.3] is

Theorem 7. Let π be a finite 2-group and $w: \pi \to Z/2$ an orientation character. Under the involution induced on $Q\pi$ by $x \to w(x)x^{-1}$ $(x \in \pi)$, the involution-invariant indecomposable summands of $Q\pi$ are either type GL or isomorphic to one of:

(1) $M_{\ell}(D)$ with involution $A \to X A^{\tau} X^{-1}$ for some $X \in M_{\ell}(D)$, where A^{τ} is τ-conjugate transpose, $X^{\tau} = \lambda X$ for $\lambda = \pm 1$, and (D,τ) is in (3.1).

(2) $M_{\ell}(D)$ with involution as in (1) and $D = D(\xi)$ for some w-$Q(\xi)$-primitive character of a subgroup $\rho \subset \pi$.

As in the orientable case, it follows that there exists an involution-invariant maximal order $\mathcal{M} \subset Q\pi$ containing $Z\pi$ which splits as $\mathcal{M} = \prod_{\nu} \mathcal{M}_{\nu}$ where $\mathcal{M}_{\nu}$ is a maximal order in an involution-invariant summand of $Q\pi$. A list of the types occurring for primitive characters can be made from (3.1) replacing Q by Z except for Ue, Sp where a maximal order in Γ_k must be chosen. Our method of calculation will rely on the sequence [HM, 1.4]:

$$(3.2) \quad \ldots \to L^h_{n+1}(\hat{\mathcal{M}}_2) \to L^p_n(Z\pi) \to L^h_n(\hat{Z}_2\pi) \oplus L^h_n(\mathcal{M}) \to L^h_h(\hat{\mathcal{M}}_2) \to \ldots$$

so we need (by Theorem 7) to compute L^h-groups for the anti-structures $(\mathcal{N}, \tau, \pm 1)$ where $\mathcal{N}$ is Morita equivalent to a simple summand of $\mathcal{M}$ and τ the involution. For the summands corresponding to $Q(\chi)$-primitive characters χ of π, this is done by using the results of [W3] together with calculations of $H^*(K_1(\mathcal{N}))$ from [HM, Section 4] to find L^h through the L^s-L^h Rothenberg sequence. For the remaining summands we need some further properties of the A,B-invariants. Let $(\mathcal{M},\alpha,u)$ denote an anti-structure on the maximal order $\mathcal{M}$, induced by an anti-structure $(Z\pi,\alpha,u)$, so that

$$(\mathcal{M},\alpha,u) = \prod_{\nu}(\mathcal{M}_{\nu},\alpha_{\nu},u_{\nu}).$$

Proposition 8.

(1) There is a commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
\cdots\to L^h_{i+1}(\hat{\mathcal{M}}_2) & \to & L^p_{i+1}(Z\pi \to \hat{Z}_2\pi) & \to & L^h_i(\mathcal{M}) & \to & L^h_i(\hat{\mathcal{M}}_2) \\
\| & & \downarrow & & \downarrow & & \| \\
L_{i+1}(\hat{\mathcal{M}}_2) & \to & L^p_i(Z\pi) & \to & L^h_i(\hat{Z}_2\pi)\oplus L^h_i(\mathcal{M}) & \to & L^h_i(\hat{\mathcal{M}}_2) \\
 & & \downarrow & & \downarrow & & \\
 & & L_i(\hat{Z}_2\pi) & \cong & L^h_i(\hat{Z}_2\pi) & &
\end{array}$$

with the middle horizontal sequence from (3.2).

(2) Let $Q\pi = \prod_{\nu} D_{\nu}$ and define

$$\Lambda_i(D_{\nu},\alpha_{\nu},u_{\nu}) = L^h_{i+1}(\mathcal{M}_{\nu} \to (\mathcal{M}_{\nu})^{\wedge}_2,\alpha_{\nu},u_{\nu})$$

Then there is a natural splitting of the top sequence in (1) and

$$L^p_{i+1}(Z\pi \to \hat{Z}_2\pi) \cong \prod_\nu \Lambda_i(D_\nu,\alpha_\nu,u_\nu).$$

We now observe that the spectrum argument for Theorem 3 is equally valid for $\hat{Z}_2\pi$ or the relative groups of $Z\pi \to \hat{Z}_2\pi$. Therefore we have A (and B) invariants defined for these groups also (using a subgroup $\rho \subset \pi$ of index 2) which are compatible with those of $Z\pi$:

$$\begin{array}{ccccccc} \to L^p_i(Z\pi) & \to & L^h_i(\hat{Z}_2\pi) & \to & L^p_i(Z\pi \to \hat{Z}_2\pi) & \to & \dots \\ \downarrow A & & \downarrow A & & \downarrow A & & \\ \to L^p_i(Z\rho,\alpha,u) & \to & L^h_i(\hat{Z}_2\rho,\alpha,u) & \to & L^p_i(Z\rho \to \hat{Z}_2\rho,\alpha,u) & \to & \dots \end{array}$$

is a commutative diagram with $(Z\rho,\alpha,u)$ the anti-structure of Theorem 2 (Note that $A = 0$ on $L^h_i(\hat{Z}_2\pi)$ since $L^h_i(\hat{Z}_2\rho) \stackrel{\approx}{\to} L^h_i(\hat{Z}_2\pi)$.) Again Lemma 5 identifies these maps A as the twisted transfer maps. This interpretation also makes sense on $L^h_i(\mathcal{M})$ since $\mathcal{N}$, a maximal involution-invariant order for $Z\rho$, can be chosen so that $\mathcal{N}$ contains the image of $\mathcal{M}$ under the usual augmentation ε: $Q\pi \to Q\rho$ where $\varepsilon(x) = 0$ if $x \notin \rho$ and then an augmentation map $\varepsilon: \mathcal{M} \to \mathcal{N}$ is defined by restriction. If $h: P \times P \to \mathcal{M}$ is the form over $\mathcal{M}$, $\varepsilon o h : P \times P \to \mathcal{N}$ is the restricted from over $\mathcal{N}$. From this definition it is clear that if $\mathcal{M}_\nu \subset \mathcal{M}$ corresponds to an absolutely irreducible character χ of π, then the image of A restricted to $L^h_i(\mathcal{M}_\nu)$ lies in the summands $L^h_i(\mathcal{N}_\nu,\alpha,u)$ corresponding to characters ξ of ρ with $\xi^* = \chi$. Now we can deal with the summands corresponding to w-$Q(\chi)$-primitive characters χ.

<u>Proposition 9</u>. Let χ be a w-$Q(\chi)$-primitive character of π and ξ a character on $\rho \subset \pi$ of index 2 with $\xi^* = \chi$ and $Q(\xi) = Q(\chi)$.

(1) The summand $D(\xi) \subset Q\rho$ corresponding to ξ is involution-invariant in the twisted anti-structure $(Q\rho, \alpha, u)$.

(2) If $(D(\chi), \tau, 1)$ corresponds to χ under the usual anti-structure on $Q\pi$, the map A followed by projection induces an isomorphism

$$A: \Lambda_i^h(D(\chi), \tau, 1) \overset{\approx}{\to} \Lambda_i^h(D(\xi), \alpha, u)$$

From this result we can see that a complete computation of $L_*^p(Z\pi)$ by our method will depend on calculation of $LN_*^p(Z\rho \to Z\pi)$ also. In fact it is enough to give this calculation when ρ is a special 2-group since then the preceeding method (which applies to $(Z\rho, \alpha, u)$ as well) gives an inductive procedure. Here we will only carry out the last step for ρ cyclic since this suffices for our application.

First we state the L^p results.

Theorem 10. Let π be a finite 2-group and $Q\pi = \prod_\nu D_\nu$ where D_ν are indecomposable, involution-invariant algebras. The groups $\Lambda_*(D_\nu)$ are zero for D_ν of type GL but for summands corresponding to the other types (3.1): (here Σ denotes the group of signatures)

(1) $\Lambda_0(D_\nu) = \Sigma$ for D_ν of type Oa, Ua, Ud or Sp.

(2) $\Lambda_1(D_\nu) = Z/2$ if D_ν has type Ub, Uc, Ue;

$\Lambda_1(D_\nu) = (Z/2)^{2^{n-2}+1}$ if D_ν has type Sp and centre of degree 2^{n-2} over Q.

(3) $\Lambda_2(D_\nu) = \Sigma$ if D_ν has type Ua or Ud;

$\Lambda_2(D_\nu) = (Z/2)^{2^{n-2}-1}$ if D_ν has type Sp and centre of degree 2^{n-2} over Q.

(4) $\Lambda_3(D_\nu) = Z/2$ if D_ν has type Oa, Ub, Uc or Ue;

$\Lambda_3(D_\nu)$ is order 2^{m+2} if D_ν is type Ob or Oc and centre of degree 2m over Q.

(5) The map $L^p_{2k}(Z\pi) \to L^h_{2k}(\hat{Z}_2\pi) = Z/2$ is onto (and splits) if k = 1, or if k = 0 and the map w: $\pi \to Z/2$ is non-trivial but does not factor through the projection $Z/4 \to Z/2$.

(6) The map $L^h_0(\hat{Z}_2\pi) \to L^h_0(Z\pi \to \hat{Z}_2\pi)$ hits diagonally (a) the elements from characters of degree 1 and type Oa if $w \equiv 1$ or (b) the elements from $L^h_0(\hat{\mathcal{M}}_2)$ for characters of degree 1 and type Oc if $w \not\equiv 1$.

<u>Remark</u>: The fact that $L^p_0(Z\pi)$ splits whenever it is onto $L^p_0(\hat{Z}_2\pi)$ follows from the fact that there is a codim 2 Arf invariant problem (Section 4) with obstruction non-zero in $L^p_0(\hat{Z}_2\pi)$ in that case.

As a corollary to this Theorem we can compute L^p for special 2-groups (Table 1). Note that when making these calculations the Morita equivalence and scaling needed to reduce D_ν to one of the types given may change the unit by -1. This is denoted by Oa^- for example in the case of the dihedral groups.

In the LN calculation for cyclic 2-groups two new types appear:

(3.3) Od: $Q(\zeta)$, $\zeta^\tau = \zeta$ $(k \geq 3)$

Uf: $Q(\zeta)$, $\zeta^\tau = -\zeta$ $(k \geq 3)$.

<u>Proposition 11</u>.

Let $L^p_{i+1}(Z\rho \to \hat{Z}_2\rho, \alpha, u) = \prod_\nu \Lambda_i(D_\nu, \alpha, u)$ where $Q\rho = \prod_\nu D_\nu$. Then for D_ν of the type Od, $\Lambda_3(D_\nu)$ has order 2^{m+2} when the centre of D_ν has degree 2m over Q and $\Lambda_i(D_\nu) = 0$ for $i \neq 3$; for D_ν of type Uf, $\Lambda_i(D_\nu) = Z/2$ for i = 1,3 and zero otherwise.

The LN groups for ρ cyclic are now given in Table 2. Note that if ρ is cyclic and type 0a, 0c or 0d is present, $L_0^h(\hat{Z}_2\rho,\alpha,u)$ injects into $L_0^p(Z\rho \to \hat{Z}_2\rho,\alpha,u)$ and hits (diagonally) the contribution to the group from $L_0^h(\hat{\mathcal{M}}_2)$ for these summands. Similarly if type $0a^-$ is present, $L_2^h(\hat{Z}_2\rho,\alpha,u)$ injects into $L_2^p(Z\rho \to \hat{Z}_2\rho,\alpha,u)$.

4. Codimension k Arf Invariants

Let π be a finite 2-group and $\rho \subset \pi$ a subgroup of index 2. If X^{n-1} is a closed PL manifold of dimension n-1 with $\pi_1 X = \pi$ and $w = w_1(X)$, we can construct some elements in $I_n^h(\pi,w)$ whose surgery obstructions are related to splitting invariants. (This construction is a special case of one which arose in work with Wu-Chung Hsiang). Let $X \to B\pi \to B(\pi/\rho) = BZ/2$ be the composite of the classifying map for $\pi_1 X$ with the reduction and form

$$f: X \to RP^{\ell}$$

for some $\ell \gg n$ by simplicial approximation. If f is made transverse regular to $RP^{\ell-k}$ for some $k > 0$ we obtain $X_k = f^{-1}(RP^{\ell-k}) \subset X$. When the fundamental class $[X_k]$ of X_k represents a non-zero class in $H_{n-k}(X;Z/2)$ let $k < [n/2]$ and choose an embedded submanifold $S_k \subset X$ of codimension k representing the Poincaré dual of $[X_k]$ under the isomorphism $H^k(X;Z/2) = HOM(H_n(X;Z/2), Z/2)$. Now let $(E,\partial E)$ denote the disk and sphere bundle of the normal bundle to $S_k \times (\frac{1}{2})$ in $X \times I$ and consider $[E,\partial E;G/TOP,*]$. Assume $n-k \equiv 2(4)$ and let $U_k \in [E,\partial E;G/TOP,*]$ be the THOM class of the normal bundle. This defines a surgery problem (rel∂E) with target E and so we obtain a normal map

$$F : W^n \to X \times I$$

which is a homeomorphism on $\partial_{\pm}W$ by replacing the interior of E with the surgery problem. The surgery obstruction $\sigma(F) \in L_n^h(\pi,w)$ for this problem is by definition the "codimension k Arf invariant". Clearly $\sigma(F) \in I_n^h(\pi,w)$ and this surgery problem exists only when $(f^*\alpha)^k \neq 0$ where $0 \neq \alpha \in H^1(RP^\ell;Z/2)$.

5. Closed Manifold Obstructions for Special 2-Groups

In this section we will use the calculations of Section 3 and the fact that the A, B invariants of Section 2 vanish on closed manifold obstructions to compute $\bar{I}_n^h(\pi,w) = \mathrm{Im}(I_n^h(\pi,w) \to L_n^p(\pi,w))$ for π a special 2-group. It will then be observed that $\bar{C}_n^h(\pi,w) = \bar{I}_n^h(\pi,w)$ by giving explicit surgery problems for each element. Essentially we show that the A, B invariants detect the elements not in $\bar{I}_n^h(\pi,w)$ by calculating the maps in the LN sequences of (1.1). Those in $\bar{I}_n^h(\pi,w)$ are all detected by the ordinary signature (arising from the map $L_0(\pi,w) \to L_0(1) = Z$ defined when $w \equiv 1$) and Arf invariants in codimensions ≤ 2.

(a) π cyclic

From Table 1, the torsion in $L_n^p(\pi,w)$ comes from $L_n^p(\hat{Z}_2\pi)$ or the representation of types Oa, Oc, Ub. For $\pi = (Z/2,\pm)$ the answer is well-known: codim. 0,1 Arf invariants ($w\equiv 1$) and codim 0,2 Arf invariants ($w\not\equiv 1$) account for all the torsion. If $\pi = (Z/4,+)$ no new classes arise but if $\pi = (Z/4,-)$ there is a codim 1 Arf invariant (and no codim. 2 Arf). Consider the splitting diagram (of sequences (1.1) combined with the usual relative sequences).

$$\begin{array}{ccc} 0 & & \\ \downarrow & & \\ L_3^p(Z/4,-) & = & (Z/2)^2 \\ \downarrow & & \downarrow \end{array}$$

$$\begin{array}{ccccccc} LN_2(Z/2 \to Z/4,-) & \to & L_2(Z/4,+) & \to & \text{rel} & \to & 0 \\ \| & & \| & & & & \\ 8Z & \rightarrowtail & 4Z \oplus Z/2 & & & & \\ & & \downarrow & & & & \\ L_1(Z/4,-) & \to & L_2(Z/2 \to Z/4,+) & & & & \\ \| & & & & & & \\ 0 & & & & & & \end{array}$$

This diagram shows that one $Z/2$ in $L_3^p(Z/4,-)$ is detected by the codim. 1 Arf invariant while the other has $A = 0$ but $B \neq 0$ so does not lie in $\bar{C}_n^h(\pi,w)$. For $(Z/2^n,+)$, $L_3^p(\pi) \overset{\approx}{\to} L_3^p(Z/2)$ by projection so the codim. 1 Arf again detects. If $\pi = (Z/2^n,-)$, the part from type Oc is detected by projection $(Z/2^n,-) \to (Z/4,-)$ and for the rest consider:

$$\begin{array}{ccccc} & & L_1(Z/2^n,-) = (Z/2)^t \quad \text{(type Ub)} & & \\ & & \downarrow & & \\ LN_o & \to & L_o(Z/2^n,+) \to L_1(Z/2^{n-1} \to Z/2^n,-) & & \end{array}$$

Since coker $(LN_o(Z/2^{n-1} \to Z/2^n,-) \to L_o(Z/2^n,+))$ is free abelian, $A \neq 0$ on all of $L_1(Z/2^n,-)$. A similar argument works in $L_3(Z/2^n,-)$ for the type Ub contribution

<u>Proposition 12</u> For $\pi = Z/2^n$ and $\ell \equiv 0,1,2,3 \pmod 4$:

$$\bar{C}_\ell(\pi,+) = Z,\ 0,\ Z/2,\ Z/2$$

and

$$\bar{C}_\ell(\pi,-) = \begin{cases} Z/2,\ 0,\ Z/2,\ 0 & \text{if } n = 1 \\ 0,\ 0,\ Z/2,\ Z/2 & \text{if } n \geq 2 \end{cases}$$

(b) π dihedral

Here for $w = (+,+)$ we must consider only L_3^p. Since $L_3^p(D2^n) = (Z/2)^{n+1}$ and $L_3^p(Z/2^{n-1}) = Z/2$ injects:

$$\begin{array}{ccc} & & L_3(Z/2^n,-) = Z/2 \\ & & \downarrow \\ & & L_3(D2^n) = (Z/2)^{n+1} \\ & & \downarrow \\ 0 \to L_2(D2^n+-) & \to & \text{rel} \\ \| & & \\ Z/2 \oplus \Sigma & & \end{array}$$

Here Σ denotes the signature part of L_2 and rel is the relative group in the vertical sequence. Therefore ker $A = (Z/2)^2$ and these are both in $C_3(D2^n)$: one from $C_3(Z/2^{n-1})$ and the other a codim. 1 Arf.

For $(D2^n, +-)$, $L_1^p = (Z/2)^{n-2}$ and $A \neq 0$ on all these. For $(D2^n, -+)$ we first calculate that

$$L_1^p(Z/2^{n-1},-) \overset{\approx}{\to} L_1^p(D2^n, -+)$$

so that we take $\rho = D2^{n-1}$ instead to compute the A-invariant. Then

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ & & L_1^p(D2^n, -+) = (Z/2)^{n-2} \\ & & \downarrow \\ L_o^p(D2^n,++) & \to & \text{rel} \\ \| & & \downarrow \\ \Sigma & & \\ & & L_o^p(D2^{n-1},++) = \Sigma' \end{array}$$

The transfer map $L_o^p(D2^n) \to L_o^p(D2^{n-1})$ is injective on the cokernel of $LN_o^p(D2^{n-1} \to D2^n) \to L_o^p(D2^n)$ so $A \neq 0$ on all of $L_1^p(D2^n, -+)$.

In $L_3^p(D2^n, -+)$ the type Uc classes are not hit from $L^p{}_3(Z/2^{n-1}, -)$ so that since $L_2^p(D2^n, --) = Z/2$ (hit from $LN_2(Z/2^{n-1} \to D2^n)$) the A-invariant detects

$$\text{coker } (L_3^p(Z/2^{n-1}, -) \to L_3^p(D2^n, -+)).$$

Finally the type Oa class is hit from $L_3^p(Z/2^{n-1}, -)$ so is in $\bar{C}_3(D2^n, -+)$.

<u>Proposition 13</u>. For $\pi = D2^n$ and $\ell \equiv 0,1,2,3 (\text{mod } 4)$:

$\bar{C}_\ell(\pi, ++) = Z,\ 0,\ Z/2,\ (Z/2)^2$

$\bar{C}_\ell(\pi, +-) = Z/2,\ 0,\ Z/2,\ 0$

$\bar{C}_\ell(\pi, -+) = Z/2,\ 0,\ Z/2,\ Z/2$

(c) <u>π semi-dihedral</u>

Since the projection $L_i^p(SD2^n) \to L_i^p(D2^{n-1})$ detects the torsion classes except from the Ob representation, it suffices to consider these in $L_3^p(SD2^n, -+)$. However these elements are not hit from

$L_3^p(SD2^{n-1}, ++)$ and the inclusion map $L_2(Q2^{n-1}, ++) \to L_2(SD2^n, ++)$ does not hit the signatures at the Ob representation. Therefore a combination of the A-invariants for $D2^{n-1} \subset SD2^n$ and $Q2^{n-1} \subset SD2^n$ (in codimension one) detects these elements. A similar arguments works for $L_1^p(SD2^n, --)$.

<u>Proposition 14</u> The projection map

$$L_i^p(SD2^n, w) \to L_i^p(D2^{n-1}, w)$$

induces an isomorphism on $\bar{C}_\ell$.

(d) <u>π quaternion</u>

First let $\pi = Q8$ and $w \equiv 1$.

From the diagram:

$$\begin{array}{ccccccc}
 & & L_3(Z/4) = Z/2 & & & & \\
 & & \downarrow & & & & \\
 & & L_3(Q8) = (Z/2)^3 & & & & \\
Z \oplus Z/2 & & \downarrow & & & & \\
\| & & & & & & \\
L_2(Q8, +-) & \to & \text{rel} & \to & Z/2 & \to & 0
\end{array}$$

we see that $\bar{C}_3(Q8, ++) = (Z/2)^2$ and the other generator of L_3 has $A \neq 0$.

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & L_1(Q8) = (Z/2)^2 & & \\ & & & & \downarrow & & \\ 0 \to & L_o(Q8, +-) & \to & \text{rel} & \to Z/2 & & \\ & \| & & & & & \\ & Z/2 & & & & & \end{array}$$

so the A-invariant detects one Z/2 in $L_1(Q8)$. The other is detected by the codim 2 Arf invariant in $L_o(Q8, +-)$ since by projection $L_o(Q8, +-) \overset{\approx}{\to} L_o(Z/2, -)$ and the splitting diagram is natural. Since $\alpha^3 = 0$ for $\alpha \in H^1(Q8; Z/2)$ the codimension 3 Arf invariant does not exist and $\bar{C}_1(Q8) = 0$. Notice that in the Cappell-Shaneson example different index 2 subgroups were used to do the iterated splittings. They exploited the fact that $\alpha^2\beta \neq 0$ for α, β generators of $H^1(Q8; Z/2)$.

For (Q8, -+) one Z/2 of $L_3^p(Q8, -+) = (Z/2)^2$ is in the image of $\bar{C}_3(Z/4, -)$ and the other is detected by the A-invariant.

<u>Proposition 15</u> For $\pi = Q8$ and $\ell = 0,1,2,3 \pmod 4$

$\bar{C}_\ell(\pi, ++) = Z,\ 0,\ Z/2,\ (Z/2)^2$

$\bar{C}_\ell(\pi, +-) = Z/2,\ 0,\ Z/2,\ Z/2$

Next let $\pi = Q2^n$ for $n \geqslant 4$. Since

$$L_2^p(Z/2^{n-1}) \to L_2(Q2^n)$$

is onto ($w \equiv 1$) and the torsion-free part of $L_2^p(Z/2^{n-1})$ can be detected by the A-invariant (modulo the image of $L_2^p(Z/2^{n-2})$), $\bar{C}_2(Q2^n, ++) = Z/2$

detected by the ordinary Arf invariant.

This can be seen considering the Frobenius inclusion

$$Q2^n \subset Z/2^{n-1} \wr Z/2 = (Z/2^{n-1} \times Z/2^{n-1}) \rtimes Z/2$$

into the wreath product. This has the property that if χ is the type Sp character on $Q2^n$ induced from ξ on $Z/2^{n-1} \subset Q2^n$ then χ extends to $\tilde{\chi}$ which is induced from $\xi \times 1$ on $Z/2^{n-1} \times Z/2^{n-1}$. Since the translates of $\xi \times 1$ in the wreath product are distinct the construction at the end of Section 5 eliminates the other elements of $L_2(Q2^n)$.

The same argument proves the $\bar{C}_o(Q2^n, +-) = Z/2$. Now in the splitting diagram ker $A \subset L_1(Q2^n, ++)$ is detected by $L_o(Q2^n, +-)$ so $\bar{C}_1(Q2^n, ++) = 0$ as for Q8. Similarly, in $L_3(Q2^n, +-)$ the image of $L_3(Z/4, -)$ gives one closed manifold class. The remaining elements in ker A are detected by $L_2(Q2^n, ++)$ so $\bar{C}_3(Q2^n, +-) = Z/2$.
For $(Q2^n, -+)$ the diagram:

$$\begin{array}{ccc} & & L_3^p(Q2^{n-1}, ++) \\ & & \downarrow \\ & & L_3^p(Q2^n, -+) \\ & & \downarrow \\ L_2^p(Q2^n, ++) & \rightarrow & \text{rel} \end{array}$$

and the fact that the Ue class in $L_3^p(Q2^n, -+)$ is not hit from $L_3^p(Q2^{n-1}, ++)$ shows that the projection

$$\bar{C}_3(Q2^n, -+) \rightarrow \bar{C}_3(D2^{n-1}, -+)$$

is an isomorphism. A similar argument proves that $\bar{C}_1(Q2^n, --) = 0$ using the splitting diagram with subgroup $(Q2^{n-1}, +-)$.

Proposition 16 Let $\pi = Q2^n$, $n > 4$ and $\ell = 0,1,2,3 \pmod 4$,

$\bar{C}_\ell(Q2^n, ++) = Z, 0, Z/2, (Z/2)^2$

$\bar{C}_\ell(Q2^n, +-) = Z/2, 0, Z/2, Z/2$

$\bar{C}_\ell(Q2^n, -+) = Z/2, 0, Z/2, Z/2$

6. Closed Manifold Obstructions for Arbitrary 2-Groups

In this section we will give the calculation of $\bar{C}_\ell(\pi,w)$ for π a finite 2-group in terms of the characters of π.

Theorem 17 Let π be a finite 2-group and $w: \pi \to Z/2$ an orientation character.

(1) If $w \equiv 1$, $\bar{C}_0 = Z$, $\bar{C}_1 = 0$, $\bar{C}_2 = Z/2$ and $\bar{C}_3(\pi) \overset{\approx}{\to} \bar{C}_3(\pi/[\pi,\pi]) \subset H_1(\pi; Z/2)$.

These are detected by signature, codim 0 Arf, and codim 1 Arf respectively.

(2) If $w \not\equiv 1$, $\bar{C}_0 = Z/2$ when w does not factor through $Z/4$, otherwise $\bar{C}_0 = 0$, $\bar{C}_1 = 0$, $\bar{C}_2 = Z/2$ and $\bar{C}_3 = (Z/2)^s$ where $s \leq \#\{$summands of $Q\pi$ of type Sp, Oa and Oc$\}$. These are detected by the codim 2, codim 0 and codim 1 Arf invariants.

Proof: Let $f : M^n \to N^n$ $(n > 5)$ represents a surgery problem of closed TOP n-manifolds with $\sigma(f) \in L^h_n(\pi,w)$. The result is first proved in dimension 4 by calculating the possible image of $[X^4,G/TOP]$ in $L^p_4(\pi,w)$ so we assume inductively that it is true for dimensions $< n$. We let

$a = i_*\sigma(f) \in L_n^p(\pi,w)$ and assume that $a = (a_\chi) \in \prod_\chi \Lambda_n(D(\chi))$ using the description of L^p in Proposition 8. This is possible since any contribution to $i_*\sigma(f)$ from $L_n(\hat{Z}_2\pi)$ can be eliminated by taking the sum of this problem with a simply-connected surgery problem or a codim 2 Arf invariant. Furthermore by Proposition 9 we can assume that $a_\chi = 0$ unless χ is induced from a primitive character.

Let χ be a character of π for which $a_\chi \neq 0$ and choose $\rho \subset \pi$ with a character ξ such that $\xi^* = \chi$, $Q(\xi) = Q(\chi)$, ξ is primitive and $\rho/\ker \xi$ a special 2-group.

Lemma 18 By the inductive assumption (and subtracting off codim k Arf invariants as before) we can assume that there exists $b \in L_n^p(\rho,w)$ such that b has image a under the map

$$L_n^p(\rho,w) \to L_n^p(\pi,w).$$

Assuming this we notice that by construction $b_\xi \neq 0$ hits a_χ and b_ξ is detected by

$$L_n^p(\rho,w) \to L_n^p(\rho/\ker \xi,\ w)$$

If $N = N_1 \cup N_2$ where $\pi_1 N_1 = \pi$ and $\pi_1 N_2 = \pi_1(\partial N_2) = \rho$, we can assume that $f = f_1 \cup f_2$ where $f_1 : M_1 = f^{-1}(N_1) \to N_1$ is a homotopy equivalence and $f_2 : M_2 = f^{-1}(N_2) \to N_2$ is a problem over ρ with obstruction b. Now define $\tilde{f}_1 : \tilde{M}_1 \to \tilde{N}_1$ (the covering with $\pi_1 = \rho$) assuming $\rho \triangleleft \pi$ and observe that the splitting problem $\partial\tilde{f}_1 : \partial\tilde{M}_1 \to \partial\tilde{N}_1$ relative to any index 2 subgroup $\rho_o \subset \rho$ vanishes in $LN_{n-2}(\rho_o \to \rho)$ because it is null-bordant using $(\tilde{M}_1, \tilde{f}_1)$ and the second description of Section 1 for LN. This splitting problem is also the boundary of $|\pi:\rho|$ copies of $f_2 : M_2 \to N_2$ where the copy corresponding to a coset $t\rho$ has fundamental group identified as $t\rho t^{-1} \subset \pi$. Since $\rho \triangleleft \pi$ the characters ξ^t determine distinct summands of $Q\rho$ and since the A-invariant splits according to the decomposition of $Q\rho$ (see the discussion following Prop. 8) it follows that $A(b_\xi) = 0$. If ρ is not normal in π we modify the argument by first identifying in pairs (using covering homeomorphisms) those boundary components of $(\tilde{M}_1, \tilde{f}_1)$ for cosets $t\rho$ such that $t\rho t^{-1} \neq \rho$. Similarly, $B(b_\xi) = 0$ and by naturality (choosing $\rho_0 \geqslant \ker \xi$) the same is true for the image of b_ξ in $L_n^p(\rho/\ker \xi, w)$. The calculations of Section 5 now imply the desired descripton of b_ξ. Since b_ξ and hence a_χ is represented by a codim k Arf invariant for $k \leqslant 2$ it can be subtracted off and the argument repeated.

<u>Proof of Lemma 18</u> Consider the splitting diagram:

$$\begin{array}{ccccc} & & L_n^p(\rho, w) & & \\ & & \downarrow & & \\ & & L_n^p(\pi, w) & & \\ & & \downarrow & \searrow^{A} & \\ L_{n-1}^p(\pi, w\phi) & \to & L_n^p(\rho \to \pi, w) & \to & LN_{n-2}(\rho \to \pi, w) \end{array}$$

Since $f : M \to N$ is a closed manifold problem there exists a normal map $g: M' \to N'$ induced from f by transversality on a characteristic codimension 1 submanifold $N' \subset N$ corresponding to the subgroup $\rho \subset \pi$ of index 2 (see Section 4). Then $i_*\sigma(g) \in L_{n-1}^p(\pi, w\phi)$ hits the image of $i_*\sigma(f)$ in $L_n^p(\rho \to \pi, w)$ and by the inductive assumption $i_*\sigma(g)$ can be represented as a sum of suitable (n-1)-dimensional (simply-connected) signature or codim k Arf invariant problems for $k \leq 2$.

However the signature problem does not exist in codimension 1 (the complement of a tubular neighbourhood of $N' \subset N$ provides a null-bordism) and the codim. 0 Arf invariant on N' gives a codim 1 Arf invariant on N. The other terms in the sum, codim 1 or 2 Arf invariants, do not give rise to non-zero elements in $L_n^p(\rho \to \pi, w)$ even when they exist because they lie in summands of $L_{n-1}^p(\pi, w\phi)$ detected by representations on subquotients of π of the form (Z/2, ±) or Z/4, -) and the calculations of Section 5 apply.

This argument shows that by adding suitable n-dimensional closed manifold problems to $f: M \to N$ we can assume that the image of $i_*\sigma(f)$ in $L_n^p(\rho \to \pi, w)$ is zero.

<u>Table 1</u> - $L_*^p(Z\pi, w)$ for π = Special 2-Group

	(‡) $\tilde{L}_0$	L_1	$\tilde{L}_2$	L_3	E(from $\hat{Z}_2\pi$)
$Z/2^n$, $\bar{x}=x^{-1}$ Ua, Oa	Σ Ua Oa	0	Σ Ua	Z/2 Oa	0 (*=0) Z/2 (*=2)
$Z/2^n$, $\bar{x}=-x^{-1}$ Ub, Oc, GL	0	Z/2 Ub	0	Z/2 Ub $(Z/2)^2$ Oc	0(n>2),Z/2(n=1) Z/2
$D2^n(+,+)$, n>3 Oa	Σ Oa	0	0	$(Z/2)^{r-1}$ Oa r=# {type Oa}	0 Z/2
$D2^n(+,-)$ Oa^-, GL	0	$(Z/2)^r$ r=# {type Oa^-}	Σ Oa^-	0	Z/2 Z/2
$D2^n(-+)$ or (--) Uc (n>4), Oa, GL	Σ Oa	Z/2 Uc	0	Z/2 Uc Oa	Z/2 Z/2
$SD2^n(++)$, n>4 Ud, Oa	Σ Oa Ud	0	Σ Ud	$(Z/2)^{r-1}$ r=# {type Oa}	0 Z/2
$SD2^n(+-)$ Ud^-, Oa^-, GL	Σ Ud	$(Z/2)^r$ r=# {type Oa^-}	Σ Oa^- Ud	0	Z/2 Z/2
$SD2^n(-+)$ Ob, Uc (n>5), Oa, GL	Σ Oa	Z/2 Uc	0	Z/2 Oa order $2^{n-4}+2^{(*)}$Ob Z/2 Uc	Z/2 Z/2
$Q2^n(++)$ Sp, Oa	Σ Oa Sp	$(Z/2)^{2^{n-3}+1}$ Sp	$(Z/2)^{2^{n-3}-1}$Sp	$(Z/2)^{r-1}$ r=# {type Oa}	0 Z/2
$SD2^n(--)$ Ob^-, Uc (n>5), Oa, GL	Σ Oa	Z/2 Uc order $2^{n-4}+2^{(*)}Ob^-$	0	Z/2 Oa Z/2 Uc	Z/2 Z/2
$Q2^n(+,-)$ Sp^-, Oa^-(n>4), GL	$(Z/2)^{2^{n-3}-1}$ Sp^-	$(Z/2)^r$ r=# {type Oa^-}	Σ Oa^- Sp^-	$(Z/2)^{2^{n-3}+1}$ Sp^-	Z/2 Z/2
$Q2^n(-+)$ or (--)(n>4) Ue, Uc (n>5), Oa, GL	Σ Oa	Z/2 Uc Ue	0	Z/2 Uc Ue Oa	Z/2 Z/2

(*) The order of the summand for type Ob^+ or Ob^- is 2^S where S is listed in the table.

(‡) $\tilde{L}_{2k} = Im(L^p_{2k+1}(Z\pi \to \hat{Z}_2\pi) \to L^p_{2k}(Z\pi))$ $\qquad E_{2k} = Im(L^p_{2k}(Z\pi) \longrightarrow L^p_{2k}(\hat{Z}_2\pi))$

Table 2 - $L_*^p(Z\rho,\alpha,u)$ for $\rho \subset \pi$ cyclic of index 2, π a special 2-group

	(‡) $\tilde{L}_0$	L_1	$\tilde{L}_2$	L_3	E(from $\hat{Z}_2\pi$)
$Z/2^n$ $\bar{x}=x^{-1}$, u=x Uj (n>2), Oa^-, Oa^+	Σ Oa^+ Uj	0	Σ Oa^- Uj	0	0 0
$Z/2^n$, $\bar{x}=x$, u=1 Od (n>3), Oc, Oa	Σ Oa	0	0	order 2^{m+2} Od(*) order 8 Oc Z/2 Oa	0 Z/2
$Z/2^n$, $\bar{x}=x^{2^{n-1}+1}$, u=1 Uf, Od, Oc, Oa	Σ Oa	Z/2 Uf	0	order 2^{m+2} Od(*) order 8 Oc Z/2 Oa Uf	0 Z/2
$Z/2^n$, $\bar{x}=x$, $u=x^{2^{n-1}}$ (n>3) Od^-, Od^+, Oc, Oa	Σ Oa	order 2^{m+1} Od^- (*)	0	order 2^{m+2} Od(*) order 8 Oc Z/2 Oa	0 0
$Z/2^n$, $\bar{x}=-x$, u=1 Uf, Ua, GL	Σ Ua	Z/2 Uf	Σ Ua	Z/2 Uf	Z/2 Z/2
$Z/2^n$, $\bar{x}=-x^{2^{n-1}+1}$, u=1 Od, Uf, Ua, GL	Σ Ua	(Z/2) Uf	Σ Ua	order 2^{m+1} Od(*) Z/2 Uf	0 Z/2
$Z/2^n$, $\bar{x}=-x$, $u=x^{2^{n-1}}$ Uf^-, Uf^+, Ua, GL	Σ Ua	Z/2 Uf^- Uf^+	Σ Ua	Z/2 Uf^- Uf^+	Z/2 Z/2
Z/2, $\bar{x}=-x$, u=x GL	0	0	0	0	Z/2 Z/2
Z/4, $\bar{x}=x$, $u=x^2$ Oc^-, Oa	Σ Oa	$(Z/2)^2$ Oc^-	0	Z/2 Oa	0 0

(*) m = # of complex places in the centre of a type Od representation

References

[BL] W. Browder and G.R. Livesay, "Fixed-point free involutions on homotopy spheres", Tohoku Math. J. 25(1973), 69-88.

[CS1] S.E. Cappell and J.L. Shaneson, "Pseudo-free actions I", Algebraic Topology: Aarhus 1978, Springer LN 763(1979), 397-447.

[CS2] S.E. Cappell and J.L. Shaneson, "A counter-example on the oozing problem for closed manifolds". Algebraic Topology: Aarhus 1978, Springer LN 763(1979), 627-634.

[F] W. Feit, Characters of Finite Groups, W.A. Benjamin, New York 1967.

[HM] I. Hambleton and R.J. Milgram, "The surgery obstruction groups for finite 2-groups". Invent. Math. 61(1980), 33-52.

[LM] S. Lopez de Medrano, Involutions on Manifolds Ergebnisse der Mathematik, Band 59, Springer-Verlag, New York-Heidelberg-Berlin 1971.

[M] S. Maumary, "Proper surgery groups and Wall-Novikov groups". Proc. of 1972 Battelle Conf. on Algebraic K-Theory, Vol. III, Springer LN 343(1973), 526-539.

[PR] E.K. Pedersen and A. Ranicki, "Projective surgery theory", Topology 19(1980), 239-254.

[Q] F. Quinn, "A geometric formulation of surgery". Proc. of Georgia Conf. on Topology and Manifolds (1969), 500-512.

[R1] A. Ranicki, "The algebraic theory of surgery I", Proc. London Math. Soc. (3) 40(1980), 87-192.

[R2] A. Ranicki, "The total surgery obstruction", Algebraic Topology: Aarhus 1978, Springer LN 763(1979), 275-315.

[R3] A. Ranicki, "Exact sequences in the algebraic theory of surgery", preprint (1980).

[T] L. Taylor, "Surgery on paracompact manifolds". Berkeley Ph.D. Thesis (1972).

[TW] L. Taylor and B. Williams, "Surgery on closed manifolds", preprint (1980).

[W1] C.T.C. Wall, Surgery on Compact Manifolds. Academic Press: London, New York 1970.

[W2] C.T.C. Wall, "Formulae for surgery obstructions", Topology 15 (1976), 189-210.

[W3] C.T.C. Wall, "Classification of hermitian forms. VI Group rings". Ann. of Math. 103(1976), 1-80.

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