ACYCLIC MAPS AND POINCARE SPACES

Ian HAMBLETON and Jean-Claude HAUSMANN

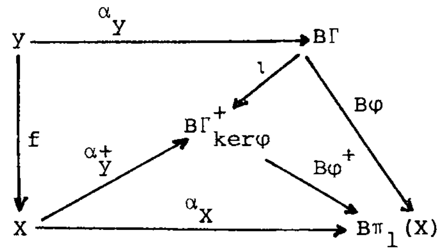
1. The "minus" problem for Poincaré spaces

Recall that a continous map $f : Y \longrightarrow Z$ is called *acyclic* if its homotopy theoretic fiber is an acyclic space, or equivalently if it induces an isomorphism on homology or cohomology with any local coefficients. If the space Y is fixed, the correspondence $f \mapsto \ker \pi_1 f$ produces a bijection between equivalence classes of acyclic maps $f : Y \longrightarrow Z$ and perfect normal subgroups of $\pi_1(Y)$. A representative $Y \longrightarrow Y_P^+$ of the class corresponding to the perfect normal subgroup P of $\pi_1(Y)$ can be obtained by a *Quillen plus construction*, which means that Y_P^+ is obtained by attaching cells of dimension 2 and 3 to Y. For details and other properties of acyclic maps, see [HH].

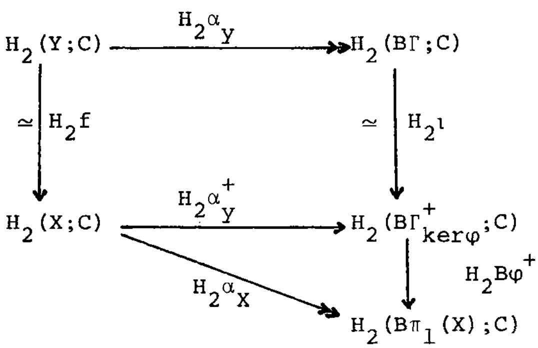
A space X is called a *Poincaré space (of formal dimension n)* if it is homotopy equivalent to a finite complex and if there exists a class $[X] \in H_n(X;\mathbb{Z})$ so that $- \cap X : H^k(X;B) \longrightarrow H_{n-k}(X;B)$ is an isomorphism for any $\mathbb{Z}\pi_1(X)$-module B. If Y is a Poincaré space and $f : Y \longrightarrow X$ an acyclic map with $\pi_1(X)$ finitely presented, then X is a Poincaré space. The homology condition is obviously satisfied for X and it only remains to prove that X is homotopy equivalent to a finite complex. As $\pi_1(X)$ is finitely presented, the group $\pi_1(X)$ is finitely presented iff $\ker\pi_1 f$ is the normal closure of finitely many elements in $\pi_1(Y)$. Hence a space Y_P^+ ($P=\ker\pi_1 f$) homotopy equivalent to X may be obtained by attaching to Y finitely many 2-cells and then the same number of 3-cells.

Let X be a Poincaré space. For each epimorphism $\varphi : \Gamma \twoheadrightarrow \pi_1(X)$ with Γ finitely presented and $\ker\varphi$ perfect, we consider the problem of finding an acyclic map $f : Y \longrightarrow X$, where Y is a Poincaré space, $\pi_1(Y) = \Gamma$ and $\pi_1 f = \varphi$. In other words : is X obtained by performing a plus construction on a Poincaré space with fundamental group Γ) (the "minus" problem for (X,φ)).

First observe that the existence of such an acyclic map $f : Y \to X$ implies some conditions on X. The following commutative diagram :



shows the existence of a lifting $\alpha^+_Y : X \to B\Gamma^+_{\ker\varphi}$ of the characteristic map $\alpha_X : X \to B\pi_1(X)$ (see [H-H, Proposition 3.1]). Moreover, recall that for any space Z, the homomorphism $H_2\alpha_Z : H_2(Z;C) \to H_2(B\pi_1(Z);C)$ is surjective for any $\mathbb{Z}\pi_1(Z)$-module C (since $B\pi_1(Z)$ is obtainable from Z by adding cells of dimension ≥ 3). Hence the following commutative diagram :



shows that for any $\mathbb{Z}\pi_1(X)$-module C, the homomorphisms $H_2\alpha^+_Y$ and $H_2B\varphi^+$ are both surjective. This, of course, implies non-trivial compatibilities between $H_2(X;C)$ and $H_2(B\Gamma;C) = H_2(\Gamma;C)$.

These first remarks suggest a more natural formulation of the above problem, using the following definition :

<u>(1.1) Definition</u> : Let X be a Poincaré space. Let us consider pairs $(\varphi,\tilde{\alpha})$, where :

1) $\varphi : \Gamma \twoheadrightarrow \pi_1(X)$ is an epimorphism of finitely presented groups with $\ker\varphi$ perfect, and

2) $\tilde{\alpha} : X \longrightarrow B\Gamma^{+}_{\ker\varphi}$ makes the following diagram commute :

$$\begin{array}{ccc} & & B\Gamma^{+}_{\ker\varphi} \\ & \overset{\tilde{\alpha}}{\nearrow} & \downarrow B\varphi^{+} \\ X & \xrightarrow{\alpha_X} & B\pi_1(X) \end{array}$$

and $H_2\tilde{\alpha} : H_2(X;C) \longrightarrow H_2(B\Gamma^{+}_{\ker\varphi};C)$ is surjective for any $\mathbb{Z}\pi_1(X)$-module C.

Such a pair $(\varphi,\tilde{\alpha})$ is *realizable* if there exists an acyclic map $f : Y \longrightarrow X$ with Y a Poincaré space, $\pi_1(Y) = \Gamma, \pi_1 f = \varphi$ and $\alpha^{+}_{Y} = \tilde{\alpha}$.

Our problem then becomes : given a Poinçaré space X and a pair $(\varphi,\tilde{\alpha})$ as in (1.1), is this pair realizable ? The answer that we are able to give to this more precise problem is contained in Theorem (1.2) below. Recall that a group G is called *locally perfect* if any finitely generated subgroup of G is contained is a finitely generated perfect subgroup of G.

<u>(1.2) Theorem</u> Let X be a Poincaré space of formal dimension $n \geq 4$.

i) a pair $(\varphi,\tilde{\alpha})$ as in (1.1) determines an element $\sigma(\varphi,\tilde{\alpha})$ in the Wall surgery obstruction group $L_n(\varphi)$. If $(\varphi,\tilde{\alpha})$ is realizable, then $\sigma(\varphi,\tilde{\alpha}) = 0$.

ii) If $\tilde{\alpha}' : X \longrightarrow B\Gamma^{+}_{\ker\varphi}$ is another lifting of α_X such that the pair $(\varphi,\tilde{\alpha}')$ satisfies to the conditions of (1,1), then $\sigma(\varphi,\tilde{\alpha}) = \sigma(\varphi,\tilde{\alpha}')$.

iii) If in addition $n \geq 5$ and $\ker\varphi$ is locally perfect, then $\sigma(\varphi,\tilde{\alpha}) = 0$ implies that $(\varphi,\tilde{\alpha})$ is realizable.

<u>(1.3) Remarks</u> : a) The Wall group used in (1.2) is the obstruction group for surgery to a homotopy equivalence (sometimes called L^h_n). Recall that the group $L_n(\ )$ fits in the exact sequence :

$$\longrightarrow L_n(\Gamma) \xrightarrow{\varphi} L_n(\pi_1(X)) \longrightarrow L_n(\varphi) \longrightarrow L_{n-1}(\Gamma) \longrightarrow$$

b) The same theory holds for simple Poincaré spaces [Wa, Chapter 2]. using simple acyclic maps (the Whitehead torsion of an acyclic map $f : Y \to X$ is well defined in $Wh(\pi_1(X))$;if this torsion vanishes, the acyclic map is called *simple*). The relevant Wall group is then $L_n^s(\varphi)$.

c) The same theory holds for non-orientable Poincaré spaces. The relevant Wall group is then $L_n(\varphi, w_1(X))$, where $w_1(X) : \pi_1(X) \to \mathbb{Z}/2\mathbb{Z}$ is the orientation character for X.

<u>Proof of (1.2)</u> : Write $B\Gamma^+$ for $B\Gamma^+_{\ker\varphi}$. Let us consider the pull-back diagram :

$$\begin{array}{ccc} T & \longrightarrow & B\Gamma \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \iota} \\ X & \xrightarrow{\tilde{\alpha}} & B\Gamma^+ \end{array}$$

The fiber of g is the same as the fiber of ι, therefore g is an acyclic map. If F is the homotopy theoretic fiber of $\tilde{\alpha}$ one has the following diagram :

$$\begin{array}{ccccccccc} \pi_2(X) & \longrightarrow & \pi_1(F) & \longrightarrow & \pi_1(T) & \longrightarrow & \pi_1(X) & \longrightarrow & 1 \\ \downarrow{\scriptstyle \pi_2\tilde{\alpha}} & & \downarrow{\scriptstyle =} & & \downarrow & & \downarrow{\scriptstyle \simeq} & & \downarrow \\ \pi_2(B\Gamma^+) & \longrightarrow & \pi_1(F) & \longrightarrow & \Gamma & \longrightarrow & \Gamma/\ker\varphi & \longrightarrow & 1 \end{array}$$

Hence $\pi_1(T) = \Gamma$ if $\pi_2\tilde{\alpha}$ is surjective. But this is the case, as can be seen by the following diagram :

$$\begin{array}{ccccc} \pi_2(X) & \xrightarrow{\simeq} & H_2(\tilde{X}) & \xrightarrow{\simeq} & H_2(X;\pi_1(X)) \\ \downarrow{\scriptstyle \pi_2\tilde{\alpha}} & & \downarrow & & \downarrow \\ \pi_2(B\Gamma^+) & \xrightarrow{\simeq} & H_2(\widetilde{B\Gamma}^+) & \xrightarrow{\simeq} & H_2(B\Gamma^+;\pi_1(X)) \end{array}$$

the right-hand vertical arrow being surjective by Part b) of (1.1).

Let Z be a space . We denote by $\Omega_n^P(Z)$ (*Poincaré bordism group*) the bordism group of maps $f : U \to Z$ where U is an oriented Poincaré space of formal dimension n. According to the theory of Quinn ([Qn], see [HV2] for proofs), these groups fit in a natural long exact sequence :

$$H_{n+1}(Z;MSG) \longrightarrow L_n(\pi_1(Z)) \longrightarrow \Omega_n^P(Z) \longrightarrow H_n(Z;MSG) \qquad (n\geq 4)$$

If Z' is a subspace of Z, one defines $\Omega_n^P(Z,Z')$ similarly, using Poincaré pairs, and on gets a corresponding sequence. Specializing to $Z = X, Z' = T$ and using the fact that $T \to X$ is an acyclic map, one gets the following commutative diagram in which the rows and columns are exact :

$$\begin{array}{ccccccc}
H_{n+1}(T;MSG) & \longrightarrow & L_n(\Gamma) & \longrightarrow & \Omega_n^P(T) & \longrightarrow & H_n(T;MSG) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{n+1}(X;MSG) & \longrightarrow & L_n(\pi_1(X)) & \longrightarrow & \Omega_n^P(X) & \longrightarrow & H_n(X;MSG) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & L_n(\varphi) & \xrightarrow{\simeq} & \Omega_n^P(X,T) & \longrightarrow & 0
\end{array}$$

This permits us to define $\sigma(\varphi,\tilde{\alpha})$ as the image of $[id_X] \in \Omega_n^P(X)$ under the composite map $\Omega_n^P(X) \to \Omega_n^P(X,T) \simeq L_n(\varphi)$.

Now, suppose that $(\varphi,\tilde{\alpha})$ is realizable by an acyclic map $f : Y \to X$ with Y a Poincaré space. Thus, f factors through a map $f : Y \to T$ representing a class in $\Omega_n^P(T)$. As f is acyclic, its mapping cylinder constitutes a Poincaré cobordism from id_X to f. Therefore, the class $[id_X]$ is mapped to zero in $\Omega_n^P(X,T)$ (since f factors through T) and $(\varphi,\tilde{\alpha}) = 0$. This proves part i) of (1.2).

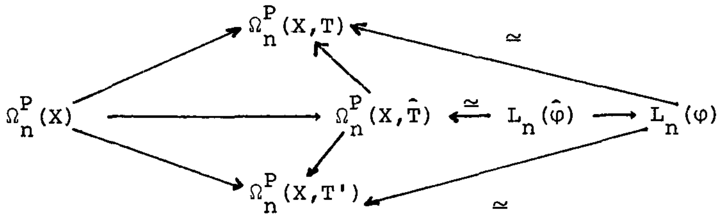
To prove ii), let us consider the pull-back diagram

$$\begin{array}{ccc}
T' & \longrightarrow & B\Gamma \\
\downarrow & & \downarrow \\
X & \xrightarrow{\tilde{\alpha}'} & B\Gamma^+
\end{array}$$

and form again the pull-back diagram

$$\begin{array}{ccc}
\hat{T} & \longrightarrow & T' \\
\downarrow & & \downarrow \\
T & \longrightarrow & X
\end{array}$$

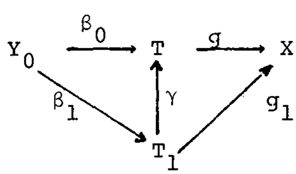
in which all the maps are now acyclic. Then the composed map $\hat{T} \to X$ is also acyclic. Denote by $\hat{\varphi}$: $\hat{\Gamma} = \pi_1(\hat{T}) \twoheadrightarrow \pi_1(X)$ the induced homomorphism. One has a commutative diagram



Therefore, $\sigma(\varphi,\tilde{\alpha})$ and $\sigma(\varphi,\tilde{\alpha}')$ are both image of a single element of $L_n(\Phi)$. This proves Part ii) of (1.2).

Let us finally prove part iii) of (1.2). If $\sigma(\varphi,\tilde{\alpha}) = 0$, then there is a map $\beta_0 : Y_0 \longrightarrow T$ representing a class in $\Omega^P_n(T)$ such that $g\circ\beta_0$ is Poincaré cobordant to id_X. To show that (φ,α) is realizable, we shall find a representative $\beta : Y \longrightarrow T$ of the class β_0 such that $\pi_1\beta$ and $\beta_* : H_*(Y;\mathbb{Z}\pi_1(X)) \longrightarrow H_*(T;\mathbb{Z}\pi_1(X))$ are isomorphisms.

By construction of the space T, the group $\ker\varphi$ acts trivially on $\pi_2(T)$ (use [HH, Proposition 5.4] to the maps ι and g). As $\ker\varphi$ is locally perfect, one can construct, as in [H2, proof of Theorem 3.1], a finite complex T_1 and a commutative diagram :



such that g_1 is an acyclic map and $\pi_1\gamma$ is an isomorphism. Thus, T_1 is a finite complex satisfying Poincaré duality with coefficients $\mathbb{Z}\pi_1(X)$ and β_1 can be covered by a map of the Spivak bundles. By surgery with coefficients for Poincaré spaces (the Cappell-Shaneson type of generalization of [Qn, Corollary 1.4]; for proofs, see [HV2]), the map β_1 determines an element $\sigma(\beta_1) \in \Gamma_n(\varphi)$, where $\Gamma_n(\varphi)$ is the Cappell-Shaneson surgery obstruction group $\Gamma^h_n(\mathbb{Z}\Gamma \longrightarrow \mathbb{Z}\pi_1(X))$ defined in [CS]. The existence of the required map $\beta : Y \longrightarrow T$ will be implied by the nullity of $\sigma(\beta_1)$.

As in [H1, §3] , it can be checked (see [HV2]) that the image of $\sigma(\beta_1)$ under the homomorphism $\Gamma_n(\varphi) \longrightarrow L_n(\pi_1(X))$ is the obstruction to $g_1 \circ \beta_1$ being Poincaré cobordant to a homotopy equivalence. The latter is obviously zero since, by construction, $g_1 \circ \beta_1 = g \circ \beta_0$ is Poincaré cobordant to id_X. Since both Γ and $\pi_1(X)$ are finitely presented, $\ker\varphi$ locally perfect is equivalent to $\ker\varphi$ being the normal closure of a finitely generated perfect group. Therefore, the homomorphism $\Gamma_n(\varphi) \longrightarrow L_n(\pi_1(X))$ is an isomorphism [H1, Theorem 1]. Then $\sigma(\beta_1) = 0$ and Part ii) of (1.2) is proved.

2. The invariant $\sigma(\varphi,\tilde{\alpha})$ as part of a total surgery obstruction theory

Let X be a Poincaré space of formal dimension $n \geq 4$. By (1.2) to each pair $(\varphi,\tilde{\alpha})$ as in (1.1), one can associate the element $\sigma(\varphi,\tilde{\alpha}) \in L_n(\varphi)$. This gives a large collection of invariants associated to X. In this context, Theorem 2.1 of [HV1] may be rephrased as follows :

(2.1) Theorem Let X be a Poincaré space of formal dimension $n \geq 5$. Let $(\varphi,\tilde{\alpha})$ be a pair as in (1.1) with $\ker\varphi$ locally perfect. If X has the homotopy type of a topological closed manifold then $\sigma(\varphi,\tilde{\alpha}) = 0$.

Thus, the elements $\sigma(\varphi,\tilde{\alpha})$ occurs as obstruction for X being homotopy equivalent to a closed topological manifold and we can except some relationship between our $\sigma(\varphi,\tilde{\alpha})$'s and the total surgery obstruction of [Ra]. We are indebted to A. Ranicki for pointing out a mistake in our first draft of this section.

Let X be a Poincaré space of formal dimension ≥ 5. According to [Ra], there is an exact sequence :

$$(2.1) \quad \ldots \to \mathcal{S}_{m+1}(X) \to H_m(X;\underline{\mathbb{L}}_0) \to L_m(\pi_1(X)) \to \mathcal{S}_m(X) \to H_{m-1}(X;\underline{\mathbb{L}}_0) \to \ldots$$

and an element $s(X) \in \mathcal{S}_n(X)$ which vanishes if and only if X is homotopy equivalent to a closed topological manifold. Here the groups are defined for $m \geq 0$ by

$$\mathcal{S}_m(X) = \pi_m(\sigma_* : X_+ \wedge \underline{\mathbb{L}}_0 \longrightarrow \underline{\mathbb{L}}_0(\pi_1(X)))$$

where σ_* is the assembly map and $\underline{\mathbb{L}}_0$ is the 1-connective covering of the spectrum $\underline{\mathbb{L}}_0(1)$ (see [Ra, p.285]; we use the notations of [Ra]). Observe that our definition of $\mathcal{S}_m(X)$ slightly differs from the one in [Ra] (we take the whole spectrum $\underline{\mathbb{L}}_0(\pi_1(X))$ instead of its 1-connective covering). This difference only affects the group $\mathcal{S}_0(X)$. Since the assembly map σ_* can be extended to $\bar{\sigma}_* : X_+ \wedge \underline{\mathbb{L}}_0(1) \longrightarrow \underline{\mathbb{L}}_0(\pi_1(X))$ we can define : $\bar{\mathcal{S}}_m(X) = \pi_m(\bar{\sigma}_*)$. This gives the exact sequences :

$$\to \bar{\mathcal{S}}_{m+1}(X) \to H_m(X;\underline{\mathbb{L}}_0(1)) \to L_m(\pi_1(X)) \to \bar{\mathcal{S}}_m(X) \to H_{m-1}(X;\underline{\mathbb{L}}_0(1))) \to$$

and

$$(2.2)\qquad \cdots \longrightarrow H_m(X;\mathbb{Z}) \longrightarrow \mathscr{S}_m(X) \xrightarrow{\lambda_m} \overline{\mathscr{S}}_m(X) \longrightarrow H_{m-1}(X;\mathbb{Z}) \longrightarrow \cdots$$

Let us define $\overline{s}(X) = \lambda_n(s(X)) \in \overline{\mathscr{S}}_n(X)$.

If $(\varphi,\tilde{\alpha})$ is any pair for X as in (1.1), consider the pull-back diagram :

$$\begin{array}{ccc} T & \longrightarrow & B\Gamma \\ {\scriptstyle g}\downarrow & & \downarrow \\ X & \xrightarrow{\tilde{\alpha}} & B\Gamma^+ \end{array}$$

which gives rise to the following diagram :

$$(2.3)\qquad \begin{array}{ccccccc} H_n(T;\mathbb{L}_0) & \longrightarrow & L_n(\Gamma) & \longrightarrow & \mathscr{S}_n(T) & \longrightarrow & H_{n-1}(T;\mathbb{L}_0) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_n(X;\mathbb{L}_0) & \longrightarrow & L_n(\pi_1(X)) & \longrightarrow & \mathscr{S}_n(X) & \longrightarrow & H_{n-1}(X;\mathbb{L}_0) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L_n(\varphi) & \xrightarrow{\simeq} & \mathscr{S}_n(X,T) & \longrightarrow & 0 \end{array}$$

in which rows and collumns are exact. One has also the corresponding diagram for $\overline{\mathscr{S}}_m(X)$. Let $\eta_m : \mathscr{S}_m(X) \longrightarrow L_m(\varphi)$ be the composed homomorphism $\mathscr{S}_m(X) \longrightarrow \mathscr{S}_m(X,T) \xleftarrow{\simeq} L_m(\varphi)$. Define $\overline{\eta}_m : \overline{\mathscr{S}}_m(X) \longrightarrow L_m(\varphi)$ accordingly, and notice that $\eta_m = \overline{\eta}_m \circ \lambda_m$.

<u>(2.4) Proposition</u> In $L_n(\varphi)$, one has the equalities :

$$\eta_n(s(X)) = \overline{\eta}_n(\overline{s}(X)) = \sigma(\varphi,\tilde{\alpha}).$$

<u>Proof</u> This follows directly from the definitions, since there is a homomorphism $\delta_X : \Omega_n^P(X) \longrightarrow \mathscr{S}_n(X)$ such that the following diagram

$$\begin{array}{ccccccc} L_n(\pi_1(X)) & \longrightarrow & \Omega_n^P(X) & \xrightarrow{\delta_X} & \mathscr{S}_n(X) & \xrightarrow{\lambda_n} & \overline{\mathscr{S}}_n(X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L_n(\varphi) & \xrightarrow{\simeq} & \Omega_n^P(X,T) & \xrightarrow[\simeq]{\delta_{X,T}} & \mathscr{S}_n(X,T) & \xrightarrow{\simeq} & \overline{\mathscr{S}}_n(X,T) \end{array}$$

commutes and $\delta_X([id_X]) = s(x)$ [Ra,pp.307-308].

(2.5) Corollary Let X be a Poincaré complex of formal dimension $n \geq 5$, and let $(\varphi, \tilde{\alpha})$ a pair as in (1.1). Suppose that the Spivak bundle for X has a TOP-reduction ξ which defines a surgery obstruction $\sigma(\xi) \in L_n(\pi_1(X))$. Then, $\sigma(\varphi, \tilde{\alpha})$ is the image of $\sigma(\xi)$ under the homomorphism $L_n(\pi_1(X)) \longrightarrow L_n(\varphi)$.

Proof By [Ra,p. 298] , the element $\sigma(\xi)$ has image $s(X)$ under the homomorphism $L_n(\pi_1(X)) \longrightarrow \mathcal{S}_n(X)$. The result thus follows from (2.4). Thus, if $\bar{s}(X) = 0$, one has $\sigma(\varphi, \tilde{\alpha}) = 0$ for any pair $(\varphi, \tilde{\alpha})$ as in (1.1). A converse to this fact might be obtained by considering some "test pairs" $(\varphi_X, \tilde{\alpha}_X)$ for X as follows : let $\mathscr{A}_i$, $i=0,1,\dots$, and $\mathscr{A} = \cup_i \mathscr{A}_i$ be the smallest classes of groups such that :

$\mathscr{A}_0$ contains the trivial group
$G \in \mathscr{A}_i$ iff at least one of the following conditions holds :

(a) there exist groups G_1, G_2 and $G_0 = G_1 \cap G_2$, all in $\mathscr{A}_{i-1}$ such that $G = G_1 *_{G_0} G_2$ and the inclusions $G_0 \subset G_i$ are $\sqrt{}$-closed in the sense of [Cl] : if $g \in G_i$ and $g^2 \in G_0$ then $g \in G_0$.

or

(b) $G = G_0 \times \mathbb{Z}$, with $G_0 \in \mathscr{A}_{i-1}$

(2.6) Proposition Let X be a finite complex of dimension n. Then there exists a pair $(\varphi_X : \Gamma_X \longrightarrow \pi_1(X), \tilde{\alpha}_X)$ satisfying 1) and 2) of (1.1) such that :

1) $\Gamma_X \in \mathscr{A}$
2) $B\Gamma_X$ is a finite complex of dimension n
3) $\tilde{\alpha}_X$ is a homotopy equivalence.

The pair $(\varphi_X, \tilde{\alpha}_X)$ is associated to a triangulation of X, according an algorithm as in [B-D-H] or [Ma]. Its construction is given in §4.

Recall that a standard conjecture is that $\tilde{K}_0(G) = 0 = Wh(G)$ for $G \in \mathscr{A}$(1). (or even for G such that BG is a finite complex).

(2.7) Theorem Suppose that $\tilde{K}_0(G) = Wh(G) = 0$ for all $G \in \mathscr{A}$. Then, for X a Poincaré space of formal dimension $n \geq 5$, the following conditions are equivalent :

(1) P. Vogel informs us that he has recently obtained a proof of this conjecture.

1) $\bar{s}(X) = 0$
2) $\sigma(\varphi,\tilde{\alpha}) = 0$ for any pair $(\varphi,\tilde{\alpha})$ for X as in (1.1)
3) $\sigma(\varphi_X,\tilde{\alpha}_X) = 0$ for some pair $(\varphi_X,\tilde{\alpha}_X)$ of (2.6).

<u>Proof</u> : Condition 1) implies Condition 2) by (2.4). The implication from 2) to 3) is straightforward. Therefore it remains to prove that 3) implies 1). As the map $\tilde{\alpha}_X$ is a homotopy equivalence, the diagram for $\mathcal{S}_m(X)$ similar to (2.3) gives the long exact sequence :

$$(2.8) \quad \dots \overline{\overline{\mathcal{S}}}_m(B\Gamma_X) \longrightarrow \overline{\overline{\mathcal{S}}}_m(X) \xrightarrow{\bar{\eta}_m} L_m(\varphi_X) \longrightarrow \overline{\overline{\mathcal{S}}}_{m-1}(B\Gamma_X) \longrightarrow \dots$$

Therefore, it suffices to establish that $\overline{\overline{\mathcal{S}}}_m(B\Gamma_X) = 0$ for $m \geq n$. As $\dim B\Gamma_X = n$, this follows from the following lemma :

<u>(2.9) Lemma</u> Let $G \in \mathcal{A}$ such that $\tilde{K}_0(P) = 0 = Wh(P)$ for any subgroup P of G with $P \in \mathcal{A}$. Then the homomorphism

$$\bar{\sigma}_m : H_m(G;\underline{\mathbb{L}}_0(1)) \longrightarrow L_m(G)$$

induced by the assembly map $\bar{\sigma}_*$ is an isomorphism for $m \geq \dim BG$ and is injective for $m = \dim BG - 1$.

<u>Proof</u> We shall prove Lemma (2.9) for $G \in \mathcal{A}_j$ by induction on j, using the classical idea of S. Cappell [C3] . The class $\mathcal{A}_0$ contains only the trivial group and $H_m(pt;\underline{\mathbb{L}}_0(1))$ is isomorphic to $L_m(1)$ for $m \geq 0$ (this is the main point where we need the spectrum $\underline{\mathbb{L}}_0(1)$ instead of $\underline{\mathbb{L}}_0$). Also $H_{-1}(pt;\underline{\mathbb{L}}_0(1)) = 0$, thus lemma (2.9) is proved for $G \in \mathcal{A}_0$.

If now $G \in \mathcal{A}_j$, then

$$\begin{array}{ccccccc}
H_m(BG_0;\underline{\mathbb{L}}_0(1)) & \to & H_m(BG_1;\underline{\mathbb{L}}_0(1)) \oplus H_m(BG_2;\underline{\mathbb{L}}_0(1)) & \to & H_m(BG;\underline{\mathbb{L}}_0(1)) & \to & H_{m-1}(BG_0;\underline{\mathbb{L}}_0(1)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L_m(G_0) & \longrightarrow & L_m(G_1) \oplus L_m(G_2) & \longrightarrow & L_m(G) & \longrightarrow & L_{m-1}(G_0)
\end{array}$$

in the first case and

$$\begin{array}{ccccccccc}
0 & \to & H_m(G_0;\underline{\mathbb{L}}_0(1)) & \to & H_m(G;\underline{\mathbb{L}}_0(1)) & \to & H_{m-1}(G_0;\underline{\mathbb{L}}_0(1)) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & L_m(G_0) & \longrightarrow & L_m(G) & \longrightarrow & L_{m-1}(G_0) & \longrightarrow & 0
\end{array}$$

in the second case, in which all the rows are exact. The exact sequences involving L-groups are those of [C1] . As dim BG_1 and dim BG_2 are $\leq$ dim BG and dim $BG_0 \leq$ dim BG-1 (in both cases), the induction step follows from the five lemma.

Using Exact sequences (2.2) and (2.3) together with Lemma (2.9), one obtains the following theorem :

(2.10) Theorem Suppose that $\tilde{K}_0(G) = 0 = Wh(G)$, for all $G \in \mathscr{A}$. Let X be a Poincaré space of formal dimension $n \geq 5$ and let $(\varphi_X, \tilde{\alpha}_X)$ be a pair as in (2.6). Then :

a) $\eta_m : \mathscr{S}_m(X) \longrightarrow L_m(\varphi_X)$ is an isomorphism for $m \geq n+2$

b) One has an exact sequence :

$$0 \longrightarrow \mathscr{S}_{n+1}(X) \xrightarrow{\eta_{n+1}} L_{n+1}(\varphi_X) \longrightarrow \mathbb{Z} \longrightarrow \mathscr{S}_n(X) \xrightarrow{\eta_n} L_n(\varphi_X)$$

Finally, we mention the following proposition which will be of interest in Remarks 4 and 5 below :

(2.11) Proposition Let G be a group as in (2.9) such that BG is a (finite) complex of dimension n . Let X be a space with $\pi_1(X) = G$ and such that the canonical map $X \longrightarrow BG$ induces an isomorphism on integral homology. Then $\mathscr{S}_m(X) = \bar{\mathscr{S}}_m(X) = 0$ for $m>n$, $\mathscr{S}_n(X) \cong \mathbb{Z}$ and $\bar{\mathscr{S}}_n(X) = 0$.

Proof This follows from Lemma (2.9) and from the comparison of the exact sequences (2.1) and (2.1 bis) for X and for BG.

2.12) Remarks 1) If one is interested in Statements (2.9), (2.10) and (2.11) only modulo 2-torsion, one can drop the assumption $\tilde{K}_0(G) = 0 = Wh(G)$ for $G \in \mathscr{A}$ as well as the condition $\sqrt{\ }$-closed in the definition of the class $\mathscr{A}$ (this would simplify §4). Indeed, the exact sequences of surgery groups used in the proof of (2.9) always exist when all the groups are tensored by $\mathbb{Z}[1/2]$.

2) From Proposition (2.11), it follows that $\mathscr{S}_m(B\mathbb{Z}^n) = 0$ for $m>n$ and $\mathscr{S}_n(B\mathbb{Z}^n) = \mathbb{Z}$. This result is mentioned in [Ra, p.310].

3) The class $\mathscr{A}$ has been chosen minimal in order to obtain (2.6) and (2.7). But Lemma (2.9) is valid for a larger class in which we allow HNN-extension (with the relevant $\sqrt{\ }$-closed condition). As in 2), one is then able to prove for instance that $\mathscr{S}_m(X) = 0$ for m>3 and $\mathscr{S}_3(X) = \mathbb{Z}$ for X belonging to a large class of sufficiently large 3-manifolds (the result is valid mod 2-torsion for all sufficiently large 3-manifolds).

4) We now construct a Poincaré space Y of formal dimension n such that $\sigma(\varphi,\tilde{\alpha}) = 0$ for all pairs $(\varphi,\tilde{\alpha})$ for Y as in (1.1) but which is not homotopy equivalent to a closed topological manifold. We assume that $\tilde{K}_0(G) = 0 = Wh(G)$ for all $G \in \mathscr{A}$ thus it suffices to prove that $\bar{s}(Y) = 0$ by (2.7).

We apply (2.6) to the case $X = S^n$. We thus obtain a group $\Gamma_n \in \mathscr{A}$ such that $B\Gamma_n$ is a finite complex of dimension n and $H_*(B\Gamma_n;\mathbb{Z}) \cong H_*(S^n;\mathbb{Z})$.

The Atiyah-Hirzebruch spectral sequence shows that $H_m(B\Gamma_n;\mathbb{L}_0) = L_m(1)$ for $1 \le m \le n$ and the homomorphism $H_m(B\Gamma_n;\mathbb{L}_0) \longrightarrow L_m(\Gamma_n)$ induced by the assembly map coincides with the inclusion $L_m(1) \rightarrowtail L_m(\Gamma_n)$. Thus, the reduced surgery group $\tilde{L}_n(\Gamma_n) = \mathrm{coker}(L_n(1) \longrightarrow L_n(\Gamma_n))$ is isomorphic to $\mathscr{S}_n(B\Gamma_n) = \mathbb{Z}$ by (2.1) and (2.11).

Let us consider the Poincaré homology sphere bordism group $\Omega_n^{PHS}(B\Gamma_n)$ defined in [H3], whose elements are represented by maps $f : \Sigma \longrightarrow B\Gamma_n$, where Σ is an oriented Poincaré space with the homology of S^n. For $n \ge 6$, the theory of [H3] gives an isomorphism :

$$\Omega_n^{PHS}(B\Gamma_n) \cong \pi_n(S^n) \oplus \tilde{L}_n(\Gamma_n) \cong \mathbb{Z} \oplus \mathbb{Z}$$

so that the class of $f : \Sigma \longrightarrow B\Gamma_n$ corresponds to the pair $(\deg f, \widetilde{f_*(\sigma)})$, where $\sigma \in L_n(\pi_1(\Sigma))$ is the surgery obstruction for any surgery problem with target Σ. As Γ_n is finitely presented and $H_1(\Gamma_n;\mathbb{Z}) = H_2(\Gamma_n;\mathbb{Z}) = 0$, it actually follows from [H3, "proof of the surjectivity of σ_n"] that for any class of $\Omega_n^{PHS}(B\Gamma_n)$ has a representative $f : \Sigma \longrightarrow B\Gamma_n$ with $\pi_1 f$ an isomorphism . Therefore, the pair $(1,k)$ with $k \neq 0$ corresponds to a map $f : Y \longrightarrow B\Gamma_n$ such that :

- f induces an isomorphism on the fundamental groups
- f induces an isomorphism on integral homology (since deg f = 1)
- Y has not the homotopy type of a closed topological manifold (otherwise k would be zero).
- $\bar{s}(Y) = 0$ (since $\mathcal{S}_n(Y) = 0$ by (2.11)).

5) The following is a version of the Novikov Conjecture : if G is a group such that BG is a Poincaré space of formal dimension n, then

a) $\mathcal{S}_m(BG) = 0$ for $m>n$ and $\mathcal{S}_n(BG) = \mathbb{Z}$
b) $s(BG) = 0$

Proposition (2.11) shows that a) is satisfied if $G \in \mathscr{A}$ (modulo the vanishing assumptions on $\tilde{K}_0$ and Wh). On the other hand, the space Y of Remark 4) above has fundamental group $\Gamma_n \in \mathscr{A}$, the same integral homology as $B\Gamma_n$ and thus satisfies a) by (2.11). But $s(Y) \neq 0$. This shows some independence between condition a) and b) and emphasizes the importance of the assumption that BG itself be a Poincaré space in the Novikov conjecture.

3. Homotopy equivalences of closed manifolds

As one might except, the results of §1 and 2 have analogues for homotopy equivalences of closed manifolds. We give here the "simple homotopy" version of this theory, which seems more natural in this framework.

(3.1) Theorem Let $j : M \longrightarrow N$ be a simple homotopy equivalence between closed manifolds of dimension $n \geq 5$. Then any pair $(\varphi, \tilde{\alpha})$ for N as in (1.1) with $\ker\varphi$ locally perfect determines an element $\sigma(j, \varphi, \tilde{\alpha}) \in L^s_{n+1}(\varphi)$ such that the following three conditions are equivalent :

a) there is a commutative diagram :

$$\begin{array}{ccccc} M_- & \xrightarrow{j_-} & N_- & \xrightarrow{\alpha_{N_-}} & B\Gamma \\ {\scriptstyle f_M}\downarrow & & {\scriptstyle f_N}\downarrow & & \downarrow{\scriptstyle \iota} \\ M & \xrightarrow{j} & N & \xrightarrow{\tilde{\alpha}} & B\Gamma^+ \end{array}$$

where M_- and N_- are closed manifolds, f_M and f_N are simple acyclic maps and j_- is a simple homotopy equivalence.

b) any commutative diagram

$$\begin{array}{ccccc} & & N_- & \xrightarrow{\alpha_{N_-}} & B\Gamma \\ & & {\scriptstyle f_N}\downarrow & & \downarrow{\scriptstyle \iota} \\ M & \xrightarrow{j} & N & \xrightarrow{\tilde{\alpha}} & B\Gamma^+ \end{array}$$

with N_- a closed manifold and f_N a simple acyclic map can be completed in a diagram as in a).

c) $\sigma(j, \varphi, \tilde{\alpha}) = 0$.

Proof Recall that in the proof of (1.2) we checked that in the pull-back diagram :

$$\begin{array}{ccc} T & \longrightarrow & B\Gamma \\ \downarrow{\scriptstyle g} & & \downarrow \\ N & \xrightarrow{\tilde{\alpha}} & B\Gamma^+ \end{array}$$

the map g is acyclic, $\pi_1(T) = \Gamma$ and $\ker\varphi$ acts trivially on $\pi_2(T)$. By [H2, Theorem 3.1], there is a commutative diagram :

$$\begin{array}{ccc} N_- & \longrightarrow & T \\ {\scriptstyle f_N}\searrow & & \swarrow{\scriptstyle g} \\ & N & \end{array}$$

such that f_N is a simple acyclic map and $\pi_1(N_-) = \pi_1(T) = \Gamma$. (This existence of f_N shows that b) implies a).)

For P a closed manifold of dimension n, let $\mathscr{S}_{TOP}(P)$ be the Sullivan-Wall set of topological structures on P [Wa, Chapter 10] According to [Ra, p.277] there is an identification $\mathscr{S}_{TOP}(P) \xrightarrow{\simeq} \mathscr{S}_{n+1}(P)$. Let $h : Q_- \longrightarrow N_-$ represent a class in $\mathscr{S}_{TOP}(N_-)$. Using a simple plus cobordism (W,Q_-,Q) (i.e. $Q_-^+ \simeq W$) one gets a simple homotopy equivalence $h^+ : Q \longrightarrow N$ whose class in $\mathscr{S}_{TOP}(N)$ is well defined. One checks that this correspondance $[h] \longrightarrow [h^+]$ is actually given by the composite :
$\mathscr{S}_{TOP}(N_-) \xrightarrow{\simeq} \mathscr{S}_{n+1}(N_-) \xrightarrow{f_{N*}} \mathscr{S}_{n+1}(N) \xrightarrow{\simeq} \mathscr{S}_{TOP}(N)$. Finally, observe that one has the following commutative diagram :

$$\begin{array}{ccc} \mathscr{S}_{n+1}(N_-) & \xrightarrow{f_{N*}} & \mathscr{S}_{n+1}(N) \\ {\scriptstyle \simeq}\searrow & & \nearrow{\scriptstyle g_*} \\ & \mathscr{S}_{n+1}(T) & \end{array}$$

The map $\mathscr{S}_{n+1}(N_-) \longrightarrow \mathscr{S}_{n+1}(T)$ is an isomorphism by the Ranicki exact sequence [Ra, p.276] indeed the map $N_- \longrightarrow T$ induces an isomorphism on the funcamental groups and on the homology.

These considerations make Theorem (3.1) straightforward if we define $\sigma(j,\varphi,\tilde{\alpha})$ to be the image of $[j] \in \mathscr{S}_{TOP}(N)$ under the composite map $\mathscr{S}_{TOP}(N) \xrightarrow{\simeq} \mathscr{S}_{n+1}(N) \xrightarrow{\eta_{n+1}} L_{n+1}(\varphi)$ (see (2.3) and (2.4)).

If $(\varphi_N,\tilde{\alpha}_N)$ is a pair for N as in (2.6), the homomorphism $\mathscr{S}_{n+1} : \mathscr{S}_{n+1}(N) \longrightarrow L_{n+1}(\varphi_N)$ is injective by (2.10). One thus obtains the analogue of (2.7) :

<u>(3.2) Theorem</u> Let $j : M \longrightarrow N$ as in (3.1). Assume that $\tilde{K}_0(G) = $ $= Wh(G) = 0$ for all $G \in \mathscr{A}$. Then, the following conditions are equivalent :

1) j is homotopic to a homeomorphism
2) $\sigma(j,\varphi,\tilde{\alpha}) = 0$ for all pair $(\varphi,\tilde{\alpha})$ for N as in (1.1)
3) $(j,\varphi_N,\tilde{\alpha}_N) = 0$ for some pair $(\varphi_N,\tilde{\alpha}_N)$ for N as in (2.6)

4. Proof of Proposition (2.6)

Our proof makes use of Statements (4.1)-(4.4) below. The proof of (4.1) is given at the end of this section.

(4.1) Lemma Let R_i ($i \in I$) be a familly of groups having a common subgroup B and let R be the amalgamated product $(*_B)_{i\in I} R_i$. Let S be a subgroup of R and let $S_i = S \cap R_i$. Suppose that the following conditions hold :

1) the union of S_i's generates S
2) S_i is $\sqrt{\ }$-closed in R_i for all i
3) if $s_i b \hat{s}_i \in B$ with $s_i, \hat{s}_i \in S_i$ and $b \in B$, then $b \in S_i$.

Then S is $\sqrt{\ }$-closed in R.

(4.2) Examples a) Condition 3) holds trivially if $B \subset S_i$ for all $i \in I$. For instance, if $B = 1$, case of a free product.

b) If B is $\sqrt{\ }$-closed in R_i for all $i \in I$, then B is $\sqrt{\ }$-closed in R (case $S_i = B$).

c) If $J \in I$ and B is $\sqrt{\ }$-closed in R_i for $i \in I \setminus J$, then the subgroup generated by $\cup_{i\in J} R_i$ is $\sqrt{\ }$-closed in R. (Take $S_i = R_i$ for $i \in J$ and $S_i = B$ for $i \notin J$).

(4.3) Lemma If G_1 and G_2 are groups in $\mathscr{A}$, so is $G_1 \times G_2$.

Proof Let $G_1 \in \mathscr{A}_m$ and $G_2 \in \mathscr{A}_n$. The proof is by induction on m+n. The statement is trivial if m+n = 0 and the induction step is easily obtained, using the isomorphisms
$G_1 \times (G_2 *_G G_3) = (G_1 \times G_2) *_{G_1 \times G} (G_1 \times G_3)$ and $G_1 \times (\mathbb{Z} \times G) = (G_1 \times G) \times \mathbb{Z}$.

(4.4) Lemma There exists an acyclic group A in $\mathscr{A}_4$ such that dim BA = 2. **(G acyclic** means that $H_*(BG;\mathbb{Z}) = 0$ where $\mathbb{Z}$ is endowed with the trivial G-action).

Proof : Let $G = \langle a,b \mid a^3 = b^5 \rangle$ (the group of the (3.5)-torus knot; one could take another (p,q)-knot with p and q relatively prime odd integers). The group G belongs to $\mathscr{A}_2$. One has G/[G,G] infinite cyclic generated by $m = a^{-1}b^2$. The commutator group [G,G] is free of rank 8 on $[a^i,b^j]$ for $i = 1,2$ and $1 \le j \le 4$. The center $\zeta(G)$ of G is infinite cyclic on a^3.

(4.4.a) Sublemma The equation $m^k x m^{-k} = x^{-1}$ is possible in G only if $x = 1$. The equation $m^k x m^{-k} = x$ is possible in G iff $x = m^i z$ with $z \in \zeta(G)$.

As the proof of (4.1), our proof of (4.4.a) uses the Serre theory of groups acting on trees. It is also posponed till the end of this section.

The element $u = [a,b]$ generates a $\sqrt{\ }$-closed subgroup U in G. Indeed, U is $\sqrt{\ }$-closed in [G,G] (since u is part of a basis of [G,G]) and [G,G] is $\sqrt{\ }$-closed in G (since G/[G,G] has no 2-torsion). On the other hand, the element m generates a subgroup M of G which is also $\sqrt{\ }$-closed. Indeed, suppose that $g^2 = m^k$. As G/[G, G] is infinite cyclic generated by m, one has $k = 2i$ and $g = ym^i$ with $y \in [G,G]$. Then, one has $m^{2i} = g^2 = ym^i ym^i = ym^i ym^{-i} m^{2i}$ which implies $m^i y m^{-i} = y^{-1}$. Thus $y = 1$ by (4.4.a).

Let G_1 and G_2 be two copies of G, with corresponding elements m_1,u_1 and m_2,u_2. By the above, the group $P = G_1 * G_2/\{m_1=u_2\}$ is in the class $\mathscr{A}_3$. By the Mayer-Vietoris sequence for amalgamated products, one checks easily that $H_*(P) = 0$ if $* \neq 0,1$ and $H_1(P) = \mathbb{Z}$, generated by m_2.

Let us consider the subgroup Q of P generated by u_1 and m_2. As $M \cap U = (1)$ in G, Q is free on u_1 and m_2 [Se, Corollary p.14]. and we have $Q \cap G_1 = U_1$ and $Q \cap G_2 = M_2$. We will prove that Q is $\sqrt{\ }$-closed in P, using (4.1) with $R_i = G_i$, $Q = S$, $S_1 = U_1$ and $S_2 = M_2$. It just remains to check Condition 3) of (4.1) which we do by showing that the equations $m^i u^s m^j = u^t$ and $u^i m^s u^j = m^t$ are possible in G only if $s = t = 1$.

Let us first consider the equation $m^i u^s m^j = u^t$. Passing to G/[G,G] shows that $j = -i$. Thus u^t is the image of u^s under an automorphism of the free group [G,G]. This implies that $t = \pm s$. One checks easily that this contradicts (4.4.a).

As for the equation $u^i m^s u^j = m^t$, one must have $s = t$ for homological reasons. The equation is then equivalent to $m^s u^j m^{-s} = u^{-i}$

which drives us back to the former case.

Let $\overline{P}$ be another copy of P. By the above, the group $A = P \times \overline{P}/\{m_2=\overline{u}_1,\ u_1=\overline{m}_2\}$ belongs to $\mathscr{A}_4$. Using the Mayer-Vietoris sequence again, one checks that A is acyclic. Observe that dim BA=2.

(4.5) Remarks on the proof of (4.4) : a) The subgroup $U_1 \subset G_1 \subset Q \subset A$ generated by u_1 is $\sqrt{\ }$-closed in A. Indeed, U_1 is $\sqrt{\ }$-closed in $Q = U_1 \times M_1$ and Q is $\sqrt{\ }$-closed in A by (4.2.b).

b) Acyclic groups can be obtained by the amalgamation of two copies of a free group F of rank 2 over a suitable subgroup 5 (see [BDH , p.11]). Problem : find such a situation where S is $\sqrt{\ }$-closed in F.

(4.6) Proof of Proposition (2.6) Following the procedure of [Ma], we consider for any polyedron L (polyedron = finite simplicial complex) the following condition $\mathscr{M}(L)$:

Condition $\mathscr{M}(L)$: There exists a map $t : (UL,TL) \longrightarrow (CL,L)$ (where CL denotes the cone over L) such that, for each connected subpolyedron M of L, one has :

a) $t|t^{-1}(CM) : t^{-1}(CM) \longrightarrow CM$ and $t|t^{-1}(M) : t^{-1}(M) \longrightarrow M$ are acyclic maps

b) $t^{-1}(CM) = B\Gamma_{CM}$ and $t^{-1}(M) = B\Gamma_M$, where Γ_M and Γ_{CM} are groups in $\mathscr{A}$; moreover, dim $B\Gamma_M$ = dim M and dim$B\Gamma_{CM}$ = dimM + 1

c) $\ker(\Gamma_M \longrightarrow \pi_1(M))$ is locally perfect

d) If M' is a connected subpolyedron of L containing M, the inclusion $t^{-1}(CM,M) \subset t^{-1}(CM',M')$ induces four homomorphisms

$$\begin{array}{ccc} \Gamma_M & \hookrightarrow & \Gamma_{CM} \\ \downarrow & & \downarrow \\ \Gamma_{M'} & \longrightarrow & \Gamma_{CM'} \end{array}$$

which are all monomorphisms and $\sqrt{\ }$-closed (a monomorphism $\gamma : G \to G'$ is $\sqrt{\ }$-closed if $\gamma(G)$ is $\sqrt{\ }$-closed in G').

We shall prove that Condition $\mathscr{M}(L)$ holds by induction on dim L .

dim L = 0 One takes t to be the identity map.

dim L = 1 One takes t to be the identity map on TL = L and on the

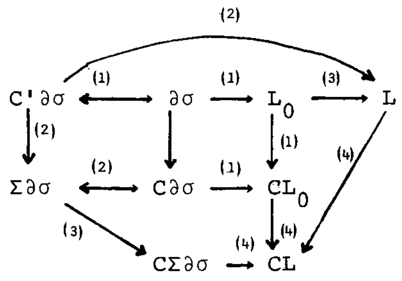
1-skeleton $UL^{(1)}$ of UL which is $L \cup C(L^{(0)})$. Let A be the acyclic group constructed for (4.4) and $u_1 \in A$ be the element considered in (4.5.a). Then BA can be taken to be a polyedron having a subpolyedron isomorphic to the boundary of a 2-simplex which represent the class u_1. Form the polyedron

$$UL = UL^{(1)} \amalg (\coprod_{\sigma} (CA)_{\sigma}) / \{\partial\sigma = (u_1)_{\sigma}\}$$

where $(BA)_{\sigma}$ is a copy of BA and σ runs over the set of 2-cells of CL. One easily check Conditions a)-d), using (4.4),(4.5.a), (4.2.b) and (4.2.c) for the latter.

Induction step : one assumes by induction that $\mathcal{M}(L)$ holds if $\dim L \le n-1$. By induction on the number of n-cells of L, it is enough to prove that $\mathcal{M}(L_0)$ implies $\mathcal{M}(L)$ when L is the union of L_0 with one n-simplex σ. As $n \ge 2$, $\partial\sigma$ is connected and one may assume that L_0 is connected.

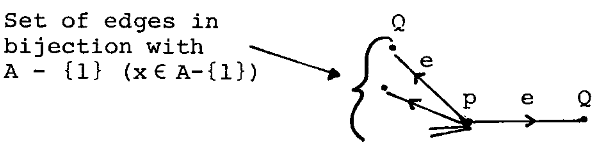
As $\mathcal{M}(L_0)$ holds, $t^{-1}(C\partial\sigma) = U\partial\sigma$ and $t^{-1}(\partial\sigma) = T\partial\sigma$ are subpolyedra of UL_0 and TL_0 respectively. Let TL be $TL_0 \cup U'\partial\sigma$, where $U'\partial\sigma$ is another copy of $U\partial\sigma$ attached to $T\partial\sigma$ and extend t to TL by sending $U'\partial\sigma$ to σ . Then $TL = B\Gamma_L$ where Γ_L is the free product $\Gamma_{L_0} * \Gamma_{C'\partial\sigma}$ with amalgamation over $\Gamma_{\partial\sigma}$ (where $C'\partial\sigma$ is another copy of $C\partial\sigma$). Observe also that $t^{-1}(L \cup CL_0) = B\Gamma_{L \cup CL_0}$, where $\Gamma_{L \cup CL_0}$ is the free product $\Gamma_{C'\partial\sigma} * \Gamma_{CL_0}$ with amalgamation over $\Gamma_{\partial\sigma}$ and that $\Gamma_{C'\partial\sigma} (*_{\Gamma_{\partial\sigma}}) \Gamma_{C_{\partial\sigma}} = \Gamma_{\Sigma(\partial\sigma)}$ is a subgroup of $\Gamma_{L \cup CL_0}$. As in [BDH, Theorem 6.1] one embedds $\Gamma_{\Sigma(\partial\sigma)}$ into the acyclic group $(A \times \Gamma_{\partial\sigma}) * \Gamma_{C\partial\sigma} = \Gamma_{C\Sigma\partial\sigma}$ (amalgamation over $\Gamma_{\partial\sigma}$; A is the acyclic group of (4.4)) by sending $g \longrightarrow g$ if $g \in \Gamma_{C\partial\sigma}$ and $g \longrightarrow aga^{-1}$ if $g \in \Gamma_{C'\partial\sigma}$, where a $A - \{1\}$. Take $UL = TL \cup UL_0 \cup m$ where m is the mapping cylinder of the above embedding and extend t to UL by sending m onto $C\sigma$. One easily check Condition a)-c) of $\mathcal{M}(L)$ (observe that $\Gamma_{C\Sigma\partial\sigma} \in \mathcal{A}$ by (4.4) and (4.3)). For Condition d), one checks that the monomorphisms $\Gamma_Y \longrightarrow \Gamma_X$ corresponding to all the inclusion $Y \longrightarrow X$ of the following diagram :



are $\sqrt{\ }$-closed. This is done as follows :

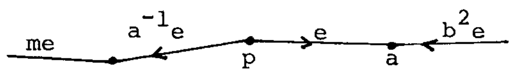
- inclusions (1) are $\sqrt{\ }$-closed because $\mathscr{M}(L_0)$ holds.
- " (2) " " " inclusions (1) are, using (4.2.b) and (4.2.c).
- if inclusion (3) is $\sqrt{\ }$-closed, then inclusions (4) are $\sqrt{\ }$closed, using several times (4.2.b) and (4.2.c). For instance, the inclusion $L \subset CL$ has to be decomposed :
  $L \subset L \cup C\partial\sigma \subset (C\Sigma\partial\sigma \cup L) \cup_{L \cup C\partial\sigma} (CL_0)$, etc.

It thus remains to prove that Inclusion (3) is $\sqrt{\ }$-closed. To simplify the notation, write Inclusion (3) under the form $G' *_H G \longrightarrow (A\times H) *_H G$ (G' a copy of G). As for the proof of (4.4.a) and (4.1), we shall use the Serre theory of amalgamated product acting on trees [Se, 4 and 5]. Recall that an amalgamated product $R_1 *_B R_2 = R$ acts on a tree T_R characterised by the following properties : there is a fundamental domain which is a segment $\overset{P}{\bullet}\!\xrightarrow{\ e\ }\!\overset{Q}{\bullet}$ isomorphic to the quotient tree $R\backslash T_R$ with isotropy groups $R_P = R_1, R_Q = R_2$ and $R_e = B$. Applying this to $R = (A\times H) *_H G$ and making the normal closure $\overline{G}$ of G act on T_R, one see that a fundamental domain isomorphic to $\overline{G}\backslash T_R$ is given by the following tree :



The isotropy group are : $R_{xe} = H$ and $R_{xQ} = xGx^{-1}$. Using [Se,§5] one deduces that $\overline{G}$ is the free product of the groups xGx^{-1} ($x \in A$) amalgamated over their common subgroup H. Therefore, the subgroup $G' *_H G$ of R which is the subgroup generated by G and aGa^{-1} is $\sqrt{\ }$-closed in $\overline{G}$ by (4.2.c) (the inclusion $H \subset A\times H$ is $\sqrt{\ }$-closed since $A \in \mathscr{A}$ and groups in $\mathscr{A}$ have no 2-torsion). On the other hand, $\overline{G}$ is $\sqrt{\ }$-closed in R since $R/\overline{G} = A$ has no 2-torsion. Therefore, $G' *_H G$ is $\sqrt{\ }$-closed in R.

Proof of Sublemma (4.4.a) Observe that the first statement is implied by the second since $m^k x m^{-k} = x^{-1}$ implies that $m^{2k} x m^{-2k} = x$. To establish the second statement, observe that the tree T_G has fundamental domain $\overset{P}{\bullet}\xrightarrow{e}\overset{Q}{\bullet}$ with isotropy groups $G_P = \langle a \rangle$, $G_Q = \langle b \rangle$ and $G_e = \zeta(G)$. One has the following situation in T_G :



By [Se, Proposition 25 §6], one deduces that the subgraph drawn above is part of an infinite chain L on which m acts by a translation of amplitude 2. Observe that the orientations of the edges of L imply that m is a generator of the oriented-automorphisms group of L. Now, if m^k commutes with x, one deduces from [Se, Propositions 25 and 27 §6] that $xL = L$ and thus $xe = m^i e$ for some i. As $G_e = \zeta(G)$, this implies that $xm^{-i} \in \zeta(G)$.

Proof of Lemma (4.1) The Serre tree T_R has here fundamental domain (isomorphic to $R\backslash T_R$) a cone on the set of vertices $\{P_i\}_{i\in I}$ (the cone vertex is called P; the edge from P_i to P is called e_i) , and the isotropy groups are $R_{P_i} = R_i$, $R_P = R_{e_i} = B$.

Let T_S be the smallest subgraph of T_R such that $\{e_i; i\in I\} \subset \{\text{Edges } T_S\}$ and $ST_S = T_S$. As S is generated by $S_i = S_{P_i}$, T_S is connected by the obvious generalisation of [Se, Lemme 2, p.49] and thus T_S is a subtree of T_R.

Let $g \in R$ such that $g^2 \in S$. As an oriented automorphism of T_R, g has either a fixed vertex or there is an infinite chain L in T on which g acts by a non-trivial translation [Se, Proposition 25 §6]. Suppose that g has a fixed vertex V. Hence $g^2 V = V$ and, as $gT_S \cap T_S \neq \emptyset$, g must fix the whole path joining V to T_S. Therefore one may suppose that $V \in T_S$ which implies that $g = tr_i t^{-1}$ with $r_i \in R_i$ (for some i) and $t \in S$. Thus, $r_i^2 = t^{-1} g^2 t \in S \cap R_i = S_i$. As S_i is $\sqrt{\ }$-closed in R_i, one has $t^{-1} g t \in S_i$ and then $g \in S$.

It then remains to check the case where g translates a chain L. As $g^2 \in S$, one has $L \subset T_S$ (otherwise $gT_S \quad T_S = \emptyset$). Therefore, by replacing if necessary g by one of its conjugate by an element of S, one may suppose that L contains the edge e_i for some $i \in I$. As $T_S \cap \mathrm{Orbit}_R(P) = \mathrm{Orbit}_S(P)$, there is $h \in S$ such that $b = h^{-1}g \in R_p = B$. One has $g^2 = hbhb \in S$ which means $bhb \in S$. As $L \subset T_S$, the vertex P_i is common to the edges e_i and $s_i e_i$ with $s_i \in S_i$. Observe that the path joining $hb(s_i e_i)$ to P_i contains $s_i e_i$, and therefore $bhb(s_i e_i) \in T_S$ implies that $bs_i e_i \in T_S$. The latter means $bs_i = \tilde{s}_i \tilde{b}$ for some $\tilde{s}_i \in S_i$ and $\tilde{b} \in B$. This contradicts Condition 3) of (4.1).

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University of Aarhus, Denmark   University of Geneva, Switzerland

Current address :

McMaster University, Hamilton
Ontario