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Smooth structures on algebraic surfaces with cyclic fundamental group

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Consider the algebraic surface $\mathbb{C}P^2 \# 9 \cdot \overline{\mathbb{C}P^2}$ obtained by blowing up 9 points in $\mathbb{C}P^2$. This surface admits an elliptic fibration over $\mathbb{C}P^1$. Applying logarithmic transformations of order p and q [1; p. 164] in two non-singular fibres, one obtains a family of new elliptic surfaces. We denote any of them by $D_{p,q}$ and call them the *Dolgachev* surfaces (some authors reserve this name for the special ones which have greatest common divisor $(p, q)=1$).

The simply connected Dolgachev surfaces $D_{p,q}$ are all homeomorphic by Freedman's classification of 1-connected closed 4-manifolds. In this paper we complete the non simply-connected case:

Theorem 1. *The Dolgachev surfaces $D_{p,q}$ and $D_{p',q'}$ are homeomorphic if and only if $(p, q)=(p', q')=k$ and, when k is even, $(p/k+q/k)\equiv(p'/k+q'/k)$ (mod 2).*

If k is odd this was proved in [10]; for $k=2$ the theorem also follows from the results given there, once the equivariant intersection form on $\pi_2(D_{p,q})$ is determined. This was carried out in [15] for $k=2$ and $(p/k+q/k)$ even.

Donaldson [5] introduced a diffeomorphism invariant for certain smooth 4-manifolds, and used it to show that $D_{2,3}$ is not diffeomorphic to $D_{1,1}$. Friedman and Morgan [9] and independently Okonek and Van de Ven [16] computed this invariant for $D_{2,q}$, q odd. They showed that all these manifolds are pairwise non-diffeomorphic. In [9] it is also proved that for given coprime integers p_1, q_1 there are only finitely many coprime pairs p, q such that $D_{p,q}$ is diffeomorphic to D_{p_1,q_1}. Recently Lübke and Okonek [12] and independently Maier [13] extended this result to arbitrary pairs p, q with prescribed greatest common divisor. Combining this result with Theorem 1 one obtains:

Corollary 2. *All the Dolgachev surfaces $D_{p,q}$ admit infinitely many smooth structures.*

The homeomorphism classification of Dolgachev surfaces is a consequence of the following more general result. We say that a manifold M has w_2-*type* (I),

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(II), or (III) if one of the following holds:

(I) $w_2(\tilde{M}) \neq 0$,

(II) $w_2(M) = 0$, or

(III) $w_2(M) \neq 0$ and $w_2(\tilde{M}) = 0$.

This is consistent with the usual notion of type I or II for bilinear forms, when we consider the intersection form on simply-connected manifolds. Note that for a complex manifold $w_2(M)$ is just the mod 2 reduction of the first Chern class $c_1(M)$.

Theorem 3. *Let M_0 and M_1 be closed, oriented, smooth 4-manifolds with fundamental group $\mathbb{Z}/k$. For $k=0(2)$ suppose that either $w_2(M_i) \neq 0$ and $4 \neq e(M_i) \geqq |\sigma(M_i)| + 4$, or $w_2(M_i) = 0$ and $e(M_i) \geqq |\sigma(M_i)| + 6$ for $i = 0, 1$.*

Then M_0 is homeomorphic to M_1, preserving the orientation, if and only if they have the same signature, Euler characteristic and w_2-type.

For $k=0(2)$ we remark that this holds also in the topological category if we add in the Kirby-Siebenmann obstruction. For k odd a homeomorphism classification without any stability assumptions was obtained in [10]. The condition here amounts to assuming that the intersection form on M_i is indefinite, or has (in the spin case) hyperbolic rank $\geqq 2$.

Corollary 4. *Let M_0 be a compact complex surface and M_1 a closed smooth 4-manifold with the same finite cyclic fundamental group and, for k even, $e(M_i) \neq 4$. Then M_0 and M_1 are homeomorphic if and only if they have the same signature, Euler characteristic and w_2-type.*

It is not known at the moment how to compute the Donaldson invariants in general. However, Donaldson proved [6] that in most cases where the manifold decomposes as a smooth non-trivial connected sum, the invariants must be trivial. On the other hand, for simply-connected algebraic surfaces he proves that they are non-trivial. Moreover Donaldson observed in [6] that comparison of these two results with Freedman's homeomorphism classification of simply-connected 4-manifolds implies the existence of 4-manifolds with more than one smooth structure. We use the same argument together with Theorem 3 to obtain:

Corollary 5. *An algebraic surface M with a non-trivial finite cyclic fundamental group and, for k even, $e(M) \neq 4$ admits at least two smooth structures.*

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Section 1. A cancellation theorem for homeomorphisms

In this section we will prove Theorem 3. This is done by combining the stable classification (i.e. up to connected sum with copies of $S^2 \times S^2$) with a cancellation theorem of Bass [3]. All manifolds considered here will be closed, connected and oriented, and homeomorphisms are assumed to preserve the orientation.

Proposition 1.1 ([11; § 3]). *Suppose that N_0 and N_1 are topological 4-manifolds with fundamental group $\mathbb{Z}/k$. If they have the same signature, Euler characteristic, w_2-type and Kirby-Siebenmann obstruction, then there is an integer r such that $N_0 \# r(S^2 \times S^2)$ is homeomorphic to $N_1 \# r(S^2 \times S^2)$.*

To apply the result of Bass we need to know that certain isometries of the equivariant intersection form on π_2 can be realized by self-homeomorphisms.

Denote the group ring $\mathbb{Z}[\pi_1]$ by Λ. We abbreviate the hyperbolic quadratic form on $\Lambda^r \otimes \Lambda^r$ by $H(\Lambda^r)$. If M has fundamental group π_1, then the intersection form $S(\tilde{M})$ on $\pi_2(M)$ gives us a hermitian form with values in Λ. On the subspace $V = \ker w_2(\tilde{M})$, this hermitian form admits a unique quadratic refinement [17; § 5].

Lemma 1.2. *Suppose that M is a topological 4-manifold with $\pi_1(M) \cong \mathbb{Z}/k$ and, if k is even, $w_2(M) \neq 0$ and $w_2(\tilde{M}) = 0$. Then for any isometry A of $H(\Lambda^r)$ there exists a self-homeomorphism g of $M \# r(S^2 \times S^2)$ such that $g_* = \mathrm{Id} \oplus A$.*

Proof. In [10] we attached to a 4-manifold M its quadratic 2-type. This consists of $\pi_1(M)$, $\pi_2(M)$, $S(\tilde{M})$ and the first k-invariant $k(M) \in H^3(\pi_1; \pi_2)$. We proved that, for π_1 a group of cohomological period 4, every isomorphism between quadratic 2-types can be realized by a homotopy equivalence.

In the situation of the Lemma, $\mathrm{Id} \oplus A$ clearly preserves the k-invariant, so it is induced by a self-homotopy equivalence g of $N = M \# r(S^2 \times S^2)$. We are finished if g is homotopic to a homeomorphism.

After identifying the normal invariants of N which have trivial signature with $H_2(N; \mathbb{Z}/2)$, we may assume that the normal invariant of g is in the image of $\pi_2(N)$. If not, the Kirby-Siebenmann obstructions of the domain and range of g would be different. By [17; p. 237] after composing g with a suitable self equivalence acting trivially on homology, we may assume that the normal invariant of g is trivial. By [18; 7.2] and the fact that $SK_1(\mathbb{Z}[\mathbb{Z}/k]) = 0$, g is a simple homotopy equivalence. Since $L_5^s(\mathbb{Z}[\mathbb{Z}/k]) = 0$ [19], the s-cobordism theorem [8] implies that g is homotopic to a homeomorphism.

Remark. In the situation of Theorem 3, *any automorphism of $H_2(M; \mathbb{Z})$, which is the identity on the torsion subgroup and is an isometry of the intersection form, can be realized by a self-homeomorphism.* This follows from the proof of Theorem 3 together with Freedman's [7] corresponding result for 1-connected 4-manifolds (for k odd a more general result was proved in [10]).

Theorem 1.3. *Let M_0 and M_1 be smooth 4-manifolds with finite fundamental group, and $M_0 \# r(S^2 \times S^2)$ homeomorphic to $M_1 \# r(S^2 \times S^2)$ for some $r \geq 0$.*

(a) *If $M_0 \approx M_0' \# 2(S^2 \times S^2)$, then M_0 is homeomorphic to M_1.*

(b) *If $M_0 \approx M_0' \# M_0'' \# (S^2 \times S^2)$, $\pi_1(M_0') \cong \mathbb{Z}/k$ is cyclic and, if k is even $w_2(M_0') \neq 0$, $w_2(\tilde{M}_0') = 0$, then M_0 is homeomorphic to M_1.*

In part (b) the manifold M_0'' must be 1-connected, but may be trivial (i.e. a 4-sphere).

Proof. Suppose that

$$h: M_0 \# r(S^2 \times S^2) \xrightarrow{\approx} M_1 \# r(S^2 \times S^2)$$

is a homeomorphism, and

$$h_*: S(\tilde{M}_0) \oplus \mathbf{H}(\Lambda^r) \xrightarrow{\cong} S(\tilde{M}_1) \oplus \mathbf{H}(\Lambda^r)$$

the induced isometry of the intersection forms (considered as Λ-hermitian forms). If $V_i = \ker w_2(\tilde{M}_i)$, then the restriction of h_* gives an isometry of Λ-quadratic forms

$$h_*: (V_0, q_0) \oplus \mathbf{H}(\Lambda^r) \longrightarrow (V_1, q_1) \oplus \mathbf{H}(\Lambda^r). \tag{1.4}$$

A *hyperbolic pair* in $(V_0, q_0) \oplus \mathbf{H}(\Lambda^r)$ is a free Λ-base (e_1, f_1) for a hyperbolic plane $\mathbf{H}(\Lambda)$ direct summand of $(V_0, q_0) \oplus \mathbf{H}(\Lambda^r)$. It is enough to prove that, under one of the conditions above, the group of self-homeomorphisms of $M_0 \# r(S^2 \times S^2)$ is transitive on the set of hyperbolic pairs in $(V_0, q_0) \oplus \mathbf{H}(\Lambda^r)$. If so, one may assume that h_* is the identity on a hyperbolic plane in $(V_0, q_0) \oplus \mathbf{H}(\Lambda^r)$, and perform surgery on the domain and range of h simultaneously to reduce r (for a detailed argument see [11, § 6]).

We wish to apply [3; IV, 3.5] to our situation. In that section Bass proves that a certain subgroup G_1 of the unitary group $U(V \oplus \mathbf{H}(P))$ of the quadratic module $V \oplus \mathbf{H}(P)$ acts transitively on the set of hyperbolic pairs in $V \oplus \mathbf{H}(P)$, when $P \cong \Lambda^n$ and $n \geqq \max(d+1, d_\Lambda + 2)$. The subgroup G_1 is generated by the transvections $\sigma_{p,a,v}$ (with $p \in P$ or $\bar{P}$, $v \in V$) [3; p. 39], and $1_V \oplus G$ where $G = \mathbf{H}(\Gamma) \cdot EU(\mathbf{H}(P))$ and $\Gamma \subseteq GL(P)$ acts transitively on the set of unimodular elements in P. The integers d and d_Λ depend on the underlying ring and the form-parameter (see [3, p. 177]). For $\Lambda = \mathbb{Z}\pi$ and the maximal form-parameter, $d = d_\Lambda = 1$ (apply the definition for d, for d_Λ use [3, p. 184]).

In case (a) when $M_0 \approx M_0' \# 2(S^2 \times S^2)$ and $r > 0$, the intersection form on $M_0 \# r(S^2 \times S^2)$ has hyperbolic rank $\geqq 3$, so Bass' Theorem applies. In case (b) when $M_0 \approx M_0' \# M_0'' \# (S^2 \times S^2)$ the rank of $P \geqq 2$, so in the minimal case the Theorem does not apply. However we will see below that the statement of [3; IV, 3.5] is only needed with $G = U(\mathbf{H}(P))$. This is a much weaker result. To prove it we must look at Bass's proof.

The proof of [3; IV, 3.5] consists of the following steps. (i) It suffices to show that G_1 acts transitively on the set of hyperbolic elements in $V \oplus \mathbf{H}(P)$. (ii) If $x = (v; p, q)$ is a hyperbolic element [3; p. 31] then after transforming x by a transvection in G_1, we may assume that (p, q) is a unimodular element [3; p. 30]. (iii) Assuming (ii), x may be transformed by $1_V \oplus G$ so that q is unimodular. (iv) After changing x by a transvection in G_1, we may assume that $v = 0$, and so $x \in \mathbf{H}(P)$. (v) The group G acts transitively on the set of hyperbolic elements in $\mathbf{H}(P)$.

With our stronger hypothesis $G = U(\mathbf{H}(P))$, steps (iii) and (v) are trivial. In step (ii) the stability assumption rank $P \geqq d+1$ is used, but since $d = 1$ in our case this is satisfied. Since we have the same transvections available the rest of the argument is identical.

With the statement of Bass' Theorem available in both our situations above, we can complete the proof by showing that the subgroup G_1 is actually realized by self-homeomorphisms of $M_0 \# r(S^2 \times S^2)$. The transvections needed are given by the self-homeomorphisms of [4; Theorem 1.5], and if $P \cong \Lambda^n$ with $n \geqq 3$, then the elementary matrices $E_n(\Lambda)$ act transitively on the set of unimodular elements of P ([2; Theorem 4.2, Theorem 11.1]). Again these together with the elements of $EU(\mathbf{H}(P))$ are induced by the self-homeomorphisms of [4; Theorem A2, p. 526]. If $n=2$, we are in case (b) of (1.3), and under these additional hypotheses, the self-homeomorphisms needed are provided by (1.2). This completes the proof.

We can now give the *Proof of Theorem 3*. The case k odd follows from [10], Theorem B together with Donaldson's result that the intersection form of a smooth 4-manifold is determined by its rank, signature and type. Thus we can assume k even. If $w_2(M_0) \neq 0$, M_0 is by 1.1 stably homeomorphic to $N_0 = N'_0 \# (S^2 \times S^2)$, where

$$N'_0 = \begin{cases} \Sigma \# \pm l_1(\mathbb{C}P^2) \# l_2(S^2 \times S^2), & \text{if } w_2(\tilde{M}_0) \neq 0, \sigma(M_0) \neq 0 \\ \Sigma \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \# l_2(S^2 \times S^2), & \text{if } w_2(\tilde{M}_0) \neq 0, \sigma(M_0) = 0 \\ \Sigma \# \pm l_1(M(E_8)) \# l_2(S^2 \times S^2), & \text{if } w_2(\tilde{M}_0) = 0. \end{cases}$$

Here Σ is a rational homology 4-sphere with $\pi_1 = \mathbb{Z}/k$, $w_2(\Sigma) \neq 0$ and appropriate Kirby-Siebenmann obstruction. A smooth Σ can be constructed as the double of a suitable thickening of a 2-dim Moore space. There exists a TOP manifold homotopy equivalent to Σ with nonzero Kirby-Siebenmann obstruction as the set of TOP homotopy smoothings surjects onto $H^2(\Sigma; \mathbb{Z}_2)$, and $Sq^2 \neq 0$ on Σ implies that the Kirby-Siebenmann obstruction is non-trivial on this set. $M(E_8)$ is the closed 1-connected topological manifold with definite intersection form of signature 8 [7]. We now apply (1.3)(b) to prove M_0 homeomorphic to N_0. Notice that the manifold N_0 decomposes as a connected sum, and the factor $\Sigma \# l_2(S^2 \times S^2)$ satisfies the required condition.

In the second case, where $w_2(M_0) = 0$, the stable type of M_0 is $N'_0 \# 2(S^2 \times S^2)$ where

$$N'_0 = \Sigma \# \pm l_1(M(E_8)) \# l_2(S^2 \times S^2),$$

Again Σ is a suitable rational homology sphere (with $w_2(\Sigma) = 0$). The result follows from (1.3)(a).

Section 2. Applications to algebraic surfaces

The proof of Theorem 1 is an immediate consequence of Theorem 3 and the following list of topological invariants for the surfaces $D_{p,q}$:

the signature: $\sigma(D_{p,q}) = -8$,
the Euler characteristic: $e(D_{p,q}) = 12$,
the fundamental group: $\pi_1(D_{p,q}) \cong \mathbb{Z}/k$, *where* $k = (p, q)$,
the second Stiefel-Whitney class: $w_2(D_{p,q}) \neq 0$, *and*
$w_2(\tilde{D}_{p,q}) = 0$ *if and only if* k *is even and* $(p/k + q/k)$ *is even.*

Since a Dolgachev surface is obtained from $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$ by a cutting and pasting process its signature and Euler characteristic are unchanged. The fundamental group is an easy consequence of the van Kampen Theorem and Rochlin's Theorem implies $w_2(D_{p,q}) \neq 0$.

We are indebted to C.T.C. Wall for the information about $w_2(\tilde{D}_{p,q})$; for the reader's convenience we include a short discussion. The intersection form on spherical elements in $H_2(D_{p,q};\mathbb{Z})$ is even if and only if $w_2(\tilde{D}_{p,q})=0$. But the spherical elements are isomorphic to $H_2(D_{p,q};\mathbb{Z})/\text{Tors}$ and the form on this group is even if and only if the canonical class K is divisible by 2 (mod. torsion). It follows from Kodaira's formula for the canonical divisor [1; p. 162], that K has the form

$$K(D_{p,q})=\frac{(pq-p-q)}{k}\cdot\omega+\tau,$$

where ω is a primitive class and τ is a torsion class in $H_2(D_{p,q};\mathbb{Z})$. From this formula we see that the canonical class is divisible by 2 (mod. torsion) if and only if $k=(p,q)$ is even and $(p/k+q/k)$ is even.

Proof of Corollary 4 and Corollary 5. Let M be a complex surface with cyclic non-trivial fundamental group. We will show that either M is homeomorphic to a Dolgachev surface which has infinitely many smooth structures (Corollary 2), or it is homeomorphic to a smooth manifold N such that $\tilde{N}$ is diffeomorphic to $X\#Y$, where neither X nor Y have negative definite intersection form. Then if M is algebraic, $\tilde{M}$ and $\tilde{N}$ are not diffeomorphic by [6], implying Corollary 5. Corollary 4 follows as it turns out that M fulfills the conditions of Theorem 3.

Note that since $\pi_1(M)$ is non-trivial, the intersection form on M is indefinite. For, if it were definite it follows from the inequality of Miyaoka and Yau that $b_2(M)=1$ [14]. But then a result of Yau implies that M is isomorphic to $\mathbb{C}P^2$ [1; p. 135]. It is also clear for an algebraic surface that $e(M)>2$.

Consider first the case where $w_2(\tilde{M})\neq 0$. Let Σ be a smooth rational homology sphere with $\pi_1(\Sigma)\cong\pi_1(M)\cong\mathbb{Z}/k$. This can be constructed as the double of a suitable thickening of a 2-complex to obtain $w_2(\Sigma)$ zero or non-zero as required. By Theorem 3, M is homeomorphic to $N:=\Sigma\# r\mathbb{C}P^2\# s\overline{\mathbb{C}P^2}$ where $r=(e(M)+\sigma(M))/2-1$ and $s=(e(M)-\sigma(M))/2-1$. Since r and s are both non-trivial, Donaldson's result implies $\tilde{M}$ is not diffeomorphic to $\tilde{N}$ (note that the intersection form of $\tilde{\Sigma}$ is indefinite as $\pi_1(M)$ is non-trivial).

Now we suppose $w_2(\tilde{M})=0$. Hence as above the intersection form of M is even and $\sigma(M)\equiv 0 (\text{mod } 8)$. Then M is a minimal surface. It follows from the Enriques-Kodaira classification [1; p. 188] that, if $\pi_1(M)$ is finite non-trivial, then $c_1^2\geqq 0$ and if $c_1^2\neq 0$ it is a minimal surface of general type.

Consider $N_0:=\Sigma\# r(\pm K)\# s(S^2\times S^2)$, K the Kummer surface, and $N_1:=\pm D_{k,k}\# r(\pm K)\# s(S^2\times S^2)$ where $k=|\pi_1(M)|$. The rational homology sphere Σ is chosen so that $w_2(\Sigma)\neq 0$ if and only if $w_2(M)\neq 0$. We are finished (by Theorem 3) if in case $\sigma(M)\equiv 0(\text{mod } 16)$ we can find r and s such that N_0 and M have same signature and Euler characteristic and similarly with N_1 instead of N_0 in case $\sigma(M)\equiv 8(\text{mod } 16)$. This can be done if and only if

$$(2.1)\qquad e(M)\geqq(11/8)|\sigma(M)|+\begin{cases}2 & \text{if } \sigma(M)\equiv 0(\text{mod } 16), \text{ or}\\ 1 & \text{if } \sigma(M)\equiv 8(\text{mod } 16)\end{cases}$$

in the two cases.

If $\sigma(M)>0$ the first inequality (which also implies the second) is by Hirzebruch's signature Theorem equivalent to

$$11c_1^2(M)\leqq 46c_2(M)-48.$$

Now, either $c_1^2(M)=0$ in which case the inequality is trivial or M is a surface of general type. In this case we use the inequality of Miyaoka and Yau [14]: $c_1^2(M) \leqq 3c_2(M)$ implying the inequality as $c_2(M)$ is at least 10. This finishes the proof if $\sigma(M)>0$.

If $\sigma(M)<0$ the inequalities follow easily from Hirzebruch's signature Theorem and the fact that if $w_2(M)=0$, $\sigma(M) \leqq -16$ by Rochlin's Theorem.

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