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DETECTION THEOREMS FOR K-THEORY AND L-THEORY

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Suppose G is a p-hyperelementary group and R is a commutative ring such that the order of G is a unit in R. Suppose J is either one of Quillen's K-theory functors or one of Wall's oriented L-theory functors. We show that $J(RG)$ can be detected by applying $J(R?)$ to the subquotients of G such that all normal abelian subgroups are cyclic. In 3.A.6 we show that such subquotients have a quite simple structure.

We also show how to detect more general L-theory functors, in particular unoriented ones and those that arise in the study of codimension one submanifolds.

Introduction

Let $\mathscr{H}_p$ denote the class of p-hyperelementary finite groups. The groups in $\mathscr{H}_p$ are semi-direct products, $G = C \rtimes P$, where C is a normal cyclic subgroup of order prime to p, and P is a p-group. Inside the class $\mathscr{H} = \bigcup \mathscr{H}_p$ of all hyperelementary groups we consider the class of *basic groups*:

$$\mathscr{B} = \{G \in \mathscr{H} \mid \text{all normal abelian subgroups of } G \text{ are cyclic}\}$$

whose structure is much simpler (see 3.A.6).

Recall that Swan [24], Lam [15] and Dress [6] have shown that when a K-theory or an L-theory functor is applied to a finite group G, it can be detected by using the hyperelementary subgroups of G. This means that the direct sum of the restriction maps from G to the subgroups of G in $\mathscr{H}$ induces an injection. In this paper we show that many of these functors can be detected by using *subquotients* of G which belong to $\mathscr{B}$ (see 1.A.12, 1.B.8 and 1.C.7). These detection results have other applications such as [4] and [13].

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Several of the sections in this paper are divided into subsections. A subsection **A** indicates that we are considering the linear case, the case which applies to K-theory. A label of **B** indicates that we are doing a quadratic version which applies to the ordinary L-theory as in [6]. The **C** subsections apply to a more esoteric quadratic theory that comes up in L-theory with arbitrary antistructures as in [26]. Those readers interested only in the linear theory may safely skip any **B** or **C** subsection. Those interested only in ordinary L-theory can safely skip any **C** subsection.

1. Background and statement of results

1.A. The linear case

Let R be a commutative ring. For any R-algebra A, we let ${}_A\mathcal{P}$ denote the category of finitely generated projective left A-modules. If A and B are R-algebras, we let ${}_B\mathcal{P}\mathcal{M}_A$ denote the category of B–A-bimodules P such that

(i) P is finitely generated projective as a left B-module, and

(ii) $rx = xr$ for all $r \in R$ and all $x \in P$.

Direct sum makes ${}_B\mathcal{P}\mathcal{M}_A$ into a symmetric monoidal category. In [19, p. 37–39], Oliver introduced the following category:

1.A.1. Definition. R-Morita is the category with objects R-algebras and

$$\mathrm{Hom}_{R\text{-Morita}}(A, B) = K_0({}_B\mathcal{P}\mathcal{M}_A).$$

Composition is given by tensor product. We also add a zero object to make R-Morita into an additive category.

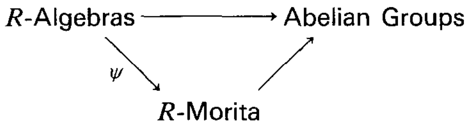
If M is an object in ${}_B\mathcal{P}\mathcal{M}_A$, then the functor

$$M \otimes_A - : {}_A\mathcal{P} \to {}_B\mathcal{P}$$

induces a homomorphism

$$M \otimes_A - : K_n(A) \to K_n(B)$$

where K_n is Quillen K-theory (see [20]). It is easy to check that the functor K_n factors as follows:



where $\psi(A) = A$ and $\psi(f : A \to B) = {}_B B_A$ with bimodule structure $b_1 \cdot b \cdot a = b_1 \cdot b \cdot f(a)$.

Another of Quillen's functors, $G_n(A)=K_n({}_A\mathcal{M})$, where ${}_A\mathcal{M}$ is the category of finitely generated, left A-modules, factors through the category where the morphisms are K_0 of the category of bimodules which are finitely generated on the left and projective on the right.

Any Morita equivalence in the classical sense (see [1, Theorem 3.5, p. 65]) yields an isomorphism in R-Morita, and in this other category.

For working with finite groups, we find a different category convenient, but before describing it we recall the following category theory:

1.A.2. *Ab-categories and the Add construction.* A category $\mathscr{C}$ is an *Ab-category* [17, p. 28] if each Hom-set has an abelian group structure on it so that composition is bilinear. Associated to an Ab-category $\mathscr{C}$ we have the free additive category Add($\mathscr{C}$) [17, p. 194, Exercise 6(a)], whose objects are n-tuples, $n=0,1,\dots,$ of objects of $\mathscr{C}$ and whose morphisms are matrices of morphisms in $\mathscr{C}$. The 0-tuple is defined so as to be a 0-object. Juxtaposition defines the biproduct. To avoid proliferation of names we will often name the Add construction of an Ab-category and then think of the Ab-category as the subcategory of 1-tuples.

A functor $F:\mathscr{A}\to\mathscr{B}$ between two Ab-categories is *additive* if the associated map $\mathrm{Hom}_{\mathscr{A}}(A_1,A_2)\to\mathrm{Hom}_{\mathscr{B}}(F(A_1),F(A_2))$ is a group homomorphism for all objects $A_1,A_2\in\mathscr{A}$. The Add construction on $\mathscr{C}$ is free in the sense that given an additive category $\mathscr{A}$ and an additive functor $F:\mathscr{C}\to\mathscr{A}$, there exists a natural extension to an additive functor $\mathrm{Add}(F):\mathrm{Add}(C)\to\mathscr{A}$. We will often use the remark that if an additive functor F is an embedding (the induced map on hom-sets is injective), then so is Add(F).

Next we recall some terminology from the theory of group actions on sets.

Given two groups, H_1 and H_2, an H_2-H_1 *biset* is a set X on which H_2 acts on the left, H_1 acts on the right and $h_2(xh_1)=(h_2x)h_1$ for all $x\in X$, $h_1\in H_1$, $h_2\in H_2$. For each point $x\in X$ we have two isotropy groups: ${}_{H_2}I(x)=\{h\in H_2\mid hx=x\}$ and $I_{H_1}(x)=\{h\in H_1\mid xh=x\}$. Given an H_3-H_2 biset X and an H_2-H_1 biset Y, recall $X\times_{H_2}Y$ is defined as $X\times Y$ modulo the relations $(x,y)\sim(xh^{-1},hy)$ for all $x\in X$, $y\in Y$ and $h\in H_2$. Clearly $X\times_{H_2}Y$ is an H_3-H_1 biset. Note that $h\in{}_{H_3}I(x,y)$ iff we can find $h_2\in H_2$ such that $h\cdot x=x\cdot h_2^{-1}$ and $y=h_2\cdot y$. These equations define a group homomorphism

$$(1.A.3)\qquad {}_{H_3}I(x,y)/{}_{H_3}I(x)\to{}_{H_2}I(y)/(I_{H_2}(x)\cap{}_{H_2}I(y))$$

which is an injection. The coset of an element $h\in{}_{H_2}I(y)$ comes from ${}_{H_3}I(x,y)$ iff $x\cdot h$ is in the same H_3-orbit as x.

1.A.4. Definition. We define a category RG-Morita as the Add construction applied to the following Ab-category. The objects are the finite groups H which are isomorphic to some subquotient of G. Define $\mathrm{Hom}_{RG\text{-Morita}}(H_1,H_2)$ as the following Grothendieck construction:

Take the collection of isomorphism classes of finite H_2-H_1 bisets X, for which $|{}_{H_2}I(x)|$ is a unit in R for all $x \in X$. Disjoint union makes this collection into a monoid. Form formal differences and set X equivalent to X' if RX is isomorphic to RX' as RH_2-RH_1 bimodules.

Define the composition

$$\mathrm{Hom}_{RG\text{-Morita}}(H_2, H_3) \times \mathrm{Hom}_{RG\text{-Morita}}(H_1, H_2) \to \mathrm{Hom}_{RG\text{-Morita}}(H_1, H_3)$$

by sending ${}_{H_3}X_{H_2} \times {}_{H_2}Y_{H_1}$ to $X \times_{H_2} Y$ as defined above. Note that (1.A.3) implies that composition is defined.

Remark. The requirement that X is equivalent to X' if RX is isomorphic to RX' as RH_2-RH_1 bimodules is perhaps less natural than requiring that X be isomorphic to X' as bisets, but in Section 4 we will want our morphism group to be a subgroup of the corresponding morphism group of R-Morita.

1.A.5. Remark. A generating set for $\mathrm{Hom}_{RG\text{-Morita}}(H_1, H_2)$ is easily found. An H_2-H_1 biset is the same thing as a left $H_2 \times H_1^{\mathrm{op}}$ set. Such a set is just a disjoint union of coset spaces of $H_2 \times H_1^{\mathrm{op}}$, and these are described by conjugacy classes of subgroups of $H_2 \times H_1^{\mathrm{op}}$. For all our serious work $|H_2 \times H_1^{\mathrm{op}}|$ will be a unit in R, so the morphism group will be generated by the collection of all these bisets.

1.A.6. Definition. The functor which sends H to the R-algebra RH and sends an H_2-H_1 biset X to the bimodule RX, is an additive functor into R-Morita, and hence extends to a functor from RG-Morita to R-Morita. We call this functor the *R-group ring functor.*

1.A.7. Remark. Clearly the map is well defined and note that $R[X \times_{H_2} Y] \cong RX \otimes_{RH_2} RY$ so the map preserves compositions. We need to see that RX is projective as a left RH_3-module. Since the orders of all the left isotropy subgroups are invertible in R, this is a standard averaging trick.

In the sequel we will write RH both for an object in R-Morita and for an object in RG-Morita since the notation displays both the group and the ring.

1.A.8. *Generalized induction and restriction maps.* Let $H_1 \subset H_2$ be finite groups. Then H_2, considered as a finite H_2-H_1 biset, gives an element in $\mathrm{Hom}_{RG\text{-Morita}}(H_1, H_2)$ called a (*generalized*) *induction* and written $\mathrm{Ind}_{H_1}^{H_2}$; H_2 considered as a finite H_1-H_2 biset yields a map in $\mathrm{Hom}_{RG\text{-Morita}}(H_2, H_1)$ called a (generalized) restriction map and written $\mathrm{Res}_{H_1}^{H_2}$.

If $H \to H/N$ is a quotient map, H/N considered as a finite H/N-H biset yields a generalized restriction map, written $\mathrm{Res}_{H/N}^{H} \in \mathrm{Hom}_{RG\text{-Morita}}(H, H/N)$; H/N considered as a finite H-H/N biset yields a generalized induction map, written $\mathrm{Ind}_{H/N}^{H} \in \mathrm{Hom}_{RG\text{-Morita}}(H/N, H)$, provided $|N|$ is a unit in R.

If we have a subquotient H/N with $H \subset K$, we can compose the two maps above to get a generalized restriction $\mathrm{Res}_{H/N}^{K} \in \mathrm{Hom}_{RG\text{-Morita}}(K, H/N)$. If $|N| \in R^{\times}$, we have a generalized induction $\mathrm{Ind}_{H/N}^{K} \in \mathrm{Hom}_{RG\text{-Morita}}(H/N, K)$. Notice that the generalized restriction goes from the group of larger order to the group of smaller order and the generalized induction goes the other way.

1.A.9. Remark. We can now give a different generating set for $\mathrm{Hom}_{RG\text{-Morita}}(H_1, H_2)$ than the one we gave in 1.A.5. The map $f: H_2 \times H_1^{\mathrm{op}} \to H_2 \times H_1$ defined by $f(h_2, h_1) = (h_2, h_1^{-1})$ defines a biset bijection between $(H_2 \times H_1^{\mathrm{op}})/S$ and $H_2 \times_S H_1$, where S is a subgroup of $H_2 \times H_1$. Hence a generating set for $\mathrm{Hom}_{RG\text{-Morita}}(H_1, H_2)$ consists of the bisets associated to a generalized restriction followed by a generalized induction $H_1 \leftarrow S \rightarrow H_2$. Such a composite is in RG-Morita iff the order of the kernel of $S \to H_1$ is a unit in R.

1.A.10. Definition. A hyperelementary group is *basic* if all its normal abelian subgroups are cyclic. We classify these groups in 3.A.6.

1.A.11. Theorem. *Let G be a p-hyperelementary group, and let R be a commutative ring such that $|G|$ is a unit in R. Then, in RG-*Morita,

(i) (The Linear Detection Theorem) *the sum of the generalized restriction maps*

$$\mathrm{Res}: R[G] \to \bigoplus \{R[H/N]: H/N \text{ is a basic subquotient of } G\}$$

is a split injection, and

(ii) (The Linear Generation Theorem) *the sum of the generalized induction maps*

$$\mathrm{Ind}: \bigoplus \{R[H/N]: H/N \text{ is a basic subquotient of } G\} \to R[G]$$

is a split surjection.

A more refined version of this result is stated and proved in Theorem 4.A.8. The result itself is proved in 4.A.9.

1.A.12. Applications. With G p-hyperelementary and $|G| \in R^{\times}$, we suppose

$$J: RG\text{-Morita} \to \mathscr{A}$$

is an additive functor. Then

$$\mathrm{Res}: J(R[G]) \to \bigoplus J(R[H/N])$$

is a split injection, and

$$\mathrm{Ind}: \bigoplus J(R[H/N]) \to J(R[G])$$

is a split surjection in $\mathscr{A}$. For example, set $J(R[G])$ equal to

(i) $K_n(R[G])$, Quillen K-theory for finitely generated projective modules,

(ii) $\mathrm{KV}_n(R[G])$, Karoubi–Villamayor K-theory (see [14, 29]),

(iii) $K_n'(R[G]) = G_n(R[G])$, Quillen K-theory for the exact category of finitely generated $R[G]$-modules,

(iv) $\mathrm{Nil}(R[G])$ (see [8]),

(v) $K_n(Z[1/m]G \to \hat{Q}_m G) \overset{\approx}{\leftarrow} K_n(ZG \to \hat{Z}_m G)$, where $m = |G|$; recall that there is an exact sequence

$$\cdots \to K_n(ZG) \to K_n(\hat{Z}_m G) \to K_n(ZG \to \hat{Z}_m G) \to \cdots,$$

(vi) $\mathrm{HH}_n(R[G])$, Hochschild homology, [5, Acknowledgements],

(vii) $\mathrm{HC}_n(R[G])$, cyclic homology, [16, Corollary 1.7].

1.A.13. Remark. All the functors except (iii) are functors out of R-Morita, and hence out of RG-Morita. Even functor (iii) is a functor out of RG-Morita.

1.A.14. Example. Recall that $\mathrm{Wh}(G) = K_1(ZG)/(\pm G^{\mathrm{ab}})$. Group homomorphisms and transfers associated to group inclusions induce maps of Wh. Composites of these maps generate the morphism groups in ZG-Morita. Since K_1 is a functor defined on Z-Morita it is easy to check that Wh is a functor on ZG-Morita. It seems unlikely that Wh is a functor on Z-Morita.

1.A.15. Non-example. In (1.A.11) we cannot drop the assumption that $|G|$ is a unit in R. For example, $K_0(Z[C(2) \times C(4)]$ is *not* detected by basic subquotients, where $C(k)$ denotes the cyclic group of order k.

In some situations we are interested in computing (rather than just detecting) functors out of RG-Morita. Call a 5-term sequence $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ *split exact* provided that there exists a map $f: C \to B$ such that $\beta \circ f = 1_C$, the identity of C; $\beta \circ \alpha = 0$, the zero map from A to C; and $\alpha \oplus f: A \oplus C \to B$ is an isomorphism.

The following theorem is proved in Section 5:

1.A.16. Theorem. *Let R be a commutative ring and G a p-hyperelementary group with $|G|$ a unit in R. Assume that G has a normal subgroup $K \cong C(p) \times C(p)$. Let $C_0, C_1, \ldots, C_p$ be the distinct cyclic subgroups of K. Let $\mathcal{Z}(G)$ denote the center of G.*

(i) *If K is central, then the following sequence is split exact in RG-Morita:*

$$0 \to RG \xrightarrow{\mathrm{Proj}} R[G/C_0] \times R[G/C_1] \times \cdots \times R[G/C_p] \xrightarrow{\beta} (R[G/K])^p \to 0.$$

(ii) *If K is not central, we may assume that $K \cap \mathcal{Z}(G) = C_0$. Let G_0 denote the centralizer of K in G. Then the following sequence is split exact in RG-Morita:*

$$0 \to RG \xrightarrow{\mathrm{Proj} \times \mathrm{Res}} R[G/C_0] \times R[G_0/C_1] \xrightarrow{\beta} R[G_0/K] \to 0.$$

The maps β are defined in Section 5: case (i) in 5.A.1. and case (ii) in 5.A.3. We will see that they live in ZG-Morita, and the sequences in Theorem 1.A.16 are 0-sequences in ZG-Morita which become split exact in RG-Morita whenever $|G|$ is a unit in R. They are definitely not exact in ZG-Morita by Non-example 1.A.15.

1.B. The Hermitian case

We begin with a discussion of quadratic form theory over a pair of rings with anti-structure. We want to develop a 'bi'-version of the usual theory so that there will be pairings mimicing those in the linear case. The concepts introduced below are just 'bi' analogues of the standard concepts in Wall's theory of quadratic forms [26, 27], and the formulae seem to be forced by the desired pairings. It seems best to just present the answers and some checks, with the rest left to the diligent reader.

Recall that a *ring with antistructure*, (A,α,u), is a ring A, an anti-automorphism $\alpha: A\to A$, and a unit $u\in A$ such that

$$\alpha^2(x)=u^{-1}xu \quad \text{for all } x\in A, \qquad \alpha(u)=u^{-1}.$$

If (A,α,u) and (B,β,υ) are rings with antistructure, then an (A,α,u)–(B,β,υ) *form* is a pair $({}_BM_A,\lambda)$ with ${}_BM_A\in {}_B\mathcal{PM}_A$ and $\lambda: M\otimes_A M^{\mathrm{t}}\to B$ is a B–B bimodule map. Here M^{t} refers to an A–B bimodule structure on M obtained from the B–A bimodule structure using α and β via the formula

$$a\bullet m\bullet b=\beta(b)\cdot m\cdot \alpha(a).$$

(We use $M^{\mathrm{t}^{-1}}$ below to denote M with the A–B bimodule structure obtained from B–A bimodule structure using α^{-1} and β^{-1}.) We will also refer to λ as a *biform*.

We say that the form is *bi-hermitian* if the following diagram commutes:

$$\begin{array}{ccc} M\otimes_A M^{\mathrm{t}} & \xrightarrow{\ \lambda\ } & B \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ M\otimes_A M^{\mathrm{t}} & \xrightarrow{\ \lambda\ } & B \end{array}$$

where $T(m_1\otimes m_2)=m_2\otimes u^{-1}\bullet m_1$ and $T(b)=\upsilon^{-1}\beta^{-1}(b)$. Note that $T^2=\mathrm{Id}$.

Given a (A,α,u)–(B,β,υ) form (M,λ), we define a new form, $T(\lambda)$, on M following [26] by

$$T(\lambda)(m_1,m_2)=\upsilon^{-1}\beta^{-1}(\lambda(m_2,m_1\cdot u)).$$

Note that $T(T(\lambda))=\lambda$, $T(\lambda)=\lambda$ iff λ is bihermitian, and $T(\lambda)=T\circ\lambda\circ T$.

Given any (A,α,u)–(B,β,υ) form, (M,λ) there is a map of B–A bimodules

$$\mathrm{ad}(\lambda): M\to \mathrm{Hom}_B(M,B)^{\mathrm{t}^{-1}}$$

defined by

$$\mathrm{ad}(\lambda)(m_1)(m_2)=\lambda(m_2,m_1).$$

We say that a form is *nonsingular* if $\mathrm{ad}(\lambda)$ is an isomorphism.

We define the *orthogonal* sum of forms as usual: if (M,λ) and (N,μ) are two (A,α,u)–(B,β,υ) forms, then $\lambda\perp\mu$ is defined by

$$(\lambda \perp \mu)(m_1 \oplus n_1, m_2 \oplus n_2) = \lambda(m_1, m_2) + \mu(n_1, n_2).$$

Note that $\lambda \perp \mu$ is nonsingular iff λ and μ are.

Another notion of sum starts with two (A, α, u)-(B, β, v) forms on M, say μ and λ. Define $(M, \mu + \lambda)$ by the formula $(\mu + \lambda)(m_1, m_2) = \mu(m_1, m_2) + \lambda(m_1, m_2)$. The set of (A, α, u)-(B, β, v) forms on M is an abelian group, denoted Sesq(M). The involution T acts on Sesq(M).

As an example, we compute this group for the free B–A bimodule.

1.B.1. Example. Let R be a commutative ring with involution $r \to \bar{r}$ and suppose A and B are R-algebras such that $\alpha(r \cdot 1) = \bar{r} \cdot 1$ and $\beta(r \cdot 1) = \bar{r} \cdot 1$. If $F = B \otimes_R A$ is the free B–A bimodule, then the map $\Phi : \mathrm{Sesq}(F) \to \mathrm{Hom}_R(A, B)$ defined by $\Phi(\lambda)(a) = \lambda(1 \otimes a, 1 \otimes 1)$ defines a $Z/2Z$-equivariant isomorphism, where $Z/2Z$ acts on $\mathrm{Hom}_R(A, B)$ by defining $T(f)(a) = v^{-1}\beta^{-1}(f(u^{-1}\alpha^{-1}(a)))$ for all $a \in A$.

Next we define the notion of a metabolic form. Given an (A, α, u)-(B, β, v) form $({}_BM_A, \lambda)$, define a form, denoted Meta(λ), on $M \oplus \mathrm{Hom}_B(M, B)^{t^{-1}}$ by

$$\mathrm{Meta}(\lambda)[(m_1, f_1), (m_2, f_2)] = \lambda(m_1, m_2) + f_2(m_1) + v^{-1}\beta^{-1}f_1(m_2 \cdot u^{-1}).$$

A *metabolic* form is any form that is isometric to Meta(λ) for some λ. A *hyperbolic* form is just a metabolic form with $\lambda = 0$. Any metabolic form is nonsingular and $T(\mathrm{Meta}(\lambda)) = \mathrm{Meta}(T(\lambda))$. Hence, the form λ is bihermitian iff Meta(λ) is.

Next we define Lagrangians. Given an (A, α, u)-(B, β, v) form $({}_BM_A, \lambda)$ we say that a bi-summand $L \subset M$ is a *Lagrangian* if the form restricted to L is 0 and if the inclusion of L into its perpendicular subspace is an isomorphism. Suppose λ is nonsingular, $M = P \oplus L$ as B–A bimodules and L is a Lagrangian. Then $\lambda \cong \mathrm{Meta}(\lambda|_P)$ where the isometry is given by $F : P \oplus L \to P \oplus \mathrm{Hom}_B(P, B)^{t^{-1}}$ defined by $F(p, m) = (p, \mathrm{ad}(\lambda)(m))$ for all $p \in P$ and $m \in L$. In particular, if λ is nonsingular, $(M, \lambda) \perp (M, -\lambda)$ is isomorphic to Meta(λ) since the diagonal copy of M is a Lagrangian.

We have the usual equation

$$\mathrm{Meta}(\lambda) \perp \mathrm{Meta}(\lambda + \gamma) \cong \mathrm{Meta}(\lambda) \perp \mathrm{Meta}(\gamma)$$

where λ and γ are biforms on the same module M. If $M^* = \mathrm{Hom}_B(M, B)^{t^{-1}}$, the isometry is given by $F : M \oplus M^* \oplus M \oplus M^* \to M \oplus M^* \oplus M \oplus M^*$ defined by $F(m, f, n, g) = (m + n, f, n, g - f - \mathrm{ad}(\lambda)(m))$. In particular, in any Grothendieck-type construction, all metabolics on the same module are equivalent.

Not all metabolics however are isometric, and we explore the relationship. Recall that $Z/2Z$ acts on Sesq(M) via T. Any bihermitian form, λ, on M determines an element

$$[\lambda] \in \hat{H}^0(Z/2Z; \mathrm{Sesq}(M))$$

and $[\lambda_1] = [\lambda_2]$ implies that Meta(λ_1) and Meta(λ_2) are isometric. Indeed, if $\lambda_2 = \lambda_1 + \phi + T(\phi)$, the map $F : M \oplus \mathrm{Hom}_B(M, B)^{t^{-1}} \to M \oplus \mathrm{Hom}_B(M, B)^{t^{-1}}$ defined by $F(m, f) = (m, f - \mathrm{ad}(\phi)(m))$ satisfies

$$\mathrm{Meta}(\lambda_2)(F(m_1, f_1), F(m_2, f_2)) = \mathrm{Meta}(\lambda_1)((m_1, f_1), (m_2, f_2)).$$

The following properties are easily checked:

(i) $[\lambda_1 + \lambda_2] = [\lambda_1] + [\lambda_2]$;

(ii) $\hat{H}^0(Z/2Z; \mathrm{Sesq}(M \oplus N)) \cong \hat{H}^0(Z/2Z; \mathrm{Sesq}(M)) \oplus \hat{H}^0(Z/2Z; \mathrm{Sesq}(N))$;

(iii) $[\lambda_1 \perp \lambda_2] = ([\lambda_1], [\lambda_2])$ under the decomposition in (ii).

Given an (A, α, u)-(B, β, υ) form $({}_BM_A, \lambda)$, and a (B, β, υ)-(C, γ, ω) form $({}_CN_B, \mu)$, define the *tensor product biform*

$$(\mu \otimes \lambda) : (N \otimes_B M) \otimes_A (N \otimes_B M)^{\mathrm{t}} \to C$$

by the formula

$$(\mu \otimes \lambda)(n_1 \otimes m_1, n_2 \otimes m_2) = \mu(n_1 \cdot \lambda(m_1, m_2), n_2).$$

Note that $T(\mu \otimes \lambda) = T(\mu) \otimes T(\lambda)$; $(\mu_1 \perp \mu_2) \otimes \lambda = \mu_1 \otimes \lambda \perp \mu_2 \otimes \lambda$; $\mu \otimes (\lambda_1 \perp \lambda_2) = \mu \otimes \lambda_1 \perp \mu \otimes \lambda_2$; if μ is bihermitian nonsingular, then $\mu \otimes \mathrm{Meta}(\lambda)$ is isometric to $\mathrm{Meta}(\mu \otimes \lambda)$; and if λ is bihermitian nonsingular, then $\mathrm{Meta}(\mu) \otimes \lambda$ is isomorphic to $\mathrm{Meta}(\mu \otimes \lambda)$. The isometry between $\mathrm{Meta}(\mu) \otimes \lambda$ and $\mathrm{Meta}(\mu \otimes \lambda)$ is given by $\mathrm{Id} \oplus F : N \otimes_B M \oplus N \otimes_B \mathrm{Hom}_B(M, B)^{\mathrm{t}^{-1}} \to N \otimes_B M \oplus (\mathrm{Hom}_C(N \otimes_B M, C))^{\mathrm{t}^{-1}}$ where F is defined by $F(n \otimes f)(n_1 \otimes m) = \mathrm{ad}(\mu)(n)(n_1 \cdot f(m))$. The map

$$G : (\mathrm{Hom}_C(N, C))^{\mathrm{t}^{-1}} \otimes_B M \to (\mathrm{Hom}_C(N \otimes_B M, C))^{\mathrm{t}^{-1}}$$

defined by $G(f \otimes m)(n \otimes m_1) = f(n \cdot \mathrm{ad}(\lambda)(m)(m_1))$ can be used as above to define an isometry between $\mu \otimes \mathrm{Meta}(\lambda)$ and $\mathrm{Meta}(\mu \otimes \lambda)$.

From these results it follows that if μ and λ are bihermitian, then so is $\mu \otimes \lambda$, and by reducing to the metabolic case it follows that the tensor product of any two bihermitian nonsingular biforms is nonsingular.

With these definitions it is straightforward to extend our linear Morita theory to the quadratic case. See also [9, 10, 12].

1.B.2. Definition. Let R be a commutative ring with involution $- : R \to R$. Then $(R, -)$-Morita is the category with

- *objects*: $(R, -)$-algebras, i.e. rings with antistructure (A, α, u) where A is an R-algebra and

$$\alpha(ra) = \bar{r}\alpha(a) \quad \text{for all } a \in A \text{ and all } r \in R;$$

- *maps*: if (A, α, u) and (B, β, υ) are $(R, -)$-algebras, then

$$\mathrm{Hom}_{(R,-)\text{-Morita}}((A, \alpha, u), (B, \beta, \upsilon))$$

is the Grothendieck group, using orthogonal sum, of all nonsingular, bihermitian (A, α, u)-(B, β, υ) forms. Composition is given by the tensor product of forms. As usual we add a zero object to make $(R, -)$-Morita into an additive category. The identity morphism in $\mathrm{Hom}_{(R,-)\text{-Morita}}((A, \alpha, u), (A, \alpha, u))$ is given by the class of the biform $\mu : A \otimes A^{\mathrm{t}} \to A$ defined by $\mu(a_1 \otimes a_2) = a_1 \alpha^{-1}(a_2)$.

The final choice of morphisms, non-singular bihermitian biforms, is dictated by our desire to have our category act on as many 'quadratic' functors as possible. See 1.B.8 for some examples.

For use below, we remark that we have *quadratic* (B,β,υ)-(A,α,u) *forms* by mimicking [26], and that these form a symmetric monoidal category under orthogonal sum, denoted $\mathsf{Quad}((B,\beta,\upsilon)$-$(A,\alpha,u))$.

Define a functor from the category of $(R,-)$-algebras and antistructure preserving R-algebra maps to $(R,-)$-Morita by sending an R-algebra with antistructure, (A,α,u) to itself and sending $f:(A,\alpha,u)\to(B,\beta,\upsilon)$ to the form $\lambda: B\otimes B^{t}\to B$ defined by $\lambda(b_1\otimes b_2)=b_1\cdot\beta^{-1}(b_2)$. The Quillen K-theory of $\mathsf{Quad}((B,\beta,\upsilon)$-$(R,-,1))$ is a functor on the category of $(R,-)$-algebras and R-algebras maps which factors through $(R,-)$-Morita via this functor.

The antistructures that we wish to deal with in the finite group case are of a very special type. We define a *geometric antistructure* on G as a 4-tuple (G,ω,θ,b), where $\omega\in\mathrm{Hom}(G,\pm 1)$, $\theta\in\mathrm{Aut}(G)$ and $b\in G$ satisfy the relations

(i) $\omega\theta(g)=\omega(g)$ for all $g\in G$,

(ii) $\theta^2(g)=b^{-1}gb$ for all $g\in G$,

(iii) $\theta(b)=b$ and $\omega(b)=+1$.

The *associated anti-automorphism* on RG is defined by the formula

$$\alpha\left(\sum r_g g\right)=\sum \bar r_g\omega(g)\theta(g^{-1}).$$

An *orientation* for a geometric antistructure is a unit $\varepsilon\in R$ such that $\bar\varepsilon=\varepsilon^{-1}$. The *associated antistructure* on RG consists of the associated anti-automorphism and the unit

$$u=\varepsilon\cdot b.$$

The case in which θ is the identity and b is the identity element in the group, denoted e, is the most important case in ordinary surgery theory, but other geometric antistructures arise in codimension 1 splitting problems (see e.g. [12, p. 55 and p. 110]).

Before defining the quadratic analogue of RG-Morita we need to introduce a hermitian structure on finite bisets. Let H_1 and H_2 be finite groups, each with a geometric antistructure, $(\theta_{H_1},\omega_{H_1},b_{H_1})$ and $(\theta_{H_2},\omega_{H_2},b_{H_2})$. Let α_1 (resp. α_2) denote the associated anti-homomorphism on RH_1 (resp. RH_2). Fix an orientation $\varepsilon\in R$ and let $u_1=\varepsilon\cdot b_{H_1}$ (resp. $u_2=\varepsilon\cdot b_{H_2}$). Define a *biset form* on a finite H_2-H_1 biset X as a pair consisting of a bijection $\theta_X:X\to X$ and a set map $\omega_X:X\to\pm 1$ which satisfy

(i) $\omega_X(kxh)=\omega_{H_2}(k)\omega_X(x)\omega_{H_1}(h)$ for all $k\in H_2$, all $x\in X$, and all $h\in H_1$,

(ii) $\theta_X(kxh)=\theta_{H_2}(k)\theta_X(x)\theta_{H_1}(h)$ for all $k\in H_2$, all $x\in X$, and all $h\in H_1$,

(iii) $\theta_X^2(x)=b_{H_2}^{-1}xb_{H_1}$ for all $x\in X$.

Associated to each biset form is a bihermitian, nonsingular (RH_2, α_2, u_2)-(RH_1, α_1, u_1) form whenever X satisfies the condition that $|{}_{H_2}I(x)| \in R^\times$ for all $x \in X$. The formula is a bit complicated but the underlying principle is easy. We want distinct orbits to be orthogonal so we can reduce to irreducible bisets. On one of these we are looking at a composition of a transfer and a projection. If the reader writes out the biform associated to each of these, the formula should follow, but once again it seems easier for exposition to just produce the formula and check the properties. To define it, first define a set map

$$\Lambda : X \times X \to RH_2$$

where we define $\Lambda(x_1, x_2)$ as follows. Let $l(x_2) = b_{H_2}\theta_X(x_2)b_{H_1}^{-1}$.

$$\Lambda(x_1,x_2) = \begin{cases} 0 & \text{if } l(x_2) \text{ and } x_1 \text{ are not in the same } H_2\text{-orbit,} \\ \dfrac{\omega(x_2)}{|{}_{H_2}I(x_1)|} \sum k & \text{otherwise,} \end{cases}$$

where we sum over the set of all $k \in H_2$ such that $k \cdot l(x_2) = x_1$. Note that this set is a coset of ${}_{H_2}I(x_1)$.

We can extend Λ to $RX \times RX$ using sesquilinearity, and it is straightforward to check that we get a bihermitian (RH_2, β, b_{H_2})-(RH_1, α, b_{H_1}) form

$$\lambda_X : RX \otimes_{RH_1} RX \to RH_2.$$

Note that λ_X is independent of the choice of orientation ε. Also note that (ω, θ, b) gives a biset form on G considered as a G-G biset. The associated form on RG is the form which gives the identity morphism in $(R,-)$-Morita.

To check that λ is nonsingular, first choose a set $\{x_i\}$ of one x_i from each H_2-orbit of X. For each x_j define an RH_2 module map $\delta_{x_j} : RX \to RH_2$ by

$$\delta_{x_j}(x_i) = \begin{cases} 0 & \text{if } i \neq j, \\ \dfrac{1}{|{}_{H_2}I(x_i)|} \sum k & \text{if } i = j. \end{cases}$$

where we sum over $k \in {}_{H_2}I(x_i)$. It is easy to see that the set $\{\delta_{x_j}\}$ is a basis for $\mathrm{Hom}_{RH_2}(RX, RH_2)$ as an RH_2-module. Since $\mathrm{ad}(\lambda)(b_{H_2}^{-1}x_j b_{H_1}) = \omega(x_j)\delta_{x_j}$, λ is nonsingular.

The set of biset forms is a monoid under disjoint union and the (RH_2, α_2, u_2)-(RH_1, α_1, u_1) form associated to the disjoint union of two biset forms is just the orthogonal sum of the (RH_2, α_2, u_2)-(RH_1, α_1, u_1) forms associated to the two biset forms.

Given an H_1-H_2 biset form (X, θ_X, ω_X) and an H_2-H_3 biset form (Y, θ_Y, ω_Y), define the ***composite biset form*** to be the H_1-H_3 biset form (Z, θ_Z, ω_Z), where $Z = X \times_{H_2} Y$, $\theta_Z(x, y) = (\theta_X(x), \theta_Y(y))$ and $\omega_Z(x, y) = \omega_X(x) \cdot \omega_Y(y)$.

A useful point to check is that the form on the composite of two biset forms is equal to the composite of the forms. With notation as in the last paragraph, we need to verify the equation

$$\lambda_Z((x_1, y_1), (x_2, y_2)) = \lambda_X(x_1 \cdot \lambda_Y(y_1, y_2), x_2).$$

Check that $l(x_2, y_2) = (l(x_2), l(y_2))$, and recall (1.A.3). If y_1 is not in the same H_2-orbit as $l(y_2)$ then $\lambda_Y(y_1, y_2) = 0$. But then $l(x_2, y_2)$ is not in the same H_3-orbit as (x_1, y_1), so both sides of our equation are 0. If $l(y_2)$ is in the same H_2-orbit as y_1, then $\lambda_Y(y_1, y_2)$ is a multiple of $\sum k$ where the sum runs over all $k \in H_2$ for which $k \cdot l(y_2) = y_1$. Fix one such k, say $\hat{k}$. Then $\lambda_X(x_1 \cdot \lambda_Y(y_1, y_2), x_2)$ is a multiple of $\sum_{h \in I} \lambda_X(x_1 \cdot h \cdot \hat{k}, x_2)$ where $I = {}_{H_2}I(y_1)$. This in turn is a multiple of $\sum \lambda_X(x_1 \cdot h \cdot \hat{k}, x_2)$ where now we sum over one representative from each coset of ${}_{H_2}I(y_1)/(I_{H_2}(x_1) \cap {}_{H_2}I(y_1))$. This is non-zero iff $x_1 \cdot \hat{k}$ and $l(x_2)$ are in the same H_3-orbit iff (x_1, y_1) and $l(x_2, y_2)$ are in the same H_3-orbit, so at least both sides of our equation vanish or not together. We leave it to the reader to keep track of multiplicities and complete the proof.

1.B.3. Definition. Let (θ, ω, b) be a geometric antistructure and let $(R, -)$ be a commutative ring with involution. Define a category (RG, θ, ω, b)-Morita as the Add construction applied to the following category. The objects are finite groups H with geometric antistructure $(\theta_H, \omega_H, b_H)$ when H is isomorphic to a subquotient K/N of G with K a θ-invariant subgroup of G; N a θ-invariant subgroup of G, normal in K, with $N \subset \ker \omega$; and $b \in K$. The geometric antistructure on G induces one on K/N and we require the isomorphism between H and K/N to take one geometric antistructure to the other.

The morphism group

$$\mathrm{Hom}_{(RG, \theta, \omega, b)\text{-Morita}}((H_1, \theta_{H_1}, \omega_{H_1}, b_{H_1}), (H_2, \theta_{H_2}, \omega_{H_2}, b_{H_2}))$$

is defined by a Grothendieck construction: take the set of isomorphism classes of finite biset forms, X, such that $|{}_{H_2}I(x)| \in R^{\times}$ for all $x \in X$. This set is a monoid under disjoint union. Form formal differences, and set (X, θ_X, ω_X) equal to (Y, θ_Y, ω_Y) provided (RX, λ_X) is isomorphic to (RY, λ_Y) as $(RH_2, \alpha_2, b_{H_2})$-$(RH_1, \alpha_1, b_{H_1})$ forms.

In the case that θ is the identity and $b = e$, we denote the above category by (RG, ω)-Morita.

1.B.4. Remark. Each orientation ε defines a functor, the R-group ring functor, from (RG, θ, ω, b)-Morita to $(R, -)$-Morita.

1.B.5. *Quadratic generalized induction and restriction maps.* The generalized induction and restriction maps defined in the linear case in (1.A.8) have quadratic analogues. If H is a θ-invariant subgroup of K with $b \in H$, then the θ and the ω for K give us an obvious biset form on K considered as either a K–H biset or an H–K biset. Hence we have induction and restriction maps which we denote as before, suppressing the biset form data in our notation.

If N is a normal subgroup of K which is θ-invariant and contained in $\ker\omega$, then K/N has an obvious geometric antistructure which also gives K/N a biset form both as a K-K/N biset and as a K/N-K biset. Hence we get generalized induction and restriction maps in (RG,θ,ω,b)-Morita whenever $|N|\in R^{\times}$.

In (RG,ω)-Morita the only conditions we need are that $N\subset\ker\omega$ and $|N|\in R^{\times}$. If ω is trivial, then we have generalized restriction maps in (RG,ω)-Morita whenever we have them in RG-Morita. These two categories are not isomorphic since the forms need not be isomorphic just because the underlying modules are.

To obtain a good structure theorem for the 'basic' groups, we restrict attention in this section to detection and generation theorems for the category (RG,ω)-Morita.

1.B.6. Definition. Suppose that G is a p-hyperelementary group equipped with an orientation character $\omega: G\to\{\pm 1\}$. Then G is *ω-basic* if all abelian subgroups of $\ker\omega$ which are normal in G are cyclic. (See (3.B.2) for a classification of these groups.)

1.B.7. Theorem. *Let (G,ω) be a hyperelementary group with an orientation character, and let R be a commutative ring with involution –, such that $|G|$ is a unit in R. Then, in (RG,ω)-Morita,*

(i) (The Quadratic Detection Theorem) *the sum of the generalized restriction maps*

$$\mathrm{Res}: R[G]\to\bigoplus\{R[H/N]: H/N \text{ is an } \omega\text{-basic subquotient of } G \text{ with } \omega \text{ trivial on } N\}$$

is a split injection, and

(ii) (The Quadratic Generation Theorem) *the sum of the generalized induction maps*

$$\mathrm{Ind}: \bigoplus\{R[H/N]: H/N \text{ is an } \omega\text{-basic subquotient of } G \text{ with } \omega \text{ trivial on } N\}\to R[G]$$

is a split surjection.

A more explicit version is available (see 4.B.7). The result itself is proved in 4.B.8.

1.B.8. Applications. We can apply 1.B.7 to any additive functor

$$J:(RG,\omega)\text{-Morita}\to\mathscr{A}$$

whenever $|G|\in R^{\times}$. As examples, set $J(R[G],\alpha_\omega,1)$ equal to:

(i) $\hat{H}^j(Z/2Z; K_n(RG))$ where the action of $Z/2Z$ on $K_n(RG)$ is induced by the functor

$$\alpha_\omega: {}_{RG}\mathscr{P}\to{}_{RG}\mathscr{P}$$

where α_ω applied to the finitely generated, projective left module P, is just the module $(\mathrm{Hom}_{RG}(P,RG))^t$,

(ii) $L_n^{(j)}(RG,\omega)$ (where for $j=2,1,0$ these are just L_n^S, L_n^K (as in [27]), and L_n^p),

(iii) $L_n^S(Z[1/m]G \to \hat{Q}_m G, \omega) \cong L_n^Y(ZG \to \hat{Z}_m G, \omega)$ where $m=|G|$, and $Y=\{\pm G^{\mathrm{ab}}, SK_1\}$ (recall that we have an exact sequence

$$\cdots \to L_n^Y(ZG,\omega) \to L_n^Y(\hat{Z}_m G,\omega) \to L_n^Y(ZG \to \hat{Z}_m G,\omega) \to \cdots$$

and these are the L_n^Y groups studied in [28]),

(iv) $L_n^K(Z[1/m]G \to \hat{Q}_m G, \omega) \cong L_n^{X_0}(ZG \to \hat{Z}_m G, \omega)$ where X_0 denotes the torsion subgroup of K_0, and the rest of the notation is the same as in (iii),

(v) $K_n(\mathrm{Quad}(RG, \alpha_\omega, \varepsilon))$, the Quillen K-theory of the symmetric monoidal category of quadratic $(RG, \alpha_\omega, \varepsilon)$-$(R, -, \varepsilon)$ forms, where $\varepsilon \in R^\times$ is central and $\bar{\varepsilon} = \varepsilon$,

(vi) $GW(G,R)$, $GU(G,R)$, or $Y(G,R)$ which are defined in [6].

To see that the functors $L_n^{(j)}(RG,\omega)$ factor through $(R,-)$-Morita, recall the definition of these functors in [26]. We see that the $L_n^p(RG,\omega)$ are the homology groups of a chain complex where the chain groups are sesquilinear forms and the boundary maps are of the form $1 \pm T$. Via tensor product, these complexes are acted on by bi-hermitian bi-forms, and hence $(R,-)$-Morita acts on $L_n^p(RG,\omega)$. The remaining $L_n^{(j)}(RG,\omega)$ are defined [31, 32] in a sufficiently functorial manner that $(R,-)$-Morita continues to act. This factorization is also discussed in [11] and [10].

Likewise the functors in (iii) and (iv) are functors out of $(R,-)$-Morita. The functors in (vi) can be checked by hand to factor through (RG,ω)-Morita.

1.C. The Witt case

In this section we explain our results for general geometric antistructures. In order to obtain a good description of the associated 'basic' groups, two changes are needed. First of all, we restrict attention to the case of 2-hyperelementary groups. Then we only get information in the Witt categories associated to the quadratic Morita categories as explained below.

We begin by defining some new maps.

1.C.1. Definition. Let (A, α, u) be a ring with antistructure, and let $c \in A$ be a unit in A. Define a new antistructure on A by *scaling by c* as follows. The new anti-automorphism is α^c and the new unit is $u^{(c)}$ defined by

$$\alpha^c(a) = c^{-1}\alpha(a)c \quad \text{for all } a \in A, \qquad u^{(c)} = u\alpha(c^{-1})c.$$

There is a $(A, \alpha^c, u^{(c)})$-(A, α, u) biform defining an isomorphism in $(R,-)$-Morita between (A,α,u) and $(A, \alpha^c, u^{(c)})$ called the *scaling isomorphism* given by

$$\lambda(a_1 \otimes a_2) = a_1 \alpha^{-1}(a_2)\alpha^{-1}(c).$$

We apply this to the oriented geometric antistructure case. Let $(G, \theta, \omega, b, \varepsilon)$ be a group with geometric antistructure, and let (α, u) denote the associated antistructure. Let $c \in G$ be an element. Define a new oriented geometric antistructure $(\theta^c, \omega, b^{(c)}, \varepsilon^{(c)})$ by

$$\theta^c(g) = c^{-1}\theta(g)c \quad \text{for all } g \in G,$$

$$b^{(c)} = b\theta(c)c, \qquad \varepsilon^{(c)} = \omega(c) \cdot \varepsilon.$$

Notice that the antistructure associated to the scaled oriented geometric antistructure is the scale by c of (α, b).

Given a map from $(A, \alpha^c, u^{(c)})$ to (B, β, υ), we get a *twisted* map from (A, α, u) to (B, β, υ) by composing with the scaling isomorphism on A using c. This construction yields a twisted restriction map. Given a map from (A, α, u) to $(B, \beta^c, \upsilon^{(c)})$, we get a *twisted* map from (A, α, u) to (B, β, υ) by composing with the scaling isomorphism on B using c^{-1}. This construction yields a generalized induction map.

We have twisted generalized induction and restriction maps from this procedure whenever we have subgroups $N \lhd H$ of G and a $c \in G$ such that H and N are θ^c-invariant, $b^{(c)} \in H$, $N \subset \ker \omega$, and $|N|$ is a unit in R in the induction case.

We also need a new category.

1.C.2. Definition. Define a category $(R, -)$-Witt as the category with the same objects as $(R, -)$-Morita and with

$$\mathrm{Hom}_{(R,-)\text{-Witt}}((A, \alpha, u), (B, \beta, \upsilon)) = \mathrm{Hom}_{(R,-)\text{-Morita}}((A, \alpha, u), (B, \beta, \upsilon))/\mathfrak{J}$$

where $\mathfrak{J}$ is the subgroup generated by the metabolic forms in $\mathrm{Hom}_{(R,-)\text{-Morita}}((A, \alpha, u), (B, \beta, \upsilon))$. Composition is defined since $\lambda \otimes \mathrm{Meta}(\mu) \cong \mathrm{Meta}(\lambda \otimes \mu)$ and $\mathrm{Meta}(\lambda) \otimes \mu \cong \mathrm{Meta}(\lambda \otimes \mu)$ for nonsingular, bihermitian forms.

Notice that there is an obvious forgetful functor from $(R, -)$-Morita to $(R, -)$-Witt, so we have generalized induction and restriction maps. Furthermore, we also have twisted induction and restriction maps.

Our first result is a detection/generation theorem in $(R, -)$-Morita that uses fewer isomorphism classes of groups but twisted maps (compare 1.B.7).

1.C.3. Theorem. *Let G be a 2-hyperelementary group with orientation ω. Then*

(i) $$(RG, \omega) \xrightarrow{\text{Res}} \bigoplus (R[H/N], \theta^c, \omega, b^{(c)}, \omega(c))$$

is a split injection in $(R, -)$-Morita, *where we sum over subquotients H/N of G such that $N \subset \ker \omega$ and H/N is either basic with θ trivial or of the form ((index 2 in a basic) $\times\, C(2)^-$) with θ acting non-trivially on the $C(2)^-$.*

(ii) $$\bigoplus (R[H/N], \theta^c, \omega, b^{(c)}, \omega(c)) \xrightarrow{\text{Ind}} (RG, \omega)$$

is a split surjection in $(R, -)$-Morita, *where we sum over the same subquotients as in* (i).

Remark. A more precise theorem is available at the end of Section 4.C where we also explain how to pick the c associated to each subquotient.

1.C.4. Definition. A 2-hyperelementary group G with geometric antistructure is called *Witt-basic* provided all abelian normal subgroups of G which are θ-invariant and contained in $\ker \omega$ are cyclic. These are classified in Theorem 3.C.1.

1.C.5. Theorem. *Let G be a 2-hyperelementary group with geometric antistructure (θ, ω, b) and orientation ε. Assume that $|G| \in R^{\times}$. Then*

(i) (The Twisted Detection Theorem) *the sum of the twisted restriction maps*

$$\mathrm{Res}: (R[G], \theta, \omega, b, \varepsilon) \to \bigoplus (R[H/N], \theta', \omega', b', \varepsilon')$$

is a split injection in $(R, -)$-Witt, *and*

(ii) (The Twisted Generation Theorem) *the sum of the twisted induction maps*

$$\mathrm{Ind}: \bigoplus (R[H/N], \theta', \omega', b', \varepsilon') \to (R[G], \theta, \omega, b, \varepsilon)$$

is a split surjection in $(R, -)$-Witt,
where in both cases we sum over triples (H, N, c) with H/N Witt-basic and for which the twisted restriction and induction maps are defined.

As usual, a more precise version is available, 4.C.4.

The functors in (1.B.8)(i), (ii), (iii) and (iv) all factor through $(R, -)$-Witt.

1.C.6. Non-example. $L_0^p(Z[C(2) \times C(4)])$ is not detected by Witt-basic subquotients, so we need $|G|$ to be a unit in R.

In Sections 6 and 7 we introduce methods for proving detection theorems for functors that do *not* satisfy the assumption that $|G|$ is a unit in R. The following theorems are applications of this method. Other applications have appeared in [13].

1.C.7. Theorem. *Suppose G is a finite 2-group. Then the sum of the generalized restriction maps is an injection*

$$\mathrm{Res}: L_n^p(ZG) \to \bigoplus_{N \trianglelefteq H \subset G} L_n^p(Z[H/N])$$

where we sum over all basic subquotients of G.

1.C.8. Theorem. *Suppose G is a finite 2-group with orientation character ω. Then the sum of the generalized restriction maps is an injection*

$$\mathrm{Res}: L_n^p(ZG, \omega) \to \bigoplus_{N \trianglelefteq H \subset G} L_n^p(Z[H/N], \omega)$$

where we sum over all subquotients for which ω is trivial on N, and for which H/N is isomorphic to

(i) *an ω-basic subquotient, or*

(ii) $C(2) \times C(4)$ *with ω non-trivial, but trivial on all elements of order 2 (we will denote this as $C(2) \times C(4)^-$), or*

(iii) $\langle t_0, t_1, g \mid t_0^2 = t_1^4 = g^2 = e,\ g t_1 g^{-1} = t_1,\ g t_0 g^{-1} = t_0 t_1^2,\ [t_0, t_1] = e \rangle$, *and* $\omega(t_0) = \omega(g) = 1$, $\omega(t_1) = -1$.

The group in (iii) is just a semidirect product $(C(2) \times C(4)^-) \rtimes C(2)$ and is also the central product over $C(2)$ of $D8$ and $C(4)^-$. We denote it hereafter by M_{16}.

2. Representations of finite groups

The first goal of this section is to define imprimitive induction and identify a special case in which it always occurs. Then we study the representation theory of basic groups, leading to a definition of the basic representation of a basic group. Finally we prove that any irreducible rational representation of a p-hyperelementary group G can be induced from some basic subquotient of G.

Let k be a field of characteristic zero. For any irreducible k-representation $\varrho : G \to \mathrm{GL}(V)$ of a finite group G we let $D_\varrho = \mathrm{End}_{kG}(V)$ be the associated division ring.

Suppose $\varrho_0 : H \to \mathrm{GL}(W)$ is a k-representation for a subgroup H, such that $kG \otimes_{kH} W \cong V$ is an irreducible k-representation of G. Then we get an injective ring map

$$\mathrm{Id}_{kG} \otimes \_ : D_{\varrho_0} = \mathrm{End}_{kH}(W) \to \mathrm{End}_{kG}(V) = D_\varrho .$$

2.1. Lemma. *With the notation above, if* $V|_H$ *contains just one copy of* W, *then* $\mathrm{Id}_{kG} \otimes \_$ *is an isomorphism.*

Proof. Let $\varepsilon : kG \otimes V|_H \to V$ be the evaluation map. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{kH}(W, V|_H) & \xrightarrow{\gamma} & \mathrm{Hom}(kG \otimes W, kG \otimes V|_H) & \xrightarrow{\varepsilon_*} & \mathrm{Hom}_{kG}(kG \otimes W, V) \\
\uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} & & \\
D_{\varrho_0} = \mathrm{Hom}_{kH}(W, W) & \xrightarrow{\delta} & \mathrm{Hom}(kG \otimes W, kG \otimes W) = D_\varrho & &
\end{array}$$

where the vertical maps α and β are induced by the inclusion of W in $V|_H$, and γ, δ are induced by $\mathrm{Id}_{kG} \otimes \_$. The hypotheses imply that α is an isomorphism. By Frobenius reciprocity (see [3, 10.8]), the composite $\varepsilon_* \circ \gamma$ is an isomorphism. Since $kG \otimes W \cong V$ is irreducible, the composite $\varepsilon_* \circ \beta$ is also an isomorphism. Thus δ is an isomorphism. □

2.2. Definition. Let ϱ be an irreducible rational representation of a finite group G. We say that ϱ is *imprimitive* if there exists a subgroup H and a rational representation η of H such that $\eta|^G = \varrho$ and the map $\mathrm{Id}_{\mathbb{Q}G} \otimes \_$ is an isomorphism. In this situation, we say that ϱ is *imprimitively induced from H* and that ϱ is *imprimitively induced from* η. If ϱ is not imprimitive, then it is *primitive.*

In Section 1 we defined a generalized induction for an irreducible rational representation on a subquotient H/N of G. First we pull back the representation on H/N to one on H, and then we induce the representation on H up to G. We say that a generalized induction is *imprimitive* whenever the induction stage is imprimitive. Usually we will just say induction even if we mean generalized induction. By examining the starting group, the reader can deduce which one is meant.

The following variant of Clifford's theorem will be useful to us:

2.3. Theorem. *Let ϱ be an irreducible Q-representation of a finite group G, and let N be a normal subgroup. Then*

$$\varrho|_N = l\cdot(\eta_1+\cdots+\eta_r)$$

where the η_i are all distinct irreducible Q-representations. The group G acts on the vector space V_ϱ, and must permute the N-invariant subspaces $l\cdot\eta_i$ transitively.

Let H be the isotropy subgroup of $l\cdot\eta_1$. Then $N\subset H$ and $|G:H|=r$. Furthermore, there is a Q-representation, $\tilde{\eta}_1$ of H with $\tilde{\eta}_1|_N=l\cdot\eta_1$; and $\tilde{\eta}_1|^G=\varrho$. This induction is always imprimitive.

Proof. All but the last two lines are a statement of the standard Clifford theorem (see [3, 11.1, p. 259]). That the induction is imprimitive follows immediately from Lemma 2.1. □

Our next result is essentially due to Witt [30].

2.4. Theorem. *Let G be a finite group which has an abelian, normal subgroup which is not cyclic and a faithful, irreducible Q-representation ϱ. Then there is a normal elementary abelian p-group A, for some prime p, of rank ≥ 2. Given any such A there is an index p subgroup E of A, such that E is not normal in G and such that ϱ is induced imprimitively from the normalizer of E.*

Proof. For some prime p the subgroup of elements of order $\leq p$ in the promised normal abelian subgroup of G which is not cyclic will be elementary abelian of rank ≥ 2. Fix such a p and note that this subgroup is an elementary abelian subgroup of rank ≥ 2 which is normal in G.

Let A denote any noncyclic normal elementary abelian subgroup of G. Recall that the irreducible Q-representations of A are determined by their kernels. The possible kernels are all of A and any index p subgroup.

Apply Theorem 2.3 and let $\varrho|_A = l\cdot(\chi_1+\cdots+\chi_r)$. Since ϱ is faithful and A is normal, the kernel of χ_1 can not be all of A, and so it is some index p subgroup E. The same argument shows that E is not normal in G. Since kernels determine representations for A, the H constructed in Theorem 2.3 is just the normalizer of E in G. By 2.3 again, the induction is imprimitive. □

As we will be working with p-hyperelementary groups, we recall some facts about their structure. First, $G=C\rtimes P$ with C cyclic of order prime to p and P a p-group. Let $\psi: P\to \mathrm{Aut}(C)$ denote the action map.

2.5. *Let H be a proper subgroup of G with index a power of p. Then the normalizer of H is strictly larger than H.*

2.6. *A p-subgroup H is normal in G iff $H \subset \ker \psi$ and H is normal in P. If H is non-trivial and normal in G, then it contains a central element of order p.*

2.7. Proposition. *Let G be a p-hyperelementary group with an abelian normal subgroup which is not cyclic. Then G contains a subgroup $K \cong C(p) \times C(p)$ which is normal in G.*

If G_0 denotes the centralizer of K in G, then

(i) *either $G = G_0$ or*

(ii) *G_0 has index p in G and the conjugation action of G/G_0 on the cyclic subgroups of K fixes one of them and is transitive on the remaining ones.*

Proof. Let E be the subgroup of elements of order $\leq p$ in the normal, non-cyclic abelian subgroup of G. By 2.6, E contains a subgroup $C_0 \cong C(p)$, central in G. Apply the same argument to E/C_0 in G/C_0 and let K be the inverse image in G of this $C(p)$ in G/C_0. Note $K \subset E$ so it is a rank 2 elementary abelian p-group, which is normal in G.

Since K is normal in G, so is G_0. Note C centralizes K since both are normal, hence the index of G_0 in G is a pth power. Consider the conjugation action of G/G_0 on K. Since K has rank 2, $\mathrm{Aut}(K) \cong \mathrm{GL}(2, F_p)$ and $|\mathrm{GL}(2, F_p)| = (p-1)(p^2-p)$, so G/G_0 is trivial or $C(p)$. In the first case there is nothing to prove, and the result in the second case is a standard result on the action of $\mathrm{Aut}(K)$ on the cyclic subgroups of K. □

We return to representation theory for p-hyperelementary groups. Theorem 2.4 and Proposition 2.7 suggest that we should study induction when we have a normal $C(p) \times C(p)$ subgroup. Let $\mathrm{Irr}_Q(G)$ denote the set of irreducible rational representations of G; if $N \subset H$ are subgroups of G, let $\mathrm{Irr}_Q(G)_{N \subset H} = \{\varrho \in \mathrm{Irr}_Q(G) \mid N = \ker \varrho \cap H\}$.

2.8. Theorem. *Let G be a non-basic p-hyperelementary group, and consider any normal subgroup $K \cong C(p) \times C(p)$. Let $C_0, \ldots, C_p$ denote the cyclic subgroups and arrange notation so that C_0 is central. Let G_0 denote the centralizer of K in G. Consider any $\varrho \in \mathrm{Irr}_Q(G)$.*

(i) *If K is central in G then $\varrho|_K = l \cdot \phi$ and $K \cap \ker \varrho = \ker \phi = K, C_0, C_1, \ldots,$ or C_p. Hence $\mathrm{Irr}_Q(G) = \mathrm{Irr}_Q(G)_{K \subset K} \amalg \mathrm{Irr}_Q(G)_{C_0 \subset K} \amalg \cdots \amalg \mathrm{Irr}_Q(G)_{C_p \subset K}$.*

(ii) *If K is not central then $K \cap \ker \varrho = K$, C_0, or $\{e\}$. If $K \cap \ker \varrho = \{e\}$, then*

$$\varrho|_{G_0} = \sum_{x \in G/G_0} \phi^x$$

and each $K \cap \ker \phi^x$ is a different C_i where $1 \leq i \leq p$. Hence $\mathrm{Irr}_Q(G) \cong \mathrm{Irr}_Q(G)_{K \subset K} \amalg$

$\mathrm{Irr}_Q(G)_{C_0 \subset K} \amalg \mathrm{Irr}_Q(G_0)_{C_1 \subset K}$, *where the embedding of* $\mathrm{Irr}_Q(G_0)_{C_1 \subset K}$ *in* $\mathrm{Irr}_Q(G)$ *sends* ϱ_0 *to* $\varrho_0|^G$, *which is always an imprimitive induction.*

Proof. Let ϕ_E denote the irreducible Q-representation of K with kernel E, and recall that the choices for E are $K, C_0, C_1, \ldots, C_p$.

Apply Clifford's theorem (2.3) to ϱ and K. If K is central, then no two distinct representations of K are conjugate, so $\varrho|_K = l \cdot \phi_E$ for some E. Hence $K \cap \ker \varrho = E$ and the result follows.

If K is not central, then the distinct representations which are conjugate are just the ones whose kernels are C_i for i with $1 \le i \le p$. Hence $\varrho|_K = l \cdot \phi$ where ϕ is either ϕ_K (iff $K \cap \ker \varrho = K$); ϕ_{C_0} (iff $K \cap \ker \varrho = C_0$); or $\phi = \sum_{i=1}^p \phi_{C_i}$ (iff $K \cap \ker \varrho = \{e\}$).

If $K \cap \ker \varrho = \{e\}$, let ϕ denote an irreducible constituent of $\varrho|_{G_0}$. Frobenius reciprocity implies that $\phi|_K$ and $\sum_{i=1}^p \phi_{C_i}$ have a common constituent. Since G_0 has a central $C(p) \times C(p)$, apply part (i) to ϕ to see that $K \cap \ker \phi = C_1, \ldots,$ or C_p.

Now apply 2.3 to ϱ restricted to G_0. By 2.5, the conjugates of ϕ have different kernels and so are distinct. Hence $\varrho|_{G_0} = l \cdot \sum_{x \in G/G_0} \phi^x$ and an easy degree argument shows that $l = 1$. □

Finally, we take up the representation theory of basic groups. As we will see shortly, basic groups are contained in the broader class defined next.

2.9. Definition. A group G is an *F-group* if it contains a self-centralizing cyclic subgroup A, i.e. A is normal and the map $G/A \to \mathrm{Aut}(A)$ induced by conjugation is injective.

The first result, observed by Fontaine [7, Lemma 3, p. 153] is

2.10. Lemma. *Any basic p-hyperelementary group is an F-group.*

Proof. To fix notation, let $G = C \rtimes P$ with C cyclic of order prime to p, and P a p-group. Let A be a maximal element of the set of normal cyclic subgroups of G containing C (ordered by inclusion). Note that G/A is a p-group, and consider the kernel of the action map $G/A \to \mathrm{Aut}(A)$. If it is non-trivial, let E be a cyclic subgroup of it. Let $B \subset G$ denote the inverse image of $E \subset G/A$ in G. Then B is clearly normal; it is abelian since any extension of a cyclic by a $C(p)$ with trivial action is abelian, and it is non-cyclic by maximality. This contradicts the fact that G is basic. □

Hence we study representations of F-groups. The key step involves the relationship between complex representations, rational representations, and Galois groups which we quickly review (or see [23, Chapter 12]).

Let ψ be an irreducible representation of G over the complex numbers C. The values of the character of ψ on the elements of G are algebraic integers, and we let

$Q(\psi)$ denote the finite extension field of the rationals Q, generated by these values. If $\tau \in \mathrm{Gal}(Q(\psi)/Q)$, then ψ^τ will denote the Galois conjugate representation, i.e. the representation whose character is just τ applied to the value of the character for ψ. The orthogonality relations for complex characters show that ψ^τ is an irreducible representation and the ψ^τ for different τ are distinct. Form the representation $\sum_{\tau \in \mathrm{Gal}(Q(\psi)/Q)} \psi^\tau$. This has a rational character but may not be the complexification of a rational representation. There does exist a minimal integer, $m_\psi > 0$, called the *Schur index*, so that

$$m_\psi \cdot \sum_{\tau \in \mathrm{Gal}(Q(\psi)/Q)} \psi^\tau$$

is the complexification of an irreducible Q-representation, and every irreducible Q-representation arises in this fashion. Finally, the division algebra D_ψ associated to ψ has center $Q(\psi)$ and index m_ψ, so $\dim_Q(D_\psi) = m_\psi^2 \cdot \dim_Q(Q(\psi))$.

Let A denote a cyclic group of some order. It has $\varphi(|A|)$ faithful irreducible complex representations, all of which are Galois conjugate. Let $a \in A$ be a generator, and let $\xi_{\langle a \rangle}$ denote the faithful irreducible complex representation which sends a to $\exp(2\pi i/|A|)$. The sum of these is the complexification of a rational representation so A has a unique irreducible faithful rational representation, denoted ϱ_A. Moreover, the automorphism group of A, $\mathrm{Aut}(A)$, acts simply transitively on the faithful irreducible complex representations of A, and there is a unique isomorphism $\mathrm{Aut}(A) \to \mathrm{Gal}(Q(\xi_{\langle a \rangle})/Q)$ which identifies the two actions on $\xi_{\langle a \rangle}$.

We apply these remarks to prove

2.11. Theorem. *Let G be an F-group with $A \subset G$ a self-centralizing cyclic subgroup. There exists a unique faithful irreducible Q-representation ϱ_G of G and ϱ_G is the only irreducible Q-representation of G which is faithful on A.*

Moreover, ϱ_G satisfies the equation $\varrho_G|_A = m \cdot \varrho_A$, where m is the Schur index of any irreducible complex constituent of ϱ_G.

Proof. Pick a generator $a \in A$. Let $\kappa = \xi_{\langle a \rangle}|^G$. By the Mackey irreducibility criterion [23, Section 7.4 Corollary], κ is irreducible provided all the conjugates of $\xi_{\langle a \rangle}$ are distinct. But G/A embeds in $\mathrm{Aut}(A)$ via the action map, and the action of $\mathrm{Aut}(A)$ on the irreducible faithful complex representations of A is faithful. Hence κ is irreducible. Moreover, $Q(\kappa)$ is the subfield of $Q(\xi_{\langle a \rangle})$ fixed by G/A considered as a subgroup of the Galois group of $Q(\xi\langle a \rangle)$ over Q via the above identifications. Hence $\mathrm{Gal}(Q(\kappa)/Q)$ is naturally identified with $\mathrm{Aut}(A)/(G/A)$.

This means that the Galois average of κ has a rational valued character and that this representation restricted to A is just the complexification of ϱ_A. Let ϱ_G denote the associated irreducible Q-representation. Frobenius reciprocity shows that

$$\varrho_A|^G = \frac{|G/A|}{m} \cdot \varrho_G,$$

and hence $\varrho_G|_A = m \cdot \varrho_A$.

Let χ be any irreducible Q-representation of G and apply Theorem 2.3 to $\chi|_A$. On A, no two distinct irreducible Q-representations can be conjugate, so $\chi|_A = l \cdot \psi$ for some irreducible Q-representation ψ of A. If $\psi \neq \varrho_A$, then χ has a non-trivial kernel. If $\psi = \varrho_A$, then $\chi = \varrho_G$ and hence ϱ_G is the unique faithful irreducible Q-representation of G. □

2.12. Definition. Let G be a basic p-hyperelementary group. By Lemma 2.10, Theorem 2.11 applies to G. We call the representation ϱ_G whose existence and uniqueness was proved in Theorem 2.11, the *basic representation of* G.

The major result in the representation theory of p-hyperelementary groups that we need is

2.13. Theorem. *Let G be a p-hyperelementary group and let ϱ be an irreducible rational representation of G. Then there exist subgroups $N_\varrho \lhd H_\varrho$ of G such that the index of H_ϱ in G is a pth power; H_ϱ/N_ϱ is a basic group; and ϱ can be induced imprimitively from the basic representation of H_ϱ/N_ϱ.*

Proof. Since imprimitive generalized induction is transitive, it is easy to see that we can induct on the subquotient structure of G, i.e. we can assume the result for all proper subquotients of G and we need only show that ϱ can be pulled back from a quotient group of G or else it can be imprimitively induced from a subgroup of prime power index.

If ϱ is not faithful, then it can be induced from a quotient group, so we may as well assume that ϱ is faithful.

If G is not basic, then there is a normal abelian non-cyclic subgroup. But in this case Theorem 2.8 shows that there is a subgroup H of index p from which we can imprimitively induce.

If G is basic and ϱ is faithful, then $\varrho = \varrho_G$ by 2.11, and 1 is a pth power. □

Remark. In Theorem 2.13, $H_\varrho/N_\varrho = \{e\}$ iff ϱ is trivial and $H_\varrho = N_\varrho = G$.

We will need some results later about the sorts of subgroups H of G from which an imprimitive induction can take place.

2.14. Proposition. *Let G be a p-hyperelementary group and let ϱ be an irreducible Q-representation. Suppose that H is a subgroup from which ϱ can be imprimitively induced. Then there exists a sequence of subgroups $H = H_0 \subset \cdots \subset H_r = G$ with each H_i of index p in the next.*

Proof. The result follows from 2.5 if we can show that the index of H in G is a pth power.

Let V_ϱ denote the vector space for ϱ, and recall that V_ϱ is a free module over the associated division algebra D_ϱ. From 2.11, it follows that $\dim_Q V_\varrho = p^r \cdot \dim_Q D_\varrho$.

Let χ be an irreducible Q-representation of H. The last argument shows that $\dim_Q V_\chi = p^s \cdot \dim_Q D_\chi$. If χ is a representation from which ϱ can be induced imprimitively, $\dim_Q D_\chi = \dim_Q D_\varrho$. Since $\dim_Q V_\varrho = |G:H| \cdot \dim_Q V_\chi$, we see that $|G:H| = p^{r-s}$. □

2.15. Proposition. *Let G be a p-hyperelementary F-group, and let N be an index p subgroup from which ϱ_G can be induced imprimitively. Then N contains a $C(p) \times C(p)$ which is normal in G.*

Proof. For notation, let $G = C \rtimes P$, with P a p-group and C cyclic of order prime to p. Let A be a self-centralizing cyclic subgroup of G, and let $A_0 = N \cap A = \ker(A \to G/N)$. Note that either $A = A_0$ or $|A:A_0| = p$. By Theorem 2.3, $\varrho|_N = \eta_1 + \cdots + \eta_p$, where the η_i are distinct and conjugate. By Theorem 2.11, $\varrho_A|^G$ is a multiple of ϱ and Frobenius reciprocity forces $\varrho_{A_0}|^N$ to contain each of the η_i.

Let L denote the centralizer of A_0 in N. Since A_0 is normal in G, so is L.

First we show that $L \neq A_0$. Suppose that A_0 were self-centralizing in N. Then by Theorem 2.11, $\varrho_{A_0}|^N$ would be a multiple of ϱ_N and the η_i could not be distinct. Hence A_0 is not self-centralizing in N. It follows that $A_0 \neq A$, so $|A:A_0| = p$ and $N/A_0 \to G/A$ is an isomorphism. Hence N/A_0 injects into $\mathrm{Aut}(A)$, so it is easy to see that $|L:A_0| = p$.

From this it follows that L is abelian, and we conclude by showing that L is not cyclic. Notice that L does not centralize A, and so A does not centralize L. Consider the action map $G/L \to \mathrm{Aut}(L)$. By projecting to $\mathrm{Aut}(A_0)$, we see that $N/L \to \mathrm{Aut}(L)$ is injective. While A does not centralize L, it does centralize A_0. This means that A/A_0 injects into $\mathrm{Aut}(L)$ but its image goes to 0 in $\mathrm{Aut}(A_0)$.

Hence, if L is cyclic, it is self-centralizing in G. The argument above that $N \cap A \neq A$ did not depend on which self-centralizing cyclic subgroup of G we began with, so repeat the argument with L. A contradiction ensues since $N \cap L = L$, and so L is not cyclic. □

2.16. Corollary. *The basic representation of a p-hyperelementary basic group is primitive.*

3. Structure of basic groups, ω-basic groups and Witt-basic groups

The goal of this section is to classify the basic groups and their quadratic relatives. We also do some quadratic representation theory that is easier to explain after we have the classification in hand. Our first goal is the classification theorem 3.A.6 below, but we begin with some lemmas.

3.A. The linear case

3.A.1. Proposition. *Let T be a finite p-group. If $[T,T]$ is not cyclic, then $[T,T]$ contains a subgroup $K \cong C(p) \times C(p)$ such that K is normal in T.*

Proof. There exists $C_0 \subseteq [T,T] \cap \mathcal{Z}(T)$ where $C_0 \cong C(p)$. Let A be a maximal member of the following set of subgroups:

$$\{B \subseteq [T,T] \mid C_0 \subseteq B,\ B \lhd T,\ B \text{ is cyclic}\}.$$

Consider

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & T & \longrightarrow & T/A & \longrightarrow & 0 \\ & & \| & & \cup & & \cup & & \\ 0 & \longrightarrow & A & \longrightarrow & [T,T] & \longrightarrow & [T/A, T/A] & \longrightarrow & 0. \end{array}$$

Since $[T,T]$ is not cyclic, $[T/A, T/A] \neq \{e\}$. Since T/A is a p-group, we can find

$$C_1 \subseteq [T/A, T/A] \cap \mathcal{Z}(T/A)$$

where $C_1 \cong C(p)$.

Let $B \subset [T,T]$ be a subgroup such that

$$0 \to A \to B \to C_1 \to 0$$

is exact. Since $C_1 \lhd T/A$, $B \lhd T$. Consider the action map $T/A \to \mathrm{Aut}(A)$. Since $\mathrm{Aut}(A)$ is abelian, C_1 is in the kernel, i.e. C_1 acts trivially on A. Hence B is abelian. By the maximality of A, B is not cyclic. Hence there exists $K \cong C(p) \times C(p) \subseteq B$.

Since K is unique in B and $B \lhd T$, $K \lhd T$. Since $B \subset [T,T]$, $K \subset [T,T]$. □

3.A.2. Proposition. *Suppose we have a diagram of groups*

$$[P,P] \subset A \subset T \subset P$$

where P is a p-group and $A \cong C(p^n)$ is self-centralizing in T. Assume that T contains no subgroup $K \cong C(p) \times C(p)$ which is normal in P.

(i) *If p is odd, the group T is cyclic.*

(ii) *If $p = 2$, the group T must be isomorphic to one of the following groups:*

$$C(2^i),\ i \geq 0; \qquad Q(2^i),\ i \geq 3; \qquad \mathrm{SD}(2^i),\ i \geq 4; \qquad D(2^i),\ i \geq 3.$$

3.A.3. Remark. The list in 3.A.2(ii) contains one 2-group which is not basic, namely $D(8)$. Notice that if

$$P = D(16) = \langle x, y \mid x^8 = y^2 = 1,\ yxy^{-1} = x^{-1} \rangle,$$

then $D(8) \cong \langle x^2, y \rangle$, $A = \langle x^2 \rangle$ is self-centralizing in $\langle x^2, y \rangle$, and $[P,P] \cong \langle x^2 \rangle$. Thus $D(8)$ must be included in the list.

The proof of 3.A.2 uses the following two lemmas:

3.A.4. Lemma. *Suppose Δ is a p-group which is a subgroup of $U=(Z/p^nZ)^\times$. Let $\alpha=1+p^{n-1}\in U$, and assume that $\alpha\notin\Delta$. If p is odd, then $\Delta=\langle 1\rangle$. If $p=2$ and $n\le 2$, then $\Delta=\langle 1\rangle$. If $p=2$ and $n>2$, then Δ is $\langle 1\rangle$, $\langle -1\rangle\cong C(2)$, or $\langle -1+2^{n-1}\rangle\cong C(2)$.*

Proof. Let ${}_pU=\{\beta\in U \mid \beta^p=1\}$. If p is odd, then U is cyclic and ${}_pU=\langle 1+p^{n-1}\rangle$. If $p=2$ and $n=1$, then $U=\langle 1\rangle$. If $p=2$ and $n=2$, then $U=\langle -1\rangle$. If $p=2$ and $n>2$, then $U=C(2)\times C(2^{n-2})$,

$$ {}_pU=\langle 1,-1,-1+p^{n-1},1+p^{n-1}\rangle, $$

and $U^p\cap {}_pU=\langle 1+p^{n-1}\rangle$. □

3.A.5. Lemma. *Suppose β is a nontrivial element of order p in $(Z/p^nZ)^\times$ (note that $n>1$). This describes an action of Z/pZ on Z/p^nZ. Then*

$$ H^2_\beta(Z/pZ;\, Z/p^nZ)\cong\begin{cases} Z/2Z & \text{if } p=2 \text{ and } \beta=-1,\\ \langle 1\rangle & \text{otherwise.}\end{cases} $$

The group $H^2_\beta(Z/pZ; Z/p^nZ)$ classifies extensions of Z/p^nZ by Z/pZ with given action. If $n>1$, the extension $0\to Z/2^nZ\to Q(2^{n+1})\to Z/2Z\to 0$ represents the nontrivial element in $H^2_\beta(Z/2Z; Z/2^nZ)$, where $\beta=-1$.

Proof. Consult [2] for the classification of group extensions and for the calculation of $H^2_\beta(Z/pZ; Z/p^nZ)$. If p is odd, the action given by β fixes no elements, so the result is clear. If $p=2$ and $\beta\neq -1$, then compute by hand that the fixed elements are all norms. If $p=2$ and $\beta=-1$, both the calculation and the claim about the extension are straightforward. □

Proof of 3.A.2. If $p=2$ and $n=1$ or 2, the result is clear. Hence, if $p=2$, we can assume $n>2$.

Claim. *There does not exist an element $x\in T$ such that $xax^{-1}=a^{1+p^{n-1}}$ for all $a\in A$.*

Proof of Claim. Suppose x exists an let $\bar{A}=\langle A,x\rangle$. Then 3.A.5 implies that $\bar{A}=A\rtimes Z/pZ$. If υ is a generator of A, then

$$ K=\langle \upsilon^{p^{n-1}},x\rangle=\{\bar{a}\in\bar{A} \mid \bar{a}^p=1\}\cong C(p)\times C(p). $$

Since $[P,P]\subset\bar{A}$, $\bar{A}$ is normal in P. Since K is characteristic in $\bar{A}$, we get that K is normal in P and x does not exist. □

To finish the proof of 3.A.2 note

(i) p odd: Lemma 3.A.4 implies $A=T$.

(ii) $p=2$: Lemma 3.A.4 implies that if $A \neq T$, then either $T/A \cong \langle -1+2^{n-1} \rangle$ or $T/A = \langle -1 \rangle$. Lemma 3.A.5 then implies that if $T/A = \langle -1+2^{n-1} \rangle$, then $T \cong \mathrm{SD}(2^{n+1})$ and if $T/A = \langle -1 \rangle$, then $T \cong D(2^{n+1})$ or $Q(2^{n+1})$. □

We can now classify the basic p-hyperelementary groups.

3.A.6. Theorem (Classification of basic p-hyperelementary groups). *Suppose $G = C \rtimes P$ is a p-hyperelementary group, where P is a p-group, C cyclic, with p prime to $|C|$. Let $\psi: P \to \mathrm{Aut}(C)$ be the map induced by conjugation.*

(i) *If p is odd, then the group G is basic if and only if* $\ker \psi$ *is cyclic.*

(ii) *If $p=2$, the group G is basic if and only if* $\ker \psi$ *is*

(a) (*cyclic*)

$$C(2^i) = \langle x \mid x^{2^i} = e \rangle, \quad i \geq 0;$$

(b) (*quaternionic*)

$$Q(2^i) = \langle x, y \mid x^{2^{i-1}} = e,\ y^2 = x^{2^{i-2}},\ yxy^{-1} = x^{-1} \rangle, \quad i \geq 3;$$

(c) (*semidihedral*)

$$\mathrm{SD}(2^i) = \langle x, y \mid x^{2^{i-1}} = y^2 = e,\ yxy^{-1} = x^{2^{i-2}-1} \rangle, \quad i \geq 4;$$

(d) (*dihedral*)

$$D(2^i) = \langle x, y \mid x^{2^{i-1}} = y^2 = e,\ yxy^{-1} = x^{-1} \rangle, \quad i \geq 4;$$

or

(e) $D(8) = \langle x, y \mid x^4 = y^2 = e,\ yxy^{-1} = x^{-1} \rangle$

and the map

$$P \to \mathrm{Out}(D(8)) \cong C(2)$$

induced by conjugation is onto.

The special case of 3.A.5 where G is a p-group is due to Roquette [22].

Proof. Let $T = \ker \psi$.

($\Rightarrow$) If G is basic, then T contains no subgroup $K \cong C(p) \times C(p)$ such that $K \lhd P$. Since $\mathrm{Aut}(C)$ is abelian, $[P, P] \subseteq \ker(\psi)$. Thus 3.1 implies $[P, P]$ is contained in a maximal normal cyclic subgroup A of T. Then A is self-centralizing in T. Apply 3.A.2. Notice that if $T \cong D(8)$ and $P \to \mathrm{Out}(D(8))$, is not onto, then T contains a subgroup isomorphic to $C(2) \times C(2)$ which is normal in P.

($\Leftarrow$) If G is not basic, then 2.7 implies that G contains a normal subgroup isomorphic to $C(p) \times C(p)$. This implies that T contains a subgroup $K \cong C(p) \times C(p)$ which is normal in P. It is easily verified that this is impossible for each of the groups listed in 3.A.6. □

3.A.7. Theorem. *If a 2-hyperelementary group G is an index 2 subgroup of a basic group, then* $\ker \psi$ *is* $C(2^i)$, $i \geq 0$; $Q(2^i)$, $i \geq 3$; $D(2^i)$, $i \geq 3$; *and* $\mathrm{SD}(2^i)$, $i \geq 4$. *In particular, any such group is an F-group.*

Remark. Note that the only non-basic groups on this list are a few cases in which $\ker \psi \cong \mathrm{D}(8)$.

Proof. If G is an index 2 subgroup of a 2-hyperelementary group $\tilde{G}$, then either $\ker \psi = \ker \tilde{\psi}$, or else $\ker \psi$ has index 2 in $\ker \tilde{\psi}$. The only case requiring comment is the $\ker \psi$ index 2 in $\ker \tilde{\psi}$ case. It is easy to list the index 2 subgroups of the cyclic, quaternionic, dihedral and semidihedral groups and to see that the only trouble could come from an index 2 subgroup of a group $\tilde{G}$ of type (e) above. But as G is normal in $\tilde{G}$, $\ker \psi \subset D(8)$ must be invariant under the map $P \to \mathrm{Out}(D(8))$ and so $\ker \psi \cong C(4)$.

The cyclic, quaternionic, dihedral and semidihedral groups each have a self-centralizing cyclic normal subgroup D. Let $A \subset G$ be the subgroup generated by the normal cyclic of order prime to 2, C, and a normal cyclic subgroup of order 2^r, D. It is easy to check that A is self-centralizing cyclic. □

3.B. The quadratic case

Before beginning the classification theorem we introduce a construction we will need.

3.B.1. Lemma. *Let G be a group with a normal subgroup $K \cong C(2) \times C(2)$ and a homomorphism $\omega : G \to \{\pm 1\}$ such that ω is non-trivial on K. Then $G = G^+ \rtimes C(2)$ where $G^+ = \ker \omega$.*

Let $\langle z \rangle = K \cap G^+$. Then there is a homomorphism, $\varepsilon : G^+ \to \pm 1$ such that $\alpha \in$ $\mathrm{Aut}(G^+)$, *the automorphism used to define the semi-direct product, is of the form*

$$\alpha(h) = \begin{cases} h & \text{if } \varepsilon(h) = 1, \\ z \cdot h & \text{if } \varepsilon(h) = -1, \end{cases}$$

for all $h \in G^+$. The element z is central. The centralizer of K in G is $\ker \varepsilon \times C(2)$; $\ker \varepsilon$ *is normal in G.*

Proof. It is clear that $\mathrm{G} = \mathrm{G}^+ \rtimes C(2)$ where the homomorphism α is given by conjugation by an element $y \in K$ with $y \neq z$ or e. Note that $\alpha(h)h^{-1}$ is in K and in G^+. Since $K \cap G^+ = \langle z \rangle$, $\alpha(h) = h$ or zh. Define $\varepsilon : G^+ \to \{\pm 1\}$ by setting $\varepsilon(h) = -1$ iff $\alpha(h) = zh$. Since $\langle z \rangle$ is normal, z is central so it is not hard to check that ε is a homomorphism.

The remaining results are clear. □

Notation. For any pair (G, ω), let $G^+ = \ker(\omega : G \to \{\pm 1\})$ and let $\ker \psi^+ = G^+ \cap \ker \psi$ where ψ can be any homomorphism defined on G.

Recall that (G, ω) is ω-basic provided that no non-cyclic abelian subgroup of G^+ is normal in G.

3.B.2. Theorem (Classification of ω-basic p-hyperelementary groups).
(i) *If p is odd, a p-hyperelementary group is ω-basic if and only if it is basic.*
(ii) *A 2-hyperelementary group (G, ω) is ω-basic if and only if either*
(a) *G is basic; or*
(b) *G is not basic, but $G = G^+ \rtimes C(2)^-$ as in* 3.B.1. *Furthermore G^+ is non-trivial and basic.*

Proof. It is clear from the definitions that basic groups are ω-basic, so we classify the ω-basic groups (G, ω) that are not basic. Definition 1.B.6 and Proposition 2.7 imply that G contains a normal subgroup $K \cong C(p) \times C(p)$ which is not contained in G^+. This means that $p = 2$ and $\omega|_K$ is split onto, so we are done with part (i) and in the case $p = 2$ we may apply Lemma 3.B.1. Write $G \cong G^+ \rtimes C(2)$ with a central $z \in G^+$ and automorphism $\alpha \in \mathrm{Aut}(G^+)$ with $\alpha(h) = h$ or zh. Furthermore, G^+ is non-empty. We are done if we can show that G^+ is basic, which we do by contradiction. Let $L \subset G^+$ be a $C(2) \times C(2)$ which is normal in G^+. We derive a contradiction by using L to construct a $C(2) \times C(2)$ in G^+ which is normal in G. From 2.6, L contains a central (in G^+) element x of order 2. If $x = z$, then L is the desired subgroup. If $x \neq z$, then $\langle z, x\rangle$ is the desired subgroup. □

The next result is an ω-analogue of 2.11. Let $\omega : G \to \{\pm 1\}$ be an orientation character. We can also view ω as a Q-representation of G via the inclusion $\{\pm 1\} \to \mathrm{GL}_1(Q)$. For any Q-representation ϱ of G, we let ϱ^ω denote $\varrho \otimes \omega$.

3.B.3. Definition. A Q-representation ϱ is *ω-invariant* if $\varrho \cong \varrho^\omega$. It is *$\omega$-irreducible* if it is ω-invariant and it cannot be expressed as a sum of nontrivial ω-invariant Q-representations. We say that an ω-irreducible Q-representation is of *type (I)* if it is irreducible as a Q-representation and *type (II)* otherwise (in which case $\varrho = \phi + \phi^\omega$).

Given a subgroup H of G and ω-irreducible Q-representations η of H and ϱ of G with $\eta|^G \cong \varrho$, we say that the induction is *ω-imprimitive* if either
(i) η and ϱ both have type (I) and the induction is imprimitive, or
(ii) $\eta = \chi + \chi^\omega$ and the induction $\chi|^G$ is imprimitive (in which case so is the induction $\chi^\omega|^G$).

We say that an ω-invariant Q-representation ϱ is *ω-primitive* if it is faithful and cannot be induced ω-imprimitively from a proper subgroup.

3.B.4. Proposition. *Let G be a group, $\omega : G \to C(2)$ a homomorphism, and $K \subset G$ a normal $C(2) \times C(2)$ with $\omega|_K$ surjective. Write $G \cong H \rtimes C(2)$ with $H = \ker \omega$. Further assume that H is an F-group. If G has a faithful irreducible Q-representation, it is unique. If there is not a faithful irreducible representation, then $G \cong H \times C(2)$ and G has precisely two irreducible Q-representations which are faithful when* *restricted to H. These are the only irreducible Q-representations of G which are faithful when restricted to A, a self-centralizing cyclic in H.*

Proof. Note that Lemma 3.B.1 applies so that the automorphism α of H giving the semi-direct product is rather special. Let z denote the element in H which is central in G and gives the automorphism α as $\alpha(h)=h$ or zh for all $h\in H$. First we show that G does not have an irreducible faithful Q-representation iff $G\cong H\times C(2)$. Consider $\varrho_H|^G$, which is faithful, and let χ be an irreducible constituent of it. By Frobenius reciprocity $\chi|_H$ has ϱ_H as a constituent, so $H\cap\ker\chi=\{\varepsilon\}$, and hence H and $\ker\chi$ commute. If $\ker\chi\neq\{e\}$, then $G=H\times C(2)$ (where the $C(2)$ is $\ker\chi$). Conversely, if $\ker\chi=\{e\}$, then χ is a faithful irreducible Q-representation of G.

Next consider the uniqueness assertions. Let ψ be an irreducible Q-representation of G, and assume that ψ is faithful when restricted to A, where A is any self-centralizing subgroup of H. Let ϕ be an irreducible constitutent of $\psi|_H$. Begin with the case $G\cong H\times C(2)$, and apply 2.3. Since the conjugation action is trivial in this case, $\psi|_H=l\cdot\phi$ and so $\phi|_A$ is faithful. By 2.11, $\phi=\varrho_H$, and we are done with the product case since ϱ_H has exactly two extensions to $H\times C(2)$.

To do the other case, notice that, since A is self-centralizing, $z\in A$, and hence A is normal in G. Let ϕ be an irreducible constituent of $\psi|_H$. Because A is a normal cyclic group, all the conjugates of ϕ have the same kernel when restricted to A, and so ϕ must be faithful when restricted to A. Theorem 2.11 implies that $\phi=\varrho_H$. A similar argument applies to any conjugate of ϕ, so from 2.3 it follows that $\psi|_H=l\cdot\varrho_H$. We are done if we can show that $\varrho_H|^G$ is irreducible or is twice an irreducible.

Let χ denote an irreducible constituent of $\varrho_H|^G$. If $\varrho_H|^G$ is Q-irreducible, then $\chi=\chi^\omega$. If $\varrho_H|^G$ is reducible, then $\varrho_H|^G$ is $\chi+\chi^\omega$, and we are done if we can show that $\chi=\chi^\omega$. If $\ker\chi\neq\{e\}$, we saw above that $G\cong H\times C(2)$ and G could not have a faithful irreducible representation. Hence we can assume that χ is faithful.

Let B be the kernel of the action map $G\to\mathrm{Aut}(A)$. Note that $A\subset B$ with cokernel at most a $C(2)$, so B is abelian. If B is cyclic, then B is self-centralizing and G is an F-group. By 2.11, $\chi=\chi^\omega$ since both are faithful irreducible Q-representations of G.

If B is not cyclic, there is a $K\cong C(2)\times C(2)$ in B which is normal in G. Since $\ker\chi=\{e\}$, K cannot be central in G, so let G_0 denote the centralizer of K in G. Note that $G_0=H_0\times C(2)$, and observe that $A\subset H_0$, so H_0 is an F-group with faithful irreducible representation ϱ_{H_0}. Let ψ be an extension of this representation to G_0. The argument in the product case shows that the only two irreducible representations of G_0 which are faithful on A_0 are ψ and ψ^ω.

Now apply 2.8. Since χ is faithful, $\chi|_{G_0}=\phi+\phi^x$, where $x\in G-G_0$; ϕ^x denotes the conjugation of ϕ by x; and $\phi\neq\phi^x$. Both ϕ and ϕ^x must be faithful on A, since their sum is. Hence ϕ is one of ψ or ψ^ω and ϕ^x is the other. So $\phi^x=\phi^\omega$. □

3.B.5. Definition. Each ω-basic group has an ω-irreducible Q-representation, called the *ω-basic representation*, and written ϱ_G. It is the unique faithful ω-irreducible Q-representation of G. It is of *type (II)* iff $G=\ker\omega\times C(2)^-$.

Remark. The necessary existence and uniqueness results have already been verified. If G is basic, the needed result follows from 2.10 and 2.11. For the groups in 3.A.6(ii)(b), Proposition 3.B.4 applies by 2.10.

3.B.6. Remark. There is no danger in writing ϱ_G, since if G is an ω-basic F-group, the ϱ_G defined in 2.11 is clearly also the ω-basic representation.

3.B.7. Remark. We leave it to the reader to show that ϱ_G is ω-primitive.

The following result implies the analogue of 2.13, namely that ω-irreducible representations can be induced up nicely from ω-basic subquotients.

3.B.8. Theorem. *Let G be a p-hyperelementary group equipped with an orientation character $\omega: G\to\{\pm 1\}$. Let ϱ be ω-irreducible. Then there are subgroups $N\lhd H$ of G with $N\subset\ker\omega$ such that H/N is ω-basic and ϱ is ω-imprimitively induced from $\varrho_{H/N}$. The index of H in G is a pth power.*

Proof. We induct by assuming the result for all proper subquotients of G.

Since ϱ is ω-invariant, $\ker\varrho\subset\ker\omega$. If $\ker\neq\{e\}$, ϱ can be pulled back from an ω-irreducible Q-representation of $G/(\ker\varrho)$, so by our inductive hypothesis we are done.

The case where $\ker\varrho=\{e\}$ proceeds as follows. If G is ω-basic, then $\varrho=\varrho_G$ by 3.B.4 and we are done again. If G is not ω-basic, then select a K in G^+ which is normal in G. Let χ be an irreducible Q-constituent of ϱ. Since ϱ is faithful and K is in G^+, it follows easily from 2.8 that K cannot be central. Theorem 2.8(ii) further implies that $\chi|_{G_0}=\psi+\psi^x$, where $x\in G-G_0$, G_0 is the centralizer of K in G, and $\psi\neq\psi^x$; indeed $K\cap\ker\psi\neq K\cap\ker\psi^x$. Since K is in G^+, $K\cap\ker\psi=K\cap\ker\psi^\omega$, so $\psi^x\neq\psi^\omega$. If $\chi=\varrho$, the type (I) case, then $\varrho=\varrho^\omega$, so $\psi^\omega=\psi$ and we can ω-imprimitively induce ϱ from G_0. If $\varrho\neq\chi$, the type (II) case, then $\psi^\omega\neq\psi$ so again we can ω-imprimitively induce ϱ from G_0. □

3.C. The Witt case

Recall (1.C.4) that a Witt-basic group is a 2-hyperelementary group in which all abelian normal subgroups that are θ-invariant are cyclic. In particular, a 2-hyperelementary group with geometric antistructure is Witt-basic iff it has no normal θ-invariant $C(2)\times C(2)$'s in $\ker\omega$. Hence the next result classifies the Witt-basic 2-hyperelementary groups.

3.C.1. Theorem (Classification of Witt-basic 2-hyperelementary groups). *Let*

(G,θ,ω,b) be a 2-hyperelementary group with geometric antistructure.

(i) *There are no normal θ-invariant $C(2)\times C(2)$'s in G iff either*

(a) *G is basic, or*

(b) *G is not basic, but* ker $\cong D(8)$, *and θ acts on $D(8)$ as a non-trivial outer automorphism.*

(ii) *There are normal θ-invariant $C(2)\times C(2)$ in G but none of them are contained in* ker(ω) *iff $G=G^+\rtimes C(2)^-$ as in* 3.B.1 *and G^+ has no normal θ-invariant $C(2)\times C(2)$'s.*

Proof. Define a new group $\tilde{G}=\langle G,x \mid xgx^{-1}=\theta(g)$ for $g\in G$, $x^2=b^{-1}\rangle$. Note that $G\subset\tilde{G}$ is of index 2. Let $\tilde{\omega}:\tilde{G}\to\{\pm 1\}$ be the homomorphism with ker $\tilde{\omega}=G$.

We begin by producing the G which do not have an θ-invariant normal $C(2)\times C(2)$. Clearly these are the groups G for which $(\tilde{G},\tilde{\omega})$ is $\tilde{\omega}$-basic. By the classification of ω-basics, G is an index 2 subgroup of a 2-hyperelementary basic group, which are listed in 3.A.7. Hence G is basic, or ker $\psi\cong D(8)$. If ker $\psi\cong D(8)$, then G still has no θ-invariant normal $C(2)\times C(2)$'s if G is basic, or if θ acts as a non-trivial outer automorphism on $D(8)$. These are the groups satisfying (i) above.

Now suppose that there are θ-invariant $C(2)\times C(2)$'s in G, none of which are in G^+. Pick a θ-invariant $C(2)\times C(2)$ and apply Lemma 3.B.1. Note that the corresponding z satisfies $\theta(z)=z$. We need to see why G^+ has no θ-invariant $C(2)\times C(2)$'s. We proceed by contradiction, so suppose that E is a $C(2)\times C(2)$, θ-invariant and normal in G^+. By 2.6 there are central (in G^+) elements of order 2 in E. We can easily find a central $x\in E$ with $\theta(x)=x$. If $x=z$ we are done. If not, the group $\langle x,z\rangle$ is a central θ-invariant $C(2)\times C(2)$ in G^+ which is normal in G, so we are done in either case. To do the converse, return for a moment to the groups with no θ-invariant normal $C(2)\times C(2)$'s. From the classification of basics (3.A.6) we see that these groups have a unique element z of order 2 in their centers, which must then satisfy $\theta(z)=z$. Hence if $G=G^+\rtimes C(2)^-$ with the automorphism α built as in 3.1 with G^+ having no θ-invariant normal $C(2)\times C(2)$'s, then there is a θ-invariant normal $C(2)\times C(2)$ in G. □

3.C.2. Definition. A Q-representation ϱ is a group homomorphism $G:\to \mathrm{GL}(V_\varrho)$, so we can define ϱ^θ by precomposing this homomorphism with θ. Define $\varrho^\alpha=(\varrho^\theta)^\omega$. A Q-representation is called *α-invariant* provided $\varrho^\alpha=\varrho$.

3.C.3. Theorem. *Each Witt-basic group has an irreducible Q-representation which is faithful and which is α-invariant. This representation is unique unless $G=G^+\times C(2)^-$ in which case there are precisely two.*

Remark. We write ϱ_G for the representation when it is unique, and call it the *Witt-basic representation*. We write ϱ_G^+ and ϱ_G^- for the two representations when there are two. We call them the *Witt-basic representations*. Note $(\varrho_G^+)^\omega=\varrho_G^-$ and vice-versa.

Proof. If G is an index 2 subgroup of a basic group, then it is an F-group by 3.A.7. Hence Witt-basic groups satisfying 3.C.1(i) have a unique faithful by 2.11. For case (ii), note that Proposition 3.B.4 applies. □

3.C.4. Theorem. *Let ϱ be an irreducible Q-representation of a 2-hyperelementary group G, with a geometric antistructure (θ, ω, b). Suppose that ϱ is α-invariant. Then, there exist subgroups $N_\varrho \lhd H_\varrho$ of G with $N_\varrho \subset \ker \omega$, and an element $c_\varrho \in G$ such that H_ϱ and N_ϱ are θ^{c_ϱ}-invariant. The scale by c_ϱ of the given antistructure on G restricts to an antistructure on H_ϱ/N_ϱ and a twisted induction and a twisted restriction are defined. Furthermore, H_ϱ/N_ϱ with its geometric antistructure is a Witt-basic group.*

If $\varrho = \varrho^\omega$, then H_ϱ/N_ϱ has a unique Witt-basic representation which induces up imprimitively to give ϱ.

If $\varrho \neq \varrho^\omega$, then H_ϱ/N_ϱ has two Witt-basic representations. One of them induces up imprimitively to give ϱ and the other induces up imprimitively to give ϱ^ω.

Proof. We say that an induction from χ on H to ϱ on G is *Witt-imprimitive* iff $\varrho^\omega = \varrho$, there is a $c \in G$ such that the geometric antistructure on G, when twisted by c, restricts to a geometric antistructure on H, and $\chi^\alpha = \chi$.

As usual, we can assume the result for proper subquotients of G. Fix an α-invariant irreducible Q-representation ϱ of G.

First we do the case in which $\varrho^\omega = \varrho$. If $\ker \varrho \neq \{e\}$, it is easy to see that $\ker \varrho$ is a θ-invariant subgroup of G^+, and so we can pull ϱ back from the quotient $G/(\ker \varrho)$, which has a geometric antistructure so that the map $G \to G/(\ker \varrho)$ is a map of groups with geometric antistructure. Suppose $\ker \varrho = \{e\}$, and that G is not Witt-basic. Then by 2.8 we can induce ϱ imprimitively from an index 2 subgroup, the centralizer, G_0 of some $K \cong C(2) \times C(2)$. Since K is θ-invariant, $b \in G_0$ and G_0 is θ-invariant. Let ψ be one of the two irreducible Q-representations of G_0 which induce up to give ϱ. If $\psi^\alpha = \psi$, then an ordinary induction is Witt-imprimitive. If $\psi^\alpha \neq \psi$, then $\psi^\alpha = \psi^x$ for some $x \in G$. If we scale by x we now get a Witt-imprimitive induction. Notice that K is θ^x-invariant, so $b^{(x)} \in G_0$. Since K is in G^+, ψ^ω and ψ have the same kernels when restricted to K. Since $\varrho^\omega = \varrho$, it is not hard to check that $\psi^\omega = \psi$.

Now we do the case $\varrho^\omega \neq \varrho$. Let $\chi = \varrho + \varrho^\omega$. Suppose that the θ-invariant K in G^+ were central. Then $K \cap \ker \varrho \neq \{e\}$ by 2.8, and $K \cap \ker \varrho = K \cap \ker \varrho^\omega$ since $K \subset G^+$. Hence χ has a kernel. If we assume that $\ker \chi \neq \{e\}$, then this subgroup is a normal, θ-invariant subgroup of G^+ so we can pass to a quotient as above. Hence, we may as well assume that $\ker \chi = \{e\}$ and that K is not central in G. Let G_0 be the centralizer of K in G. Just as in the last paragraph, we can induce ϱ Witt-imprimitively from a representation ψ on G_0. It follows that ϱ^ω is induced from ψ^ω using exactly the same twist. □

4. The detection and generation theorems

We review the usual idempotent decomposition of QG. The simple factors of QG are in one to one correspondence with the irreducible rational representations of G, and the central simple idempotent associated to a $\varrho \in \mathrm{Irr}_Q(G)$ is given by the formula

$$e_\varrho = \frac{a_\varrho}{|G|} \sum_{g \in G} \mathrm{tr}(\varrho(g^{-1})) \cdot g$$

where a_ϱ is the complex dimension of an irreducible constituent of the complexification of ϱ, and $\mathrm{tr}(\varrho(g^{-1}))$ is just the character of ϱ applied to g^{-1} (see [33, p. 4, Proposition 1.1]).

Notice that, if $|G| \in R^\times$, then $e_\varrho \in RG$, and

$$RG = \bigoplus_{\varrho \in \mathrm{Irr}_Q(G)} e_\varrho RG.$$

In R-Morita we also get a decomposition. Let $[e_\varrho]$ represent the RG–RG bimodule $e_\varrho RG$ in R-Morita, so

$$[e_\varrho] \in \mathrm{Hom}_{R\text{-Morita}}(RG, RG).$$

We have the usual idempotent equations:

(i) $$[e_\varrho] \cdot [e_\psi] = \begin{cases} 0 & \text{if } \varrho \not\equiv \psi, \\ [e_\varrho] & \text{if } \varrho = \psi; \end{cases}$$

(ii) $$1_{RG} = \sum_{\varrho \in \mathrm{Irr}_Q(G)} [e_\varrho].$$

There are two standard maps in R-Morita: the *diagonal map* $\Delta : A \to \bigoplus A$ and the *fold*, or *sum map* $\Sigma : \bigoplus A \to A$.

We can rephrase (ii) as

(iii) The following diagram commutes:

$$\begin{array}{ccc} RG & \xrightarrow{\ \Delta\ } & \bigoplus RG \\ {\scriptstyle 1_{RG}} \Big\| & & \Big\downarrow {\scriptstyle \bigoplus [e_\varrho]} \\ RG & \xleftarrow{\ \Sigma\ } & \bigoplus RG. \end{array}$$

4.A. *The linear case*

The first goal is to prove that the maps $[e_\varrho]$ which are defined to be in R-Morita have natural lifts to RG-Morita, Theorem 4.A.5. After some initial technical discussion, we prove a key commutativity result, Proposition 4.A.4. The promised strong forms of the linear detection and generation theorems then follow fairly easily.

4.A.1. Lemma. *If $|G| \in R^{\times}$, then the R-group ring functor RG-Morita $\to$ R-Morita is injective.*

Proof. Injective means that $\mathrm{Hom}_{RG\text{-Morita}}(H_1, H_2) \to Hom_{R\text{-Morita}}(RH_1, RH_2)$ is injective. By Bass [1, Proposition 1.3, p. 346], RX and RX' are equal in $\mathrm{Hom}_{R\text{-Morita}}(RH_1, RH_2)$ iff there is an RH_2–RH_1 bimodule, C, which is projective as an RH_2-module, such that $RX \oplus C \cong RX' \oplus C$ as RH_2–RH_1 bimodules.

Since $R[H_2] \otimes R[H_1]^{\mathrm{op}}$ is a free bimodule, we can find a bimodule surjection $f: (R[H_2] \otimes R[H_1]^{\mathrm{op}})^n \to C$ for some finite n. Since $|H_1| \cdot |H_2|$ is a unit in R, C is projective as a bimodule since it is projective as an R-module and we can average any R module splitting of f to a bimodule splitting. Hence we can assume that the C above is free. But the free bimodule is just our functor applied to the H_2–H_1 biset $H_2 \oplus H_1$, and so X and X' were already equivalent in RG-Morita. □

We introduce some terminology to enable us to deal efficiently with all our various notions of irreducibility.

4.A.2. Definition. A Q-representation ϱ of a finite group G is called *unital* if, whenever we write $\varrho = \sum \psi_i$, $\psi_i \not\cong \psi_j$ unless $i = j$. A collection of unital Q-representations $\{\varrho_i\}$ is called *complete* iff every irreducible Q-representation of G occurs in exactly one of the ϱ_i.

4.A.3. *Extensions of notation and terminology.* If $\varrho = \sum \psi_i$ is unital, then define $e_\varrho = \sum e_{\psi_i} \in RG$; $[e_\varrho] = \sum [e_{\psi_i}]$ and a representing bimodule is $\bigoplus e_{\psi_i} RG = e_\varrho RG \subset RG$. We say that ϱ is imprimitively induced from χ on $H \subset G$, provided $\chi = \sum \phi_i$ and each ψ_i is induced imprimitively from ϕ_i. (Note that $\chi\,|^G = \varrho$.) Extend the notion of imprimitive induction to subquotients as we did in the irreducible case.

Notice that an ω-irreducible representation is unital, and an ω-imprimitive induction is imprimitive.

The proofs of the next two lemmas have the same form. We leave it to the reader to check that the defined map really is a bimodule map as claimed. Moreover, since $\bigotimes_{Z[1/m]} R$ preserves isomorphisms, it suffices to prove the result for $R = Z[1/m]$ where $m = |G|$. First we show that the defined map is onto; then we show that the domain of the map is torsion-free; and then we show that the two ranks are the same.

4.A.4. Lemma. *Let $N \lhd H$ be groups, and let ϱ be a unital Q-representation of H that is pulled back from a Q-representation $\bar{\varrho}$ on H/N. Then the map of $R[H/N]$–$R[H/N]$ bimodules*

$$f: R[H/N] \underset{RH}{\otimes} e_\varrho RH \underset{RH}{\otimes} R[H/N] \to e_{\bar{\varrho}} R[H/N]$$

defined by $f(\bar{h}_1 \otimes e_\varrho h_2 \otimes \bar{h}_3) = \bar{h}_1 e_{\bar{\varrho}} \pi(h_2)\bar{h}_3$ *for all* $\bar{h}_1, \bar{h}_3 \in H/N$ *and all* $h_2 \in H$ (π *denotes the projection* $\pi: H \to H/N$) *is an isomorphism whenever* $|H| \in R^\times$.

Proof. Clearly the map is onto. Since $e_\varrho RH$ is a summand of RH as an RH-RH bimodule, the domain of our map is a summand of $R[H/N] \otimes_{RH} RH \otimes_{RH} R[H/N] \cong R[H/N]$ and so is torsion-free.

Define $\sigma: R[H/N] \to RH$ by

$$\sigma(\bar{h}) = \frac{1}{|N|} \sum h$$

where the sum runs over the elements in H in the coset of $\bar{h}$. The map σ is a ring map which splits the projection and which takes $e_\varrho RH$ isomorphically onto $e_{\bar{\varrho}} R[H/N]$. But it is easy to see that the map $e_\varrho RH \to R[H/N] \otimes_{RH} e_\varrho RH \otimes_{RH} R[H/N]$ which takes $e_\varrho \cdot h$ to $1 \otimes e_\varrho \cdot h \otimes 1$ induces a surjection, so $R[H/N] \otimes_{RH} e_\varrho RH \otimes_{RH} R[H/N]$ and $e_{\bar{\varrho}} R[H/N]$ have the same rank. □

4.A.5. Lemma. *Let H be a normal subgroup of G and let ϱ be a unital Q-representation of G which is induced imprimitively from η on H. Then the natural RG–RG bimodule map*

$$\hat{\imath}: RG \underset{RH}{\otimes} e_\eta RH \underset{RH}{\otimes} RG \to e_\varrho RG$$

defined by $\hat{\imath}(g_1 \otimes e_\eta h \otimes g_2) = e_\varrho \cdot g_1 \cdot e_\eta h \cdot g_2$ *is an isomorphism whenever* $|H| \in R^\times$.

Proof. From 2.3 we have the idempotent equation

$$e_\varrho = \sum_{j=1}^{r} e_{\eta^{x_j}}$$

where $\{x_j \in G\}$ are a set of coset representatives for G/H and $r = |G/H|$. Note that $e_{\eta^{x_j}} = x_j e_\eta x_j^{-1}$, so

$$\sum_{j=1}^{r} \hat{\imath}(x_j \otimes e_\eta \otimes x_j^{-1} g) = e_\varrho \cdot g$$

for all $g \in G$, and our map is onto.

Next note $RG \cong \bigoplus x_i RH \cong \bigoplus RHx_j$, so

$$RG \underset{RH}{\otimes} e_\eta RH \underset{RH}{\otimes} RG \cong \bigoplus_{i,j} x_i RH \underset{RH}{\otimes} e_\eta RH \underset{RH}{\otimes} RHx_j \quad \text{as } R \text{ modules}$$

$$\cong \bigoplus_{i,j} x_i \otimes e_\eta RH \otimes x_j.$$

This shows that $RG \otimes_{RH} e_\eta RH \otimes_{RH} RG$ is torsion-free, and that its rank is $r^2 \cdot \mathrm{rank}_R\, e_\eta RH$. Imprimitive induction implies $\mathrm{rank}_R\, e_\varrho RG = r^2 \cdot \mathrm{rank}_R\, e_\eta RH$. □

4.A.6. Proposition. *Let $N \lhd H$ with $H \subset G$ where G is p-hyperelementary: suppose* *that ϱ is a unital Q-representation of G that is induced imprimitively from ψ on H/N. Suppose $|H|\in R^{\times}$. Then, in R-*Morita, *the following diagram commutes*:

$$\begin{array}{ccc} RG & \xrightarrow{[e_\varrho]} & RG \\ {\scriptstyle \mathrm{Res}_G^{H/N}}\downarrow & & \uparrow{\scriptstyle \mathrm{Ind}_G^{H/N}} \\ R[H/N] & \xrightarrow{[e_\psi]} & R[H/N]. \end{array}$$

Proof. Begin by assuming that ϱ is Q-irreducible. We can factor the restriction and induction maps as maps from H/N to H and then from H to G. Since the induction is imprimitive, we can further factor the inclusion $H\subset G$ into a sequence of normal inclusions by 2.14.

Hence it suffices to prove that the diagram commutes for two special cases: namely a quotient group, G/N of G and a normal subgroup, H of G. The way that we tell that our diagrams commute in R-Morita is to write down the bimodules representing the two different sequences of compositions and see that the two resulting bimodules are isomorphic. For the quotient group case, this is just Lemma 4.A.4 and for the normal subgroup case it is just Lemma 4.A.5.

Since the diagram commutes for irreducible ϱ it is easy to extend to the case of a sum of different irreducibles. □

4.A.7. Theorem. *Let G be a p-hyperelementary group, and let ϱ be a unital Q-representation of it. Let $|G|\in R^{\times}$. Then there is a unique map in RG-*Morita *which hits $[e_\varrho]$ in R-*Morita. *We will denote this map in RG-*Morita *also by $[e_\varrho]$.*

Proof. Since $|G|\in R^{\times}$, the R-group ring functor embeds RG-Morita into R-Morita by Lemma 4.A.1, so the uniqueness result is clear. To prove existence, it suffices to do the irreducible case. We can assume that the result holds for all groups which are proper subquotients of G. If ϱ has a kernel, then from Proposition 4.A.4 it follows that the composite $RG\to R[G/N]\xrightarrow{[e_\eta]} R[G/N]\to RG$ is just $[e_\varrho]$. Since the first and last maps in the composite are naturally in RG-Morita, so is $[e_\varrho]$. A similar argument holds if ϱ can be imprimitively induced from a proper subgroup using Lemma 4.A.5.

In the case where ϱ is faithful and cannot be induced imprimitively from a proper subgroup, then G is basic and $\varrho=\varrho_G$ by Theorem 2.13. The $[e_\psi]$ for all the representations of G except ϱ_G can be assumed to be in RG-Morita, and 1_{RG} comes from the G–G biset G and so is in RG-Morita. Since the sum of all the $[e_\psi]$'s is 1_{RG} in R-Morita we can *define* $[e_\varrho]$ in RG-Morita so that the sum of all the $[e_\psi]$'s is 1_G in RG-Morita. □

4.A.8. Linear detection and generation theorem. *Let G be a p-hyperelementary* *group, and assume that $|G|$ is a unit in R. Suppose given a complete set of unital representations of G, say $\{\varrho_i\}$. Suppose further that we are given subquotients $\{H_i/N_i\}$ with Q-representations ψ_i and suppose that each ϱ_i is imprimitively induced from ψ_i. Then, in RG-*Morita, *the following composite is the identity*:

$$RG \xrightarrow{\text{Res}} \bigoplus R[H_i/N_i] \xrightarrow{\times[e_{\psi_i}]} \bigoplus R[H_i/N_i] \xrightarrow{\text{Ind}} RG.$$

Proof. The result follows easily in R-Morita from the idempotent equation (equation (iii) in the introduction to Section 4) and Proposition 4.A.6. It then holds in RG-Morita by Lemma 4.A.1 and Theorem 4.A.7. □

4.A.9. Proof of Theorem 1.A.11. By 2.13, for each irreducible Q-representation ϱ we can find subquotients H_ϱ/N_ϱ which are basic groups and so that ϱ is induced imprimitively from the basic representation. Apply 4.A.8 to this collection. □

The last result in this section translates some of the idempotent results from above into RG-Morita.

4.A.10. Theorem. *Proposition* 4.A.6 *holds in RG-*Morita. *Moreover, suppose given subgroups $N \lhd H$ of G; $\varrho \in \mathrm{Irr}_Q(G)$ and a unital representation η on H/N. The composition*

$$RG \xrightarrow{[e_\varrho]} RG \xrightarrow{\text{Res}} R[H/N] \xrightarrow{[e_\eta]} R[H/N]$$

*is trivial in RG-*Morita *if ϱ is not a constituent of* $\mathrm{Ind}_{H/N}^{G}(\eta)$.

Proof. The maps in Proposition 4.A.6 are in RG-Morita by 4.A.7 and the diagram commutes in RG-Morita by 4.A.1 and 4.A.6.

For the last result we may assume that η is irreducible and that we are working in R-Morita. Let ϕ on H be the pullback of the representation η. A representing bimodule for our map is $e_\eta R[H/N] \otimes_{RH} e_\varrho RG$. By 4.A.4, $e_\phi RH \otimes_{RH} e_\varrho RG$ surjects onto it, so we prove $e_\phi RH \otimes_{RH} e_\varrho RG = 0$.

It follows from the construction of the idempotent decompositions that the composite $e_\phi RH \subset RH \subset RG \to e_\varrho RG$ is the 0-map under our hypotheses, so, in the ring RG, $e_\phi \cdot e_\varrho = 0$. But $e_\phi RH \otimes_{RH} e_\varrho RG$ is an RH–RG bimodule summand of $RH \otimes_{RH} e_\varrho RG = e_\varrho RG$ and the image is generated by $e_\phi \otimes e_\varrho = e_\phi \otimes e_\phi \cdot e_\varrho = 0$. □

4.B. The quadratic case

The goals and the strategy are the same as for the linear case.

4.B.1. Lemma. *If $|G| \in R^\times$, then the R-group ring functor* (RG, ω)-Morita $\to$ $(R, -)$-Morita *is injective. If in addition 2 is a unit in R, then the R-group ring functor* (RG, θ, ω, b)-Morita $\to$ $(R, -)$-Morita *is injective.*

Proof. As in the linear case, it is no trouble to prove that if (RX, λ_X) is equivalent to (RY, λ_Y) in $(R, -)$-Morita, then there is a metabolic form on a free bimodule, say (C, λ), so that $(RX, \lambda_X) \perp (C, \lambda)$ is isomorphic to $(RY, \lambda_Y) \perp (C, \lambda)$. The problem is that the metabolic form on the free bimodule may not come from a biset form.

One biset form on the rank 1 free H_2-H_1 biset, $X = H_2 \times H_1$ is defined by

$$\theta_X(k, h) = (\theta_{H_2}(k) b_{H_2}^{-1}, b_{H_1} \theta_{H_1}(h))$$

and

$$\omega_X(k, h) = \omega_{H_2}(k) \cdot \omega_{H_1}(h).$$

The only other one just takes ω to be minus the ω above. The orthogonal sum of these two forms is a metabolic form, denoted $\mathrm{Meta}(\lambda_{\mathrm{free}})$. We can define another biset form on $X \amalg X$ as follows: $\theta_{X \amalg X}(x_1, x_2) = (\theta_X(x_2), \theta_X(x_1))$ and $\omega_{X \amalg X}(x_1, x_2) = \omega_X(x_1) \cdot \omega_X(x_2)$, where θ_X and ω_X are the ones constructed above. In the associated form on $RX \oplus RX$, each copy of RX is a Lagrangian, so this form is hyperbolic.

If $|G|$ is odd, and the antistructures are standard, use 1.B.1 to compute that $\hat{H}^0(Z/2Z; \mathrm{Hom}_R(RH_1, RH_2)) \cong Z/2Z$ and that $[\lambda_{\mathrm{free}}]$ is the generator. It follows easily from the formulae (i), (ii) and (iii) below 1.B.1 that any metabolic on a free bimodule is equivalent to one coming from a free biset form.

If 2 is a unit in R, then all metabolics are hyperbolic and we are done again. □

4.B.2. Definition. We can associate to each group G with oriented geometric antistructure the biset form on G which is the identity in our category. The associated form is defined by

$$\lambda(g_1, g_2) = \omega(g_2) \cdot g_1 \cdot \theta^{-1}(g_2).$$

We can restrict this form to any of the $e_\varrho RG$. If ϱ is α-invariant, then we get a nonsingular bihermitian form on $e_\varrho RG$. If $\varrho \neq \varrho^\alpha$, then we get a nonsingular bihermitian form on $e_{\varrho + \varrho^\alpha} RH$ which is easily seen be hyperbolic.

The proofs of the next two lemmas consist of verifying that an explicit map preserves an explicit form. They are omitted.

4.B.3. Lemma. *Let $N \lhd H$ be groups, and let ϱ be a unital α-irreducible Q-representation of H that is pulled back from a Q-representation $\bar{\varrho}$ on H/N. Suppose that $N \subset \ker \omega$ and that N is θ-invariant. Then the map of $R[H/N]$-$R[H/N]$ bimodules f defined in Lemma* 4.A.4. *is an isometry whenever $|H| \in R^\times$.*

4.B.4. Lemma. *Let H be a θ-invariant, normal subgroup of G with $b \in H$, G p-hyperelementary and let ϱ be a unital α-invariant Q-representation of G which is induced imprimitively from η on H with η α-invariant. The RG–RG bimodule map $\hat{\imath}$ defined in Lemma* 4.A.5 *is an isometry whenever $|H| \in R^\times$.*

4.B.5. Proposition. *Let $N \lhd H$ with $H \subset G$ where G is p-hyperelementary. Let*

*(θ, ω, b) be a geometric antistructure and suppose that H and N are θ-invariant and $N \subset \ker \omega$. Suppose that ϱ is an α-invariant unital Q-representation of G that is induced imprimitively from ψ on H/N. Suppose $b \in H$, so there is an induced geometric antistructure on H/N and suppose that ψ is α-invariant. Suppose $|H| \in R^{\times}$. Then, in $(R,-)$-*Morita*, the following diagram commutes:*

$$\begin{array}{ccc} RG & \xrightarrow{[e_\varrho]} & RG \\ {\scriptstyle \mathrm{Res}_G^{H/N}} \downarrow & & \uparrow {\scriptstyle \mathrm{Ind}_G^{H/N}} \\ R[H/N] & \xrightarrow{[e_\psi]} & R[H/N]. \end{array}$$

Proof. The proof is much the same as in the linear case (Proposition 4.A.6). Of course we use Lemmas 4.B.3 and 4.B.4 instead of their linear versions. By Proposition 2.14 we can find a sequence of subgroups between H and G, each normal in the next, but we need to have them θ-invariant as well. If H_1 is θ-invariant and normal in H_2, then consider the group generated by H_2 and $\theta(H_2)$. This group is certainly θ-invariant, and H_1 is still normal in it. Finish as in the linear case. □

4.B.6. Theorem. *Let G be a p-hyperelementary group with a geometric antistructure, for which θ is the identity. Let ϱ be an ω-invariant unital Q-representation of it. Let $|G| \in R^{\times}$. In (RG, ω)-*Morita* there is a unique map which hits $[e_\varrho]$ in $(R,-)$-*Morita*. We will denote this map in (RG, ω)-*Morita* also by $[e_\varrho]$.*

Proof. By Lemma 4.B.1, the R-group ring functor is an embedding, so the uniqueness result is clear. As in the linear case (Theorem 4.A.7) we can reduce to the case in which ϱ is ω-irreducible. We can further assume that the result holds for all groups which are proper subquotients of G. If ϱ has a kernel, or can be induced imprimitively from a proper subgroup, use Proposition 4.B.5 and finish as in the linear case.

In the case where ϱ is faithful and cannot be induced imprimitively from a proper subgroup, then G is ω-basic and $\varrho = \varrho_G$ by Definition 3.B.5. The $[e_\psi]$ for all the representations of G except ϱ_G can be assumed to be in (RG, ω)-Morita, and 1_{RG} comes from the G–G biset form G and so is in (RG, ω)-Morita. Since the sum of all the $[e_\psi]$'s in 1_{RG} in $(R,-)$-Morita we can *define* $[e_\varrho]$ in (RG, ω)-Morita so that the sum of all the $[e_\psi]$'s is 1_G in (RG, ω)-Morita. □

4.B.7. Quadratic detection and generation theorem. *Let G be a p-hyperelementary group, and assume that $|G|$ is a unit in R. Suppose given a geometric antistructure in which θ is the identity. Let $\{\varrho_i\}$ be a complete collection of ω-invariant unital Q-representations. Let $\{N_i \lhd H_i\}$ be a collection of subquotients of G with $N_i \subset \ker \omega$ for all i. Assume that ϱ_i is induced imprimitively from an ω-invariant unital repre-*

sentation ψ_i. Then, in (RG,ω)-Morita, *the following composite is the identity*:

$$RG \xrightarrow{\text{Res}} \bigoplus R[H_i/N_i] \xrightarrow{\bigoplus [e_{\psi_i}]} \bigoplus R[H_i/N_i] \xrightarrow{\text{Ind}} RG.$$

Proof. The corresponding result in $(R,\text{-})$-Morita follows easily from the idempotent equation (equation (iii) in the introduction to Section 4) and Proposition 4.B.5. By Theorem 4.B.6 and Lemma 4.B.1 the result also holds in (RG,θ,ω,b)-Morita. □

4.B.8. Proof of Theorem 1.B.7. By Theorem 3.B.8 each ω-irreducible Q-representation can be induced from the ω-basic Q-representation on an ω-basic subquotient by an imprimitive induction. Apply Theorem 4.B.7. □

4.C. The Witt case

Our first goal is the proof of the detection/generation theorem, 1.C.5, but we begin with some definitions and lemmas.

4.C.1. Definition. We call a ring with antistructure, (A,α,u), *hyperbolic* provided $A=A_1\times A_2$ as rings, and $\alpha(A_1\times 0)=0\times A_2$.

4.C.2. Lemma. *If (A,α,u) is a hyperbolic ring with antistructure and (B,β,v) is any ring with antistructure, then any B–A or A–B nonsingular, bihermitian biform is hyperbolic.*

Proof. Let $\lambda: M\otimes_B M^{\mathrm{t}}\to A$ be an A–B nonsingular, bihermitian biform. Define $M_1=(1,0)M$ and $M_2=(0,1)M$. Note $M=M_1\oplus M_2$ since $M_1\cap M_2=\{0\}$. This is because $(1,0)$ acts as the identity on M_1, and as 0 on M_2.

Next note that $\lambda|_{M_1}$ is trivial. Indeed, $\lambda(m_1,\bar{m}_1)=\lambda((1,0)\cdot m_1,(1,0)\cdot\bar{m}_1)=\lambda((1,0)\cdot m_1,\bar{m}_1\bullet(0,1))=(1,0)\lambda(m_1,\bar{m}_1)(0,1)=0$. A similar argument shows that $\lambda|_{M_2}$ is trivial.

Contemplation of the isomorphism $\mathrm{ad}(\lambda)$ shows that λ is hyperbolic with respect to M_1 and M_2.

A similar argument works for the B–A case. □

4.C.3. Lemma. *Let G be a 2-hyperelementary group with oriented geometric antistructure $(\theta,\omega,b,\varepsilon)$. Suppose that $|G|\in R^{\times}$. Let ψ be a unital Q-representation of G such that $\psi^{\alpha}=\psi$. Assume that every irreducible Q-representation ϱ of G which satisfies $\varrho^{\alpha}=\varrho$ is a constituent of ψ. Then, in* $(R,\text{-})$-Witt,

$$1_{(RG,\theta,\omega,b,\varepsilon)}=[e_{\psi}].$$

Proof. Given the hypotheses, it is easy to find a unital representation χ, such that $\psi+\chi+\chi^{\alpha}$ is unital and contains every irreducible Q-representation of G. Then

$RG = e_\psi RG \times e_\chi RG \times e_{\chi^\alpha} RG$. The ring $e_\chi RG \times e_{\chi^\alpha} RG$ is hyperbolic in the induced antistructure, so the result follows from Lemma 4.C.2. □

We have our usual theorem.

4.C.4. Theorem. *Let G be a 2-hyperelementary group with oriented geometric antistructure $(\theta, \omega, b, \varepsilon)$. Suppose that $|G| \in R^\times$. Let ψ_i be a collection of unital Q-representations of G such that $\psi_i^\alpha = \psi_i$. Suppose there are subgroups $N_i \lhd H_i$ of G with Q-representations ϕ_i such that ψ_i is induced imprimitively from ϕ_i. Suppose that $N_i \subset \ker \omega$. Suppose that for each i there is a $c_i \in G$ such that H_i and N_i are $\theta^{c_i} = \theta_i$-invariant and ϕ_i is α_i-invariant. Suppose $b_i = b^{(c_i)} \in H_i$. Finally, suppose that each irreducible Q-representation ϱ of G which is α-invariant occurs in exactly one ψ_i.*

Then, in $(R, -)$-Witt, *the following composite is the identity*:

$$(RG, \theta, \omega, b, \varepsilon) \xrightarrow{\text{Res}} \bigoplus (R[H_i/N_i], \theta_i, \omega, b_i, \varepsilon_i)$$
$$\xrightarrow{\times [e_{\phi_i}]} \bigoplus (R[H_i/N_i], \theta_i, \omega, b_i, \varepsilon_i) \xrightarrow{\text{Ind}} (RG, \theta, \omega, b, \varepsilon)$$

where a subscript i indicates that we have changed the antistructure by scaling by c_i before restricting to the subquotient.

Proof. The proof by now should be clear. □

4.C.5. Proof of 1.C.5. The proof of 1.C.5 follows from 3.C.4 and 4.C.4. □

We conclude this section with a proof of 1.C.3, as well as a remark about 4.C.4. Notice that both the ω-basics and the Witt-basics come in three types:

(i) basic groups,
(ii) basic groups $\times$ $C(2)^-$,
(iii) the rest.

Any type (iii) group G has a unique faithful Q-representation ϱ_G which can be induced imprimitively from a representation χ on an index 2 subgroup of the form $H \times C(2)^-$, where H is an index 2-subgroup of a basic group. The reason that G is still on our list is that $\chi^\alpha \neq \chi$. There is an element $c \in G$ however, so that if we scale by c, χ is α^c-invariant.

To prove 1.C.3, we first apply the (RG, ω)-Morita theorem, 1.B.7, and then use the above observation to eliminate type (iii) groups at the expense of introducing twisted maps.

Notice that in 4.C.4 we could also eliminate the type (iii) groups. A further simplification occurs in $(R, -)$-Witt. Notice that some of the type (ii) groups are hyperbolic and hence can also be eliminated. This occurs whenever the θ associated to the group acts trivially on the central $C(2) \times C(2)$.

5. Some split exact sequences in Morita categories

In this section we want to prove that the 5-term sequences in 1.A.16 are split exact. We will do this by showing that they are contractible. Given a sequence in an additive category

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0,$$

it is a *0-sequence* if $\beta \circ \alpha = 0$. It is *contractible* provided there exist maps $f: C \to B$ and $g: B \to A$ such that

(5.0) (i) $g \circ f = 0$, (ii) $\beta \circ f = 1_C$, (iii) $g \circ \alpha = 1_A$, (iv) $\alpha \circ g + f \circ \beta = 1_B$.

It is an easy exercise to check that contractible implies split exact (and even vice-versa).

5.A. *The linear case*

We are going to prove Theorem 1.A.16. The proof divides into two cases depending on whether the K is central or not. We begin with the central case.

In our 5-term sequence for this case, $A = RG$, $B = \bigoplus_{i=0}^{p} R[G/C_i]$ and $C = (R[G/K])^p$. The map $\alpha : RG \to \bigoplus_{i=0}^{p} R[G/C_i]$ is just the product of the individual projections $G \to G/C_i$. We define the map β.

5.A.1. Definition. Define $\beta : \bigoplus_{i=0}^{p} R[G/C_i] \to (R[G/K])^p$ as follows: for $1 \le i \le p$, $\beta|_{R[G/C_i]}$ is just the projection: to define $\beta|_{R[G/C_0]}$ we define its negative to be the composite $R[G/C_0] \to R[G/K] \to (R[G/K])^p$, where the first map is the projection and the second map is the diagonal.

Notice that β is defined in ZG-Morita and that $\beta \circ \alpha = 0$ even in ZG-Morita.

Next we define $f: (R[G/K])^p \to \bigoplus_{i=0}^{p} R[G/C_i]$ by describing its projection to each factor $R[G/C_i]$. The projection to $R[G/C_0]$ is the 0-map, and for $1 \le i \le p$ the projection to $R[G/C_i]$ is the composite $(R[G/K])^p \to R[G/K] \to R[G/C_i]$ where the first map is projection onto the ith factor and the second map is generalized induction associated to the projection. (Note that this map is only defined if p is a unit in R.)

The definition of $g: \bigoplus_{i=0}^{p} R[G/C_i] \to RG$ is next. We define it as the sum of maps $g_i : R[G/C_i] \to RG$: g_0 is the generalized induction map; for $1 \le i \le p$, g_i is the composite $R[G/C_i] \xrightarrow{e} R[G/C_i] \to RG$ where the second map is the generalized induction associated to the projection and where e is $1_{R[G/C_i]}$ minus the composite $R[G/C_i] \to R[G/K] \to R[G/C_i]$ of the projection and the corresponding generalized induction. Notice that all the g_i are defined whenever p is a unit in R, and equation 5.0(i) holds.

5.A.2. Lemma. *Let G be a finite group and N a normal subgroup. Then the following diagram commutes in RG-Morita whenever $|N| \in R^{\times}$:*

$$\begin{array}{ccc} R[G/N] & \xrightarrow{1_{R[G/N]}} & R[G/N] \\ {\scriptstyle \text{Proj}}\downarrow & & \uparrow{\scriptstyle \text{Ind}} \\ RG & \xrightarrow{1_{RG}} & RG. \end{array}$$

Proof. The proof is sufficiently similar to the proof of Lemma 4.A.4 that it is omitted. □

Using the definitions of the maps and the lemma, it is easy to check that equations 5.0(ii) and (iv) hold whenever p is a unit in R.

Finally, we assume that $|G|$ is a unit in R. It is not hard to check that $g\circ\alpha = 1_A$ using 2.8(i) and 4.A.6.

We turn now to the case in which K is not central in G. Our 5-term sequence for this case has $A = R[G]$, $B = R[G/C_0] \oplus R[G_0/C_1]$ and $C = R[G_0/K]$. The map $\alpha: A \to B$ is the sum of the projection map $RG \to R[G/C_0]$ and the generalized restriction $RG \to R[G_0/C_1]$. We define β.

5.A.3. Definition. Define a map $\beta: R[G/C_0] \oplus R[G_0/C_1] \to R[G_0/K]$ as the sum of two maps: $R[G_0/C_0] \to R[G_0/K]$ is the projection and the map $R[G/C_0] \to R[G_0/K]$ is the negative of the composite $R[G/C_0] \xrightarrow{\text{Res}} R[G_0/C_0] \xrightarrow{\text{Proj}} R[G_0/K]$.

Notice that β is defined in ZG-Morita.

5.A.4. Lemma. *Let H be a subgroup of G, and let $N \subset H$ be normal in G. Then, in ZG-Morita, the following diagram commutes:*

$$\begin{array}{ccc} RG & \xrightarrow{\text{Res}} & RH \\ {\scriptstyle \text{Proj}}\downarrow & & \downarrow{\scriptstyle \text{Proj}} \\ R[G/N] & \xrightarrow{\text{Res}} & R[H/N]. \end{array}$$

Proof. The proof consists of showing that the projection map, $R[H/N] \otimes_{RH} RG \to R[H/N] \otimes_{R[H/N]} R[G/N]$, is an isomorphism. It is left to the reader. □

Using Lemma 5.A.4 it is easy to see that $\beta\circ\alpha = 0$ in ZG-Morita as we claim.

Next we define the map $f: R[G_0/K] \to R[G/C_0] \oplus R[G_0/C_1]$ as the sum of two maps. The map from $R[G_0/K] \to R[G_0/C_1]$ is just the projection, and the other map is the 0-map. The map $g: R[G/C_0] \oplus R[G_0/C_1] \to RG$ is the sum of two maps. The map $R[G/C_0] \to RG$ is the induction associated to the projection, and the map

$R[G_0/C_1] \to RG$ is the following composite: $R[G_0/C_1] \xrightarrow{e} R[G_0/C_1] \xrightarrow{q} R[G_0] \xrightarrow{\text{Ind}} RG$ where e is $1_{R[G_0/C_1]}$ minus the composite $R[G_0/C_1] \xrightarrow{\text{Proj}} R[G_0/K] \xrightarrow{\bar{q}} R[G_0/C_1]$ where q and $\bar{q}$ are the inductions associated to the obvious projections. Notice that to define f and g it is only necessary to invert p. With just p inverted, it is easy to check that (5.0)(i), (ii) and (iv) hold.

Finally, by inverting $|G|$, we can use 2.8 and Proposition 4.A.6 to check (5.0)(iii).

5.B. The quadratic case

5.B.1. Theorem. *Let G be a 2-hyperelementary group with oriented geometric antistructure $(\theta, \omega, b, \varepsilon)$. Let $K \cong C(2) \times C(2)$ be a θ-invariant normal subgroup of G such that $K \subset \ker \omega$. Let C_0, C_1, C_2 denote the cyclic subgroups of K.*

(ia) *If K is central and θ acts as the identity on it, then the following sequence is split exact in* (RG, θ, ω, b)-Morita

$$0 \to (RG, \theta, \omega, b, \varepsilon) \xrightarrow{\text{Proj}} (R[G/C_0], \bar{\theta}, \bar{\omega}, \bar{b}, \varepsilon) \oplus (R[G/C_1], \bar{\theta}, \bar{\omega}, \bar{b}, \varepsilon) \oplus$$

$$(R[G/C_2], \bar{\theta}, \bar{\omega}, \bar{b}, \varepsilon) \xrightarrow{\beta} (R[G/K], \bar{\theta}, \bar{\omega}, \bar{b}, \varepsilon)^2 \to 0.$$

(ib) *If K is central and θ does not act as the identity on it, let C_0 denote the subgroup fixed by θ. Then*

$$(RG, \theta, \omega, b, \varepsilon) \to (R[G/C_0], \bar{\theta}, \bar{\omega}, \bar{b}, \varepsilon)$$

ia an equivalence in $(R, -)$-Witt.

(iia) *If K is not central, we may assume that $K \cap \mathcal{Z}(G) = C_0$. Let G_0 denote the centralizer of K in G. Assume that θ acts trivially on K. Then the following sequence is split exact in* (RG, θ, ω, b)-Morita:

$$0 \to (RG, \theta, \omega, b, \varepsilon) \xrightarrow{\text{Proj} \oplus \text{Res}} (R[G/C_0], \bar{\theta}, \bar{\omega}, \bar{b}, \varepsilon) \oplus (R[G_0/C_1], \bar{\theta}, \bar{\omega}, \bar{b}, \varepsilon)$$

$$\xrightarrow{\beta} (R[G_0/K], \bar{\theta}, \bar{\omega}, \bar{b}, \varepsilon) \to 0.$$

(iib) *Assume that K is not central and that θ acts non-trivially on K. Then $C_0 = K \cap \mathcal{Z}(G)$ is θ-invariant, and there is a $c \in G$ such that conjugation by c permutes C_1 and C_2. The following sequence is split exact in* $(R, -)$-Morita:

$$0 \to (RG, \theta, \omega, b, \varepsilon)$$

$$\xrightarrow{\text{Proj} \oplus \text{Res}} (R[G/C_0], \bar{\theta}, \bar{\omega}, \bar{b}, \varepsilon) \oplus (R[G_0/C_1], \bar{\theta}^c, \bar{\omega}, \bar{b}^{(c)}, \omega(c) \cdot \varepsilon)$$

$$\xrightarrow{\beta} (R[G_0/K], \bar{\theta}, \bar{\omega}, \bar{b}, \varepsilon) \to 0$$

where a bar over a symbol indicates that it is the natural restriction of the corresponding symbol on G to the subquotient.

The maps β are described below. As in 1.A.16, all the displayed maps are defined in (ZG, θ, ω, b)-Morita and the sequences are 0-sequences. They just may not be exact until $|G|$ is inverted as Non-example 1.C.6 shows.

Proof. The proof here divides into four cases. Recall that the R-group ring functor defines a functor from RG-Morita to R-Morita, so we have the linear diagrams in R-Morita as well. Also recall that all our groups are 2-hyperelementary.

Case (ia). In this case each C_i is θ-invariant, so each of the maps that we wrote down in the linear case (i) is also naturally a map in (RG, θ, ω, b)-Morita, and the proof is similar to the linear case: prove the quadratic version of Lemma 5.A.2 whenever N is a θ-invariant subgroup in $\ker \omega$ and then finish exactly as we did for the linear case.

Case (ib). This is the case that forces us to move out of (RG, ω)-Morita. It is possible to define twisted biforms and work in an 'RG-Witt' category, but it does not seem worth the effort.

The point is that the all the representations in $\mathrm{Irr}_Q(G)_{C_1 \subset K}$ are taken to representations in $\mathrm{Irr}_Q(G)_{C_2 \subset K}$, so in $(R, -)$-Witt they can be ignored. By 2.8, the projection map $G \to G/C_0$ induces an isomorphism on the remaining factors.

Case (iia). Once again, all the maps we wrote down in the linear case (ii) are naturally maps in (RG, θ, ω, b)-Morita and so the proof goes just as before.

Case (iib). To explain the problem here note that the map $RG \to R[G_0/C_1]$ is not a quadratic map because C_1 is not θ-invariant. However, C_1 is not normal either, so we can find $c \in G$ such that conjugation by c interchanges C_1 and C_2, and hence θ^c leaves C_1 fixed, and indeed, θ^c acts as the identity on K. Hence we can apply Case (iia) to the oriented geometric antistructure $(\theta^c, \omega, b^{(c)}, \varepsilon^{(c)})$ where $\varepsilon^{(c)} = \omega(c) \cdot \varepsilon$. Since

$$\begin{array}{ccccc}
(RG, \theta, \omega, b, \varepsilon) & \xrightarrow{\text{Proj}} & (R[G/C_0], \bar{\theta}, \bar{\omega}, \bar{b}, \varepsilon) & \xrightarrow{\text{Proj}} & (R[G/K], \bar{\theta}, \bar{\omega}, \bar{b}, \varepsilon) \\
\downarrow \text{twist} & & \downarrow \text{twist} & & \downarrow \text{twist} \\
(RG, \theta^c, \omega, b^{(c)}, \varepsilon^{(c)}) & \xrightarrow{\text{Proj}} & (R[G/C_0], \bar{\theta}^c, \bar{\omega}, \bar{b}^{(c)}, \varepsilon^{(c)}) & \xrightarrow{\text{Proj}} & (R[G/K], \bar{\theta}^c, \bar{\omega}, \bar{b}^{(c)}, \varepsilon^{(c)})
\end{array}$$

commutes, we easily derive the required results. □

6. On the computation of the restriction map

We want to define a partial ordering on the set of irreducible Q-representations of a p-hyperelementary group. We say that $\phi < \varrho$ if $\ker \varrho \subset \ker \phi$ and one of the following holds:

(i) $\deg \chi_\phi < \deg \chi_\varrho$, where χ denotes an irreducible complex constituent of the subscript, or

(ii) $\deg \chi_\phi = \deg \chi_\varrho$ and $Q(\chi_\phi)$ is properly contained in $Q(\chi_\varrho)$.

The following result is useful for computing some generalized restriction maps:

6.1. Theorem. *Suppose that $\varrho \in \mathrm{Irr}_Q(G)$ is such that there is a subquotient S of G, which has an $\eta \in \mathrm{Irr}_Q(S)$ such that ϱ is induced imprimitively from η. Let $\phi \in \mathrm{Irr}_Q(G)$ and $\tau \in \mathrm{Irr}_Q(S)$ be arbitrary elements. Suppose the composite $RG \xrightarrow{[e_\phi]} RG \xrightarrow{\mathrm{Res}} RS \xrightarrow{[e_\tau]} RS$ is non-trivial. We have the following two results:*

(i) *If $\tau = \eta$, then $\phi = \varrho$.*

(ii) *If $\tau < \eta$, then $\phi < \varrho$.*

Proof. Begin by assuming that $\tau = \eta$. There is a basic subquotient F of S so that τ is imprimitively induced from ϱ_F. But then F is also a subquotient of G and ϕ is induced imprimitively from ϱ_F. Part (i) now follows from 4.A.10. It also follows from 4.A.10 that $RG \xrightarrow{[e_\phi]} RG \xrightarrow{\mathrm{Res}} RS \xrightarrow{[e_\tau]} RS$ is trivial unless $\tau|^G$ contains ϕ as a constituent.

We now assume that $\tau < \eta$. Part (ii) will be shown to follow from the result that $\tau|^G$ contains ϕ as a constituent. To fix notation, let $H \subset G$ be the subgroup mapping onto S. Since we know the kernel of an induced representation in terms of the kernel of the original representation, we see that $\ker \varrho = \ker \eta|^G \subset \ker \tau|^G$. But, if $\tau|^G$ contains ϕ as a constituent, $\ker \tau|^G \subset \ker \phi$, and we have a first part of what we must prove.

Let χ_τ be an irreducible constituent of τ, and similarly we have χ_ϕ, χ_η and χ_ϱ. If $\chi_\tau|^G$ is reducible, then clearly $\phi < \varrho$ (indeed $\deg \chi_\varrho = \deg \chi_\eta|^G = |G:H| \deg \chi_\eta \geq |G:H| \deg \chi_\tau = \deg \chi_\tau|^G$ and $\deg \chi_\tau|^G > \deg \chi_\phi$). Hence we need only consider the case for which $\chi_\tau|^G = \chi_\phi$. If $\deg \chi_\tau < \deg \chi_\eta$ then again $\phi < \varrho$.

Hence we may as well assume that $\chi_\tau|^G = \chi_\phi$ and $\deg \chi_\tau = \deg \chi_\eta$. The first equation implies that $Q(\chi_\tau) \supseteq Q(\chi_\phi)$. Since $\deg \chi_\tau = \deg \chi_\eta$, we must have $Q(\chi_\tau)$ is properly contained in $Q(\chi_\eta)$. Since $Q(\chi_\eta) = Q(\chi_\varrho)$, once again $\phi < \varrho$. □

This result can be applied in several places to prove absolute detection theorems. We begin by proving a general detection theorem and then discussing several situations. First we introduce some notation.

Given two unital representations ϕ and ϱ of G, we say $\phi < \varrho$ provided each irreducible rational constituent of ϕ is less than each irreducible rational constituent of ϱ.

Let F_1 be a additive functor defined on $(Z[1/m]G, \omega)$-Morita into an abelian category $\mathscr{A}$. Let F_2 be a functor defined on the category (ZG, ω)-Morita into $\mathscr{A}$. (F_2 need not be additive.) Consider F_1 also to be defined on (ZG, ω)-Morita, and let $\partial : F_1 \to F_2$ be a natural transformation. Let $N \lhd H$ be subgroups of G with $N \subset \ker \omega$.

Fix an ω-invariant unital representation η of H/N. We say the triple (H, N, η) is ∂-*good* iff

$$\ker \partial \subset F_1(Z[1/m][H/N], \omega) \xrightarrow{[e_\tau]} F_1(Z[1/m][H/N], \omega)$$

is injective, where τ is the maximal unital representation with $\tau < \eta$. (Note that τ is ω-invariant.)

6.2. Image detection theorem. *With notation as above, fix a p-hyperelementary group G, and let $m = |G|$. Let K denote a normal subgroup of G with $K \subset \ker \omega$, and let $\pi: G \to G/K$ be the projection. Let $\mathscr{S}$ be a complete (Definition 4.A.2) set of unital representations of G, each of which is ω-invariant. Suppose there is one representation, $\varrho_K \in \mathscr{S}$, which contains precisely the irreducible Q-representations of G whose kernels contain K. For every other $\varrho \in \mathscr{S}$ suppose given a subquotient $N_\varrho \triangleleft H_\varrho$ and a unital representation $\eta = \eta_\varrho$ such that ϱ is imprimitively induced from η. Finally, suppose that for each $\varrho \neq \varrho_K$, the triple $(H_\varrho, N_\varrho, \eta_\varrho)$ is ∂-good.*

Consider the commutative square

$$\begin{array}{ccc} F_1(G,\omega) & \xrightarrow{d_1} & F_1(G/K,\omega) \oplus \bigoplus_{\mathscr{S}} F_1(H_\varrho/N_\varrho,\omega) \\ \partial \downarrow & & \downarrow \\ F_2(G,\omega) & \xrightarrow{d_2} & F_2(G/K,\omega) \oplus \bigoplus_{\mathscr{S}} F_2(H_\varrho/N_\varrho,\omega). \end{array}$$

Finally, assume

(i) $\pi: \ker(F_1(G,\omega) \to F_2(G,\omega)) \to \ker(F_1(G/K,\omega) \to F_2(G/K,\omega))$ *is onto.*

Then $d_2|_{\operatorname{Im}\partial}$ is one to one.

Addendum. *We may replace $\mathscr{S}$ in the above sum by the subset $\mathscr{S}'$ where (H_ϱ, N_ϱ) is in $\mathscr{S}'$ iff $F_1([e_\eta])$ does not induce the 0-map on $F_1(Z[1/m]H_\varrho/N_\varrho)$.*

Proof. We may as well assume we are working in a subcategory of the category of abelian groups. Let $x \in \ker(d_2) \cap \operatorname{Im}\partial$, and select $y \in F_1(G)$ with $\partial(y) = x$. The assumption on the map π between the kernels means that one can select y such that it maps to 0 in $F_1(G/K)$. We will show that this y is 0 which proves the theorem.

Let Ω be the set of ω-irreducible representations of G. We can use the quadratic detection theorem 1.B.7(i) to write

$$y = \bigoplus_{\phi \in \Omega} y_\phi$$

where $y_\phi = F_1([e_\phi])(y)$, and $y = 0$ iff each $y_\phi = 0$. The proof that $y = 0$ is by contradiction. Choose a $\phi \in \Omega$ such that $y_\phi \neq 0$ and if $\psi \in \Omega$ with $\psi < \phi$, then $y_\phi = 0$. This we can clearly do.

Let $\varrho \in \mathscr{S}$ be the unique representation which has ϕ as a constitutent, and note $\varrho \neq \varrho_K$. Let Y_ϱ be the image of y in $F_1(R[H_\varrho/N_\varrho],\omega)$. From 4.A.8, $F_1(\mathrm{Ind}^G_{H_\varrho/N_\varrho})(F_1([e_\eta])(Y_\varrho)) = \bigoplus_\phi y_\phi$ where the sum runs over the constituents of ϱ. In particular, $F_1([e_\eta])(Y_\varrho) \neq 0$.

Hence $\varrho \in \mathscr{S}'$ and therefore $(H_\varrho, N_\varrho, \eta)$ is ∂-good. Since $Y_\varrho \in \ker \partial$, this means $F_1([e_\tau])(Y_\varrho) \neq 0$. But $Y_\varrho = F_1(\mathrm{Res}^G_{H_\varrho/N_\varrho})(y)$ by definition, so there exists a $\psi \in \Omega$ such that $F_1([e_\tau])(F_1(\mathrm{Res}^G_{H_\varrho/N_\varrho})(y_\psi)) \neq 0$. Hence $y_\psi \neq 0$ and from Proposition 4.A.10 and 6.1 we see that $\psi < \phi$. This is a contradiction. □

We give two examples based on the two functors $F_1(G) \cong L^p(ZG \to \hat{Z}_2G,\omega) \cong L^K(Z[\frac{1}{2}]G \to \hat{Q}_2G)$ (see [12, 1.]) and $F_2(G) \cong L^p(ZG,\omega)$ for finite 2-groups. If ω is trivial, we take $K \cong G$ and let $\mathscr{S}$ be a set of basic subquotients, one for each representation which is not trivial. It follows easily from [12, p. 115, Example 1] that all basic 2-groups are ∂-good for the corresponding basic representation, except for the trivial group. Since $\{e\}$ never occurs as a quotient group for the elements in $\mathscr{S}$, all the H_η/N_η's in $\mathscr{S}$ are ∂-good. Since $L^p(\hat{Z}_2G) \to L^p(\hat{Z}_2[G/K])$ is an isomorphism, (i) is clearly satisfied, and the map

$$L^p(ZG) \xrightarrow{d_2} L^p(Z) \oplus \bigoplus_{\mathscr{S}} L^p(Z[H_\eta/N_\eta])$$

is a monomorphism.

If ω is not trivial, take $K \cong [G,G]$. By [12, p. 115, Example 2], all ω-basic groups which are not basic except $C(4)^-$ are ∂-good. If $C(4)^-$ appears in a set of ω-basic subquotients where the corresponding H is a proper subgroup of G, we can induce the corresponding representation from a subquotient of order 16. This group of order 16 has a $C(4)^-$ subquotient for which the faithful irreducible Q-representation on $C(4)$ induces up imprimitively. The only group of order 16 with this property is the group M_{16} of 1.C.8. It is also not hard to check that $L^p(ZG^{(\mathrm{ab})},\omega)$ is detected by $C(2) \times C(4)^-$ quotients so we see that $L^p(ZG,\omega)$ is detected by ω-basic subquotients which are not $C(4)^-$ plus one $C(2) \times C(4)^-$ quotient for each '$C(4)^-$ quotient representation' and one subquotient M_{16} for each remaining '$C(4)^-$ representation'.

Finally, we correct the proof of [13, Theorem 5.4], which is wrong for the case $i=2$. Here again we take K to be $[G,G]$, and note that [13, Theorem 4.5 and Lemma 5.2] imply that the collection $\mathscr{S}'$ above consists of dihedral subquotients, which are ∂-good. Theorem 6.2 supplies the necessary result to reduce to a routine diagram chase.

7. Another approach to detection theorems

The idea in this section is to prove detection theorems in situations in which the order of G is not a unit in R. Let W be a functor out of (RG,ω)-Morita into an abelian category. In general, one wants to produce a list of 2-hyperelementary groups, G, such that, if G is *not* on the list, then the sum of the generalized restriction maps

$$W(RG,\omega)\to \bigoplus W(R[H/N],\bar{\omega})$$

is injective, where the product runs over all *proper* subquotients of G.

In this generality it is difficult to make further progress. One way to proceed is to assume that our functor fits into a long exact sequence

$$\cdots \longrightarrow Y_{n+1}\longrightarrow W_n \longrightarrow X_n \xrightarrow{\psi_n} Y_n \longrightarrow W_{n-1}\longrightarrow \cdots .$$

If G is not ω-basic, then we can apply W_*, X_* and Y_* to either the 0-sequence in 5.B.1(ia) or the one in 5.B.1(iia). We get a commutative diagram like that in the next lemma with $A_{*,n}=W_n$, $B_{*,n}=X_n$ and $C_{*,n}=Y_n$. The vertical maps ψ_n in 7.1 will be sums of maps $\psi_n : X_n\to Y_n$ above.

7.1. Long snake lemma. *Suppose given a commutative diagram in an abelian category*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A_{1,n} & \xrightarrow{f_n} & A_{2,n} & \xrightarrow{g_n} & A_{3,n} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_{1,n} & \longrightarrow & B_{2,n} & \longrightarrow & B_{3,n} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \psi_n} & & \\
0 & \longrightarrow & C_{1,n} & \longrightarrow & C_{2,n} & \longrightarrow & C_{3,n} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A_{1,n-1} & \xrightarrow{f_{n-1}} & A_{2,n-1} & \xrightarrow{g_{n-1}} & A_{3,n-1} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where the vertical columns are long exact sequences, each B and each C row is exact and each A row is a 0-sequence (for all $n\in Z$). Then there is a connecting homomorphism $\delta_n : A_{3,n}\to A_{1,n-1}$ such that

$$\cdots \longrightarrow A_{1,n}\xrightarrow{f_n} A_{2,n}\xrightarrow{g_n} A_{3,n}\xrightarrow{\delta_n} A_{1,n-1}\xrightarrow{f_{n-1}} A_{2,n-1}\xrightarrow{g_{n-1}} A_{3,n-1}\longrightarrow \cdots$$

is a long exact sequence.

If $B_{3,n} \to C_{3,n}$ is injective or $B_{2,n} \to C_{2,n}$ is trivial, then δ_n is trivial.

Proof. A diagram chase. □

Remark. Notice that when δ_{n+1} is trivial, we detect $W_n(RG, \omega)$ by proper subquotients.

7.2. Example. Take $W_n(ZG, \omega) = L_n^p(ZG, \omega)$, $X_n(ZG, \omega) = L_n^p(\hat{Z}_2 G, \omega)$ and $Y_n(ZG, \omega) = L_n^p(ZG \to \hat{Z}_2 G, \omega)$. Let G be a 2-group. The C row is exact by Application 1.B.8(iv) and Theorem 5.B.1(ia) or (iia).

Davis and Milgram [4] applied these techniques to the following example:

7.3. Example. Take $W_n(ZG, \omega) = L_n^h(QG, \omega)$, $X_n(ZG, \omega) = L_n^{K \to h}(QG, \omega)$ and $Y_n(ZG, \omega) = L_{n-1}^K(QG, \omega)$. Let G be a 2-group. This W_n is a functor out of (ZG, ω)-Morita because the modules defining the maps have the required freeness [11, Proposition 5.6]. The C row is exact by Theorem 5.B.1(ia) or (iia) plus the fact that the round L-theory is a functor out of (QG, ω)-Morita.

The functors used in both of these examples have an additional feature. We say that a functor F satisfies Condition 7.4 provided

7.4. Condition. Any projection map $G \to G/N$ where $N \subset \ker \omega$ induces an isomorphism $F(RG, \omega) \to F(R[G/N], \omega)$.

7.5. Lemma. *If a functor F satisfies Condition* 7.4, *then the sequence obtained by applying F to the* 0*-sequence in* 5.B.1(ia) *or* (iia) *is exact.*

Proof. Easy. □

7.6. Remark. In both Example 7.2 and 7.3, the X functor satisfies Condition 7.4. For Example 7.2, see [12, 12]. For Example 7.3, see [11, Proposition 3.2].

7.7. Proof of Theorem 1.C.7. Consider Example 7.2 with ω trivial. By [12, Example 1, p. 115], the map ψ_n is trivial ($n \not\equiv 0 \pmod 4$) or is injective ($n \equiv 0 \pmod 4$). Then by Lemma 7.1, δ_{n+1} is trivial. □

For other applications we produce a refinement of this technique.

7.8. Theorem. *Let G be a finite* 2*-group and let $\cdots \to W_n \to X_n \to Y_n \to \cdots$ be a long exact sequence of functors out of* $(ZG, -)$-Morita. *Suppose that Y applied to the sequence in* 5.B.1(ia) *or* (iia) *is exact, and suppose that X satisfies Condition* 7.4.

Finally, suppose that the map ψ_{n+1} is injective if ω factors through $C(4)^-$ and is 0 otherwise. Then δ_{n+1} is trivial unless G is ω-basic, $G \cong C(2) \times C(4)^-$, or M_{16}.

Proof. We can begin by assuming that G is not ω-basic. The proof divides into two cases, as in Section 5. Begin with the case in which G has a central $K \cong C(2) \times C(2)$ contained in G^+.

The goal here is to prove that either δ_{n+1} is trivial or $G = C(2) \times C(4)^-$. If $\omega_{G/K}$ factors through $C(4)^-$, then Lemma 6.1 implies δ_{n+1} is trivial. If G is abelian of rank ≥ 3, then it is possible to choose a central K so that $\omega_{G/K}$ factors through $C(4)^-$. So hereafter assume $\omega_{G/K}$ does not factor through $C(4)^-$ and that, if G is abelian it is of rank 2.

If at least two of the ω_{G/C_i} do not factor through $C(4)^-$, then a diagram chase shows that δ_{n+1} is trivial. (It is helpful to recall the definition of the map β from Section 5.A.1.)

We henceforth assume that $\omega_{G/K}$ does not factor through $C(4)^-$ and that at least two of the ω_{G/C_i} do. If G is non-abelian, then let $C_0 \subset K \cap [G, G]$. Choose C_1 so that ω_{G/C_1} does factor through $C(4)^-$. Since $C_0 \subset [G, G]$, $\omega_{G/K}$ also factors through $C(4)^-$, which is a contradiction.

If G is abelian, it is of rank 2 and hence of the form $C(2^j) \times C(2^i)^-$. Note that $i \leq 2$ since $\omega_{G/K}$ does not factor through $C(4)^-$. Next note that $i \geq 2$ and $j = 1$ since otherwise at most one of the ω_{G/C_i} factors through $C(4)^-$.

The remaining case is the one in which we have a normal K, but no central one. If $\omega_{G_0/K}$ factors through $C(4)^-$, then Lemma 6.1 implies δ_{n+1} is trivial, so henceforth assume that $\omega_{G_0/K}$ does not factor through $C(4)^-$. If ω_{G_0/C_1} does not factor through $C(4)^-$, then another diagram chase shows that δ_{n+1} is trivial, so we now assume ω_{G_0/C_1} does factor through $C(4)^-$.

Note that $\mathcal{Z}_2(G_0) = K$, since if $\mathcal{Z}_2(G_0)$ were larger, there would be an $E \cong C(2) \times C(2) \subset \mathcal{Z}_2(G_0)$ which would be central in G. If E were not in G^+, then G_0 would be $G_0^+ \times C(2)^-$ which is impossible.

We wish to argue that G_0 must be abelian. Note first that $\mathcal{Z}_2(G) \cap G_0^+ = C_0$ since there are no central $C(2) \times C(2)$'s in G^+. It follows that $[G_0, G_0] \cap \mathcal{Z}_2(G) = C_0$. But this is not possible since then $\omega_{G_0/K}$ would factor through $C(4)^-$.

Now we know that G_0 is a rank 2 abelian. We know that ω_{G_0/C_1} does factor through $C(4)^-$. The conjugation action of G on G_0 gives an isomorphism between G_0/C_1 and G_0/C_2 which preserves the ω's. As in the central case, it now follows that $G_0 \cong C(2) \times C(4)^-$.

Now G is an extension of $C(2) \times C(4)^-$ by a $C(2)$. Consider the subgroup G^+ which is easily seen to be a non-abelian group of order 8 with a normal $C(2) \times C(2)$, hence it is $D(8)$. It is easy to show that the extension for G is semi-direct and we can choose an element $g \in G^+$ of order 2 giving the splitting.

Finally, we determine the action map: $\alpha(h) = g \cdot h \cdot g^{-1}$ for all $h \in C(2) \times C(4)^-$. Let t_0 and t_1 be generators for $C(2) \times C(4)^-$ with $\omega(t_1) = -1$ so t_1 has order 4 and we choose t_0 to have order 2 and be in ker ω. Note $\alpha(t_1^2) = t_1^2$, so $\alpha(t_0) = t_0 \cdot t_1^2$, since the action is nontrivial on the $C(2)\times C(2)\subset C(2)\times C(4)^-$. Clearly $\alpha(t_1)=t_1^{\pm 1}$ or $t_0\cdot t_1^{\pm 1}$. This second possibility cannot occur since α has order 2 on $C(2)\times C(4)^-$. If $\alpha(t_1)=t_1^{-1}$, then we an replace t_1 with $t_0\cdot t_1$ on which α acts trivially. □

7.9. Proof of 1.C.8. The ψ_{n+1} maps for the functors in Example 7.2 are described in [12, Example 2, p. 115]. If $n+1\not\equiv 0 \pmod 4$, then Lemma 7.1 proves the result. If $n+1\equiv 0 \pmod 4$, then Theorem 7.8 finishes the proof. □

Remark. The Davis–Milgram example, Example 7.3, also follows from Lemma 7.1 and Theorem 7.8.

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