K-Theory 7: 537–574, 1993.

On the Computation of the Projective Surgery Obstruction Groups

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(Received: October 1993)

Abstract. The computation of the projective surgery obstruction groups $L_n^p(\mathbf{Z}G)$, for G a hyperelementary finite group, is reduced to standard calculations in number theory, mostly involving class groups. Both the exponent of the torsion subgroup and the precise divisibility of the signatures are determined. For G a 2-hyperelementary group, the $L_n^p(\mathbf{Z}G)$ are detected by restriction to certain subquotients of G, and a complete set of invariants is given for oriented surgery obstructions.

Key words. Surgery on manifolds, Hermitian K-theory.

0. Introduction

The projective surgery obstruction groups were first introduced by S. P. Novikov [30] in the context of Hermitian K-theory and the topology of infinite cycle covers of compact manifolds. These groups arise as the codimension 1 summands in a splitting theorem for the Wall surgery obstruction groups of a Laurent polynomial extension [39, 12], or more generally in the classification of non-compact manifolds [40, 41, 27, 33]. The algebraic description of projective L-theory and the splitting theorem were given a systematic exposition by A. A. Ranicki [34, 36], including a definition of the lower L-groups by analogy with the lower K-groups of Bass.

With the extensive development of 'bounded' or 'controlled' topology in the last decade, the role of projective and lower surgery obstruction groups has increased in importance. For example, the concrete structure of these groups is relevant to the classification of linear representations of finite groups up to topological conjugacy [5, 6, 18] and the recent survey article [13] describes other applications. The purpose of the present paper is to provide a reference for further computations in this area.

*Partially supported by NSERC grant A4000.
**Partially supported by NSF grant DMS-8610730(1) and the Danish Research Council.

Here we treat only the projective surgery obstruction groups and defer the discussion of lower L-groups to a subsequent paper.

The problem of computing the projective surgery obstruction groups $L_n^p(\mathbf{Z}G)$ has already been extensively studied for the case where G is a finite hyper-elementary group (see, for example, [2, 3, 7, 20, 23, 24, 32, 44]). These computations may be considered as a further development and application of the classical theory of quadratic forms over fields. In many cases a particular geometric problem provided the motivation for considering, a special class of groups in detail. This occurred for example in the space form problem [26, 10, 15, 4], or in the problem of surgery obstructions on closed manifolds [14, 17]. Unfortunately, the most complete statements in the literature for 2-hyperelementary groups (in [3, 23, 24]) are based on an incorrect calculation [1, Cor. 4a] of the map $L_i^h(\hat{\mathbf{Z}}_2\pi) \to L_i^h(\hat{\mathbf{Q}}_2\pi)$.

In this paper, we reorganize the calculations so that it is easy to incorporate the correct results about this map (from [16, 1.16]) and then focus on two aspects of these computations which have not received close attention. The first aspect is *functoriality*: is the calculation arranged so that one can determine (even in principle) the induced maps on L-groups arising from group homomorphisms? The second aspect is *effectiveness*: given a character table for G, how much of the calculation is derived via an algorithm from this information, and what additional data is needed?

From this point of view, the situation is satisfactory at the moment only for G a finite 2-group [20]. In Section 2 we establish the analogous results for the relative group $L_n^p(\mathbf{Z}G \to \hat{\mathbf{Z}}_2G)$ when G is 2-hyperelementary, and tabulate the answers (see Table 2 at the end of Section 2). Note that when G is not a 2-group, the relative group does not instantly decompose *in a natural way* according to the character table of G (compare [3, 23, 24] where the reduction to character theory is not functorial). To obtain this we need the splitting of [15, Section 6]. In the technique of calculation, our only other innovation is to take up a suggestion of C. T. C. Wall [44, p. 259]: 'The statements for $\mathrm{CL}_i^K(S)$ could probably be simplified by using relative groups.' Indeed we replace Wall's $\mathrm{CL}_n^K(S)$, which measure the deviation from the Hasse principle, by the relative groups $L_n^K(S \to S_A)$ where S denotes a central simple algebra (with involution) and S_A is its adelic completion.

Our functorial description of the relative groups $L_n^p(\mathbf{Z}G \to \hat{\mathbf{Z}}_2G)$ and [16, 4.6] leads to an efficient 'computation' of $L_n^p(\mathbf{Z}G)$ for hyperelementary groups in the sense that we have reduced it to standard calculations in number theory, mostly involving class groups. In the process we have settled all the extension questions and the precise divisibility of the signatures. Section 5 lists the necessary steps and the answers. One consequence is the following corollary.

COROLLARY 5.21. *The torsion subgroup in $L_n^p(\mathbf{Z}G, \alpha, u)$ has exponent 4 for any geometric antistructure. In the special case of the standard oriented anti-structure, the torsion subgroup has exponent 2.*

By Dress induction [11], [45, 2.1.2], the L-theory for a general finite group is computed by restriction to the collection of 2-hyperelementary subgroups of G. More precisely, the result is a computation of $L^p_n(\mathbf{Z}G)$ in terms of L^p of 2-hyperelementary groups, *provided that* the maps induced by group homomorphisms are also calculable. This paper can be viewed as a step towards determining these induced maps (see [20] if G is a 2-group).

We now give two qualitative results, valid for the standard oriented antistructure (defined by the involution $g \mapsto g^{-1}$, for $g \in G$). First, it turns out that we can reduce further to the class of basic 2-hyperelementary groups using the methods of [21] and the main result of [16]. Recall that a *basic* group is one for which all normal Abelian subgroups are cyclic. More explicitly [21, 3.A.6], a basic 2-hyperelementary group $G = \mathbf{Z}/m \rtimes \sigma$, where m is odd and σ a 2-group, has

$$\sigma_1 = \ker(t\colon \sigma \to (\mathbf{Z}/m)^{\times})$$

cyclic, dihedral, semi-dihedral or quaternion.

THEOREM A. *Let G be a 2-hyperelementary group. Then the sum of all the (generalized) restriction maps*

$$L^p_n(\mathbf{Z}G) \to L^p_n(\mathbf{Z}[\bar{G}]) \oplus \sum \{L^p_n(\mathbf{Z}[H/N])\colon H/N \text{ a basic subquotient of } G\}$$

is a natural (split) injection, where $\bar{G} = G/[\sigma_1, \sigma_1]$ and $\mathbf{Z}G$ has the standard oriented antistructure.

A generalized restriction map is restriction followed by the map induced by a surjection of groups. The usefulness of Theorem A is likely to be in deciding whether a surgery obstruction is zero or non-zero, in terms of the information available over the sub-quotients.

Our second general result deals more specifically with the problem of deciding whether the surgery obstruction of a geometric problem is nonzero. Let $\lambda^p(f, b) \in L^p_n(\mathbf{Z}G)$ be the surgery obstruction of a degree 1 normal map $f\colon M \to X$, $b\colon \nu_M \to \xi$, where M^n is a closed n-manifold and X is a finitely dominated Poincaré space of dimension n [33]. Associated to this situation, there are certain primary invariants: the multisignature, Arf invariants, semi-characteristic $\chi_{\frac{1}{2}}$, and (cohomology) finiteness obstruction $\sigma_*(X) \in H^0(\tilde{K}_0(\mathbf{Z}G))$. These are defined at the beginning of Section 4 in terms of certain natural maps of L-groups.

Here and in the rest of the paper, we use the notation $H^i(A)$ as a short form for the Tate cohomology group $H^i(\mathbf{Z}/2; A)$ with coefficients in a $\mathbf{Z}/2$-module A. In order to have compatibility with the maps in the arithmetic sequence, we will use the $\mathbf{Z}/2$-module structure on $\tilde{K}_0(\mathbf{Z}G)$ induced by $[P] \mapsto -[P^*]$ for any projective module P. Then our final invariant for detecting the surgery obstruction, defined on the kernel of the primary invariants, is the δ-invariant:

$$\delta(f, b) \in H^{n+1}(\mathrm{Wh}(\hat{\mathbf{Q}}_2G)/\mathrm{Wh}'(\mathbf{Z}G))/\{L^h_{n+1}(\hat{\mathbf{Z}}_2G) \oplus d^*H^n(\tilde{K}_0(\mathbf{Z}G))\}.$$

This is defined in detail in (4.3) following [15, Section 5].

THEOREM B. *For any oriented degree 1 normal map (f, b), the projective surgery obstruction $\lambda^p(f, b)$ is detected by the multi-signature, Arf invariants, semicharacteristic, cohomology finiteness obstruction and the δ-invariant.*

We remark that both the finiteness obstruction ([26, 28, 46]) and the δ-invariant (see [15, Sections 4, 9]) are computable using Reidemeister torsion.

1. Reduction to Simple Algebras

The geometrically significant surgery obstruction groups are $L_n^{\tilde{X}}(\mathbf{Z}G)$, and these are algebraic L-groups ([34]) with decorations in an involution-invariant subgroup $\tilde{X} \subseteq \tilde{K}_0(\mathbf{Z}G)$ or $\tilde{X} \subseteq \mathrm{Wh}(\mathbf{Z}G)$. The most important examples for this paper are $L_n^p(\mathbf{Z}G)$ or $L_n^h(\mathbf{Z}G)$, where $\tilde{X} = \tilde{K}_0(\mathbf{Z}G)$ or $\tilde{X} = \mathrm{Wh}(\mathbf{Z}G)$ respectively. In order to compute these groups in terms of the character theory of G and the number theory associated to the centre fields in $\mathbf{Q}G$, it is convenient to use the Morita invariant 'round' algebraic L-theory $L_n^X(A, \alpha, u)$ for a ring A with antistructure, based on involution-invariant subgroups X of $K_1(A)$ (see [19, Section 2]). In addition, we will use the 'round' projective L-groups $L_n^P(\mathbf{Z}G)$ based on the subgroup $\tilde{K}_0(\mathbf{Z}G) \subseteq K_0(\mathbf{Z}G)$. These are related to the usual projective L-groups by the exact sequence [19, 3.2], [15, 3.8]

$$0 \to L_{2k}^p(\mathbf{Z}G) \to L_{2k}^P(\mathbf{Z}G) \to \mathbf{Z}/2 \to L_{2k-1}^P(\mathbf{Z}G) \to L_{2k-1}^p(\mathbf{Z}G) \to 0. \tag{1.1}$$

The map into $\mathbf{Z}/2$ is given by the rank (mod 2) of the underlying projective module.

Now let $G = \mathbf{Z}/m \rtimes \sigma$ be a 2-hyperelementary group, where m is odd and σ is a finite 2-group. Let $R = \mathbf{Z}[G]$, $S = \mathbf{Q}[G]$ and for $d \mid m$, let $R(d) = \mathbf{Z}[\zeta_d]^t\sigma$ be the twisted group ring quotient of R defined by sending a generator of $\mathbf{Z}/m$ to ζ_d. The 'round' L-groups of these rings based on the full K_1 are denoted L^K. These were introduced by C. T. C. Wall and used extensively in his calculations of L-groups [45, Section 1.1].

The following is a version of Wall's 'main exact sequence' [45].

THEOREM 1.2 ([15, 6.13, 7.2]). *There is a natural splitting*

$$L_n^p(\mathbf{Z}G) = \sum^{\oplus} \{L_n^p(\mathbf{Z}G)(d) : d \mid m\}$$

such that

(i) *for $d \neq 1$, $L_n^p(\mathbf{Z}G)(d) \cong L_n^P(\mathbf{Z}G)(d)$,*

(ii) *$L_n^p(\mathbf{Z}G)(d)$ maps isomorphically under restriction to $L_n^p(\mathbf{Z}[\mathbf{Z}/d \rtimes \sigma])(d)$,*

(iii) *for each $d \mid m$, there is a long exact sequence*

$$\cdots \to L_{n+1}^K(\hat{S}(d)) \to L_n^P(\mathbf{Z}G)(d) \to \prod_{l \nmid d} L_n^K(\hat{R}_l(d)) \oplus L_n^K(S(d)) \to L_n^K(\hat{S}(d)) \to \cdots$$

where $S(d) = R(d) \otimes \mathbf{Q}$ and $\hat{S}(d)$ is the completion $S(d) \otimes \hat{\mathbf{Z}}$ of $S(d)$.

Remark 1.3. In order to obtain the splitting above from the results of [15, Section 6] it suffices to show that the maps in the arithmetic sequence are compatible with the usual Mackey functor structure on each term, defined by induction and restriction with respect to odd-index subgroups of G. Most of these maps are just 'change of ring' maps in L-theory, and the required compatibility is clear from the definitions given by Dress [11] or Ranicki [35]. The connecting homomorphism in the arithmetic sequence is the composite of two 'change of ring' maps for relative L-groups and the boundary map in the exact sequence of a pair (see [35, p. 485]). The compatibility for the connecting homomorphism with induction and restriction now follows immediately from the algebraic description in [35, p. 109] for relative L-groups. The same proof works for geometric anti-structures (see Section 3) of the form (R, β_0, b_0) with $b_0 \in \sigma$, since these admit induction and restriction maps of anti-structures for the set of odd-index subgroups containing b_0.

Such exact sequences for computing L-groups come from the arithmetic square, [35, Section 6], where the basic form is

$$\cdots \to L_n^P(R) \to L_n^K(\hat{R}) \oplus L_n^K(S) \to L_n^K(\hat{S}) \to L_{n-1}^P(R) \to \cdots$$

One of the points we wish to emphasize is that most of the difficulties involved in computing $L_n^p(\mathbf{Z}G)$ concern the group $L_n^h(\hat{\mathbf{Z}}_2G)$. We therefore reorganize the calculation by considering the exact sequence

$$\cdots \to L_{i+1}^{p,h}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2G) \to L_i^p(\mathbf{Z}G) \to L_i^h(\hat{\mathbf{Z}}_2G) \xrightarrow{\psi_i} L_i^{p,h}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2G) \cdots \tag{1.4}$$

By excision

$$L_i^{p,h}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2G) \cong L_i^K(\hat{R}_{\mathrm{odd}} \oplus S \to \hat{S}), \tag{1.5}$$

where $\hat{R}_{\mathrm{odd}}$ is the product of the l-adelic completions of R at all odd primes l.

We now introduce the groups

$$\mathrm{CL}_i^K(S) = L_i^K(S \to S_A) \tag{1.6}$$

where $S_A = (S \otimes \hat{\mathbf{Z}}) \oplus (S \otimes \mathbf{R})$ is the adelic completion of S. The exact sequence in (1.2) can be rewritten by adding in the L-groups of the completion of $S(d)$ at the infinite primes: $T(d) = S(d) \otimes \mathbf{R}$. Then by the arithmetic sequence and (1.5) we have the for each $d \mid m$, a long exact sequence

$$\ldots \mathrm{CL}_{i+1}^K(S(d)) \to L_{i+1}^{p,h}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2G)(d)$$
$$\to \prod_{l \nmid 2d} L_i^K(\hat{R}_l(d)) \oplus L_i^K(T(d)) \xrightarrow{\gamma_i(d)} \mathrm{CL}_i^K(S(d)) \tag{1.7}$$

Since $l \nmid 2d$ each factor (and the maps $\gamma_i(d)$) splits according to the decomposition of $S(d)$ into simple algebras. By quadratic Morita equivalence [20, 2.5], [19, Section 5], we are reduced to the L-groups of (skew) fields or the rings of integers in the centre (with some antistructures).

In the next section we will tabulate the domain, range, kernel and cokernel of $\gamma_i(d)$. Then there is an exact sequence

$$0 \to \operatorname{coker} \gamma_i(d) \to L_i^{p,h}(R \to \hat{R}_2)(d) \to \ker \gamma_{i-1}(d) \to 0 \tag{1.8}$$

determining the relative group up to extensions.

In the rest of this section we summarize some of the standard facts about quadratic forms on simple algebras with centre field continuous, local (of characteristic 0), and finite. For our purposes, the main references are [43] and [44]. Since we are mainly interested in the applications to surgery theory, we will restrict ourselves to the simple algebras which arise from the rational group rings of finite groups. This assumption will simplify the arguments at various points. More precisely, if D denotes such a skew field with centre E, and $A \subseteq E$ the ring of integers, then E is an Abelian extension of $\mathbf{Q}$. We fix an odd integer d such that $\hat{D}_l$ is *split*, and E_l is an *unramified* extension of $\hat{\mathbf{Q}}_l$ for all finite primes l with $l \nmid 2d$. We also assume that D has 'uniformly distributed invariants': the Hasse invariants and Schur indices of D at all primes $l \in E$ over a fixed rational prime are equal. This holds for the algebras arising from group rings by the Benard–Schacher Theorem [48, Th. 6.1].

In addition to listing the values of the groups, we mention explicit invariants (such as signature and discriminant) used to detect them. From these facts we compute the CL_i^K (see Table I) and prepare for the computation of the maps γ_i. The reader, if inclined to check the tables given later, should first assemble a complete table of the L^S to L^K Rothenberg sequences from the information given below ([45, 1.2] is useful for L^S). Recall that L^S is the 'round' L-theory based on the subgroup $0 \subseteq K_1(D)$. We remark that whenever the discriminant $L_i^K(D, \alpha, u) \to H^i(K_1(D))$ is an isomorphism, the map $L_i^S(D, \alpha, u) \to L_i^K(D, \alpha, u)$ is zero.

If (D, α, u) denotes an antistructure on a division algebra with centre E (and $A \subseteq E$ the ring of integers), then we distinguish as usual types U, Sp and O (see [45, Section 1.2]). We further subdivide into types OK if $D = E$, type OD if $D \neq E$ and similarly for type Sp. If an involution-invariant factor is the product of two simple rings interchanged by the involution, this is type GL. Such factors make no contribution to L-theory. When the anti-structure is understood, we will say 'D has type…' for short. Recall that $L_i^K(D, \alpha, -u) = L_{i+2}^K(D, \alpha, u)$ and types O and Sp are interchanged, so we usually just list type O.

(1.9) CONTINUOUS FIELDS

For continuous fields ($E = \mathbf{R}$ or $\mathbf{C}$) the signature gives an explicit isomorphism of $L_0^K(\mathbf{C}, c, 1)$, $L_2^K(\mathbf{C}, c, 1)$, $L_0^K(\mathbf{R}, 1, 1)$ and $L_0^K(\mathbf{H}, c, 1)$ onto $2\mathbf{Z}$ (the types are U, U, O and Sp); in all these cases except for $(\mathbf{H}, c, 1)$ the discriminant map $L_0^K(E) \to H^0(E^\times)$ is onto. Indeed the groups $H^i(K_1(\mathbf{H})) = 0$ so $L_i^S(\mathbf{H}, c, 1) = L_i^K(\mathbf{H}, c, 1)$. The discriminant also gives an isomorphism for $L_1^K(\mathbf{R}, 1, 1) = \mathbf{Z}/2$ and $L_1^K(\mathbf{C}, 1, 1) = \mathbf{Z}/2$. The other L^K-groups are zero. In the final calculation we wish to keep track of the divisibility of the

the signatures. The notation $2\mathbf{Z}$ stands for an infinite cyclic group of signatures taking on any even value.

(1.10) LOCAL FIELDS

Over local fields (of characteristic 0), in type U: $L_{2i}^K(D) \cong H^0(E^\times) = \mathbf{Z}/2$ via the discriminant and $L_{2i+1}^K(D) = 0$. In type OD, $L_0^K(D) \cong H^0(E^\times)$ and the others are zero. In type OK, $L_1^K(E) \cong H^1(E^\times) = \mathbf{Z}/2$ by the discriminant and $L_0^K(E)$ is an extension of $H^0(E^\times)$ by $\mathbf{Z}/2$ described in (1.12), while $L_2^K(E) = L_3^K(E) = 0$. The natural map $L_1^S(E) \to L_1^K(E)$ is zero (see [44, 3.5]).

(1.11) FINITE FIELDS

For finite fields in type U, $L_i^S = L_i^K = 0$, and in type O characteristic 2, $L_i^S = L_i^K = \mathbf{Z}/2$ for each i. For type O odd characteristic, the discriminant gives isomorphisms $L_0^K \cong \mathbf{Z}/2$, $L_1^K = \mathbf{Z}/2$ and $L_2^K = L_3^K = 0$. The groups $L_i^S = 0$ for $i = 0, 3$ and $L_1^S = L_2^S = \mathbf{Z}/2$. The map $L_1^S \to L_1^K$ is zero.

PROPOSITION 1.12. *In type* OK *over a local field E, $L_0^K(E, 1, 1)$ is given by the extension*

$$0 \to \mathbf{Z}/2 \to L_0^K(E, 1, 1) \to H^0(E^\times) \to 0.$$

This extension is split $\Leftrightarrow -1 \in E^{\times 2}$.

Proof. Only the last part remains to be proved here. See [31, 63:20] for an alternate argument. There is an isomorphism

$$\kappa\colon L_0^S(E, 1, 1) \xrightarrow{\approx} \mathrm{CL}_0^S(E) \cong \mathbf{Z}/2$$

for the local field E given by

$$\kappa(q) = S_E(q) S_E(\tfrac{1}{2} r(q) \mathbb{H})$$

where S_E denotes the Hasse–Witt invariant over E [29, p. 80], [31, Section 63B], here applied to a form of determinant 1 ($\in E^\times/E^{\times 2}$) and rank $r(q) \equiv 0 \pmod 4$. The hyperbolic form of rank 2 is denoted $\mathbb{H}$. Since

$$S_E(q \perp q') = S_E(q) \cdot S_E(q')$$

for forms of rank $\equiv 0 \pmod 4$ and every element of L_0^S is represented by such a form, it is easy to check that κ is well-defined. To extend this homomorphism to $L_0^K(E, 1, 1)$, consider the Clifford algebra associated to a quadratic space [43, p. 133]. This gives an isomorphism

$$C\colon L_0^K(E, 1, 1) \xrightarrow{\approx} G\mathrm{Br}^+(E),$$

where the right-hand side is the graded Brauer group: i.e. $\{\pm 1\} \rtimes E^\times/E^{\times 2}$ with the multiplication

$$(c, d) \cdot (c', d') = \left(cc'\left(\frac{d, d'}{E}\right), dd'\right).$$

This is a split extension if and only if the 2-cocycle defined by the Hasse symbol $((d, d')/E)$ is a coboundary. Since the Hasse symbol gives a nonsingular symmetric bilinear form on a $\mathbf{Z}/2$-vector space $V = E^\times/E^{\times 2}$, [31, 63.13],

$$S_E: V \times V \to \{\pm 1\},$$

$((d, d')/E)$ is a coboundary only when

$$\left(\frac{d, d}{E}\right) = +1 \quad \text{for all } d \in E^\times.$$

But

$$\left(\frac{d, d}{E}\right) = \left(\frac{d, -1}{E}\right)$$

so this happens if and only if $-1 \in E^{\times 2}$. Assuming this, we choose a function $f: E^\times/E^{\times 2} \to \{\pm 1\}$ such that $f(1) = +1$,

$$f(d)f(d')f(dd') = \left(\frac{d, d'}{E}\right), \quad \text{for all } d,\, d' \in E^\times/E^{\times 2},$$

and $\mathrm{Arf}(V, f) = 0$.

Now an extension of κ

$$\tilde{\kappa}_f: L_0^K(E, 1, 1) \to \mathbf{Z}/2$$

can be defined by $\tilde{\kappa}_f(q) = S_E(q)f(d(q))$, where $d(q) =$ determinant of q. Since

$$\begin{aligned}
\tilde{\kappa}_f(q \perp q') &= S_E(q \perp q')f(d(q)d(q')) \\
&= S_E(q)S_E(q')\left(\frac{d(q), d(q')}{E}\right) f(d(q)d(q')) \\
&= S_E(q)f(d(q))S_E(q')f(d(q')) \\
&= \tilde{\kappa}_f(q)\tilde{\kappa}_f(q')
\end{aligned}$$

this gives a homomorphism. Also, $\tilde{\kappa}(q \perp \mathbb{H}) = \tilde{\kappa}(q)$, so it is well defined on the L-group. □

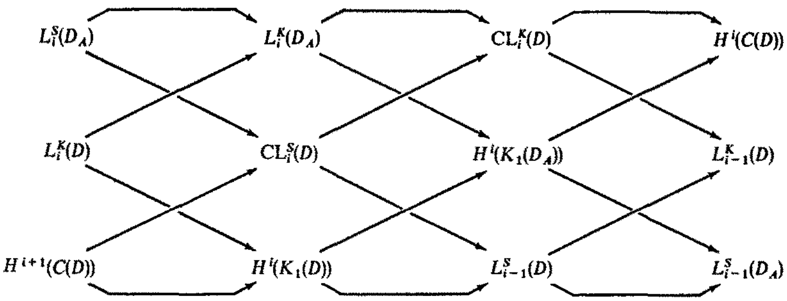
The choice f of a quadratic refinement in the above proof is not unique. For a nondyadic field, for example, $E^\times/E^{\times 2}$ has rank 2 and basis $\{\Delta, \pi\}$ with π a prime element and Δ a generator of units modulo squares. There are two choices of f with trivial Arf invariant, namely either $f(\Delta) = -1$, $f(\pi) = 1$ or $f(\Delta) = 1$, $f(\pi) = -1$. Let us note that the composition

$$L_0^K(A, 1, 1) \to L_0^K(E, 1, 1) \xrightarrow{\kappa_f} \{\pm 1\}$$

is $f(\Delta)$, where A is the integers of E.

This completes the discussion of the L^K-groups in type OK, for E a local field. The remaining terms needed in (1.7) are $L_i^K(\hat{R}_l(d))$ for some finite prime $l \nmid 2d$. We can reduce modulo the radical to the case of finite fields [44].

Next we will calculate the groups $\mathrm{CL}_i^K(D)$ for each type of anti-structure, with the final result listed in Table I below. Let $C(D) = K_1(D_A)/K_1(D)$. The weak approximation theorem [31, p. 8] combined with the calculation of K_1 using reduced norms ([37, 33.15], [47]) implies that $C(D) = C(E) = E_A^\times/E^\times$. We consider the diagram:



(1.13)

together with the results of [43, 44, 22]. The diagram contains four interlocking long exact sequences. This follows easily from naturality of L-theory under change of K-theory [35, 6.2]. The groups $\mathrm{CL}_i^S = 0$ in type U, and for type O are [44]:

$$\mathrm{CL}_i^S(D) = \mathbf{Z}/2,\ H^0(C(E)),\ H^1(C(E)),\ 0 \quad (i \equiv 0, 1, 2, 3 \pmod 4). \tag{1.14}$$

In type O the only place to check is the exact sequence

$$0 \to \mathrm{CL}_1^K(D) \to H^1(C(E)) \xrightarrow{\delta} \underset{\mathbf{Z}/2}{\underset{\|}{\mathrm{CL}_0^S(D)}} \to \mathrm{CL}_0^K(D) \to H^0(C(E)) \to 0$$

To understand maps involving $H^i(C(D))$, we will use the short exact sequences $(i = 0, 1)$

$$0 \to H^i(E^\times) \to H^i(E_A^\times) \to H^i(C(D)) \to 0, \tag{1.15}$$

following from the global square theorem [31, 65C]. Note that for $D \neq E$, $C(E) \cong C(D)$ and the involution is trivial. Therefore $H^1(E_A^\times) = \Pi_v\{\pm 1\}_v$ and $H^1(C(D))$ is this product modulo $-1 \in E^\times$. The map $H^1(\hat{E}_A^\times) \to H^1(C(D))$ is the obvious projection.

LEMMA 1.16. *The map* δ *is zero in type* OK *and nonzero in type* OD.

Proof. To check this one needs to know that, in type O, the map

$$L_0^S(\hat{D}_l) \to L_0^S(D_A) \to \mathrm{CL}_0^S(D) \cong \mathbf{Z}/2$$

is an isomorphism for any l finite ([44, p. 259]) and onto for l an infinite real place where D splits ([43, p. 140]).

Now in type OK note that the map $H^1(\hat{E}_l^\times) \to L_0^S(\hat{E}_l^\times)$ is zero for any prime l, since in odd dimensions the nonzero L^K-groups listed above are all isomorphic via discriminant to $H^1(E_l^\times)$. On the other hand the map $H^1(\hat{E}_A^\times) \to H^1(C(E))$ is onto, so δ is onto.

In type OD where D is nonsplit at some finite prime l, the map $H^1(\hat{E}_l^\times) \to L_0^S(\hat{D}_l)$ is an isomorphism and so is $L_0^S(\hat{D}_l) \to \mathrm{CL}_0^S(D)$.

Finally, in type OD where D is nonsplit only at infinite primes, we have the isomorphism $L_1^K(D_A) \xrightarrow{\approx} H^1(K_1(D_A))$ and

$$\mathrm{cok}(H^1(K_1(D_A)) \to H^1(C(D))) \cong (\mathbf{Z}/2)^{r-1},$$

where r is the number of primes at which D is nonsplit, here equal to the number of real places. In type OD, Kneser (see [38, p. 370]) gives the formula $r-2$ for the rank of the elementary 2-group $\ker(L_0^K(D) \to L_0^K(D_A))$. This implies the exact sequence

$$0 \to L_1^K(D_A) \to \mathrm{CL}_1^K(D) \to (\mathbf{Z}/2)^{r-2} \to 0.$$

If $\delta = 0$, $\mathrm{CL}_1^K(D)$ would surject onto $H^1(K_1(D_A))$ and then in turn onto $(\mathbf{Z}/2)^{r-1}$, but the surjection would factor through $(\mathbf{Z}/2)^{r-2}$, which is impossible. □

Before we can tabulate the results for the CL^K we must determine the map δ and settle an extension question.

LEMMA 1.17. *In type* OD *the composite*

$$H^1(E_A^\times) \to H^1(C(D)) \xrightarrow{\delta} \mathrm{CL}_0^S(D) \cong \mathbf{Z}/2$$

maps $\langle -1 \rangle_l \in H^1(E_l^\times)$ *trivially if and only if* $\hat{D}_l$ *is split.*

Proof. Unless l is an infinite place at which $\hat{D}_l$ is nonsplit, this follows easily from the facts listed in the beginning of the proof of (1.16). Note that the map

$$H^0(K_1(D)) \to H^0(K_1(D_A))$$

is not injective precisely when D is nonsplit at infinity (an element in $K_1(D) \cong E^*$ represents zero in $H^0(K_1(D))$ only when it is the square of some totally positive element). Its kernel is $(\mathbf{Z}/2)^{r_1-1}$, where r_1 is the number of real primes in the totally real field E. Hence $H^0(K_1(D_A)) \to H^1(C(D))$ is not onto in that case.

If D is nonsplit only at the infinite places, the fact that δ is nontrivial and Galois invariance implies the result. In the remaining case we will look at the maps in (1.13) more closely.

Let D be the quaternion algebra over E spanned by λ and μ, with $\lambda^2 = a$, $\mu^2 = b$ and $\lambda\mu = -\mu\lambda$. Then there exists a form q over D with discriminant $d(q) = v^2$, for some $v \in E^\times$ with $v_l < 0$ at only one infinite place l and Hilbert symbol $(a, v)_p = -1$ precisely at a nonempty set $\mathscr{S}$ of primes, with $|\mathscr{S}|$ even, where D is nonsplit. In fact, let $q = \langle \lambda \rangle \perp \langle a^{-1} v \lambda \rangle$ where $\lambda^2 = a$ (see [38, Chap. 10, p. 370]).

Then since $\ker(L_0^K(D) \to L_0^K(D_A))$ is represented by forms $\langle \lambda \rangle \perp \langle -u\lambda \rangle$ with Hilbert symbol $(a, u)_p = -1$ only at an even number of places where D is nonsplit, it follows that

$$[q] \in \ker(L_0^K(D) \to L_0^K(D_A)).$$

We choose an element $x \in \mathrm{CL}_1^K(D)$ which maps to $[q]$ under the boundary map, and call z the image of x in $H^1(C(E))$. It follows that z is represented by $\Pi\{\langle -1\rangle_p : p \in \mathscr{S}\}$, modulo indeterminacy at the primes where D splits, *cf.* (1.13).

Next, define a class $y \in H^1(C(E))$ by the idèle which agrees with v at finite primes and equals 1 at infinite primes. Note that y represents the image of $\langle -1\rangle_l \in H^1(\hat{E}_l^\times)$ since $y/[v]$ is non-trivial only at the prime l and $C(D) \cong C(E)$.

Then both z and y map to $[\bar{v}^2] \in H^0(K_1(D))$ and, hence, $y - z$ is the image of the element $w = \Pi\{\langle -1\rangle_p : p \in \mathscr{S}, p \neq l\}$ under the map $H^1(K_1(D_A)) \to H^1(C(E))$. As in the first part of the proof, it now follows that $\delta(y) = \delta(y - z) \neq 0$ since $|\mathscr{S} - \{l\}|$ is odd. □

LEMMA 1.18. *In type* OK, *the sequence*

$$0 \to \mathbf{Z}/2 \to \mathrm{CL}_0^K(E) \to H^0(C(E)) \to 0$$

is split $\Leftrightarrow -1 \in E^{\times 2}$.

Proof. At a finite prime l, by (1.12) the upper sequence in the diagram

$$\begin{array}{ccccccccc} 0 \to & L_0^S(\hat{E}_l, 1, 1) & \to & L_0^K(\hat{E}_l, 1, 1) & \to & H^0(\hat{E}_l^\times) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \mathrm{CL}_0^S(E) & \longrightarrow & \mathrm{CL}_0^K(E) & \to & H^0(C(E)) & \to 0 \end{array} \tag{1.19}$$

splits $\Leftrightarrow -1 \in \hat{E}_l^{\times 2}$. Since $L_0^S(\hat{E}_l, 1, 1) \to \mathrm{CL}_0^S(E)$ is an isomorphism for l finite, the lower sequence must be nonsplit whenever the upper sequence is. By the global square theorem, if $-1 \notin E^{\times 2}$ there is a finite prime l for which $-1 \notin \hat{E}_l^{\times 2}$ and the lower sequence is non-split.

Conversely, if $-1 \in E^{\times 2}$, then $H^0(E_A^\times) = H^0(\hat{E}^\times)$ and (1.15) implies that $H^0(\hat{E}^\times)$ maps onto $H^0(C(E))$. It follows from [44, Section 4] and the vanishing of L^S-groups of local rings of integers that $L_0^S(\hat{E})$ is the direct sum of the groups $L_0^S(\hat{E}_l, 1, 1)$ from (1.19). Hence the sequence $0 \to L_0^S(\hat{E}) \to L_0^K(\hat{E}) \to H^0(\hat{E}^\times) \to 0$ remains exact and so $L_0^K(\hat{E}, 1, 1)$ maps onto $\mathrm{CL}_0^K(E)$. However the upper exact sequence splits, so $\mathrm{CL}_0^K(E)$ has exponent 2.

In the following table, $C(E) = E_A^\times / E^\times$. The results are now summarized in Table I.

Table I

$\mathrm{CL}_i^K(D)$	Type OK	Type OD	Type U
$i = 3$	0	0	0
$i = 2$	0	0	$H^0(C(E))$
$i = 1$	$H^1(C(E))$	$\ker(\delta: H^1(C(E)) \to \mathbf{Z}/2)$	0
$i = 0$	$\mathbf{Z}/2 \tilde{\times} H^0(C(E))$	$H^0(C(E))$	$H^0(C(E))$

2. The Computation of the Relative Group

We are now ready to compute the map $\gamma_i(d)$ for each involution-invariant factor of $S(d)$. Such a factor is a matrix algebra over a skew field, and by Morita equivalence it suffices to study γ_i for and antistructure (D, α, u) on a skew field D. Its centre E is an Abelian extension of $\mathbf{Q}$ with ring of integers $A \subseteq E$. We fix an odd integer d such that $\hat{D}_l$ is *split*, and E_l is an *unramified* extension of $\hat{\mathbf{Q}}_l$ for all finite primes l with $l \nmid 2d$. We also assume that the Hasse invariants and Schur indices of D at all primes $l \in E$ over a fixed rational prime are equal.

First we consider type U where $H^0(C(E)) = \mathbf{Z}/2$ lies in the sequence

$$0 \to H^0(E^\times) \to H^0(E_A^\times) \to H^0(C(E)) \to 0.$$

At finite primes $L_i^K(\hat{A}_l) = L_i^K(\hat{A}_l/\mathrm{Rad}) = 0$, since the right-hand side is the sum of L^K-groups of finite fields. At the infinite places we have the signature group $L_{2i}^K(D_\infty)$. This is non-zero when D_∞ remains type U (a change to type **GL** is possible) and the fixed field $E_0 \subseteq E$ of the involution is real. We call this type U(C) and otherwise U(**GL**). In type U(C), each factor $2\mathbf{Z}$ maps surjectively to $H^0(C(E)) = \mathbf{Z}/2$ so $\operatorname{cok} \gamma_{2i} = 0$ and $\ker \gamma_{2i} = \Sigma$, where Σ is a subgroup of index 2 in a direct sum of factors $2\mathbf{Z}$, one for each complex place.

Next we consider type O. It is convenient to introduce the 'discriminant part' $\tilde{\gamma}_i$ of γ_i for a factor $(D, \alpha, u) = (E, 1, 1)$ of type OK to fit into the following commutative diagram:

$$\begin{array}{ccc} \prod_{l \nmid 2d} L_i^K(\hat{A}_l) \times L_i^K(E_\infty) & \xrightarrow{\gamma_i} & \mathrm{CL}_i^K(E) \\ \downarrow & & \downarrow \\ H^i(\hat{A}_{2d'}^\times) \times H^i(E_\infty^\times) & \xrightarrow{\tilde{\gamma}_i} & H^i(C(E)) \end{array} \tag{2.1}$$

where $\hat{A}_{2d'}^\times = \Pi_{l \nmid 2d} \hat{A}_l^\times$. Below we will also use the notation $\hat{A}_{2d}^\times = \Pi_{l \mid 2d} \hat{A}_l^\times$. Since $\tilde{\gamma}_i$ has the same kernel and cokernel as the map (see (1.14))

$$H^i(\hat{A}_{2d'}^\times) \times H^i(E_\infty^\times) \times H^i(E^\times) \to H^i(E_A^\times),$$

we are led to consider the following commutative diagram (for $i = 0$):

$$\begin{array}{ccccccccc} 0 \to & \ker \tilde{\gamma}_0 & \to & H^0(\hat{A}_{2d'}^\times) \times H^0(E_\infty^\times) \times H^0(E^\times) & \to & H^0(\hat{E}_A^\times) & \to & \operatorname{cok} \tilde{\gamma}_0 & \to 0 \\ & \downarrow & & \downarrow & & \| & & \downarrow & \\ 0 \to & E^{(2)}/E^{\times 2} & \to & H^0(\hat{A}^\times) \times H^0(E_\infty^\times) \times H^0(E^\times) & \to & H^0(\hat{E}_A^\times) & \to & H^0(\Gamma(E)) & \to 0 \\ & \downarrow & & \downarrow & & & & & \\ & H^0(\hat{A}_{2d}^\times) & = & H^0(\hat{A}_{2d}^\times) & & & & & \end{array} \tag{2.2}$$

Here $E^{(2)}$ denotes the elements of E with even valuation at all finite primes and $\Gamma(E)$ is the ideal class groups defined by

$$1 \to E^\times/A^\times \to \hat{E}^\times/\hat{A}^\times \to \Gamma(E) \to 1.$$

To obtain the middle sequence, add $H^0(\hat{A}_{2d}^{\times})$ to the domain of $\tilde{\gamma}_0$, then the map to $H^0(E_{\hat{A}}^{\times})$ has the same kernel and cokernel as $H^0(E^{\times}) \to H^0(I(E))$ where $I(E) = \hat{E}^{\times}/\hat{A}^{\times}$ is the ideal group of E.

From (2.2) we obtain the following exact sequence

$$0 \to \ker \tilde{\gamma}_0 \to E^{(2)}/E^{\times 2} \xrightarrow{\Phi} H^0(\hat{A}_{2d}^{\times}) \to \operatorname{cok} \tilde{\gamma}_0 \to H^0(\Gamma(E)) \to 0 \tag{2.3}$$

for the computation of $\tilde{\gamma}_0$ in type OK. In type OD when (D, α, u) is non-split at all infinite primes, the term $H^0(E_{\infty}^{\times})$ is missing from the top row of (2.2). This produces instead:

$$0 \to \ker \tilde{\gamma}_0 \to E^{(2)}/E^{\times 2} \xrightarrow{\Phi'} H^0(\hat{A}_{2d}^{\times}) \oplus H^0(E_{\infty}^{\times}) \to \operatorname{cok} \tilde{\gamma}_0 \to H^0(\Gamma(E)) \to 0 \tag{2.4}$$

For the map $\tilde{\gamma}_1$ in type OK a similar but easier analysis gives $\ker \tilde{\gamma}_1 = 0$ and an exact sequence

$$0 \to H^1(A^{\times}) \to H^1(\hat{A}_{2d}^{\times}) \to \operatorname{cok} \tilde{\gamma}_1 \to 0. \tag{2.5}$$

In type OD, nonsplit at infinite primes, $H^1(E_{\infty}^{\times})$ is added to the middle term.

It remains now to obtain $\ker \gamma_i$ and $\operatorname{cok} \gamma_i$ from the results above. We introduce subtypes:

Type O:
OK(**R**) if E has a real embedding,
OK(**C**) if E has no real embedding,
OD(**H**) if D is nonsplit at infinite primes,
OD(**R**) if D is split at infinite primes and E has a real embedding,
OD(**C**) if D is split at infinite primes and E has no real embedding.

Type U:
U(**C**) if D_{∞} has type U,
U(**GL**) if D_{∞} has type **GL**.

We remark that in type U(**C**) the centre field of D_{∞} at each infinite place is the complex numbers with complex conjugation as the induced involution. Type U(**GL**) algebras are isomorphic to matrix rings over $\mathbf{C} \times \mathbf{C}$ or $\mathbf{R} \times \mathbf{R}$, at each infinite place, with the induced involution interchanging the two factors of **C** or **R**.

In case OK(**R**), $\mathrm{CL}_0^K(E)$ is an extension of $H^0(C(E))$ by $\mathrm{CL}_0^S(E) = \mathbf{Z}/2$. This is in the image of $L_0^S(\mathbf{R}, 1, 1) \subseteq L_0^S(E_{\infty}, 1, 1)$ so does not appear in $\operatorname{cok} \gamma_0$. It follows that $\operatorname{cok} \gamma_0 = \operatorname{cok} \tilde{\gamma}_0$ and $\ker \gamma_0$ is an extension of $\ker \tilde{\gamma}_0$ by $\tilde{\Sigma}$, where $\tilde{\Sigma} = \ker(L_0^S(E_{\infty}, 1, 1) \to \mathbf{Z}/2)$ is a free Abelian group of rank equal to r_1, the number of real places of E. In general, the extension for $\ker \gamma_0$ is nonsplit. This implies that the signature divisibility in the corresponding summand Σ of $L_1^{p,h}(R \to \hat{R}_2)$ differs from that in $\tilde{\Sigma}$, which is a subgroup of index 2 in a direct sum $(4\mathbf{Z})^{r_1}$.

In case OK(**C**), $L_0^K(E_{\infty}, 1, 1) = 0$ and we will determine in (2.8) when the $\mathbf{Z}/2$ in $\mathrm{CL}_0^K(E)$ can be hit from $\ker \tilde{\gamma}_0$ in the 'snake' homomorphism associated to (2.1).

Let $\Theta_E: E^{(2)}/E^{\times 2} \to H^0(E_\infty^\times)$ denote the reduction map at the infinite places, and r_E be the 2-rank of image $(\Phi_E | \ker \Phi)$. Define

$$\Phi'': \ker \Theta_E \to H^0(\hat{A}_{2d}^\times) \tag{2.6}$$

to be the restriction of Φ to $\ker \Theta_E$. By comparison with (2.4) we see that $\ker \Phi'' \cong \ker \Phi'$. In addition, there is an exact sequence

$$0 \to \ker \Phi' \to \ker \Phi \to \operatorname{im} \Theta_E \to \operatorname{cok} \Phi'' \to \operatorname{cok} \Phi \to 0, \tag{2.7}$$

where the middle map is induced by $\Theta_E | \ker \Phi$, which will allow us to compute the 2-ranks of these groups in terms of fundamental invariants of E such as class groups and $\operatorname{im} \Theta_E$.

LEMMA 2.8. *In type* OK(**R**), $\ker \gamma_0 = \Sigma \oplus \ker \Phi'$ *and* $\operatorname{cok} \gamma_0 = \operatorname{cok} \Phi \oplus H^0(\Gamma(E))$. *The signature group* $\Sigma \cong (8\mathbf{Z}) \oplus (4\mathbf{Z})^{r_1 - r_E - 1} \oplus (2\mathbf{Z})^{r_E}$.

In type OK(**C**), $\ker \gamma_0 = \ker \Phi$ *and*

$$\operatorname{cok} \gamma_0 = \mathbf{Z}/2 \tilde{\times} (\operatorname{cok} \Phi \oplus H^0(\Gamma(E)).$$

The extension is split $\Leftrightarrow -1 \in E^{\times 2}$.

Proof. First we consider type OK(**C**), via the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathbf{Z}/2 & = & \mathbf{Z}/2 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 \to & \ker \gamma_0 & \to & H^0(\hat{A}_{2d'}^\times) & \xrightarrow{\gamma_0} & \mathbf{Z}/2 \tilde{\times} H^0(C(E)) & \longrightarrow & \operatorname{cok} \gamma_0 & \to 0 \\
 & \downarrow & & \| & & \downarrow & & \downarrow & \\
0 \to & \ker \tilde{\gamma}_0 & \to & H^0(\hat{A}_{2d'}^\times) & \xrightarrow{\tilde{\gamma}_0} & H^0(C(E)) & \longrightarrow & \operatorname{cok} \tilde{\gamma}_0 & \to 0 \\
 & & & & & \downarrow & & & \\
 & & & & & 0 & & &
\end{array}$$

leading to the 'snake' sequence,

$$0 \to \ker \gamma_0 \to \ker \tilde{\gamma}_0 \xrightarrow{\partial} \mathbf{Z}/2 \to \operatorname{cok} \gamma_0 \to \operatorname{cok} \tilde{\gamma}_0 \to 0.$$

From (2.3), $\operatorname{cok} \tilde{\gamma}_0 = \operatorname{cok} \Phi \oplus H^0(\Gamma(E))$ and $\ker \tilde{\gamma}_0 = \ker \Phi$, so it remains to check that $\partial = 0$.

Choose an element $\langle \alpha \rangle = \Pi_{l \nmid 2d} \langle \alpha_l \rangle$ in $\ker \tilde{\gamma}_0$. By (2.3) there exists $a \in E^\times$ such that

$$\begin{array}{lll} a^{-1}\alpha_l \in \hat{E}_l^{\times 2} & \text{if } l \nmid 2d, & l \text{ finite,} \\ a \in \hat{E}_l^{\times 2} & \text{if } l \mid 2d, & \text{or } l \text{ infinite.} \end{array} \tag{2.9}$$

Extend the definition of $\langle \alpha \rangle$ by setting $\alpha_l = 1$ if $l \mid 2d$ or l infinite and consider the quadratic form

$$q = \langle a^{-1} \rangle \perp \langle -1 \rangle \perp \langle \alpha \rangle \perp \langle -1 \rangle$$

representing an element in $L_0^K(\hat{E}_A, 1, 1)$. Since q has discriminant 1, for each prime l we can compute $\kappa_l(q) = S_{\hat{E}_l^\times}(q) S_{\hat{E}_l^\times}(2\mathbb{H})$ using the properties of the Hilbert symbol [31, 63B] to determine whether $\partial\langle a\rangle$ is nonzero. If $l \nmid 2d$,

$$\kappa_l(q) = \left(\frac{a^{-1}, -1}{l}\right)\left(\frac{\alpha, -1}{l}\right)\left(\frac{-a^{-1}, -\alpha}{l}\right)\left(\frac{-1, -1}{l}\right) = \left(\frac{a, -1}{l}\right).$$

If $l \mid 2d$ or l infinite, $q = \mathbb{H} \oplus \mathbb{H}$ so $\kappa_l(q) = 1$. Therefore, under the reciprocity map

$$\kappa\colon L_0^S(\hat{E}_A, 1, 1) \to \mathrm{CL}_0^S(E) \cong \mathbf{Z}/2$$

given by $\kappa(q) = \Pi_l\, \kappa_l(q)$, we get

$$\kappa(q) = \prod_l \left(\frac{a, -1}{l}\right) = +1$$

since $a \in E^\times$.

Next we consider type OK(**R**) by using a similar diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \tilde{\Sigma} & \longrightarrow & \bigoplus_{l\mid\infty} 4\mathbf{Z} & \longrightarrow & \mathbf{Z}/2 & \longrightarrow 0 & & \\
& \downarrow & & \downarrow & & \downarrow & & & \\
0 \to & \ker\gamma_0 & \longrightarrow & H^0(\hat{A}_{2d'}^\times) \oplus \bigoplus_{l\mid\infty} 2\mathbf{Z} & \xrightarrow{\gamma_0} & \mathrm{CL}_0^K(E) & \to & \operatorname{cok}\gamma_0 & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \| & \\
0 \to & \ker\tilde{\gamma}_0 & \to & H^0(\hat{A}_{2d'}^\times) \oplus H^0(E_\infty^\times) & \xrightarrow{\tilde{\gamma}_0} & H^0(C(E)) & \to & \operatorname{cok}\tilde{\gamma}_0 & \to 0 \\
& & & & & \downarrow & & & \\
& & & & & 0 & & &
\end{array}$$

Clearly the torsion subgroup of $\ker\gamma_0$ injects into $\ker\Phi' \subseteq \ker\Phi = \ker\tilde{\gamma}_0$. Conversely, if $\langle\alpha\rangle \in \ker\Phi'$, the same calculation of Hasse symbols as above shows that $\langle\alpha\rangle$ is the image of a torsion element of $\ker\gamma_0$.

We also observe that if $x \in \tilde{\Sigma}$ is an indivisible element such that $x = 2y$, $y \in \ker\gamma_0$, then the image of y in $H^0(E_\infty^\times)$ is nontrivial. It follows that a generator of the factor $8\mathbf{Z}$ is not divisible in $\ker\gamma_0$, and the remaining divisibility assertions are clear. □

For type OD the discussion must be modified to take the situation at infinite primes into account in (2.2), but $\mathrm{CL}_0^K(D) = H^0(C(E))$ so the corresponding result to (2.8) is not as elaborate.

LEMMA 2.10. *In type* OD(**H**),

$$\ker\gamma_0 = \ker\Phi', \qquad \operatorname{cok}\gamma_0 = \operatorname{cok}\Phi' \oplus H^0(\Gamma(E)), \quad \text{and} \quad \ker\gamma_2 = \Sigma.$$

The divisibility of the signature at each infinite place is $2\mathbf{Z}$. *In type* OD($\mathbf{R}$), $\ker \gamma_0 = \Sigma \oplus \ker \Phi'$ *and* $\operatorname{cok} \gamma_0 = \operatorname{cok} \Phi \oplus H^0(\Gamma(E))$. *The signature group* $\Sigma \cong (4\mathbf{Z})^{r_1(E)-r_E} \oplus (2\mathbf{Z})^{r_E}$. *In type* OD($\mathbf{C}$), $\ker \gamma_0 = \ker \Phi'$ *and* $\operatorname{cok} \gamma_0 = \operatorname{cok} \Phi \oplus H^0(\Gamma(E))$.

We now carry out a similar procedure to analyse $\ker \gamma_1$ and $\operatorname{cok} \gamma_1$, starting from (2.1) and (2.5). Note that in type OK,

$$\mathrm{CL}_1^K(E) \cong H^1(C(E))$$

so that $\gamma_1 = \tilde{\gamma}_1$. In type OD there is a (split) exact sequence

$$0 \to \mathrm{CL}_1^K(D) \to H^1(C(E)) \xrightarrow{\delta} \mathbf{Z}/2 \to 0.$$

In type OD($\mathbf{H}$), the diagram

$$\begin{array}{ccccccccc}
 & & H^1(E^\times) & =\!= & H^1(E^\times) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 \to & \ker \tilde{\tilde{\gamma}}_1 \to & H^1(\hat{A}^\times_{2d'}) \oplus H^1(E^\times) & \xrightarrow{\tilde{\tilde{\gamma}}_1} & H^1(E^\times_{A}) & \to & \operatorname{cok} \tilde{\tilde{\gamma}}_1 & \to 0 \\
 & \| & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \ker \tilde{\gamma}_1 \longrightarrow & H^1(\hat{A}^\times_{2d'}) & \xrightarrow{\tilde{\gamma}_1} & H^1(C(E)) & \to & \operatorname{cok} \tilde{\gamma}_1 & \to 0 \\
 & & \downarrow & & \downarrow & & & \\
 & & 0 & & 0 & & &
\end{array} \tag{2.11}$$

shows that $\tilde{\gamma}_1$ has the same kernel and cokernel as the map $\tilde{\tilde{\gamma}}_1$ (this has already been used in the derivation of (2.5) for type OD($\mathbf{H}$)). But $\tilde{\gamma}_1$ is related to γ_1 by the diagram,

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
0 \to \ker \gamma_1 & \to & H^1(\hat{A}^\times_{2d'}) & \xrightarrow{\gamma_1} & \mathrm{CL}_1^K(D) & \to & \operatorname{cok} \gamma_1 \to 0 \\
\| & & \| & & \downarrow & & \downarrow \\
0 \to \ker \tilde{\gamma}_1 & \to & H^1(\hat{A}^\times_{2d'}) & \xrightarrow{\tilde{\gamma}_1} & H^1(C(E)) & \to & \operatorname{cok} \tilde{\gamma}_1 \to 0 \\
 & & & & {\scriptstyle\delta}\downarrow & & \downarrow \\
 & & & & \mathbf{Z}/2 & = & \mathbf{Z}/2
\end{array}$$

It follows that $\ker \gamma_1 = 0$ and in type OD($\mathbf{H}$) (after substituting the formula (2.5) for $\operatorname{cok} \tilde{\gamma}_1$),

$$0 \to \operatorname{cok} \gamma_1 \to \frac{H^1(\hat{A}^\times_{2d}) \oplus H^1(E^\times_\infty)}{H^1(A^\times)} \xrightarrow{\Delta'} \mathbf{Z}/2 \to 0 \tag{2.12}$$

The map Δ' is defined by

$$\Delta'(\langle -1 \rangle_l) = \begin{cases} -1, & \hat{D}_l \text{ nonsplit,} \\ +1, & \hat{D}_l \text{ split.} \end{cases}$$

This definition arises from (1.17) and (2.11). Note that in types OD(**R**) or OD(**C**), $H^1(E_\infty)$ is not present in (2.12) and

$$0 \to \operatorname{cok} \gamma_1 \to \frac{H^1(\hat{A}_{2d}^\times)}{H^1(A^\times)} \xrightarrow{\Delta} \mathbf{Z}/2 \to 0 \tag{2.13}$$

describes $\operatorname{cok} \gamma_1$ (the map Δ is defined by the same formula as for Δ').

The results are summarized in Table II (where any groups not displayed are zero!). Recall that for a simple factor of type Sp, the groups $\operatorname{cok} \gamma_i$ and $\ker \gamma_i$ are equal to $\operatorname{cok} \gamma_{i+2}$ and $\ker \gamma_{i+2}$ respectively. In the table within the type OK section, the subcases are for types OK(**R**) ad OK(**C**), respectively. A similar convention applies to the other sections. The extensions in type OK for $i = 0$ are split if and only if $-1 \in E^{\times 2}$. The symbol Σ denotes the subgroup in $\ker \gamma_i$ detected by signatures. The rest of the notation in the table was defined earlier: $C(E)$ just before Table I in Section 1, $\Gamma(E)$ just after diagram (2.2), Φ, Φ' in (2.3), (2.4), and Δ, Δ' in (2.12), (2.13).

Remark 2.14. In the exact sequence (1.8)

$$0 \to \operatorname{cok} \gamma_i(d) \to L_i^{p,h}(R \to \hat{R}_2)(d) \to \ker \gamma_{i-1}(d) \to 0$$

decomposes as a direct sum over the simple involution–invariant factors of $S(d)$, so Table II determines $L_i^{p,h}(R \to \hat{R}_2)(d)$ up to extension. But we will show in (Proposition 4.7) that (1.8) is actually split, so that we have

$$L_i^{p,h}(R \to \hat{R}_2)(d) \cong \operatorname{cok} \gamma_i(d) \oplus \ker \gamma_{i-1}(d).$$

Furthermore, it follows from Table II that the torsion subgroups of these relative groups have exponent four at most, and have exponent two if no type OK(**C**) factors

Table II

	$\Pi_{l \nmid 2d} L_i^K(\hat{A}_l) \times L_i^K(E_\infty)$	$\mathrm{CL}_i^K(E)$	$\ker \gamma_i^K$	$\operatorname{cok} \gamma_i^K$
Type OK$\binom{\mathbf{R}}{\mathbf{C}}$				
$i = 0$	$H^0(\hat{A}_{2d'}^\times) \times \begin{cases} \oplus 2\mathbf{Z} \\ 0 \end{cases}$	$\mathbf{Z}/2 \tilde{\times} H^0(C(E))$	$\begin{cases} \Sigma \oplus \ker \Phi' \\ \ker \Phi \end{cases}$	$\begin{cases} \operatorname{cok} \Phi \oplus H^0(\Gamma(E)) \\ \mathbf{Z}/2 \tilde{\times} (\operatorname{cok} \Phi \oplus H^0(\Gamma(E))) \end{cases}$
$i = 1$	$H^1(\hat{A}_{2d'}^\times) \times H^1(E_\infty)$	$H^1(C(E))$	0	$H^1(\hat{A}_{2d}^\times)/H^1(A^\times)$
Type OD(**H**)				
$i = 0$	$H^0(\hat{A}_{2d'}^\times) \times 0$	$H^0(C(E))$	$\ker \Phi'$	$\operatorname{cok} \Phi' \oplus H^0(\Gamma(E))$
$i = 1$	$H^1(\hat{A}_{2d'}^\times) \times 0$	$\ker \delta$	0	$\ker \Delta'$
$i = 2$	$0 \times \oplus 2\mathbf{Z}$	0	Σ	0
Type OD$\binom{\mathbf{R}}{\mathbf{C}}$				
$i = 0$	$H^0(\hat{A}_{2d'}^\times) \times \begin{cases} \oplus 2\mathbf{Z} \\ 0 \end{cases}$	$H^0(C(E))$	$\begin{cases} \Sigma \oplus \ker \Phi' \\ \ker \Phi \end{cases}$	$\operatorname{cok} \Phi \oplus H^0(\Gamma(E))$
$i = 1$	$H^1(\hat{A}_{2d'}^\times) \times H^1(E_\infty)$	$\ker \delta$	0	$\ker \Delta$
Type U$\binom{\mathbf{C}}{\mathrm{GL}}$				
$i = 0, 2$	$\begin{cases} 0 \times \oplus 2\mathbf{Z} \\ 0 \times 0 \end{cases}$	$H^0(C(E))$	$\begin{cases} \Sigma \\ 0 \end{cases}$	$\begin{cases} 0 \\ \mathbf{Z}/2 \end{cases}$

are present. Such factors do not occur, for example, in the standard oriented antistructure on $\mathbf{Z}G$, for any finite group G, and hence the projective obstruction groups $L_*^p(\mathbf{Z}G)$ for *oriented* surgery problems have torsion of exponent $\leqslant 4$. In Section 5 we will show that the torsion subgroup actually has exponent two in this case, and give the precise answer for any geometric anti-structure.

We now recall some notation from number theory, needed to compute some of the entries in Table II. Let $\Gamma(E, d) = \tilde{K}_0(A[1/d])$ denote the ray class group of the field E, and $\Gamma^*(E, d)$ the strict ray class group. These groups are usually described by the sequences

$$0 \to A[1/d]^\times \to E^\times \to \prod_{l \nmid d} \hat{E}_l^\times/\hat{A}_l^\times \to \Gamma(E, d) \to 0,$$

or

$$0 \to A[1/d]^* \to E^* \to \prod_{l \nmid d} \hat{E}_l^\times/\hat{A}_l^\times \to \Gamma^*(E, d) \to 0.$$

Here $E^* \subseteq E^\times$ is the subgroup of elements which are positive at all real places. There is an exact sequence (assuming that E has trivial involution)

$$0 \to H^0(A[1/d]^\times) \to E_{d'}^{(2)}/E^{\times 2} \to H^1(\Gamma(E, d)) \to 0 \tag{2.15}$$

where $E_{d'}^{(2)} \subseteq E^\times$ is the subgroup of elements with even valuation at all primes not dividing d. The map into $H^1(\Gamma(E, d))$ sends $e \in E_{d'}^{(2)}/E^{\times 2}$ to $\Sigma_{l \nmid d} \frac{1}{2} v_l(e) \cdot l$.

Let $\gamma(E, d)$ (resp. $\gamma^*(E, d)$) denote the 2-rank of the (strict) class group of $A[1/d]$. The 2-ranks of $\Gamma(E)$, $\Gamma^*(E)$ are denoted γ and γ^* respectively. In general it is a well-known and difficult problem in number theory to compute these ranks. We will now see that they appear as the ranks of certain groups in Table II, and hence the final computation of the relative L-groups is reduced to number theory (compare [4] for related calculations).

Let g_p denote the number of primes in A over the rational prime p and $g_d(E) = \Sigma_{p \mid d} g_p$. Then $g_d(E)$ is the rank of $H^0(\hat{E}_d^\times/\hat{A}_d^\times)$. Let r_1, r_2 denote the number of real and complex places of E, so that $[E:\mathbf{Q}] = r_1 + 2r_2$.

The 2-ranks of the kernel and cokernel of the maps

$$\Phi_E\colon E^{(2)}/E^{\times 2} \to H^0(\hat{A}_{2d}^\times), \tag{2.16}$$

and

$$\Phi_E'\colon E^{(2)}/E^{\times 2} \to H^0(\hat{A}_{2d}^\times) \oplus H^0(E_\infty^\times) \tag{2.17}$$

from (2.3) and (2.4) can now be given in terms of more familiar invariants. Recall that in (2.17) the field E is totally real. The 2-rank of $H^0(\hat{A}_{2d}^\times)$ is just $g_{2d} + [E:\mathbf{Q}] = g_{2d} + r_1 + 2r_2$.

PROPOSITION 2.18.

(i) *The 2-rank of* $\ker \Phi_E$ *(resp.* $\ker \Phi'_E$*) is* $\gamma^*(E, 2d)$ *(resp.* $\gamma(E, 2d)$*).*
(ii) *The 2-rank of* $\operatorname{cok} \Phi_E$ *(resp.* $\operatorname{cok} \Phi'_E$*) is* $g_{2d}(E) + r_2 + \gamma^*(E, 2d)) - \gamma(E)$ *(resp.* $g_{2d}(E) + r_1 + \gamma(E, 2d)) - \gamma(E)$*).*
(iii) *The 2-rank* r_E *of* $\operatorname{im}(\Theta_E | \ker \Phi_E)$ *is* $\gamma^*(E, 2d) - \gamma(E, 2d)$. *If* $r_1 \neq 0$ *then* $r_E \leqslant (r_1 - 1)$.

Remark 2.19. *If d is an odd prime and* $E \subset \mathbf{Q}(\zeta_d)$ *is a subfield such that* $\langle 2 \rangle$ *generates the Galois group of E over* $\mathbf{Q}$, *then* $\gamma(E, 2d) = \gamma(E, d) = \gamma(E)$.

Proof. We have just collected some of the results described in more detail in [45, Section 4.6]. Since $E/\mathbf{Q}$ is a Galois extension either r_1 or r_2 is zero, and for Φ' note that $r_2 = 0$. Now by (2.15) and the Dirichlet unit theorem, the 2-rank of $E^{(2)}/E^{\times 2}$ is $(r_1 + r_2 + \gamma(E))$. Then the rank of $\ker \Phi$ (resp. $\ker \Phi'$) is $\gamma^*(E, 2d)$ (resp. $\gamma(E, 2d)$) by [45, 4.6.1], and (i) and (ii) follow directly. For (iii), note that the definition of Θ_E and (2.6) imply that $\ker \Phi' = \ker \Phi''$. If $r_E = r_1$, then $r_1 + \gamma(A, 2d) = \gamma^*(A, 2d)$ and $\operatorname{cok} \Phi' = \operatorname{cok} \Phi$. From the exact sequence

$$\cdots \to \prod_{l|\infty} \{\pm 1\} \to \Gamma^*(E, 2d) \to \Gamma(E, 2d) \to 0$$

we would conclude that the map from the group of signs at the infinite places maps injectively into the strict ray class group. However, this map factors through the map $\Pi\{\pm 1\} \to \Gamma^*(E)$ and this always has a non-trivial kernel (from the global unit $-1 \in A^\times$). □

An Example: Finite 2-Groups

Here complete calculations already appear in [20, Section 3, p. 80]. To check our results against the tables there note that $\Gamma(E)$ and $\Gamma^*(E)$ have odd order (Weber's Theorem) for all the centre fields appearing in $\mathbf{Q}G$ and $g_2(E) = 1$. Hence, Φ and Φ' are injective with $\operatorname{cok} \Phi_E$ of 2-rank $1 + r_2$ (resp. $\operatorname{cok} \Phi'_E$ of 2-rank $1 + r_1$). As above, the degree $[E, \mathbf{Q}] = r_1 + 2r_2$, where r_1 denotes the number of real places of E and r_2 the number of complex places.

In [20] the basic antistructures on the simple components of $\mathbf{Q}\pi$ are labelled Γ_N, F_N, R_N, H_N, UI and UII. These have type OK(**C**), OK(**C**), OK(**R**), OD(**H**), U(**C**) and U(**GL**) respectively in our notation. In our tables, the distinction between Γ_N and F_N is whether $-1 \in E^{\times 2}$. Let ζ_N denote a primitive 2^Nth root of unity. The centres E for the type O factors are $\mathbf{Q}(\zeta_{N+1})$, $\mathbf{Q}(\zeta_{N+2} - \zeta_{N+2}^{-1})$, $\mathbf{Q}(\zeta_{N+2} + \zeta_{N+2}^{-1})$, $\mathbf{Q}(\zeta_N + \zeta_N^{-1})$ so that (r_1, r_2) equals $(0, 2^{N-1})$, $(0, 2^{N-1})$, $(2^N, 0)$, $(2^{N-2}, 0)$, respectively. Therefore using Table II and the exact sequence (1.8), for the contribution of the components to $L_i^K(R \to \hat{R}_2)$ we get Table III.

Table III

	OK(C), $-1 \in E^{\times 2}$	OK(C), $-1 \notin E^{\times 2}$	OK(R)	OD(H)	U(C)	U(GL)
$i=3$	0	0	0	Σ	Σ	0
$i=2$	0	0	0	0	0	$\mathbf{Z}/2$
$i=1$	0	0	Σ	$(\mathbf{Z}/2)^{r_1-1}$	Σ	0
$i=0$	$\mathbf{Z}/2 \oplus (\mathbf{Z}/2)^{r_2+1}$	$\mathbf{Z}/4 \oplus (\mathbf{Z}/2)^{r_2}$	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^{r_1+1}$	0	$\mathbf{Z}/2$

Notice that $\ker \gamma_{i-1}$ contributed only the signature group Σ (free Abelian of rank equal to the number of real places in the fixed field of the involution), so no extension problems arise. Also from (2.5), $\operatorname{cok} \gamma_1 = 0$ in type OK because $d = 1$ and there is only one prime over 2. However from (2.12), in type OD(H), the group $\operatorname{cok} \gamma_1$ is nontrivial in general with 2-rank $(r_1 - 1)$. In [20] the groups L^p-groups for $\mathbf{Z}G$ were completely determined, for G a finite 2-group with any geometric anti-structure. To carry this out for more general groups G we need to compute the maps ψ_i: $L_i^h(\hat{\mathbf{Z}}_2 G) \to L_i^{p,h}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2 G)$ and settle some extension questions. This will be done in the next three sections.

3. The Proof of Theorem A

Our starting point is the main result of [16, 1.16] which computes the map

$$\Psi_i: L_i^h(\hat{\mathbf{Z}}_2 G) \to L_i^h(\hat{\mathbf{Q}}_2 G). \tag{3.1}$$

We are assuming that $R = \mathbf{Z}G$ has an oriented or nonoriented geometric antistructure (R, β, b) given by $\beta(g) = w(g)\theta(g^{-1})$, where θ is an automorphism of $G = \mathbf{Z}/d \rtimes \sigma$ with $\theta^2(g) = bgb^{-1}$, $b \in G$ and $w: G \to \{\pm 1\}$ is an orientation character (see [16, 1.4] or [20, p. 110] for the relations). If $T \in \mathbf{Z}/d$ denotes a generator, then θ induces an automorphism of $\mathbf{Z}/d$ given by $\theta(T) = T^{\vartheta}$ (this corrects a misprint in the formula in [16, p. 148, *l*.-9]). To complete the proof of Theorem A we will later specialize to the case when θ is the identity as usual in surgery theory.

In [16, (1.7), (2.4)–(2.7)] we showed that whenever the domain $L_i^h(\hat{R}_2, \beta, b)$ of Ψ_i is nonzero, there exists a global 'scaling' isomorphism so that the scaled anti-structure (R, β_0, b_0) has $b_0 \in \sigma$. Then by Remark 1.3 it follows that the d-components are defined and that the splitting of Theorem 1.2 applies to our situation. In particular, since the map Ψ_i splits as in (1.2), it is enough to give the answer for the d-component.

Recall that if $\sigma_1 = \ker(t: \sigma \to (\mathbf{Z}/d)^{\times})$, then any irreducible complex character of G is induced up from $\chi \otimes \xi$ on $\mathbf{Z}/d \times \sigma_1$ where χ is a linear character of $\mathbf{Z}/d$ and ξ is an irreducible character of σ_1. The representations in the semisimple algebra $S(d)$ are the ones involving faithful linear characters χ of $\mathbf{Z}/d$. These representations are divided into types O, Sp and U, and we say that the *order* of a linear character ξ is the order of its image $\xi(\sigma_1)$. Let $S(d, \xi)$ denote the simple factor of $S(d)$ associated to an involution-invariant character $(\chi \otimes \xi)^*$, induced up from $\chi \otimes \xi$.

THEOREM 3.2. ([16, 1.16]) *If $d>1$ and there is no element $g_0 \in \sigma$ satisfying $t(g_0) = -\vartheta^{-1}$, then $L_i^K(\hat{R}_2(d), \beta, b) = 0$. Otherwise if $d>1$ pick such a g_0 (or when $d=1$ set $g_0 = 1$), and let $m = i + (1 - w(g_0))$. For each irreducible complex character ξ of σ_1 the composite*

$$L_i^K(\hat{R}_2(d), \beta, b) \xrightarrow{\Psi_i(d)} L_i^K(\hat{S}_2, \beta, b)(d) \xrightarrow{proj.} L_i^K(\hat{S}_2(d, \xi), \beta, b)$$

is injective or zero and detected by the discriminant. It is injective if and only if the character ξ is

(a) *linear type O (and $m \equiv 0$ or 1 (mod 4))*
(b) *linear type Sp (and $m \equiv 2$ or 3 (mod 4))*
(c) *linear type U (and m even), order 2^l and $\xi(b_0^{2^{l-1}}) = -1$. Here the types refer to the antistructure $(\hat{\mathbf{Q}}_2[\sigma_1], \beta_0, b_0)$, with $\beta_0(a) = g_0\beta(a)g_0^{-1}$ and $b_0 = g_0\beta(g_0^{-1})bw(g_0) \in \sigma_1$.*

Remark. A type I linear character ξ has type O (resp. Sp) if $\xi(b_0) = 1$ (resp. $\xi(b_0) = -1$). If σ_1 has a linear character ξ of type 3.2(c), then (by projecting onto the $\mathbf{Z}/2$ quotient of $\xi(\sigma_1)$) it also has linear characters of type O and Sp.

The next step is to use this calculation to obtain information about the map $\psi_i: L_i^h(\hat{\mathbf{Z}}_2G) \to L_i^{p,h}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2G)$ in (1.4). Notice that the d-component of this map factors through the natural map $L_i^K(\hat{\mathbf{Q}}_2G)(d) \to \operatorname{cok} \gamma_i(d)$. The composite of $\Psi_i(d)$ with this natural map is denoted $\bar{\psi}_i(d)$. We remark that the subtype in (3.2)(c) is U(**C**), and hence the image of $\psi_i(d)$ projected into one of these factors is zero by Table II.

COROLLARY 3.3. *Let $G = \mathbf{Z}/m \rtimes \sigma$ a 2-hyperelementary group, with a geometric antistructure. For any $d \mid m$ the map $\psi_i(d): L_i^h(\hat{\mathbf{Z}}_2G)(d) \to L_i^{p,h}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2G)(d)$ is detected by the projection, $G \to \bar{G} = \mathbf{Z}/m \rtimes \sigma/[\sigma_1, \sigma_1]$.*

In this situation, we can finish the proof of Theorem A by using the relative detection theorem for $L_i^{p,h}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2G)(d)$ given in [21, 1.B.7] to obtain an absolute detection theorem for $L_i^p(\mathbf{Z}G)(d)$. Although Theorem A is stated only for the standard oriented antistructure, the same technique gives a version for the nonoriented case as well.

To carry this out, we will apply another result of [21] to the functors

$$F_1(G, w) = L_{i+1}^{p,h}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2G)(d) \quad \text{and} \quad F_2(G, w) = L_i^p(\mathbf{Z}G)(d)$$

for d fixed, and $\partial: F_1(G, w) \to F_2(G, w)$ is the boundary map in the long exact sequence of L-groups.

The result is stated in the following setting. Let F_1 be an additive functor defined on $(\mathbf{Z}[1/m]G, w)$-Morita (see [21, 1.B.3]) into an Abelian category $\mathscr{A}$. Let F_2 be a functor defined on $(\mathbf{Z}G, w)$-Morita into $\mathscr{A}$. Consider F_1 also to be defined on $(\mathbf{Z}G, w)$-Morita, and let $\partial: F_1 \to F_2$ be a natural transformation. Let $N \lhd H$ be subgroups of G with $N \subset \ker w$. Fix a w-invariant unital representation η of H/N (see

[21, 4.A.2] for the definition). We say that the triple (H, N, η) is *∂-good* if the composite

$$\ker \partial \subseteq F_1(\mathbf{Z}[1/2d][H_\rho/N_\rho], w) \xrightarrow{[e_\tau]} F_1(\mathbf{Z}[1/2d][H_\rho/N_\rho], w)$$

is injective, where τ is the maximal unital representation with $\tau < \eta$.

THEOREM 3.4 ([21, 6.2]). *Fix a p-hyperelementary group G and let $m = |G|$. Let K denote a normal subgroup of G with $K \subset \ker w$, and let $\bar{\pi}\colon G \to G/K$ be the projection. Let $\mathcal{S}$ be a complete set of unital representations of G, each of which is w-invariant. Suppose there is one representation, $\rho_K \in \mathcal{S}$, which contains precisely the irreducible* **Q**-*representations of G whose kernels contain K. For every other $\rho \in \mathcal{S}$ suppose given a subquotient $N_\rho \lhd H_\rho$ and a unital representation $\eta = \eta_\rho$ such that ρ is imprimitively induced from η. Finally, suppose that for each $\rho \neq \rho_K$, the triple $(H_\rho, N_\rho, \eta_\rho)$ is ∂-good.*

Consider the commutative square

$$\begin{array}{ccc} F_1(G, w) & \xrightarrow{d_1} & F_1(G/K, w) \oplus \bigoplus_{\mathcal{S}} F_1(H_\rho/N_\rho, w) \\ \partial\big\downarrow & & \big\downarrow \\ F_2(G, w) & \xrightarrow{d_2} & F_2(G/K, w) \oplus \bigoplus_{\mathcal{S}} F_2(H_\rho/N_\rho, w) \end{array}$$

If $\pi\colon \ker(F_1(G, w) \to F_2(G, w)) \to \ker(F_1(G/K, w) \to F_2(G/K, w))$ is onto, then $d_2|_{\operatorname{Im}\partial}$ is one to one.

The Proof of Theorem A: Let the subgroup K referred to in the statement of (3.4) be $K = [\sigma_1, \sigma_1]$ and let ρ_K denote the sum of all the irreducible **Q**-representations of G, faithful on $\mathbf{Z}/d$, whose kernels contain K. For every other representation ρ in a complete set of w-invariant unitals $\mathcal{S}$ we must find a subquotient $N_\rho \lhd H_\rho$ and a unital representation η_ρ such that ρ is imprimitively induced from η_ρ and the triple $(H_\rho, N_\rho, \eta_\rho)$ is ∂-good.

For each $\rho \in \mathcal{S}$ whose kernel does not contain K we choose an w-basic subquotient H/N for which the faithful representation induces imprimitively to ρ. If ρ is induced from a proper subquotient with cyclic $\ker t$, it is ∂-good unless $\ker t$ is order 2, or in type O of order 4. In these exceptional cases, we can choose ∂-good subquotients with $\ker t$ dihedral of order 8 or the group M_{16} respectively (see [21, 1.C.8] and the discussion at the end of [21, Section 6]). The remaining condition of [21, 6.2] is satisfied since the map $L_i^h(\hat{\mathbf{Z}}_2 G) \to L_i^h(\hat{\mathbf{Z}}_2 \bar{G})$ is an isomorphism. □

There is a version of Theorem 3.4 for functors out of $(\mathbf{Z}[1/m]G, \theta, w, b)$-Morita: the relevant representation theory is done in [21, 3.C.4]. This leads to another generalization of Theorem A for arbitrary geometric antistructures, by using [21, 3.C.1, 4.C.4] to identify the Witt-basic groups and give a detection theorem for the relative groups. The details are very similar and the proof will be left to the reader. The statement of [21, 4.C.4] uses the concept of an 'oriented geometric antistructure' $(\theta, w, b, \varepsilon)$ on a group ring RG. Here the word 'oriented', defined on [21, p. 256], refers to a certain choice of unit ε in the coefficient ring and *not* to a property of the orientation character $w: G \to \{\pm 1\}$. To avoid confusion, we will always take $\varepsilon = +1$ and suppress this terminology. By a *Witt-basic subquotient* of $(\mathbf{Z}G, \theta, w, b)$ we mean an antistructure $(\mathbf{Z}[H/N], \theta', w', u')$ where H/N is a subquotient of G such that (i) $N \in \ker w$ with w' the induced map, (ii) there exists a $c \in G$ such that H, N are invariant under $\theta' = c \circ \theta \circ c^{-1}$, and (iii) the unit $u' = w(c)b'$ where $b' = c\theta(c)b \in H$. Associated to a Witt-basic subquotient, there is a twisted restriction map [21, p. 261] on L-theory.

THEOREM 3.5. *Let G be a 2-hyperelementary group with a geometric antistructure (θ, w, b), and let $\beta(g) = w(g)\theta(g^{-1})$. Then the sum of all the twisted restriction maps*

$$L_n^p(\mathbf{Z}G, \beta, b) \to L_n^p(\mathbf{Z}[\bar{G}], \bar{\beta}, \bar{b}) \oplus \sum \{L_n^p(\mathbf{Z}[H/N], \beta', u')\}$$

is a natural (split) injection, where $\bar{G} = G/[\sigma_1, \sigma_1]$ and we sum over the Witt-basic subquotients of $(\mathbf{Z}G, \beta, b)$.

4. The Proof of Theorem B

The primary invariants listed before the statement of Theorem B compute the image of $\lambda^p(f, b)$ under certain maps of L-groups. For instance,

multi-signature: $L_{2i}^p(\mathbf{Z}G) \to L_{2i}^h(\mathbf{R}G)$

Arf invariant: $L_{2i}^p(\mathbf{Z}G) \to L_{2i}^h(\hat{\mathbf{Z}}_2G)$

semi-characteristic: $L_{2i+1}^p(\mathbf{Z}G) \to L_{2i+1}^h(\hat{\mathbf{Z}}_2G)$

(cohomology) finiteness obstruction: $L_i^p(\mathbf{Z}G) \to H^{i-1}(\tilde{K}_0(\mathbf{Z}G))$

We use here the involution $[P] \mapsto -[P^*]$ on $\tilde{K}_0(\mathbf{Z}G)$.

In each case the usefulness of these invariants depends on their computability from the information available about the surgery problem.

The δ-invariant is a secondary invariant, defined on the kernel of the last three primary ones (omitting the multisignature). Before giving the definition, we point out that the formulation in [35, 2.5] for any morphism of rings with involution $A \to B$ of the relative L-groups $L_*^{X,Y}(A \to B)$ in the exact sequence

$$\cdots \to L_n^X(A) \to L_n^Y(B) \to L_n^{X,Y}(A \to B) \to L_{n-1}^X(A) \to \cdots$$

as relative quadratic Poincaré cobordism groups has the following consequence:

LEMMA 4.1. *The $*$-invariant subgroups defined by*

$$U = \operatorname{Im}(\tilde{K}_1(A) \to \tilde{K}_1(B)) \subseteq \tilde{K}_1(B),$$

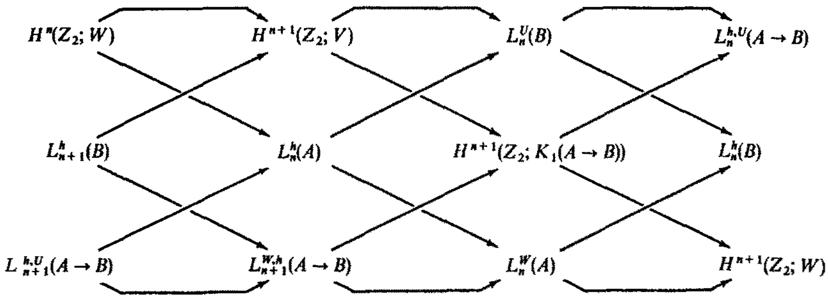
$$V = \tilde{K}_1(B)/U = \operatorname{Im}(\tilde{K}_1(B) \to K_1(A \to B)) \subseteq K_1(A \to B),$$

$$W = \ker(\tilde{K}_0(A) \to \tilde{K}_0(B)) \subseteq \tilde{K}_0(A)$$

yield a short exact sequence of $\mathbf{Z}/2$-modules

$$0 \to V \to K_1(A \to B) \to W \to 0$$

and the following commutative diagram



(4.1)

of exact sequences in L-theory. Diagram chasing induces a homomorphism

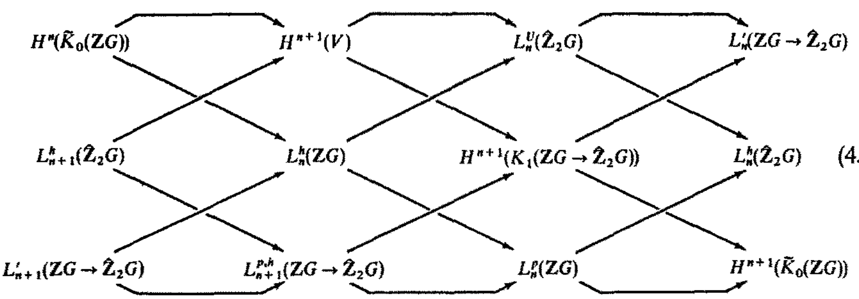
$$\delta \colon \ker(L^W_n(A) \to L^h_n(B) \oplus H^{n+1}(Z_2; W)) \to \operatorname{coker}(L^h_{n+1}(B) \oplus H^n(Z_2; W) \to H^{n+1}(Z_2; V)).$$

We will now apply this result for $A = \mathbf{Z}G$ and $B = \hat{\mathbf{Z}}_2 G$, to obtain a diagram involving the geometrically significant L-groups (as defined in Section 1), and to explicitly define the δ-invariant. Similar diagrams appear also in [15, 3.7, 3.11].

Our intermediate L-theory decorations are given by $U = \operatorname{Im}(\mathrm{Wh}'(\mathbf{Z}G) \to \mathrm{Wh}'(\hat{\mathbf{Z}}_2 G))$, or more precisely by its pre-image $\tilde{U} \subseteq \tilde{K}_1(\hat{\mathbf{Z}}_2 G))$, together with $V = \mathrm{Wh}'(\hat{\mathbf{Z}}_2 G)/U$ and $W = \tilde{K}_0(\mathbf{Z}G)$.

Recall that the 'prime' notation denotes the image of these Whitehead groups in $\mathrm{Wh}(\mathbf{Q}G)$ or $\mathrm{Wh}(\hat{\mathbf{Q}}_2 G)$ respectively [15, Section 3]. We can see by reduced norms that the natural map $\mathrm{Wh}(\mathbf{Q}G) \to \mathrm{Wh}(\hat{\mathbf{Q}}_2 G)$ is an injection. It follows that $\mathrm{Wh}'(\mathbf{Z}G)$ injects into $\mathrm{Wh}'(\hat{\mathbf{Z}}_2 G))$, and so $U \cong \mathrm{Wh}'(\mathbf{Z}G)$.

Now Lemma 4.1 gives a commutative diagram:



(4.2)

The definition of the δ-invariant needed for Theorem B (a special case of the map given in Lemma 4.1) is based on the homomorphisms

$$\begin{array}{c} L^h_n(\mathbf{Z}G) \to L^U_n(\hat{\mathbf{Z}}_2 G) \leftarrow H^{n+1}(\mathrm{Wh}'(\hat{\mathbf{Z}}_2 G)/\mathrm{Wh}'(\mathbf{Z}G)) \\ \qquad\qquad\qquad\qquad\qquad\qquad {\scriptstyle j_*}\big\downarrow \\ \qquad\qquad\qquad\qquad H^{n+1}(\mathrm{Wh}(\hat{\mathbf{Q}}_2 G)/\mathrm{Wh}'(\mathbf{Z}G)) \end{array} \tag{4.3}$$

More explicitly, an element $x \in L_n^p(\mathbf{Z}G)$ in the kernel of the cohomology finiteness obstruction can be lifted to $L_n^h(\mathbf{Z}G)$ and then mapped into $L_n^U(\hat{\mathbf{Z}}_2G)$. If in addition, the original element had trivial Arf invariant or semicharacteristic, then $x \in \ker(L_n^p(\mathbf{Z}G) \to L_n^h(\hat{\mathbf{Z}}_2G))$. By commutativity of (4.2), we can further lift from $L_n^U(\hat{\mathbf{Z}}_2G)$ into

$$H^{n+1}(V) \cong H^{n+1}(\mathrm{Wh}'(\hat{\mathbf{Z}}_2G)/\mathrm{Wh}'(\mathbf{Z}G)),$$

and the δ invariant is *defined* as the image of this lifted element under the natural map

$$H^{n+1}(\mathrm{Wh}'(\hat{\mathbf{Z}}_2G)/\mathrm{Wh}'(\mathbf{Z}G)) \xrightarrow{j_*} H^{n+1}(\mathrm{Wh}(\hat{\mathbf{Q}}_2G)/\mathrm{Wh}'(\mathbf{Z}G)).$$

In order to obtain a *well-defined invariant* on the kernel of the primary invariants listed above for $L_n^p(\mathbf{Z}G)$, we again refer to diagram (4.2). It follows that we must divide out the image under j_* of the indeterminacy

$$I = L_{n+1}^h(\hat{\mathbf{Z}}_2G) \oplus d^*H^n(\tilde{K}_0(\mathbf{Z}G)) \subseteq H^{n+1}(\mathrm{Wh}'(\hat{\mathbf{Z}}_2G)/\mathrm{Wh}'(\mathbf{Z}G))$$

where the first term is mapped to $H^{n+1}(V)$ by the discriminant, and the second term is just the image of the connecting homomorphism:

$$d^*: H^n(\tilde{K}_0(\mathbf{Z}G)) \to H^{n+1}(\mathrm{Wh}'(\hat{\mathbf{Z}}_2G)/\mathrm{Wh}'(\mathbf{Z}G))$$

in Tate cohomology associated with the sequence

$$0 \to \mathrm{Wh}'(\hat{\mathbf{Z}}_2G)/\mathrm{Wh}'(\mathbf{Z}G) \to \mathrm{Wh}'(\mathbf{Z}G \to \hat{\mathbf{Z}}_2G) \to \tilde{K}_0(\mathbf{Z}G) \to 0.$$

We next check that the map j_* on $H^{n+1}(V)$ does not lose any information after dividing out the indeterminacy, and note that the advantage of composing with j_* is to produce a range for the δ-invariant which is computable by reduced norms.

LEMMA 4.4. *Let* $V = \mathrm{Wh}'(\hat{\mathbf{Z}}_2G)/\mathrm{Wh}'(\mathbf{Z}G)$ *and* $V' = \mathrm{Wh}(\hat{\mathbf{Q}}_2G)/\mathrm{Wh}'(\mathbf{Z}G)$. *The map* j_* *in* (4.3) *induces an injection*

$$H^{n+1}(V)/I \to H^{n+1}(V')/j_*(I)$$

Proof. The composite

$$K_1(\mathbf{Z}G \to \hat{\mathbf{Z}}_2G) \xrightarrow{l} K_1(\mathbf{Z}G \to \hat{\mathbf{Q}}_2G) \to K_1(\mathbf{Q}G \oplus \hat{\mathbf{Z}}_{\mathrm{odd}}G \to \hat{\mathbf{Q}}G)$$

is an isomorphism by excision. Consider the diagram

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Wh}'(\hat{\mathbf{Z}}_2G)/\mathrm{Wh}'(\mathbf{Z}G) & \to & K_1(\mathbf{Z}G \to \hat{\mathbf{Z}}_2G) & \to & \tilde{K}_0(\mathbf{Z}G) & \to 0 \\ & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle l} & & \| & \\ 0 \to & \mathrm{Wh}(\hat{\mathbf{Q}}_2G)/\mathrm{Wh}'(\mathbf{Z}G) & \to & K_1(\mathbf{Z}G \to \hat{\mathbf{Q}}_2G) & \to & \tilde{K}_0(\mathbf{Z}G) & \to 0 \end{array}$$

and compare the induced sequences on Tate cohomology. Since the induced map l_* is an injection on Tate cohomology, the commutative diagram

$$\begin{array}{ccccc} H^n(\tilde{K}_0(\mathbf{Z}G)) & \xrightarrow{d^*} & H^{n+1}(V) & \to & H^{n+1}(K_1(\mathbf{Z}G \to \hat{\mathbf{Z}}_2 G)) \\ \| & & \downarrow{j_*} & & \downarrow{l_*} \\ H^n(\tilde{K}_0(\mathbf{Z}G)) & \xrightarrow{d_0^*} & H^{n+1}(V') & \to & H^{n+1}(K_1(\mathbf{Z}G \to \hat{\mathbf{Q}}_2 G)) \end{array}$$

implies that j_* induces an injection

$$H^{n+1}(V)/d^*H^n(\tilde{K}_0(\mathbf{Z}G)) \to H^{n+1}(V')/j_*d^*H^n(\tilde{K}_0(\mathbf{Z}G)).$$

Now the proof of Lemma 4.4 is completed by comparing the following exact sequences via j_*:

$$\begin{array}{ccccccc} L^h_{n+1}(\hat{\mathbf{Z}}_2 G) & \longrightarrow & H^{n+1}(V)/d^*H^n(\tilde{K}_0(\mathbf{Z}G)) & \longrightarrow & H^{n+1}(V)/I & \longrightarrow & 0 \\ \| & & \downarrow{j_*} & & \downarrow{l_*} & & \\ L^h_{n+1}(\hat{\mathbf{Z}}_2 G) & \to & H^{n+1}(V')/j_*d^*H^n(\tilde{K}_0(\mathbf{Z}G)) & \to & H^{n+1}(V')/j_*(I) & \to & 0 \end{array}$$

□

From Lemma 4.4 and (4.2), the proof of Theorem B amounts to showing that the image of the composite

$$L'_{n+1}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2 G) \xrightarrow{i_*} L^{p,h}_{n+1}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2 G) \to L^p_n(\mathbf{Z}G) \tag{4.5}$$

is mapped injectively by the multi-signature

$$L^p_n(\mathbf{Z}G) \to L^h_n(\mathbf{R}G).$$

This may be checked on the d-component, where i_* in (4.5) splits further according to the representations of $\mathbf{Q}G$ which are faithful on $\mathbf{Z}/d$.

The groups $L'_{n+1}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2 G)(d)$ have been studied in [45, 25] by a sequence similar to (1.7). The comparison can easily be made by considering the commutative diagram:

$$\begin{array}{ccccccc} \to \mathrm{CL}^S_{i+1}(S(d)) & \to & L'_{i+1}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2 G)(d) & \to & \prod_{l \nmid 2d} L^S_i(\hat{R}_l(d)) \oplus L^S_i(T(d)) & \to & \mathrm{CL}^S_i(S(d)) \\ \downarrow{1} & & \downarrow{2} & & \downarrow{3} & & \downarrow \\ \to \mathrm{CL}^K_{i+1}(S(d)) & \to & L^{p,h}_{i+1}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2 G)(d) & \to & \prod_{l \nmid 2d} L^K_i(\hat{R}_l(d)) \oplus L^K_i(T(d)) & \to & \mathrm{CL}^K_i(S(d)) \end{array} \tag{4.6}$$

From Section 1 we know that the map 1 is nonzero only on type OK factors with $i + 1 \equiv 0 \pmod 4$, where there is a short exact sequence

$$\begin{array}{l} 0 \to \mathrm{CL}^s_0(D) \to \mathrm{CL}^K_0(D) \to H^0(C(D)) \to 0 \\ \qquad\quad \| \\ \qquad\quad \mathbf{Z}/2 \end{array}$$

In type OK(**R**) from Table II, the image of 1 does not survive to $L_0^{p,h}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2G)(d)$. In the standard oriented anti-structure there are no type OK(**C**) factors.

Note that the map 3 is trivial on the terms $L_i^S(\hat{R}_l(d)) \to L_i^K(\hat{R}_l(d))$ and an injection on the term $L_i^S(T(d))$.

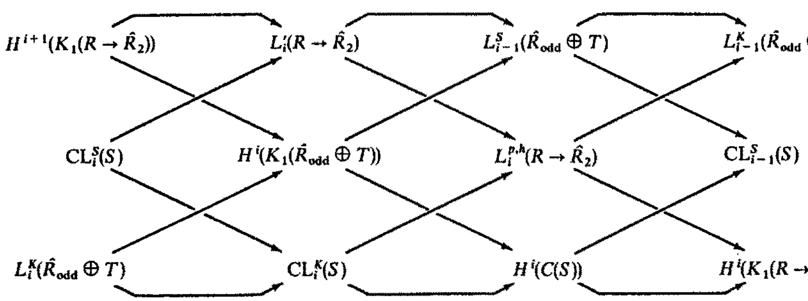
The remaining point to check is the possibility that there is an element in the image of 2 which comes from $\mathrm{CL}_{i+1}^K(S(d))$. If we consider an element in some factor of $L'_{i+1}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2G)(d)$ which is non-trivial in $L_i^S(\hat{R}_{\mathrm{odd}}(d))$ but trivial in $L_i^S(T(d))$, then the factor has type O and $i \equiv 1, 2 \pmod 4$ by (1.11). On the other hand, the groups $\operatorname{cok} \gamma_{i+1}$ are nonzero in type O only when $i + 1 \equiv 0,\ 1 \bmod 4$, or $i \equiv 0, 3 \bmod 4$. □

Remark. The argument applies more generally to any geometric antistructure with no type OK(**C**) factors. Such factors, however, are common in nonoriented situations (*e.g.* $G = \mathbf{Z}/4$ with $w \not\equiv 1$). To extend Theorem B to cover the nonoriented case, we must add restriction to the subquotients detecting the type OK(**C**) factors to the list of invariants. The relevant subquotients are 2-hyperelementary groups $\mathbf{Z}/d \rtimes \sigma$ with $\sigma_1 = \mathbf{Z}/4$ and $w|\sigma_1 \not\equiv 1$. If G is one of these groups to begin with, we could define a third level invariant on the kernel of the δ-invariant by taking the Hasse invariant at each type OK(**C**) factor.

PROPOSITION 4.7. *The sequence* (1.8) *is split exact, so that*

$$L_i^{p,h}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2G) \cong \operatorname{cok} \gamma_i \oplus \ker \gamma_{i-1}.$$

Proof. From Table II this is obvious unless $i \equiv 1 \pmod 4$ and $S(d)$ contains a factor (D, α, u) of type O. To handle these cases we use another braid diagram:



(4.8)

In order to avoid introducing more notation, we will also use '$L_i^{p,h}(R \to \hat{R}_2)$' for summand of this group corresponding to a simple factor of $S(d)$. This should cause no confusion. For a type O factor the maps in the above braid

$$H^i(C(D)) \to \mathrm{CL}_{i-1}^S(D)$$

are isomorphisms for $i \equiv 2,\ 3 \pmod 4$, as we checked in calculating $\mathrm{CL}_i^K(D)$ from diagram (1.13).

Since $L_0^S(\hat{R}_{\mathrm{odd}} \oplus T) = L_0^S(T)$, it follows that there are short exact sequences

$$0 \to \begin{Bmatrix} \Sigma \\ 0 \end{Bmatrix} \to L_1^{p,h}(R \to \hat{R}_2) \to H^1(K_1(R \to \hat{R}_2)) \to \begin{Bmatrix} 0 \\ \mathbf{Z}/2 \end{Bmatrix} \to 0,$$

where the upper row in the bracketed terms refers to types OK(**R**) or OD(**R**), and the lower row to types OK(**C**), OD(**H**) and OD(**C**). Since the middle group $H^1(K_1(R \to \hat{R}_2))$ has exponent 2, and the torsion subgroup of $L_1^{p,h}(R \to \hat{R}_2)$ injects into it, we are done. □

5. The L^p-theory of Hyperelementary Groups

In this section we will summarize and complete the calculation of $L_i^p(\mathbf{Z}G, \beta, b)$ for an arbitrary geometric antistructure $(\mathbf{Z}G, \beta, b)$ on a 2-hyperelementary group G, with $b \in G$. We point out that these groups are sufficient to compute (in the sense of Dress induction) the L theory localized at 2 for all finite groups, at least for the standard oriented or nonoriented antistructures. In these cases it may be useful to have a direct calculation of the L^p-groups of p-hyperelementary groups for p odd. It was remarked in [45, 2.4.1] that any such group is the direct product of a cyclic 2-group and a group of odd order, and we obtain the corresponding result to [45, 2.4.2].

PROPOSITION 5.1. *Let $G = \sigma \times \rho$, where σ is an abelian 2-group and ρ has odd order. Define $\Sigma_{2k+1} = 0$ and $\Sigma_{2k} = \mathrm{im}(L_{2k}^p(\mathbf{Z}G, w) \to L_{2k}^p(\mathbf{R}\sigma \to \mathbf{R}G, w))$, where $w\colon G \to \{\pm 1\}$ is an orientation character. Then $L_i^p(\mathbf{Z}G, w) = L_i^p(\mathbf{Z}\sigma, w) \oplus \Sigma_i$. The last summand is free abelian and detected by signatures at the type* U *representations of G which are nontrivial on ρ.*

Proof. The inclusion $\mathbf{Z}\sigma \to \mathbf{Z}G$ induces an isomorphism on the 2-adic L-groups, and the relative group $L_{2k}^p(\mathbf{R}\sigma \to \mathbf{R}G, w)$ is detected by the representations of G which are nontrivial on ρ. These all have type U or GL. Now the projection map $\mathbf{Z}G \to \mathbf{Z}\sigma$ together with the obvious map to Σ_i is clearly onto, and induces a 2-local isomorphism (by reducing to the 2-hyperelementary subgroups of G, which are cyclic). Since the 2-localization map on L-groups is an injection, we are done. □

Now suppose that $G = \mathbf{Z}/d \rtimes \sigma$ is a 2-hyperelementary group equipped with a geometric antistructure. The steps involved in computing $L_i^p(\mathbf{Z}G, \beta, b)$ from the long exact sequence (1.4) are as follows:

(1) Determine the types, Schur indices and centre fields for all the rational representations of $G = \mathbf{Z}/d \rtimes \sigma$, following the method given in [16, p. 148], or [20, Appendix I]. For the d-component (1.2) we need to consider only those representations which are faithful on $\mathbf{Z}/d$.

(2) Refer to Table II for the contribution of each simple involution-invariant factor of $\mathbf{Q}G$ to the relative group $L_i^{p,h}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2 G, \beta, b)$.

(3) Compute the necessary ranks of the class groups following (2.18), (2.8) and (2.10) to complete the calculation of $\ker \gamma_i(d)$ and $\operatorname{cok} \gamma_i(d)$. By (4.7) the d-component of the relative group is just

$$L_i^{p,h}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2 G)(d) = \operatorname{cok} \gamma_i(d) \oplus \ker \gamma_{i-1}(d).$$

(4) Apply (3.2) to compute the maps $\Psi_i(d)\colon L_i^h(\hat{\mathbf{Z}}_2G, \beta, b)(d) \to L_i^h(\hat{\mathbf{Q}}_2G, \beta, b)(d)$. For $d > 1$ the domain of $\Psi_i(d)$ is $g_2(\mathbf{Z}/2)$, where g_2 is the number of dyadic primes in the field $\mathbf{Q}(\zeta_d)^\sigma$. For $d = 1$, the domain of Ψ_i is zero for i odd and $\mathbf{Z}/2$ for i even.

So far these steps are fully described in Sections 1–3, and we have no more to say about them. The remaining steps will be carried out in this section, under the assumption that $d > 1$. The case $d = 1$ has been fully discussed in [20].

(5) We note that the maps

$$\psi_i(d)\colon L_i^h(\hat{\mathbf{Z}}_2G)(d) \to L_i^{p,h}(\mathbf{Z}G \to \hat{\mathbf{Z}}_2G)(d)$$

factor through

$$\bar{\psi}_i(d)\colon L_i^h(\hat{\mathbf{Z}}_2G)(d) \to \operatorname{cok} \gamma_i(d)$$

and compute the kernel and cokernel of the maps $\bar{\psi}_i(d)$. This step also computes the kernel and cokernel of $\psi_i(d)$ since

$$\ker \psi_i(d) = \ker \bar{\psi}_i(d), \quad \text{and} \quad \operatorname{cok} \psi_i(d) \cong \operatorname{cok} \bar{\psi}_i(d) \oplus \ker \gamma_{i-1}(d)$$

by (4.7) again.

(6) We settle the remaining extension problems in

$$0 \to \operatorname{cok} \psi_{i+1}(d) \to L_i^p(\mathbf{Z}G, \beta, b)(d) \to \ker \psi_i(d) \to 0,$$

to finish the calculation.

We now begin a discussion of steps 5 and 6.

5.2. COMPUTATION OF THE MAPS $\bar{\psi}_i(d)$

We begin by remarking that if $\Psi_i(d) = 0$, then $\bar{\psi}_i(d) = 0$ also. This can happen only if $i \equiv 2, 3 \pmod 4$. Suppose that $S(d, \xi)$ denotes a summand of $S(d)$ for which the image of $\Psi_i(d)$ is nontrivial (see (3.2)). Then ξ is a linear type O, Sp or special type U(**C**) character (when i is even) of σ_1. Let $\operatorname{cok} \gamma_i(d, \xi)$ denote the corresponding factor of $\operatorname{cok} \gamma_i(d)$ and $\bar{\psi}_i(d, \xi)$ denote the composition

$$L_i^K(\hat{R}_2(d)) \xrightarrow{\bar{\psi}_i(d)} \operatorname{cok} \gamma_i(d) \to \operatorname{cok} \gamma_i(d, \xi)$$

with the projection map. We will first determine the maps $\bar{\psi}_i(d, \xi)$ for each character ξ above.

In type U the images of the Arf invariant elements in $L_{2i}^K(\hat{R}_2(d)) = g_2(\mathbf{Z}/2)$ are all nontrivial in $CL_{2i}^K(D) = H^0(C(E)) = \mathbf{Z}/2$, since the map $L_{2i}^K(\hat{S}_l(d, \xi)) \to H^0(C(E))$ is nontrivial for each prime $l \mid 2$ [8, pp. 172–175].

Recall that in type O there is a subgroup of rank $g_2, g_2(1 - 4\delta) \subseteq H^0(\hat{A}_{2d}^\times)$ where δ is a 2-local integer whose reduction to the residue field has nonzero trace in $\mathbb{F}_2$.

PROPOSITION 5.3. *Suppose that $\Psi_i(d)$ is injective into a factor $S(d,\xi)$ with skew field D and centre field E.*

(i) *In type* U(**C**) *the image of the map $\bar{\psi}_i(d,\xi)$ is zero.*

(ii) *In type* O *the image of the map $\bar{\psi}_i(d,\xi)$ is zero unless $i=0$ or 1. The kernel of $\bar{\psi}_0(d,\xi)$ has 2-rank $\gamma^*(E,d)-\gamma^*(E,2d)$ if D is split at infinite primes (resp. $\gamma(E,d)-\gamma(E,2d)$ if D is nonsplit at infinite primes). The image of $\bar{\psi}_1(d,\xi)$ has 2-rank g_2 (i.e. $\bar{\psi}_1(d,\xi)$ is injective) unless* (a) *the type is* OK *and $d=1$, where the image is zero, or* (b) *the type is* OD(**R**) *or* OD(**C**) *and $g_d(E)=1$, where the image has 2-rank g_2-1.*

Proof. For (i) we have already noted that the factor must be type U(**C**) for $\Psi_i(d)$ to be nonzero. In that case, $\operatorname{cok}\gamma_i(d,\xi)=0$. For (ii) in the case when $i=0$, we recall that the image of $\Psi_0(d)$ is isomorphic to $g_2(1-4\delta)\subseteq H^0(\hat{E}_2^\times)$ by [16, 2.12, 3.9]. Now the kernel of the map $\bar{\psi}_0(d,\xi)$ in type OK (D split at infinite primes) is easily determined from the following diagram

$$\begin{array}{ccccccccc}
 & & & & g_2(1-4\delta) & = & L_0^K(\hat{R}_2(d)) & & \\
 & & & & \downarrow & & \downarrow{\scriptstyle \bar{\psi}_0(d,\xi)} & & \\
0\to\ker\Phi & \to & E^{(2)}/E^{\times 2} & \xrightarrow{\Phi} & H^0(\hat{A}_{2d}^\times) & \longrightarrow & \operatorname{cok}\gamma_0(d,\xi) & & (5.4)\\
\downarrow & & \| & & \downarrow & & \downarrow & & \\
0\to\ker\bar{\Phi} & \to & E^{(2)}/E^{\times 2} & \xrightarrow{\bar{\Phi}} & H^0(\hat{A}_{2d}^\times)/g_2(1-4\delta) & \to & \operatorname{cok}\bar{\psi}_0(d,\xi) & &
\end{array}$$

When D is nonsplit at infinite primes we must replace Φ by Φ', adding $H^0(E_\infty^\times)$ to the range of the reduction maps. The result listed in (ii) follows from the fact that the 2-rank of $\ker\bar{\Phi}$ or $\ker\bar{\Phi}'$ is $\gamma^*(E,d)$ or $\gamma(E,d)$, respectively, [45, p. 56]. In case (iii) we use [16, 2.12, 3.10] to conclude that the image of $\Psi_1(d)$ is isomorphic to $g_2(\langle -1\rangle)\subseteq H^0(\hat{E}_2^\times)$. Then the result follows by counting ranks in (2.12) and (2.13). In case (b), $d>1$ (since these types do not occur for 2-groups) so the domain of Δ in (2.13) has rank $(\Sigma_{p|2d}\, g_p)-1=g_2$, and $\ker\Delta$ has rank g_2-1. □

For the maps $\bar{\psi}_i(d)$ we must compare the kernels of the various $\bar{\psi}_i(d,\xi)$ for all the linear type O or Sp characters ξ of σ_1. Clearly $\ker\bar{\psi}_i(d)=\cap\ker\bar{\psi}_i(d,\xi)$.

LEMMA 5.5. *Let ξ_1,ξ_2 be linear type* O (*resp. type* Sp) *characters of the antistructure $(\hat{\mathbf{Q}}_2\sigma_1,\beta_0,b_0)$, with associated centre fields E_j $(j=1,2)$ for $S(d,\xi_j)$. If $i=0$ or $i=1$ (resp. $i=2$ or 3), suppose that $\Psi_i(d)$ is injective into $S(d,\xi_1)$ and $E_1\subseteq E_2$. Then* $\ker\bar{\psi}_i(d,\xi_1)\subseteq\ker\bar{\psi}_i(d,\xi_2)$.

Proof. By [16, 2.12, 2.19] the image of $\bar{\psi}_i(d,\xi_j)$ is either $\langle 1-4\delta\rangle$ or $\langle -1\rangle$ $(i\equiv 0,1 \pmod 4)$ in $H^i(\hat{A}_l(\xi_j))$ for each $l\,|\,2$. Since both of these are mapped injectively under inclusions $E_1\subset E_2$, the result follows from (5.4). □

Since the trivial character ξ_0 of σ_1 has type OK, and has the minimal possible centre field $E_0=\mathbf{Q}(\zeta_d)^\sigma$, we obtain:

COROLLARY 5.6. *Assume that $d>1$. The map $\psi_1(d)$ is injective, and* $\ker\psi_0(d)=\ker\bar{\psi}_0(d,\xi_0)$ *where ξ_0 denotes the trivial character of σ_1. The map $\psi_3(d)$ is zero if σ_1*

has no linear type Sp *characters, and otherwise* $\ker \psi_3(d) \subseteq \mathbf{Z}/2$ *with equality only when all linear type* Sp *characters are of type* Sp$D(\mathbf{R})$ *or* Sp$D(\mathbf{C})$ *and* $g_d(E) = 1$ *for their centre fields.*

Proof. The first and last statements follow from Proposition 5.3, and the second from Lemma 5.5. □

Remark 5.7. Suppose that $\xi_0, \xi_1, \ldots, \xi_r$ are the linear type O (resp. type Sp) characters of the antistructure $(\hat{\mathbf{Q}}_2\sigma_1, \beta_0, b_0)$, with associated centre fields E_j $(j = 0, 1, \ldots, r)$ for $S(d, \xi_j)$. Then $\operatorname{cok} \bar{\psi}_i(d)$ can be computed by the long exact sequence

$$0 \to \ker \bar{\psi}_i(d) \to \prod \ker \bar{\psi}_i(d, \xi_j) \to \left(\prod_j L_i^K(\hat{A}_2(\xi_j))\right) \Big/ L_i^K(\hat{R}_2(d))$$
$$\to \operatorname{cok} \bar{\psi}_i(d) \to \prod_j \operatorname{cok} \bar{\psi}_i(d, \xi_j) \to 0.$$

EXAMPLE 5.8. Many of the 2-hyperelementary group arising in the space form problem have σ_1 cyclic generated by b_0 [25]. More generally, suppose that the unit $b_0 \in \sigma_1$ generates a direct factor of $\sigma_1/[\sigma_1, \sigma_1]$. Then the unique minimal field for the type Sp characters is $\mathbf{Q}(\chi_d)^\sigma$, appearing as the centre field for the character ξ_0 of σ_1 which projects onto the direct factor $\langle b_0 \rangle$ and then maps b_0 to -1. As in Corollary 5.6, the map $\psi_3(d)$ then is injective and $\ker \psi_2(d) = \ker \bar{\psi}_2(d, \xi_0)$.

To state our final conclusion about the maps $\bar{\psi}_2(d)$, we need to generalize the notion of ('unramified') quadratic extensions from fields to group algebras. Recall that 'unramified' refers to all finite primes, and 'strictly' unramified refers to all places including the infinite ones.

Let $\mathbf{R}(\pi)$ denote the complex representation ring of π, and $\mathbf{R}(\pi)(d)$ the subring generated by the representations which are faithful on $\mathbf{Z}/d$. It is convenient to denote by $\mathbf{R}_{\mathrm{Sp}}(\pi)$ the $\mathbf{Z}$-span of the type Sp characters of $(\mathbf{Z}\pi, \beta_0, b_0)$. More generally, $\mathbf{R}_\Sigma(\pi)$ will denote the span of a given set Σ of irreducible characters. Let $\pi_0 = \mathbf{Z}/d \times \sigma_0$, where $\sigma_0 = \langle b_0 \rangle$. Following [16, 2.12], we have character homomorphisms

$$A_2^r : \mathbf{R}(\pi_0)(d) \to \bar{\mathbf{Q}}_2^\times,$$

where $\bar{\mathbf{Q}}$ is the separable closure of the rationals, and $r \in \mathbf{Z}/d^\times/\langle 2 \rangle$. The cohomology classes of these character homomorphisms are just the images of the Arf invariant one planes under the discriminant

$$L_2^K(\hat{\mathbf{Z}}_2 \otimes \mathbf{Z}[\zeta_d]\sigma_0, \beta_0, b_0) \to \hat{H}^0(\operatorname{Hom}_{\Omega_2}(\mathbf{R}(\pi_0)(d), \bar{\mathbf{Q}}_2^\times), \beta_0).$$

There is an induced map (project into the type Sp factors)

$$\hat{H}^0(\operatorname{Res}_*^{\mathrm{Sp}}) : \hat{H}^0(\operatorname{Hom}_{\Omega_2}(\mathbf{R}(\pi_0)(d), \bar{\mathbf{Q}}_2^\times), \beta_0) \to \hat{H}^0(\operatorname{Hom}_{\Omega_2}(\mathbf{R}_{\mathrm{Sp}}(\pi)(d), \bar{\mathbf{Q}}_2^\times), \beta_0)$$

and we denote by

$$V_2(\pi, d) \subseteq \hat{H}^0(\operatorname{Hom}_{\Omega_2}(\mathbf{R}_{\mathrm{Sp}}(\pi)(d), \bar{\mathbf{Q}}_2^\times), \beta_0)$$

the subgroup (of 2-rank $g_2(d)$) generated by the images of the character homomorphisms $\{A_2^\zeta\}$ under this map.

For our semisimple algebra $S(d) = \mathbf{Q}[\zeta_d]^t\sigma$ over $\mathbf{Q}$, this subgroup will provide a 'coherence' condition with respect to a set Σ of irreducible characters in our definition of quadratic extensions. To relate global and local quantities we use the reduction map

$$\Phi_{(S,\Sigma)}\colon \hat{H}^0(\mathrm{Hom}_\Omega(\mathbf{R}_{\mathrm{Sp}}(\pi)(d), \bar{\mathbf{Q}}(2d)^{(2)}), \beta_0)$$
$$\to \prod_{p|2d} \hat{H}^0(\mathrm{Hom}_{\Omega_p}(\mathbf{R}_{\mathrm{Sp}}(\pi)(d), \bar{\mathbf{Q}}_p^\times), \beta_0) \oplus \hat{H}^0(\mathrm{Hom}_\Omega(\mathbf{R}_\Sigma(\pi)(d), \mathbf{C}^\times), \beta_0).$$

Here $\bar{\mathbf{Q}}(2d)$ is the union of all Abelian extensions of the rational numbers $\mathbf{Q}$ which are unramified at all finite primes $p \nmid 2d$, and $\bar{\mathbf{Q}}(2d)^{(2)}$ denotes the nonzero elements with even valuation at all finite primes $p \nmid 2d$. The domain of $\Phi_{(S,\Sigma)}$ is the Tate cohomology of a subgroup of the Fröhlich 'Hom' description for $\hat{H}^0(K_1(S(d)), \beta_0)$. Using this identification, a unit $u \in S(d)$ lies in the domain of $\Phi_{(S,\Sigma)}$ provided that its reduced norm lies in $E(2d)^{(2)}$ for each centre field E in a type Sp factor of $S(d)$.

DEFINITION 5.10. The algebra $S(d)[\sqrt{u}] = S(d)[t]/(t^2 - u)$ is said to be an (Sp, Σ)- coherent quadratic extension of $S(d)$ if

(i) $u \in S(d)$ is a central unit fixed by β_0, projecting to 1 at each factor not of type Sp, whose reduced norm in each factor of $S(d)$ has even valuation at all primes $p \nmid 2d$, and

(ii) the class $\Phi_{(S,\Sigma)}(u)$ is in the image of $\hat{H}^0(\mathrm{Res}_*^{\mathrm{Sp}})$ at the dyadic primes, trivial at the primes $p \mid d$, and trivial at infinite primes in the factors given by Σ.

The extension is called *unramified* if

(iii) the class $\Phi_{(S,\Sigma)}(u) \in V_2(\pi, d)$.

Note that condition (iii) implies that the extension of centre fields $E[\sqrt{u}]/E$ is quadratic unramified for each simple factor of $S(d, \xi)$ of type Sp, and strictly unramified in the factors corresponding to the characters in Σ. Similarly, we have the notion of a Sp-coherent quadratic extension which is *decomposed* at a prime whenever this is the case for the centre field extensions.

DEFINITION 5.11. Let n be an integer and Σ a set of type Sp characters. Let $\gamma_{\mathrm{Sp}}(S, n, \Sigma)$ be the number of distinct unramified Sp-coherent quadratic extensions of $S(d)$ which are decomposed at the primes dividing n and strictly unramified at Σ. If Σ is the set of all type Sp characters we denote this rank by $\gamma_{\mathrm{Sp}}(S, n)$.

PROPOSITION 5.12. *Suppose that $\Psi_2(d)$ is injective. Then $\gamma_{\mathrm{Sp}}(S, d, \Sigma) - \gamma_{\mathrm{Sp}}(S, 2d, \Sigma)$ is the 2-rank of* $\ker \psi_2(d)$, *where Σ is the set of characters of type* Sp*D* (*i.e. with simple factors nonsplit at infinite primes*).

Proof. A similar diagram to (5.4) including all the characters of $S(d)$ relates $\ker \Phi_{(S,\Sigma)}$ to $\ker \bar{\psi}_2(d)$. Now with our definitions, the 2-rank of $\ker \Phi_{(S,\Sigma)}$ is just the number of distinct unramified Sp-coherent quadratic extensions of $S(d)$ which are decomposed at the primes dividing $2d$ and strictly unramified at Σ. The modified map $\bar{\Phi}_{(S,\Sigma)}$ obtained by dividing out $V_2(\pi, d)$ in the range has kernel whose rank is $\gamma_{\mathrm{Sp}}(S, d, \Sigma)$. □

Remark 5.13. These 2-ranks are computable using class field theory in the case when $S(d)$ has only one type Sp character (compare Proposition 5.3). In general, the above result should be considered as a reduction of our computation to a specific arithmetical question. It would be interesting to know whether the answer could again be related to ray class groups of $R(d) = \mathbf{Z}[\zeta_d]^t\sigma$. In this discussion we have concentrated on $\ker \psi_2(d)$ since this was not determined above. We remark that there is a similar formulation for $\ker \psi_0(d)$ using type O characters throughout in place of type Sp characters, i.e. we use $\mathbf{R}_{\mathrm{O}}(\pi)$ in the definitions of $\Phi_{(S,\Sigma)}$ and $\hat{H}^0(\mathrm{Res}_*^{\mathrm{O}})$ to introduce unramified O-coherent quadratic extensions, and ranks $\gamma_{\mathrm{O}}(S, d, \Sigma)$. In the next subsection we will need these maps again.

5.14. FINAL EXTENSION PROBLEMS

We now consider step 6, the extensions determining the groups $L_i^p(\mathbf{Z}G, \beta, b)(d)$.

Since $\operatorname{cok} \psi_{i+1}(d) \cong \operatorname{cok} \bar{\psi}_{i+1}(d) \oplus \ker \gamma_i(d)$ and $\ker \psi_i(d) = \ker \bar{\psi}_i(d)$, there are two potential sources of nontrivial extensions. To study the ones involving an extension of $\ker \bar{\psi}_i(d)$ by $\ker \gamma_i(d)$, we introduce the notation

$$u_i(d)\colon \prod_{l \nmid d} L_i^K(\hat{R}_l(d)) \oplus L_i^K(T(d)) \to \mathrm{CL}_i^K(S(d))$$

for the map from the arithmetic sequence. Then we have an exact sequence

$$0 \to \ker \gamma_i(d) \to \ker u_i(d) \to \ker \bar{\psi}_i(d) \to 0. \tag{5.15}$$

The extensions in this sequence can be determined from the diagram

$$\begin{array}{ccccccc}
0 \to \ker \gamma_i(d) & \to & \prod_{l \nmid 2d} L_i^K(\hat{R}_l(d)) \oplus L_i^K(T(d)) & \xrightarrow{\gamma_i(d)} & \mathrm{CL}_i^K(S(d)) \\
\downarrow & & \downarrow & & \| \\
0 \to \ker u_i(d) & \to & \prod_{l \nmid d} L_i^K(\hat{R}_l(d)) \oplus L_i^K(T(d)) & \xrightarrow{u_i(d)} & \mathrm{CL}_i^K(S(d)) \\
\downarrow & & \downarrow & & \\
L_i^K(\hat{R}_2(d)) & = & L_i^K(\hat{R}_2(d)) & &
\end{array} \tag{5.16}$$

Since the middle vertical sequence is split with torsion subgroup of exponent two, the extensions in (5.15) are limited by those in the sequence

$$0 \to \Lambda_i(d) \to L_i^K(T(d)) \to \mathrm{CL}_i^K(S(d)).$$

But for i odd, the kernel $\Lambda_i(d) = 0$ and so (5.15) is split. For $i = 0$ the kernel is only nontrivial at each type OK(**R**) factor, where it $\Lambda_0(d, \xi) = 8\mathbf{Z} \oplus (4\mathbf{Z})^{r_1(\xi)-1}$ with $r_1(\xi)$ the number of real places in the centre field of the factor. The extension for $\ker u_0(d)$ is therefore nontrivial over the 8Z summands in the linear type OK(**R**) factors of the signature group of $\ker \gamma_0(d)$. These extensions arise already in the classical theory of quadratic forms over $\mathbf{Z}$ with odd determinant from the (mod 4) relationship between the signature and the Arf invariant [29]. When $i = 2$ and $\Psi_2(d)$ is injective, then a similar relation exist between the extensions and $8\mathbf{Z}$ summands from linear type $\mathrm{Sp}K(\mathbf{R})$ factors. If $\Psi_2(d) = 0$, then (5.15) is split.

Let $r_1(S) = \Sigma\{r_1(\xi)\colon \xi \text{ of type OK}\}$, and Θ_S be the reduction map

$$\Theta_S\colon \hat{H}^0(\mathrm{Hom}_\Omega(\mathbf{R}_O(\pi)(d), \bar{\mathbf{Q}}(2d)^{(2)}), \beta_0) \to \hat{H}^0(\mathrm{Hom}_\Omega(\mathbf{R}_\Sigma(\pi)(d), \mathbf{C}^\times), \beta_0)$$

over the infinite primes in the centre fields corresponding to the set Σ of all linear type $\mathrm{OK}(\mathbf{R})$ characters. Recall now that the map $\bar{\Phi}_{(S,\Sigma)}$ is obtained from $\Phi_{S,\Sigma}$ by dividing out the subgroup $V_0(\pi, d)$ in the range. Let $r_0(S)$ denote the 2-rank of image $(\Theta_S \mid \ker \bar{\Phi}_{(S,\Sigma)})$, where Σ is the set of type OD characters.

We will now summarize our computation of $\ker u_i(d)$. The signature group is a direct sum of 2-divisible, 4-divisible and 8-divisible summands, and the torsion subgroup is of exponent 2. For $\ker \gamma_0(d)$ the signature group is computed in Lemma 2.8 and Lemma 2.10. The map $\bar{\Phi}'_S$ is obtained from $\bar{\Phi}_{(S,\Sigma)}$ by taking Σ as the set of all type O characters.

PROPOSITION 5.17.

(i) *For i odd,* $\ker u_i(d) = \ker \gamma_i(d) \oplus \ker \bar{\psi}_i(d)$.
(ii) *If $\Psi_0(d)$ is injective, the torsion subgroup of* $\ker u_0(d)$ *is* $\ker \bar{\Phi}'_S$ *with 2-rank $\gamma_0(S, d)$, and the signature group is the direct sum of* $(8\mathbf{Z})^{r_1(S) - r_0(S)} \oplus (4\mathbf{Z})^{r_0(S)}$ *together with the 2-divisible and 4-divisible part of* $\ker \gamma_0(d)$. *If $\Psi_2(d)$ is injective, the analogous conclusion holds for* $\ker u_2(d)$ *with type* Sp *replacing type* O.
(iii) *If $\Psi_i(d) = 0$, then* $\ker u_i(d) = \ker \gamma_i(d) \oplus g_2(\mathbf{Z}/2)$.
(iv) *The torsion subgroup of* $\ker u_i(d)$ *has exponent two.*

Our final problem is to determine the extensions in the sequence

$$0 \to \operatorname{cok} u_{i+1}(d) \to L_i^p(\mathbf{Z}G)(d) \to \ker u_i(d) \to 0 \tag{5.18}$$

It is not difficult to produce the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{cok} \bar{\psi}_{i+1}(d) & \to & \operatorname{cok} \psi_{i+1}(d) & \to & \ker \gamma_i(d) & \to & 0 \\
 & & \cong\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{cok} u_{i+1}(d) & \to & L_i^p(\mathbf{Z}G)(d) & \to & \ker u_i(d) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \ker \psi_i(d) & = & \ker \bar{\psi}_i(d) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

to relate the new maps with the others we have been using. The result is

PROPOSITION 5.19. *There is a splitting*

$$L_i^p(\mathbf{Z}G)(d) \cong \operatorname{cok} \bar{\psi}_{i+1}(d) \oplus \ker u_i(d).$$

Proof. We must show that the remaining extensions between $\ker \bar{\psi}_i(d)$ and $\operatorname{cok} \bar{\psi}_{i+1}(d)$ are trivial. For $i = 1$ there is nothing to prove, since $\bar{\psi}_1(d)$ is injective. For i even, we first form the composite extension

$$S' = \bigcup \{S[\sqrt{u}] : u \in A\},$$

where A is a set of central units $u \in S$ which closely 2-adically approximates one of the non-squares $(1 - 4\delta)$, in each centre field of $\hat{S}_2$ fixed by β_0. We take a large enough collection A to cover all the possibilities. Now let R' be the order in S' obtained by extending the centre of R by the ring of integers in the centre fields of S'. We extend the involution β_0 by the identity. Now we have the diagram

$$\begin{array}{ccccccccc} 0 \to & \operatorname{cok} \bar{\psi}_{i+1}(d) & \to & L_i^p(R)(d) & \to & \ker u_i(d) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \operatorname{cok} \bar{\psi}'_{i+1}(d) & \to & L_i^p(R')(d) & \to & \ker u'_i(d) & \to 0 \end{array}$$

where the 'prime' decoration refers to the arithmetic sequence for R'. Now the map induced by inclusion $L_i^K(\hat{R}_2(d)) \to L_i^K(\hat{R}'_2(d))$ is zero by construction, since we have included in S' the square roots of all possible discriminants of the Arf invariant one elements in $\hat{R}_2(d)$. However, the map induced on $\operatorname{cok} \bar{\psi}_{i+1}(d)$ is an injection, since i is even (see Table II). Therefore, the torsion in the upper sequence has exponent two and the sequence is split.

For $i = 3$ we consider the arithmetic square

$$\begin{array}{ccc} R(d) & \longrightarrow & \hat{R}_2(d) \\ \downarrow & & \downarrow \\ \hat{R}_{\text{odd}}(d) \oplus S(d) \oplus T(d) & \longrightarrow & S_A(d) \end{array}$$

where the rings in the lower row split into factors corresponding to the irreducible characters ξ of σ_1. The extension problem can be considered as one of lifting a 'patching' isomorphism over $S_A(d)$ between the images of elements of $L_3^K(\hat{R}_2(d))$ and $L_3^K(\hat{R}_{\text{odd}}(d)) \oplus L_3^K(S(d))$. Now the nontrivial elements in $L_3^K(\hat{R}_2(d))$ are represented by automorphisms

$$\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in each factor of $\hat{S}_2(d)$. Since only type O or U(GL) factors contribute to cok $\gamma_0(d)$, we must look at the trivialization of σ at one of these factors over $S_A(d)$. But τ can be written as the product

$$\tau = \begin{pmatrix} 1 & 0 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

of elementary unitary and hyperbolic automorphisms and this relation can be lifted over the corresponding factors of the lower left corner $\hat{R}_{\text{odd}}(d) \oplus S(d) \oplus T(d)$. Therefore the extension for $L_3^p(\mathbf{Z}G)(d)$ is split. □

Remark 5.20. Our computations do not agree with those of Bak–Kolster in [3, Theorem 4.1], and we have already pointed out the source of the discrepancy in [16, p. 143]. For example, for the group $G = \mathbf{Z}/3 \rtimes \mathbf{Z}/4$ with $\ker t = \mathbf{Z}/2$, we get $L_1^p(\mathbf{Z}G) = (\mathbf{Z}/2)^2$ and Bak–Kolster get $L_1^p(\mathbf{Z}G) = (\mathbf{Z}/2)^3$. To extract this answer from [3], note that the statements of both parts of [3, Theorem 4.1] are written incorrectly (compare the case of L_1^p for a finite 2-group [20, Section 3, p. 80]). What was intended can be seen from reading the proof on [3, p. 62], after correcting the misprint in the first line of the displayed formulas for coker $\partial^{-\lambda}$ at the bottom of the page: in the first line $\lambda = 1$ should be $\lambda = -1$.

COROLLARY 5.21. *The torsion subgroup of* $L_i^p(\mathbf{Z}G, \beta, b)$ *has exponent four, for any finite group G. If* $i \not\equiv 3 \pmod 4$ (resp. $i \not\equiv 1 \pmod 4$) *or the geometric antistructure induced on any hyperelementary subgroup has no factors of type* OK(**C**) (*resp. type* SpK(**C**)), *then the torsion subgroup of* $L_i^p(\mathbf{Z}G, \beta, b)(d)$ *has exponent two.*

5.22. THE ARF SUBGROUP OF $L_{2i}^p(\mathbf{Z}G, \beta, b)$

In [9], Clauwens introduced the Arf subgroup Arf_{2i}^h of $L_{2i}^h(\mathbf{Z}G, w)$ as the kernel of the map to the L-theory of even symmetric forms. We might wonder how much of the group $L_{2i}^h(\hat{\mathbf{Z}}_2 G, w)$ is in the image of the Arf subgroup (this affects extensions since the Arf subgroup has exponent 2). More generally we could try to explicitly give generators for the torsion subgroup of $L_{2i}^p(\mathbf{Z}G, \beta, b)$, or at least the elements with Arf invariant one, using the image of this Arf subgroup. Note that this question is related to the computation of $\ker(L_{2i}^p(\mathbf{Z}G, \min) \to L_{2i}^p(\mathbf{Z}G, \max))$, by applying [1, Theorem 1 & Cor. 4b].

The following example (provided by Bob Oliver) shows that the Clauwens subgroup will not give enough elements for this purpose. Let σ_1 be a central extension with normal subgroup $H = (\mathbf{Z}/2)^6$ and $\sigma_1/H = (\mathbf{Z}/2)^4$. The six possible commutators of lifts of the nontrivial elements of σ_1/H give a basis for H. Then we define an extension $\sigma = \langle \sigma_1, g_0 \rangle$ of σ_1 by setting g_0^2 equal to any nonsquare of σ_1 lying in H. Let g_0 act trivially on the elements of σ_1. Then for $G = \mathbf{Z}/d \rtimes \sigma$ with the standard oriented antistructure, the map $\Psi_2 = 0$ so there are Arf invariant one elements in the L-group but $\text{Arf}_2^h = 0$.

Acknowledgement

We would like to thank Tony Bak, Steffen Bentzen, Manfred Kolster, Erkki Laitinen, Bob Oliver, and Andrew Ranicki for useful conversations.

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