

# EQUIVARIANT PRINCIPAL BUNDLES OVER SPHERES AND COHOMOGENEITY ONE MANIFOLDS

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## 1. Introduction

Let  $\Pi$  and  $G$  be Lie groups. A principal  $(\Pi, G)$ -bundle is a locally trivial, principal  $G$ -bundle  $p: E \rightarrow X$  such that  $E$  and  $X$  are left  $\Pi$ -spaces. The projection map  $p$  is  $\Pi$ -equivariant and  $\gamma(e \cdot g) = (\gamma e) \cdot g$ , where  $\gamma \in \Pi$  and  $g \in G$ , acts on  $e \in E$  by the principal action. Equivariant principal bundles, and their natural generalizations, were studied by T. E. Stewart [24], T. tom Dieck [5; 6, I (8.7)], R. Lashof [15, 16] together with P. May [17] and G. Segal [18].

These authors study equivariant principal bundles by homotopy-theoretic methods. There exists a classifying space  $B(\Pi, G)$  for principal  $(\Pi, G)$ -bundles [5]; so the classification of equivariant bundles in particular cases can be approached by studying the  $\Pi$ -equivariant homotopy type of  $B(\Pi, G)$ . If the structural group  $G$  of the bundle is *abelian*, then the main result of [18] states that equivariant bundles over a  $\Pi$ -space  $X$  are classified by the ordinary homotopy classes of maps  $[X \times_{\Pi} E\Pi, BG]$ . In practice, this program leads to an obstruction theory rather than a classification. See, however, the results of Lashof in the special cases where  $\Pi$  acts semi-freely [14] or transitively [16] on the base space  $X$ .

Another approach to equivariant principal bundles uses the ‘local’ invariants arising from isotropy representations at singular points of  $(X, \Pi)$ , together with equivariant gauge theory [1, 8, 9, 10]. By an isotropy representation at a  $\Pi$ -fixed point  $x_0 \in X$  we mean the homomorphism  $\alpha_{x_0}: \Pi \rightarrow G$  defined by the formula

$$\gamma \cdot e_0 = e_0 \cdot \alpha(\gamma)$$

where  $e_0 \in p^{-1}(x_0)$ . The homomorphism  $\alpha$  is independent of the choice of  $e_0$  up to conjugation in  $G$ . The relationship between the local invariants and the homotopy classification (in the form of a Localization Theorem?) deserves further study.

In this paper, we use the second approach for  $\Pi = \mathrm{SO}(n)$  acting in the standard way on  $X = S^n$ . In this concrete situation, we obtain a complete classification by relatively elementary geometric methods. It turns out that the local isotropy representations at the north and south poles of  $S^n$  explicitly determine the classification of  $(\mathrm{SO}(n), G)$  principal bundles over  $S^n$  (for short  $(n, G)$ -bundles). One surprising consequence is that the set  $\mathcal{E}(n, G)$  of  $(n, G)$ -bundles is finite for  $n \geq 3$ . In contrast, the set of (non-equivariant) principal  $G$ -bundles over  $S^n$  is often infinite. A detailed statement of these results is given in the next section and their proofs, essentially self-contained, are explained in §§ 3 to 6. Several examples are given in § 7. In § 8 we show how these results fit into the more general setting of equivariant  $(\Pi, G)$ -bundles over certain  $\Pi$ -manifolds studied by

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K. Jänich [13] and E. Straume [25]. In particular, we obtain a classification of  $(\Pi, G)$ -bundles over manifolds with cohomogeneity 1.

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## 2. Statement of results

Let  $S^n$  be the  $n$ -dimensional sphere of radius 1 in  $\mathbb{R}^{n+1}$ . We consider the action on  $S^n$  of the group  $\mathrm{SO}(n)$ , by orthogonal transformations fixing the poles  $(0, \dots, 0, \pm 1)$ .

Let  $G$  be a Lie group. We denote by  $\mathcal{R}(n, G)$  the set of smooth homomorphisms from  $\mathrm{SO}(n)$  to  $G$  modulo the conjugations by elements of  $G$ . Unless specified, all maps between manifolds are smooth of class  $C^\infty$ .

By a  $G$ -principal bundle  $\eta$  over  $S^n$ , we mean, as usual, a smooth map  $p: E \rightarrow S^n$  from a manifold  $E = E(\eta)$  and a free right action  $E \times G \rightarrow E$  so that  $p(z \cdot g) = p(z)$  with the standard local triviality condition. The isomorphism classes of  $G$ -bundles over  $S^n$  are in bijection with  $\pi_{n-1}(G)/\pi_0(G)$ , the quotient of the homotopy group  $\pi_{n-1}(G)$  (based at the identity element  $e$  of  $G$ ) by the action of  $\pi_0(G)$  induced by the conjugation of  $G$  on itself. The bijection associates to a bundle  $\eta$  the class  $C(\eta) := [\partial(\mathrm{id}_{S^n})] \in \pi_{n-1}(G)/\pi_0(G)$ , where  $\partial: \pi_n(S^n) \rightarrow \pi_{n-1}(G)$  is the boundary operator in the homotopy exact sequence of  $\eta$  [23, Theorem 18.5].

An  $\mathrm{SO}(n)$ -equivariant principal  $G$ -bundle  $\xi$  over  $S^n$  (or an  $(n, G)$ -bundle for short) is a  $G$ -principal bundle  $\xi^b$  over  $S^n$  together with a left action  $\mathrm{SO}(n) \times E(\xi) \rightarrow E(\xi)$  commuting with the free right action of  $G$  and such that the projection to  $S^n$  is  $\mathrm{SO}(n)$ -equivariant (we write  $E(\xi)$  for  $E(\xi^b)$ ). Two  $(n, G)$ -bundles  $\xi_1$  and  $\xi_2$  are *isomorphic* if there exists a diffeomorphism  $h: E(\xi_1) \rightarrow E(\xi_2)$  which is both  $\mathrm{SO}(n)$  and  $G$ -equivariant and which commutes with the projections to  $S^n$ , inducing the identity on  $S^n$ . We will compute the set  $\mathcal{E}(n, G)$  of isomorphism classes of  $(n, G)$ -bundles.

Let  $\xi$  be an  $(n, G)$ -bundle. Choose points  $a, b \in E(\xi)$  such that  $p(a) = (0, \dots, -1)$  and  $p(b) = (0, \dots, 1)$ . Let  $\alpha$  and  $\beta$  be the maps from  $\mathrm{SO}(n)$  to  $G$  determined by the formulae  $A \cdot a = a \cdot \alpha(A)$  and  $A \cdot b = b \cdot \beta(A)$ . We shall prove in Lemma 3.2 that  $\alpha$  and  $\beta$  are smooth homomorphisms and that their classes in  $\mathcal{R}(n, G)$  depend only on  $[\xi] \in \mathcal{E}(n, G)$ . We call  $\alpha$  and  $\beta$  the *isotropy representations* (associated to  $a$  and  $b$ ). This defines a map  $J: \mathcal{E}(n, G) \rightarrow \mathcal{R}(n, G) \times \mathcal{R}(n, G)$  by  $J(\xi) := ([\alpha], [\beta])$ . When  $n = 2$  and  $G$  is connected,  $J(\xi)$  is a complete invariant which, in particular, determines the (non-equivariant) isomorphism class of  $\xi^b$ . More precisely, let  $\psi: \mathcal{R}(2, G) \times \mathcal{R}(2, G) \rightarrow \pi_1(G)$  be the map determined by  $\psi(\alpha, \beta)(z) := [\alpha(z)\beta(z)^{-1}]$ . To define  $\psi$  we use the identification  $\mathrm{SO}(2) = S^1$  and note that  $\psi$  is well defined if  $G$  is connected.

**THEOREM A.** *Suppose that  $G$  is a connected Lie group. Then,*

- (i) *the map  $J: \mathcal{E}(2, G) \rightarrow \mathcal{R}(2, G) \times \mathcal{R}(2, G)$  is a bijection;*
- (ii) *if  $J(\xi) = ([\alpha], [\beta])$ , then  $\psi(\alpha, \beta) = C(\xi^b)$ .*

We shall now generalize Theorem A by allowing  $n \geq 2$  and  $G$  to be any Lie group. In general,  $J$  is then neither injective nor surjective and  $C(\xi^b)$  is not determined by  $J(\xi)$ . Consider  $\mathrm{SO}(n-1)$  as the subgroup of  $\mathrm{SO}(n)$  fixing the last coordinate. The restriction  $[\mu] \mapsto [\mu|_{\mathrm{SO}(n-1)}]$  gives a map  $\mathrm{Res}: \mathcal{R}(n, G) \rightarrow \mathcal{R}(n-1, G)$ . Denote by

$\mathcal{R}(n, G) \times_{(n-1)} \mathcal{R}(n, G)$  the set of  $([\alpha], [\beta]) \in \mathcal{R}(n, G) \times \mathcal{R}(n, G)$  such that  $\text{Res}[\alpha] = \text{Res}[\beta]$ . If  $\varphi: H \rightarrow G$  is a group homomorphism, we denote by  $Z_\varphi \subset G$  the centralizer of  $\varphi(H)$ .

**THEOREM B.** *Let  $G$  be any Lie group. Then the following hold.*

(i) *The image of  $J$  is  $\mathcal{R}(n, G) \times_{(n-1)} \mathcal{R}(n, G)$ .*

(ii) *Let  $\alpha, \beta: \text{SO}(n) \rightarrow G$  be two smooth homomorphisms such that  $\alpha|_{\text{SO}(n-1)} = \beta|_{\text{SO}(n-1)} =: \gamma$ . Then  $J^{-1}([\alpha], [\beta])$  is in bijection with the set of double cosets  $\pi_0(Z_\alpha) \backslash \pi_0(Z_\gamma) / \pi_0(Z_\beta)$ .*

**2.1. REMARK.** The compatibility statement in Part (i) of Theorem B was also observed by K. Grove and W. Ziller [7, Proposition 1.6]. In §8, Theorem B is extended to a more general setting, to include equivariant principal bundles over ‘special’  $\Pi$ -manifolds in the sense of Jänich [13]. In particular, this provides a classification of the equivariant bundles considered by Grove and Ziller.

Since  $\text{SO}(1)$  is trivial, Theorem B reduces to Part (i) of Theorem A when  $n = 2$ . To determine  $C(\xi^b)$  as in Part (ii) of Theorem A, we must choose particular representatives of  $[\alpha]$  and  $[\beta]$  (in general,  $J(\xi)$  does not determine  $\xi^b$ : see examples 7.2 and 7.5). An *isotropic lifting* for  $\xi$  is a smooth curve  $\tilde{c}: [-1, 1] \rightarrow E(\xi)$  lifting the meridian arc  $c(t) = (0, \dots, \cos(\frac{1}{2}\pi t), \sin(\frac{1}{2}\pi t))$  and such that  $B \cdot \tilde{c}(t) = \tilde{c}(t)\alpha(B)$  for all  $B \in \text{SO}(n-1)$ . Isotropic lifting always exists (see Lemma 3.5). Choosing  $a := \tilde{c}(-1)$  and  $b := \tilde{c}(1)$  leads to isotropy representations  $\alpha, \beta: \text{SO}(n) \rightarrow G$  such that  $\alpha|_{\text{SO}(n-1)} = \beta|_{\text{SO}(n-1)}$ . The map  $\psi(\alpha, \beta): \text{SO}(n) \rightarrow G$  constructed as in Theorem A then satisfies  $\psi(\alpha, \beta)(AB) = \psi(\alpha, \beta)(A)$  when  $B \in \text{SO}(n-1)$ . It thus induces a map

$$\bar{\psi}(\alpha, \beta): S^{n-1} \cong \text{SO}(n)/\text{SO}(n-1) \rightarrow G.$$

Note that  $\bar{\psi}$  is well defined since  $\alpha$  and  $\beta$  are actual homomorphisms and not conjugacy classes.

**PROPOSITION C.** *Let  $\xi$  be an  $(n, G)$ -bundle. Let  $\alpha, \beta: \text{SO}(n) \rightarrow G$  be the isotropy representation associated to the end points of an isotropic lifting. Then,  $[\bar{\psi}(\alpha, \beta)] = C(\xi^b)$  in  $\pi_{n-1}(G)/\pi_0(G)$ .*

We shall prove two consequences of Theorem B and Proposition C which emphasize the contrast between the cases  $n = 2$  and  $n \geq 3$ .

**PROPOSITION D.** *Let  $\eta$  be a principal  $G$ -bundle over  $S^2$  with  $G$  a non-trivial Lie group. Then there exist infinitely many  $\xi \in \mathcal{E}(2, G)$  such that  $\xi^b \cong \eta$ .*

**PROPOSITION E.** *For  $G$  a compact Lie group, the set  $\mathcal{E}(n, G)$  is finite when  $n \geq 3$ .*

These results are proved in §5, while the earlier sections are devoted to preliminary material. In §6, we determine which  $(n, G)$ -bundles come from an  $\text{SO}(n+1)$ -equivariant bundle. Examples are given in §7.

## 3. Preliminary constructions

3.1. *J is well defined.* This follows from the following lemma.

3.2. LEMMA. *Let  $\xi$  be an  $(n, G)$ -bundle. Let  $a, b \in E(\xi)$  such that  $p(a) = (0, \dots, -1)$  and  $p(b) = (0, \dots, 1)$ . Let  $\beta$  and  $\alpha$  be the maps from  $\mathrm{SO}(n)$  to  $G$  determined by the formulae  $A \cdot a = a \cdot \alpha(A)$  and  $A \cdot b = b \cdot \beta(A)$ . Then  $\alpha$  and  $\beta$  are smooth homomorphisms and their class in  $\mathcal{R}(n, G)$  depends only on  $[\xi] \in \mathcal{E}(n, G)$ .*

*Proof.* Let  $A, B \in \mathrm{SO}(n)$ . One has

$$\begin{aligned} a \cdot \alpha(BA) &= (BA) \cdot a = B \cdot (A \cdot a) = B \cdot (a \cdot \alpha(A)) \\ &= (B \cdot a) \cdot \alpha(A) = a \cdot (\alpha(B)\alpha(A)). \end{aligned}$$

Therefore,  $\alpha$  and, similarly,  $\beta$  are homomorphisms. They are smooth because the action of  $\mathrm{SO}(n)$  is smooth. If  $a'$  is another choice instead of  $a$ , there exists  $g \in G$  such that  $a' = a \cdot g$  and one has

$$(3.3) \quad a \cdot (g\alpha'(A)) = a' \cdot \alpha'(A) = A \cdot a' = A \cdot a \cdot g = a \cdot (\alpha(A)g),$$

whence  $\alpha'(A) = g^{-1}\alpha(A)g$ . This proves that the class of  $(\alpha, \beta)$  in  $\mathcal{R}(n, G) \times \mathcal{R}(n, G)$  does not depend on the choice of  $a$  and  $b$ . Now, if  $h: E(\xi) \xrightarrow{\cong} E(\xi')$  is an  $(\mathrm{SO}(n), G)$ -equivariant diffeomorphism over the identity of  $S^n$ , then, by choosing  $a' := h(a)$  and  $b' := h(b)$ , one has  $(\alpha', \beta') = (\alpha, \beta)$ . The proof of Lemma 3.2 is then complete.  $\square$

3.4. *Isotropic liftings.* Let  $I := [-1, 1]$  and  $c: I \rightarrow S^n$  be the parametrisation of the meridian arc  $c(t) = (0, \dots, \cos(\frac{1}{2}\pi t), \sin(\frac{1}{2}\pi t))$ . Let  $\tilde{c}: I \rightarrow E = E(\xi)$  be a (smooth) lifting of  $c$ . As  $c(t)$  is fixed by  $\mathrm{SO}(n-1)$ , one has  $B \cdot \tilde{c}(t) = \tilde{c}(t) \cdot \alpha_t(B)$ , for  $B \in \mathrm{SO}(n-1)$ . As in the proof of Lemma 3.2, one checks that this gives a smooth path  $\alpha_t$  ( $t \in I$ ) of homomorphisms from  $\mathrm{SO}(n-1)$  to  $G$ , which depends on the lifting  $\tilde{c}$ . Call  $\tilde{c}$  *isotropic* if  $\alpha_t$  is constant:  $\alpha_t(B) = \alpha(B)$  for all  $B \in \mathrm{SO}(n-1)$ .

3.5. LEMMA. *Any  $(n, G)$ -bundle admits an isotropic lifting.*

We shall make use of connections on  $(n, G)$ -bundles which are  $\mathrm{SO}(n)$ -invariant. These can be obtained by averaging any connection (see [1, p.522]), since the space of connections is an affine space. If a curve  $u(t)$  in  $E(\xi)$  is horizontal for an  $\mathrm{SO}(n)$ -invariant connection, then  $u(t) \cdot g$  and  $A \cdot u(t)$  are horizontal. Lemma 3.5 then follows from the following.

3.6. LEMMA. *Let  $\xi$  be an  $(n, G)$ -bundle endowed with an  $\mathrm{SO}(n)$ -invariant connection. Then, any lifting  $\tilde{c}$  of  $c$  which is horizontal is isotropic.*

*Proof.* If  $\tilde{c}$  is a horizontal lifting, then so are  $B \cdot \tilde{c}$  and  $\tilde{c} \cdot \alpha(B)$  for  $B \in \mathrm{SO}(n-1)$ . As  $B \cdot \tilde{c}(-1) = \tilde{c}(-1) \cdot \alpha(B)$ , one has  $B \cdot \tilde{c}(t) = \tilde{c}(t) \cdot \alpha(B)$  for all  $t \in I$ .  $\square$

3.7. *The  $(n, G)$ -bundles  $\xi_{\alpha, \beta}$ . If  $X$  is a topological space, the unreduced suspension  $\Sigma X$  is*

$$\Sigma X := I \times X / \{(-1, x) \sim (-1, x') \text{ and } (1, x) \sim (1, x'), \forall x, x' \in X\}.$$

We denote by  $C_-X$  the image of  $[-1, 1) \times X$  in  $\Sigma X$  and by  $C_+X$  that of  $(-1, 1] \times X$ .

Let  $(\alpha, \beta)$  be a pair of smooth homomorphisms from  $\mathrm{SO}(n)$  to  $G$ . Define the space  $\widehat{E}_{\alpha, \beta}$  by

$$\begin{aligned} \widehat{E}_{\alpha, \beta} := I \times \mathrm{SO}(n) \times G / \{ & (-1, A', g) \sim (-1, A, \alpha(A^{-1}A')g) \text{ and} \\ & (1, A', g) \sim (1, A, \beta(A^{-1}A')g), \forall A \in \mathrm{SO}(n) \}. \end{aligned}$$

The space  $\widehat{E}_{\alpha, \beta}$  admits an obvious free right action of  $G$  and a map  $p: \widehat{E}_{\alpha, \beta} \rightarrow \Sigma \mathrm{SO}(n)$ . This makes a principal  $G$ -bundle over  $\Sigma \mathrm{SO}(n)$ ; indeed, trivializations on  $C_{\pm} \mathrm{SO}(n)$  are given by

$$(3.8) \quad \begin{aligned} \widehat{\varphi}_-: [t, A, g] &\mapsto ([t, A], \alpha(A)g) \quad \text{if } -1 \leq t < 1, \\ \widehat{\varphi}_+: [t, A, g] &\mapsto ([t, A], \beta(A)g) \quad \text{if } -1 < t \leq 1. \end{aligned}$$

The change of trivializations is

$$(3.9) \quad \widehat{\varphi}_- \circ \widehat{\varphi}_+^{-1}([t, A], g) = ([t, A], \alpha(A)\beta(A)^{-1}g).$$

Now, suppose that  $\alpha|_{\mathrm{SO}(n-1)} = \beta|_{\mathrm{SO}(n-1)}$ . Form the space  $E_{\alpha, \beta}$  as the quotient

$$E_{\alpha, \beta} := \widehat{E}_{\alpha, \beta} / \{ [t, AB, g] \sim [t, A, \alpha(B)g], \forall B \in \mathrm{SO}(n-1) \}.$$

Let  $\varepsilon: \mathrm{SO}(n) \rightarrow S^{n-1}$  be the map which associates to a matrix its last column; it is also the projection  $\varepsilon: \mathrm{SO}(n) \rightarrow \mathrm{SO}(n)/\mathrm{SO}(n-1) \cong S^{n-1}$ . There are a map  $p: E_{\alpha, \beta} \rightarrow \Sigma S^{n-1}$ , given by  $p([t, A, g]) = [t, \varepsilon(A)]$ , and a free  $G$ -action, given by  $[t, A, g] \cdot g_1 := [t, A, gg_1]$ . As above, we check that this defines a  $G$ -principal bundle over  $\Sigma S^{n-1}$ ; the trivializations  $\widehat{\varphi}_{\pm}$  descend to trivializations  $\check{\varphi}_{\pm}$  over  $C_{\pm} S^{n-1}$ .

The map  $[t, A] \mapsto A \cdot c(t)$  descends to a homeomorphism  $f: \Sigma S^{n-1} \cong S^n$ . By replacing  $p$  by  $f \circ p$ , we obtain a (topological) principal  $G$ -bundle

$$\xi_{\alpha, \beta}: E_{\alpha, \beta} \xrightarrow{p} S^n.$$

Let  $S_{\pm}^n$  be the punctured spheres  $S_{\pm}^n := f(C_{\pm} S^{n-1})$ . The trivializations given by the compositions

$$(3.10) \quad \varphi_{\pm}: p^{-1}(S_{\pm}^n) \xrightarrow{\check{\varphi}_{\pm}} C_{\pm} S^{n-1} \times G \xrightarrow{f \times \mathrm{id}} S_{\pm}^n \times G$$

are homeomorphisms from  $p^{-1}(S_{\pm}^n)$  onto manifolds. The change of trivialization is a diffeomorphism, being obtained by conjugating that of (3.9) by  $f$ . Therefore,  $\varphi_{\pm}$  produce a smooth manifold structure on  $E_{\alpha, \beta}$ . The map  $p$  and the  $G$  action are smooth. One checks that the map

$$\mathrm{SO}(n) \times \widehat{E}_{\alpha, \beta} \rightarrow \widehat{E}_{\alpha, \beta} \quad \text{given by } C \cdot [t, A, g] := [t, CA, g]$$

descends to a smooth  $\mathrm{SO}(n)$ -action on  $E_{\alpha, \beta}$  which makes  $\xi_{\alpha, \beta}$  an  $(n, G)$ -bundle.

**3.11. Proof of Part (i) of Theorem B.** Let  $\xi$  be an  $(n, G)$ -bundle. By Lemma 3.5 there exists an isotropic lifting  $\tilde{c}: I \rightarrow E(\xi)$  of  $c$ . Choosing  $a := \tilde{c}(-1)$  and  $b := \tilde{c}(1)$  produces a representative  $(\alpha, \beta)$  of  $J(\xi)$  with  $\alpha|_{\mathrm{SO}(n-1)} = \beta|_{\mathrm{SO}(n-1)}$ . Therefore, the image of  $J$  is contained in  $\mathcal{R}(n, G) \times_{(n-1)} \mathcal{R}(n, G)$ .

Conversely, a class  $P \in \mathcal{R}(n, G) \times_{(n-1)} \mathcal{R}(n, G)$  is represented by a pair  $(\alpha, \beta)$  with  $\alpha|_{\mathrm{SO}(n-1)} = \beta|_{\mathrm{SO}(n-1)}$ . Let  $\mathbf{1}$  be the identity matrix in  $\mathrm{SO}(n)$  and  $e$  be the

unit element of  $G$ . Computing  $J(\xi_{\alpha,\beta})$  with the points  $a := [-1, \mathbf{1}, e]$  and  $b := [1, \mathbf{1}, e]$  in  $E_{\alpha,\beta}$  shows that  $J(\xi_{\alpha,\beta}) = P$ .  $\square$

#### 4. The map $\tilde{J}_\gamma$

Let  $\gamma: \text{SO}(n-1) \rightarrow G$  be a smooth homomorphism. Define a set  $\mathcal{R}_\gamma(n, G)$  as follows: an element of  $\mathcal{R}_\gamma(n, G)$  is represented by a pair  $(\alpha, \beta)$  of smooth homomorphisms from  $\text{SO}(n)$  to  $G$  such that  $\alpha|_{\text{SO}(n-1)} = \beta|_{\text{SO}(n-1)} = \gamma$ . Two pairs  $(\alpha_1, \beta_1)$  and  $(\alpha_2, \beta_2)$  represent the same element of  $\mathcal{R}_\gamma(n, G)$  if there is a smooth path  $\{g_t \mid t \in [-1, 1]\}$  in the centralizer  $Z_\gamma$  of  $\gamma(\text{SO}(n-1))$  such that  $\alpha_2(A) = g_{-1}\alpha_1(A)g_{-1}^{-1}$  and  $\beta_2(A) = g_1\beta_1(A)g_1^{-1}$ . There is an obvious map  $j: \mathcal{R}_\gamma(n, G) \rightarrow \mathcal{R}(n, G) \times_{(n-1)} \mathcal{R}(n, G)$ .

Part (i) of Theorem B, already proven in (3.11), permits us to define a map  $\bar{J}: \mathcal{E}(n, G) \rightarrow \mathcal{R}(n-1, G)$  by  $\bar{J}(\xi) := \text{Res}[\alpha] = \text{Res}[\beta]$ . We shall now compute the preimage  $\bar{J}^{-1}([\gamma])$ .

4.1. PROPOSITION. *Let  $\gamma: \text{SO}(n-1) \rightarrow G$  be a smooth homomorphism. Then there exists a bijection  $\tilde{J}_\gamma: \bar{J}^{-1}([\gamma]) \cong \mathcal{R}_\gamma(n, G)$  such that  $j \circ \tilde{J}_\gamma = J$ .*

The proof divides into several steps.

4.2. *Definition of  $\tilde{J}_\gamma$ .* Let  $\xi$  be an  $(n, G)$ -bundle with  $\bar{J}(\xi) = [\gamma]$ . Choose, using Lemma 3.5, an isotropic lifting  $\tilde{c}_0: I \rightarrow E(\xi)$  of  $c$ . As  $\bar{J}(\xi) = [\gamma]$ , the constant path  $\alpha_t^0: \text{SO}(n-1) \rightarrow G$  associated to  $\tilde{c}_0$  is conjugated to  $\gamma$ : there exists  $g \in G$  such that  $\alpha_t^0(B) = g\gamma(B)g^{-1}$ . Let  $\tilde{c} := \tilde{c}_0 \cdot g$ . As in equation (3.3), one checks that  $\tilde{c}$  is  $\gamma$ -isotropic, that is,  $\alpha_t = \gamma$ . Choosing  $a := \tilde{c}(-1)$  and  $b := \tilde{c}(1)$  then produces a pair  $(\alpha, \beta)$  of smooth homomorphisms from  $\text{SO}(n)$  to  $G$  which represents a class  $\tilde{J}_\gamma(\xi)$  in  $\mathcal{R}_\gamma(n, G)$ .

To see that  $\tilde{J}_\gamma$  is well defined, let  $\tilde{c}'$  be another  $\gamma$ -isotropic lifting of  $c$ . The smooth path  $t \mapsto g_t \in G$  defined by  $\tilde{c}'(t) = \tilde{c}(t) \cdot g_t$  satisfies

$$\gamma(B) = \alpha'_t(B) = g_t^{-1}\alpha_t(B)g_t = g_t^{-1}\gamma(B)g_t$$

for all  $B \in \text{SO}(n-1)$ . Therefore,  $g_t \in Z_\gamma$ . One has  $\alpha'(A) = g_{-1}^{-1}\alpha_t(A)g_{-1}$  and  $\beta'(A) = g_1^{-1}\beta_t(A)g_1$ , for all  $A \in \text{SO}(n)$ , which proves that  $\tilde{J}_\gamma(\xi)$  does not depend on the choice of a  $\gamma$ -isotropic lifting.

Now, if  $h: E(\xi) \cong E(\xi')$  is an  $(\text{SO}(n), G)$ -equivariant diffeomorphism over the identity of  $S^n$  and  $\tilde{c}: I \rightarrow E(\xi)$  is a  $\gamma$ -isotropic lifting for  $\xi$ , then  $c' := h \circ \tilde{c}$  is a  $\gamma$ -isotropic lifting for  $\xi'$  giving  $(\alpha', \beta') = (\alpha, \beta)$ . This proves that  $\tilde{J}_\gamma$  is well defined.

4.3. *Surjectivity of  $\tilde{J}_\gamma$ .* Let  $(\alpha, \beta)$  represent a class  $P$  in  $\mathcal{R}_\gamma(n, G)$ . One checks that  $\tilde{J}_\gamma(\xi_{\alpha,\beta}) = P$ , using the fact that the path  $t \mapsto [t, \mathbf{1}, e]$  is a  $\gamma$ -isotropic lifting for  $\xi_{\alpha,\beta}$ .

4.4. *Injectivity of  $\tilde{J}_\gamma$ .* Let  $\gamma: \text{SO}(n-1) \rightarrow G$  be a smooth homomorphism, and let  $\xi$  be an  $(n, G)$ -bundle with  $\bar{J}(\xi) = [\gamma]$ . There exists  $a \in E(\xi)$  with  $p(a) = (0, \dots, -1)$  and  $B \cdot a = a \cdot \gamma(B)$  for all  $B \in \text{SO}(n-1)$ . Choose an  $\text{SO}(n)$ -invariant connection on  $\xi$  and let  $\tilde{c}$  be a horizontal lifting of  $c$  with  $\tilde{c}(-1) = a$ . By Lemma 3.6,  $\tilde{c}$  is  $\gamma$ -isotropic. If  $\beta: \text{SO}(n) \rightarrow G$  is defined by  $A \cdot \tilde{c}(1) = \tilde{c}(1) \cdot \beta(A)$ , then  $(\alpha, \beta)$  represents  $\tilde{J}_\gamma(\xi)$ .

Consider the map  $\widehat{\lambda}: \widehat{E}_{\alpha,\beta} \rightarrow E(\xi)$  given by

$$\widehat{\lambda}([t, A, g]) := A \cdot \widetilde{c}(t) \cdot g.$$

The map  $\widehat{\lambda}$  descends to a continuous map  $\lambda: E_{\alpha,\beta} \rightarrow E(\xi)$  which is both  $\mathrm{SO}(n)$  and  $G$ -equivariant and which covers the identity of  $S^n$ . Therefore  $\xi$  and  $\xi_{\alpha,\beta}$  are isomorphic as topological  $(n, G)$ -bundles. What remains to prove is that  $\lambda$  is a diffeomorphism, which is clear except possibly in a neighbourhood of the fibers  $E_{\pm}$  above the north and south poles.

The connection on  $\xi$  provides a smooth trivialization of  $\xi$  restricted to the punctured sphere  $S^n_-$  (see §3.7) in the following way. Consider the map  $s_-: p^{-1}(S^n_-) \rightarrow E_-$  assigning to  $z$  the end point in  $E_-$  of the horizontal path through  $z$  above the meridian arc through  $p(z)$ . Define the  $G$ -equivariant map  $\sigma_-: p^{-1}(S^n_-) \rightarrow G$  by  $s_-(z) = a \cdot \sigma_-(z)$ . The required trivialization  $\tau_-: p^{-1}(S^n_-) \rightarrow S^n_- \times G$  is  $\tau_-(z) := (p(z), \sigma_-(z))$ .

Take the trivialization  $\varphi_-$  for  $\xi_{\alpha,\beta}$  defined in formulae (3.10) of §3.7. As  $\widetilde{c}$  is horizontal, one has

$$\tau_- \circ \lambda \circ \varphi_-^{-1}(x, g) = (x, g).$$

This, and the same for  $E_+$ , prove that  $\lambda$  is a diffeomorphism. We have thus established that if  $\widetilde{J}_\gamma(\xi)$  is represented by  $(\alpha, \beta)$  then the  $(n, G)$ -bundle  $\xi$  is isomorphic to  $\xi_{\alpha,\beta}$ , which proves the injectivity of  $\widetilde{J}_\gamma$ .

The proof of Proposition 4.1 is now complete.  $\square$

## 5. Proof of the main results

This section contains the proofs of the results stated in §2.

**5.1. Proof of Theorem B.** Part (i) has already been proven in §3.11. We shall now prove Part (ii). Let  $\alpha, \beta: \mathrm{SO}(n) \rightarrow G$  be two smooth homomorphisms such that  $\alpha|_{\mathrm{SO}(n-1)} = \beta|_{\mathrm{SO}(n-1)} = \gamma$ . The pair  $(\alpha, \beta)$  defines a class  $[\alpha, \beta] \in \mathcal{R}_\gamma(n, G)$ . The group  $Z_\gamma \times Z_\gamma$  acts on  $\mathcal{R}_\gamma(n, G)$  by

$$(g, h) \cdot [\alpha, \beta] := [g\alpha g^{-1}, h\beta h^{-1}].$$

The set  $J^{-1}([\alpha], [\beta]) \subset \bar{J}^{-1}([\gamma])$  is in bijection with an orbit of the above action via the bijection  $\widetilde{J}_\gamma: \bar{J}^{-1}([\gamma]) \cong \mathcal{R}_\gamma(n, G)$  of Proposition 4.1. Let ‘ $\sim$ ’ be the equivalence relation on  $Z_\gamma \times Z_\gamma$  defined by  $(g_1, h_1) \sim (g_2, h_2)$  if and only if  $(g_1, h_1)[\alpha, \beta] = (g_2, h_2)[\alpha, \beta]$ . Let

$$\phi: Z_\gamma \times Z_\gamma \rightarrow \pi_0(Z_\alpha) \setminus \pi_0(Z_\gamma) / \pi_0(Z_\beta)$$

be the map defined by  $\phi(g, h) := [g^{-1}h]$ . Part (ii) of Theorem B then follows from the following lemma.

**5.2. LEMMA.** *The equivalence  $(g_1, h_1) \sim (g_2, h_2)$  holds if and only if  $\phi(g_1, h_1) = \phi(g_2, h_2)$ .*

*Proof.* Suppose that  $(g_1, h_1) \sim (g_2, h_2)$ . This means that there exist  $s_-, s_+ \in Z_\gamma$ , with  $[s_-] = [s_+]$  in  $\pi_0(Z_\gamma)$ , such that the equality

$$(g_1 \alpha g_1^{-1}, h_1 \beta h_1^{-1}) = (s_- g_2 \alpha g_2^{-1} s_-^{-1}, s_+ h_2 \beta h_2^{-1} s_+^{-1})$$

holds in  $Z_\gamma \times Z_\gamma$ . This implies that

$$g_1 = s_- g_2 A \quad \text{and} \quad h_1 = s_+ h_2 B$$

with  $A \in Z_\alpha$  and  $B \in Z_\beta$  (the centralizers of the images of  $\alpha$  and  $\beta$ ). Therefore  $g_1^{-1} h_1 = A^{-1} g_2^{-1} s_-^{-1} s_+ h_2 B$ , which implies that  $\phi(g_1, h_1) = \phi(g_2, h_2)$ .

To prove the converse, observe that

- (i)  $(g, h) \sim (Cg, Ch)$  for  $C \in Z_\gamma$ ,
- (ii)  $(g, h) \sim (gA, hB)$  for  $A \in Z_\alpha$  and  $B \in Z_\beta$ ,
- (iii)  $(g, h) \sim (g, uh)$  for  $u$  in the identity component of  $Z_\gamma$ .

Suppose that  $\phi(g_1, h_1) = \phi(g_2, h_2)$ . This means that there are  $A \in Z_\alpha$ ,  $B \in Z_\beta$  and  $u$  in the identity component of  $Z_\gamma$  such that  $g_1^{-1} h_1 = A^{-1} g_2^{-1} u h_2 B$  ( $u$  can be put in the middle since the identity component of  $Z_\gamma$  is a normal subgroup of  $Z_\gamma$ ). One then has

$$(g_1, h_1) \sim (e, g_1^{-1} h_1) = (e, A^{-1} g_2^{-1} u h_2 B) \sim (g_2 A, u h_2 B) \sim (g_2, h_2). \quad \square$$

*Proof of Proposition C.* Let  $\xi$ ,  $\alpha$  and  $\beta$  be as in the statement of Proposition C. Let  $\gamma := \alpha|_{\text{SO}(n-1)} = \beta|_{\text{SO}(n-1)}$ . Then,  $\xi \in \tilde{J}^{-1}([\gamma])$  and, by § 4.2, one has  $\tilde{J}_\gamma(\xi) = [\alpha, \beta]$  in  $\mathcal{R}_\gamma(n, G)$ . By § 4.4,  $\xi = \xi_{\alpha, \beta}$ . Therefore,  $C(\xi^b) = C(\xi_{\alpha, \beta}^b) = [\tilde{\psi}(\alpha, \beta)]$ , where the last equality comes from equation (3.9) of § 3.7 and the fact that  $C(\xi^b)$  can be represented by its characteristic map [23, Theorem 18.4].  $\square$

*Proof of Theorem A.* Since  $\text{SO}(1)$  is trivial,  $Z_\gamma = G$  which is supposed to be connected. Therefore, Part (i) is a particular case of Part (i) of Theorem B. Let  $c$  map  $I$  to  $S^n$  parametrizing the meridian arc, as in § 3.4. Let  $\alpha, \beta: \text{SO}(2) \rightarrow G$  be two homomorphisms representing  $J(\xi)$ . One can find  $a$  and  $b$  so that  $\alpha$  and  $\beta$  are the isotropy representations associated to  $a$  and  $b$ . As  $G$  is connected, the submanifold  $P_0 := p^{-1}(c(I))$  of  $E(\xi)$  is connected and there is a smooth lifting  $\tilde{c}$  of  $c$  such that  $\tilde{c}(-1) = a$  and  $\tilde{c}(1) = b$ . As  $\text{SO}(1)$  is trivial,  $\tilde{c}$  is isotropic. Part (ii) of Theorem A then follows from Proposition C.  $\square$

*Proof of Proposition D.* Recall that any element of  $\pi_1(G, e)$  can be represented by a homomorphism (a geodesic in a maximal compact subgroup  $K$  of  $G$ , with a  $K$ -bi-invariant Riemannian metric, being a 1-parameter subgroup [11, Chapter IV, § 6]). Therefore, if  $\eta$  is a  $G$ -bundle over  $S^2$ , there exists a homomorphism  $\alpha: S^1 \rightarrow G$  such that  $C(\eta) = [\alpha]$ . For  $q \in \mathbb{N}$ , let  $\alpha_q: S^1 \rightarrow G$  be given by  $\alpha_q(z) := \alpha(z)^q$ . If  $\alpha$  is not trivial, the classes  $[\alpha_q]$  are all distinct in  $\mathcal{R}(2, G)$ . Indeed, the set  $\mathcal{R}(2, G)$  is in bijection with lattice points in a Weyl chamber of the Lie algebra of a maximal torus of  $G$  and the point representing  $\alpha_q$  is  $q$  times those representing  $\alpha$ .

Suppose first that  $\eta$  is not trivial. Hence,  $\alpha$  is not trivial and  $[\alpha_{q+1}, \alpha_q]$  are all different classes in  $\mathcal{R}_\gamma(2, G)$  with  $[\tilde{\psi}(\alpha_{q+1}, \alpha_q)] = C(\eta)$ . The result then follows from Propositions C and 4.1.

When  $\eta$  is trivial, one takes any non-trivial homomorphism  $\alpha: \text{SO}(2) \rightarrow G$ . The classes  $[\alpha_q, \alpha_q]$  in  $\mathcal{R}_\gamma(2, G)$  represent infinitely many distinct  $\text{SO}(2)$ -equivariant  $G$ -bundles  $\xi_q$  with trivial  $\xi_q^b$ .  $\square$

*Proof of Proposition E.* If  $n \geq 3$ , the group  $\text{SO}(n)$  is semi-simple and the set



$\mathcal{R}(n, G)$  is finite. The latter follows from the following known results:

- (i) a homomorphism is determined by its tangent map at the identity (as a homomorphism of Lie algebras);
- (ii) the Lie algebra of  $G$  contains only finitely many semi-simple Lie subalgebras, up to inner automorphism [21, Proposition 12.1];
- (iii) there are only finitely many homomorphisms between two semi-simple Lie algebras, modulo inner automorphisms.

If  $G$  is compact, then the group  $Z_\gamma$  is compact and  $\pi_0(Z_\gamma)$  is finite. Proposition E then follows from Theorem B. □

5.3. REMARK. To remove the hypothesis ‘ $G$  compact’ from Proposition E, it is enough to consider the case  $G$  connected. Indeed,  $\mathcal{R}_\gamma(n, G)$  is a quotient of  $\mathcal{R}_\gamma(n, G_e)$ , where  $G_e$  is the connected component of  $e$ . One would then need the following kind of result: if  $H$  is a compact Lie subgroup of a connected Lie group  $G$ , then  $\pi_0(Z(H))$  is finite. We do not know whether this is true.

### 6. $\mathrm{SO}(n + 1)$ -equivariant bundles

In this section, we describe the  $(n, G)$ -bundles which are  $\mathrm{SO}(n + 1)$ -equivariant  $G$ -bundles, for the natural action of  $\mathrm{SO}(n + 1)$  on  $S^n$ . Let  $\delta: \mathrm{SO}(n) \cong \mathrm{SO}(n)$  be the conjugation by the diagonal  $(n \times n)$ -matrix  $\mathrm{diag}(1, \dots, 1, -1)$  (or, equivalently,  $\mathrm{diag}(-1, \dots, -1, 1)$ ). If  $\alpha: \mathrm{SO}(n) \rightarrow G$  is a smooth homomorphism, observe that  $\mathrm{Res}[\alpha] = \mathrm{Res}[\alpha \circ \delta]$  in  $\mathcal{R}(n - 1, G)$ .

6.1. THEOREM. *Let  $\xi$  be an  $(n, G)$ -bundle. If  $\xi$  comes from an  $\mathrm{SO}(n + 1)$ -equivariant  $G$ -bundle then  $J(\xi)$  is of the form  $([\alpha], [\alpha \circ \delta])$ .*

*For any  $[\alpha] \in \mathcal{R}(n, G)$  there is a unique  $\xi \in \mathcal{E}(n, G)$  which comes from an  $\mathrm{SO}(n + 1)$ -equivariant  $G$ -bundle and is such that  $J(\xi) = ([\alpha], [\alpha \circ \delta])$ .*

*Proof.* For  $\theta \in [0, \pi]$ , let  $R_\theta \in \mathrm{SO}(n + 1)$  be the rotation of angle  $\theta$  in the plane of the last two coordinates. Let  $R := R_\pi$ , the diagonal matrix with entries  $(1, \dots, 1, -1, -1)$ .

Let  $\xi$  be an  $\mathrm{SO}(n + 1)$ -equivariant bundle. Choose  $a, b \in E(\xi)$ , with  $p(a) = (0, \dots, -1)$  and let  $b := R \cdot a$ . For  $A \in \mathrm{SO}(n)$ , one has  $R^{-1}AR = \delta(A)$  and

$$\begin{aligned} (6.2) \quad b\beta(A) &= A \cdot b = A \cdot (R \cdot a) = R \cdot (R^{-1}AR) \cdot a \\ &= R \cdot a\alpha(\delta(A)) = b\alpha(\delta(A)), \end{aligned}$$

whence  $\beta = \alpha \circ \delta$ , which proves Part (i).

Let  $\alpha: \mathrm{SO}(n) \rightarrow G$  be a smooth homomorphism and set  $\gamma := \alpha|_{\mathrm{SO}(n-1)}$ . Suppose that  $\xi$  is an  $\mathrm{SO}(n + 1)$ -equivariant  $G$ -bundle with  $a \in E(\xi)$  such that  $p(a) = (0, \dots, -1)$  and  $A \cdot a = a\alpha(A)$  for  $A \in \mathrm{SO}(n)$ . Then  $\xi \in \tilde{J}^{-1}([\gamma])$ . Let  $c: I \rightarrow E(\xi)$  be the curve  $c(t) := R_{\theta(t)} \cdot a$ , where  $\theta(t) := \frac{1}{2}\pi(t - 1)$ . Using the relation  $R_\theta^{-1}BR_\theta = B$  for  $B \in \mathrm{SO}(n - 1)$ , one checks, as in equation (6.2), that  $c$  is a  $\gamma$ -isotropic lifting. By §4.2 and equation (6.2), one has  $\tilde{J}_\gamma(\xi) = ([\alpha], [\alpha \circ \delta])$  in  $\mathcal{R}_\gamma(n, G)$ . By Proposition 4.1, this proves that  $\xi$  is determined by  $\alpha$ , which proves the uniqueness statement of Part (ii).

It remains to construct, for a smooth homomorphism  $\alpha: \mathrm{SO}(n) \rightarrow G$ , an  $\mathrm{SO}(n + 1)$ -equivariant  $G$ -bundle with  $J(\xi) = ([\alpha], [\alpha \circ \delta])$ . Consider the map

$p: \text{SO}(n+1) \rightarrow S^n$  sending a matrix to its last column vector. This makes an  $\text{SO}(n+1)$ -equivariant  $\text{SO}(n)$ -bundle (the principal  $\text{SO}(n)$ -bundle associated to the tangent bundle of  $S^n$ ). Let  $\xi$  be the  $G$ -bundle obtained by the Borel construction, using the homomorphism  $\alpha \circ \delta: \text{SO}(n) \rightarrow G$ ,

$$(6.3) \quad E(\xi) := \text{SO}(n+1) \times G / \{(BA, g) = (B, \alpha(\delta(A))g)\}.$$

This is an  $\text{SO}(n+1)$ -equivariant  $G$ -bundle. Choosing  $a := (R, e)$  and  $b := (I, e)$ , one sees that

$$A \cdot a = (AR, e) = (R\delta(A), e) = (R, \alpha(A)e) = \alpha\alpha(A)$$

and

$$A \cdot b = (A, e) = (I, \alpha(\delta(A))e) = b\alpha(\delta(A)).$$

Therefore,  $\xi$  is an  $\text{SO}(n+1)$ -equivariant  $G$ -bundle with  $J(\xi) = ([\alpha], [\alpha \circ \delta])$ .  $\square$

6.4. REMARK. Theorem 6.1 and its proof show that any  $\text{SO}(n+1)$ -equivariant  $G$ -bundle is derived from the tangent bundle to  $S^n$  by the Borel construction (formula (6.3)). This can be compared with [10, § 6].

### 7. Examples and applications

NOTATION. If  $X$  is a set, we denote by  $\Delta X$  the diagonal in  $X \times X$ .

7.1.  $n = 2$  and  $G = U(m)$ . A homomorphism  $\alpha: S^1 \rightarrow U(m)$  has, up to conjugacy, a unique diagonal form  $\alpha(z) = \text{diag}(z^{p_1}, \dots, z^{p_m})$ , with  $p_1 \geq \dots \geq p_m$ . The same holds for  $\beta$ . By Theorem A,  $\mathcal{E}(2, U(m))$  is then in bijection with the set of pairs  $(p, q)$  of  $m$ -tuples of non-increasing integers. In  $\pi_1(U(m)) = \mathbb{Z}$ , one has  $[\alpha] = \sum_{i=1}^m p_i$  and  $[\beta] = \sum_{i=1}^m q_i$ , so, by Proposition C,

$$C(\xi^b) = \sum_{i=1}^m (p_i - q_i).$$

If one wishes instead to characterize  $\xi^b$  by its first Chern number  $c(\xi^b) \in H^2(S^2) = \mathbb{Z}$ , then  $c(\xi^b) = -C(\xi^b)$  [19, p. 445].

For instance, if  $\tau$  is the unit tangent bundle over  $S^2$  with the natural action, then  $\alpha(z) = z^{-1}$ ,  $\beta(z) = z$ , and so  $C(\tau) = -2$  and  $c(\tau) = 2 = \chi(S^2)$ .

Note that if  $\xi$  comes from an  $\text{SO}(3)$ -bundle, then, by Theorem 6.1, one has  $q_i = -p_i$  (as for  $\tau$  above). In particular  $c(\xi^b)$  must be even (see also [10, (6.3)]).

7.2.  $n = 2$  and  $G = O(2)$ . For  $q \in \mathbb{Z}$ , let  $\alpha_q: \text{SO}(2) \rightarrow O(2)$  be the homomorphism  $A \rightarrow A^q$ . The set  $\mathcal{R}(2, O(2))$  is in bijection with  $\mathbb{N}$  given by  $\alpha_q \mapsto |q|$ . The same recipe produces a bijection  $\pi_1(O(2), e) / \pi_0(O(2)) \cong \mathbb{N}$ . Let  $\xi_{p,q} := \xi_{\alpha_p, \alpha_q}$  be the  $(2, O(2))$ -bundles constructed in § 3.7. By Proposition 4.1 and § 4.3, each  $\mathcal{E}(2, O(2))$  is represented by some  $\xi_{p,q}$ , with the only relation  $\xi_{p,q} = \xi_{-p,-q}$ . One has  $J(\xi_{p,q}) = (|p|, |q|)$  and  $C(\xi_{p,q}) = |p - q|$ . Therefore,  $J^{-1}(r, s)$  contains one element if  $rs = 0$  and two otherwise.

7.3.  $n = 2$  and  $G = \text{SO}(m)$  with  $m \geq 3$ . A maximal torus of  $\text{SO}(m)$  is formed by matrices containing 2-blocks concentrated around the diagonal, and so is isomorphic to  $\text{SO}(2)^k$  where  $k = \lfloor \frac{1}{2}m \rfloor$ . As in § 7.1, by Theorem A,  $\mathcal{E}(2, \text{SO}(m))$

is then in bijection with the set of pairs  $(p, q)$  of  $k$ -tuples of non-increasing integers. The bundle  $\xi^b$  is determined by its second Stiefel–Whitney number  $w(\xi^b) \in \mathbb{Z}_2$  which is then given by

$$w(\xi^b) = \sum_{i=1}^{\lfloor m/2 \rfloor} (p_i - q_i) \pmod{2}.$$

Again,  $\xi$  comes from an  $\mathrm{SO}(3)$ -equivariant bundle if and only if  $q_i = -p_i$  and then  $\xi^b$  is trivial.

7.4.  $n = 2k + 1 \geq 3$  and  $G$  is a compact classical group other than  $\mathrm{SO}(2m)$ . The important thing is that  $\mathrm{SO}(n-1)$  contains a maximal torus of  $\mathrm{SO}(n)$ . Therefore, by [3, Chapter 6, Corollary 2.8], for any embedding  $\psi: G \hookrightarrow U(m)$ , the representations  $\psi \circ \alpha$  and  $\psi \circ \beta$  are conjugate in  $U(m)$ . For  $G$  a compact classical group other than  $\mathrm{SO}(2m)$ , this implies that  $\alpha$  and  $\beta$  are conjugate in  $G$  [20, Proposition 8, p. 56]. Therefore, the image of  $J$  is the diagonal  $\Delta \mathcal{R}(n, G)$ . We do not know whether this is true for  $G = \mathrm{SO}(2m)$  (see [20, Remark, p. 57] for a possible source of counter-examples).

7.5.  $n = 2k + 1 \geq 3$  and  $G = \mathrm{SO}(n)$ . The set  $\mathcal{R}(n, \mathrm{SO}(n))$  has just two elements, represented by the trivial homomorphism and the identity  $\mathrm{id}$  of  $\mathrm{SO}(n)$ . If  $\iota: \mathrm{SO}(n-1) \subset \mathrm{SO}(n)$  denotes the inclusion, then  $Z_{[\iota]}$  contains two elements, represented by the identity matrix and the diagonal matrix  $D := \mathrm{diag}(-1, \dots, -1, 1)$ . Let  $\delta$  be the inner automorphism of  $\mathrm{SO}(n)$  given by the conjugation by  $D$ . By Theorem B, the set  $\mathcal{E}(n, \mathrm{SO}(n))$  for  $n = 2k + 1 \geq 3$  then contains three elements as follows.

(i) The trivial bundle  $S^n \times \mathrm{SO}(n)$  with the action  $A \cdot (z, B) = (A \cdot z, B)$ . The isotropy representations are both trivial.

(ii) The trivial bundle  $S^n \times \mathrm{SO}(n)$  with the action  $A \cdot (z, B) = (A \cdot z, AB)$ . It is characterized by  $\bar{J}(\xi) = [\iota]$  and  $\tilde{J}_i(\xi) = ([\mathrm{id}], [\mathrm{id}])$ . This does not come from an  $\mathrm{SO}(n+1)$ -equivariant bundle.

(iii) The principal  $\mathrm{SO}(n)$ -bundle  $\mathcal{F}S^n$  associated with the tangent bundle of  $S^n$ . It is characterized by  $\bar{J}(\mathcal{F}S^n) = [\iota]$  and  $\tilde{J}_i(\mathcal{F}S^n) = ([\mathrm{id}], [\delta])$ . This comes from an  $\mathrm{SO}(n+1)$ -equivariant bundle.

Observe that  $([\mathrm{id}], [\delta]) = ([\delta], [\mathrm{id}])$  in  $\mathcal{R}_i(n, \mathrm{SO}(n))$ . By Proposition C, this implies that  $C(\mathcal{F}S^n) = -C(\mathcal{F}S^n)$ . This proves again the classical fact that  $C(\mathcal{F}S^{2k+1}) \in \pi_{2k}(\mathrm{SO}(2k+1))$  is of order 2 [4, Corollary IV.1.11].

7.6.  $n = 2k \geq 6$  and  $G = \mathrm{SO}(n)$ . The set  $\mathcal{R}(n, \mathrm{SO}(n))$  contains three elements, represented by the trivial homomorphism, the identity  $\mathrm{id}$  of  $\mathrm{SO}(n)$ , and the conjugation  $\delta$  by the diagonal matrix with entries  $(-1, \dots, -1, 1)$ . The non-trivial homomorphisms restrict to the inclusion  $\iota: \mathrm{SO}(n-1) \subset \mathrm{SO}(n)$ . The group  $Z_\iota$  is trivial. Therefore,  $J$  is injective and the set  $\mathcal{E}(n, \mathrm{SO}(n))$  for  $n = 2k$  then contains five elements, as follows.

(i) The trivial bundles  $S^n \times \mathrm{SO}(n)$  with the actions  $A \cdot (z, B) = (A \cdot z, B)$ ,  $A \cdot (z, B) = (A \cdot z, AB)$  and  $A \cdot (z, B) = (A \cdot z, \delta(A)B)$ . Their images under  $J$  give the diagonal  $\Delta \mathcal{R}(n, \mathrm{SO}(n))$ .

(ii) The principal  $\mathrm{SO}(n)$ -bundle  $\mathcal{F}S^n$  associated with the tangent bundle of  $S^n$ . One has  $J(\mathcal{F}S^n) = ([\mathrm{id}], [\delta])$ .

(iii) The  $(n, G)$ -bundle  $-\mathcal{F}S^n$  with  $J(-\mathcal{F}S^n) = ([\delta], [\mathrm{id}])$ . Its underlying  $\mathrm{SO}(n)$ -principal bundle is stably trivial with Euler number  $-2$ .

The trivial bundle with action  $A \cdot (z, B) = (A \cdot z, B)$ , as well as the bundles in (ii) and (iii), are the ones coming from  $\mathrm{SO}(n+1)$ -equivariant bundles.

7.7.  $n = 4$  and  $G = \mathrm{SO}(3)$ . The groups  $\mathrm{SO}(4)$  and  $\mathrm{SO}(3)$  are built up out of the unit quaternions  $S^3$  by  $\mathrm{SO}(4) \cong (S^3 \times S^3) / \{(1, 1), (-1, -1)\}$  and  $\mathrm{SO}(3) \cong S^3 / \{\pm 1\}$ . Recall that these isomorphisms are constructed as follows: the orthogonal transformation  $A_{p,q} \in \mathrm{SO}(4)$  associated to  $(p, q) \in S^3 \times S^3$  is  $A_{p,q}(x) := px\bar{q}$ , where  $x \in \mathbb{H}$  is a quaternion and  $\mathbb{H}$  is made isomorphic to  $\mathbb{R}^4$  by choosing  $(i, j, k, 1)$  as a basis. The correspondence  $p \rightarrow A_{p,p}$  then induces the inclusion  $\iota: \mathrm{SO}(3) \subset \mathrm{SO}(4)$ . The automorphism  $\delta$  of  $\mathrm{SO}(4)$  defined in § 6 is just the conjugation by  $D := \mathrm{diag}(-1, \dots, -1, 1)$ . One checks easily that  $\delta(A_{p,q}) = A_{q,p}$ , since  $Dx = \bar{x}$  is the quaternion conjugation.

The non-equivariant isomorphism class of an  $\mathrm{SO}(3)$ -principal bundle  $\eta$  is characterized by  $C(\eta) \in \pi_3(\mathrm{SO}(3)) = \pi_3(S^3) = \mathbb{Z}$ . It is also determined by its first Pontrjagin number  $p(\eta) \in 4\mathbb{Z}$ , with the relation  $p(\eta) = 4C(\eta)$ .

The set  $\mathcal{R}(4, \mathrm{SO}(3))$  contains three elements represented by the trivial homomorphism and those induced by the projections  $S^3 \times S^3 \rightarrow S^3$  given by  $\sigma_1(p, q) := p$  and  $\sigma_2(p, q) := q$ . The last two restrict over  $\mathrm{SO}(3)$  to the identity  $\mathrm{id}$  of  $\mathrm{SO}(3)$ . The group  $Z_\iota$  being trivial, the map  $J$  is injective. This shows that the set  $\mathcal{E}(4, \mathrm{SO}(3))$  contains five elements.

(i) The trivial bundles  $S^4 \times \mathrm{SO}(3)$  with the actions  $A \cdot (z, B) = (A \cdot z, B)$  and  $A \cdot (z, B) = (A \cdot z, \sigma_i(A)B)$  for  $i = 1, 2$ . Their images by  $J$  give the diagonal  $\Delta\mathcal{R}(4, \mathrm{SO}(3))$ .

(ii) The principal  $\mathrm{SO}(3)$ -bundle  $\mathcal{H}: \mathbb{R}P^7 \rightarrow S^4$  coming from the quaternionic Hopf bundle  $S^7 \rightarrow S^4$ ; the  $\mathrm{SO}(4)$ -action comes from the  $\mathrm{SU}(2) \times \mathrm{SU}(2)$ -action on  $S^7$  given by  $(p, q) \cdot (z_1, z_2) = (pz_1, qz_2)$ . We have  $J(\mathcal{H}) = ([\sigma_1], [\sigma_2])$  and  $p(\mathcal{H}^b) = 4$ .

(iii) The  $(n, G)$ -bundle  $-\mathcal{H}$  with  $J(-\mathcal{H}) = ([\sigma_2], [\sigma_1])$  and  $p(\mathcal{H}^b) = -4$ .

The trivial bundle with action  $A \cdot (z, B) = (A \cdot z, B)$ , as well as the bundles in (ii) and (iii), are the ones coming from  $\mathrm{SO}(5)$ -equivariant bundles.

7.8.  $n = 4$  and  $G = \mathrm{SO}(4)$ . With the notation of § 7.7, the set  $\mathcal{R}(4, \mathrm{SO}(4))$  contains five elements represented by:

(i) the trivial homomorphism;

(ii) those induced by  $\sigma_1(p, q) := (p, p)$  and  $\sigma_2(p, q) := (q, q)$ ;

(iii) the identity  $\mathrm{id}$  of  $\mathrm{SO}(4)$ ;

(iv) the homomorphism  $\delta(p, q) := (q, p)$ .

The non-trivial homomorphisms all restrict to  $\iota$  over  $\mathrm{SO}(3)$ . The group  $Z_\iota$  is trivial and then  $J$  is injective. The non-equivariant isomorphism class of an  $\mathrm{SO}(4)$ -principal bundle  $\eta$  is characterized by  $C(\eta) \in \pi_3(\mathrm{SO}(4)) = \pi_3(S^3 \times S^3) = \mathbb{Z} \oplus \mathbb{Z}$ . More usually, one takes the pair of integers  $(p(\eta), e(\eta))$  formed by the first Pontrjagin number (in  $2\mathbb{Z}$ ) and the Euler number of  $\eta$ . The linear map which sends  $C(\eta)$  to  $(p(\eta), e(\eta))$  has matrix  $\begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}$ . This can be checked using the examples below.

The set  $\mathcal{E}(4, \text{SO}(3))$  contains the following seventeen elements.

(i) The trivial bundles  $S^4 \times \text{SO}(3)$  with the five actions using the above representations. Their images by  $J$  are just the diagonal elements  $\Delta \mathcal{R}(4, \text{SO}(4))$ .

(ii) The principal  $\text{SO}(4)$ -bundle  $\widehat{\mathcal{H}}$  whose total space is  $E(\widehat{\mathcal{H}}) := \mathbb{R}P^7 \times_{\text{SO}(3)} \text{SO}(4)$ . One has  $J(\widehat{\mathcal{H}}) = ([\sigma_1], [\sigma_2])$  and  $C(\widehat{\mathcal{H}}^b) = (1, 1)$ ; therefore its characteristic classes are  $(p(\widehat{\mathcal{H}}^b), e(\widehat{\mathcal{H}}^b)) = (4, 0)$ . One also has its ‘inverse’  $-\widehat{\mathcal{H}}$ , with  $J(-\widehat{\mathcal{H}}) = ([\sigma_2], [\sigma_1])$  and  $(p(\widehat{\mathcal{H}}^b), e(\widehat{\mathcal{H}}^b)) = (-4, 0)$ .

(iii) The principal  $\text{SO}(4)$ -bundle  $\mathcal{T}S^4$  associated with the tangent bundle of  $S^4$ . One has  $J(\mathcal{T}S^4) = ([\text{id}], [\delta])$ . Therefore,  $C(\mathcal{T}S^4) = (1, -1)$  and  $(p(\mathcal{T}^b), e(\mathcal{T}^b)) = (0, 2)$ . Again, one can consider its inverse.

(iv) The  $(n, G)$ -bundles  $\xi_i$  ( $i = 1, 2$ ) with  $J(\xi_i) = ([\text{id}], [\sigma_i])$  and their inverses  $-\xi_i$ . They satisfy  $C(\xi_1) = (0, -1)$  and  $C(\xi_2) = (1, 0)$ , or, equivalently,

$$(p(\xi_1^b), e(\xi_1^b)) = (-2, 1) \quad \text{and} \quad (p(\xi_2^b), e(\xi_2^b)) = (2, 1).$$

(v) The  $(n, G)$ -bundle  $\xi_{i,\delta}$  ( $i = 1, 2$ ) with  $J(\xi_i) = ([\delta], [\sigma_i])$  and their inverses. They satisfy  $C(\xi_{1,\delta}) = (-1, 0)$  and  $C(\xi_{2,\delta}) = (0, 1)$ , or, equivalently,

$$(p(\xi_{1,\delta}^b), e(\xi_{1,\delta}^b)) = (-2, -1) \quad \text{and} \quad (p(\xi_{2,\delta}^b), e(\xi_{2,\delta}^b)) = (2, -1).$$

Only the trivial bundle with action  $A \cdot (z, B) = (A \cdot z, B)$  and the bundles in (ii) and (iii) come from  $\text{SO}(5)$ -equivariant bundles.

7.9.  $G = U(m)$ . In order to have non-trivial  $(n, U(m))$ -bundles, one must have  $\dim U(m) > \dim \text{SO}(n)$ . We check that we are then in the stable range, where, by Bott periodicity,

$$\pi_{n-1}(U(m)) \approx \pi_{n-1}(U(m+k)) \approx \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \mathbb{Z} & \text{if } n \text{ is even.} \end{cases}$$

*Problem.* Which integers occur as  $C(\xi^b)$  for an  $(n, U(m))$ -bundle  $\xi$ ?

### 8. A more general setting

The orthogonal action of  $\text{SO}(n)$  on  $S^n$  is an example of the *special*  $\Pi$ -manifolds defined by Jänich [13, 1.2]. Other examples include the ‘cohomogeneity 1’ actions studied by E. Straume [25] (see [7] for a recent application and other references). In this section we give the classification of equivariant  $(\Pi, G)$ -bundles over special  $\Pi$ -manifolds. We will assume in this section that  $\Pi$  and  $G$  are both compact Lie groups.

Let  $X$  be a smooth, connected, closed  $n$ -dimensional manifold with a smooth  $\Pi$ -action. Choose a  $\Pi$ -invariant Riemannian metric on  $X$ , and then each tangent space  $T_x X$  contains a  $\Pi_x$ -invariant subspace  $V_x$  perpendicular to the orbit  $\Pi \cdot x$ . Then  $X$  is called a *special*  $\Pi$ -manifold if for each  $x \in X$  the representation of  $\Pi_x$  on the normal space  $V_x$  is the direct sum of a trivial representation and a transitive representation.

It follows that the orbit space  $M = X/\Pi$  admits a natural structure as a topological manifold with boundary [13, 1.3], with dimension equal to  $n - \dim(\Pi/H)$  where  $H$  is the principal isotropy type. Under the ‘functional’

smooth structure [2, VI.6], the orbit space  $M$  is a smooth manifold with boundary. The pair  $(X, \pi: X \rightarrow M)$  is called a special  $\Pi$ -manifold over  $M$ .

Special  $\Pi$ -manifolds over  $M$  were classified by Jänich [13, 3.2], and independently by W.-C. Hsiang and W.-Y. Hsiang [12] (see also [2, V.5, VI.6]).

Let  $\partial M_A = \{B_\alpha\}_{\alpha \in A}$  denote the set of boundary components of  $M$ . An *admissible* isotropy group system  $(H, U_A)$  over  $M$  consists of a closed subgroup  $H$  of  $\Pi$  and a set  $U_A = \{U_\alpha\}_{\alpha \in A}$  of closed subgroups in  $\Pi$  containing  $H$ , with the property that for each  $\alpha \in A$  there exists a transitive representation in which  $H$  appears as the isotropy group of a non-zero vector. Let  $\Gamma = N(H)/H$  and  $\Omega_\alpha = N(U_\alpha) \cap N(H)/H$  for each  $\alpha \in A$ . The idea of the classification is to re-construct  $X$  from the unique principal  $\Gamma$ -bundle  $P$  over  $M$  such that  $P|_{M_0} = \{x \in X \mid \Pi_x = H\}$ , and a reduction of the structural group of  $P|_{B_\alpha}$  to  $\Omega_\alpha$  over each of the boundary components  $B_\alpha$ .

Let  $B_\alpha \times [0, 1]$  be a collar neighbourhood of some component  $B_\alpha$  in  $M$ , and let  $Y_\alpha = \pi^{-1}(B_\alpha \times [0, 1])$  denote its pre-image in  $X$ . The key fact is the following identification of  $Y_\alpha$  as a  $\Pi$ -space.

8.1. THEOREM (Tube Theorem [2, V.4.2]). *Let  $Y = Y_\alpha$ ,  $B = B_\alpha$  and  $\Omega = \Omega_\alpha$ . There exist a right  $\Omega$ -principal bundle  $Q = Q_\alpha$  over  $B$ , and a  $\Pi$ -equivariant diffeomorphism*

$$M_\psi \times_\Omega Q \cong Y$$

*commuting with the projection to  $[0, 1]$ , where  $M_\psi$  denotes the mapping cylinder of the canonical projection  $\psi: \Pi/H \rightarrow \Pi/U$ .*

Let  $X_0 = X - \bigcup_\alpha \pi^{-1}(B_\alpha \times [0, \frac{1}{2}])$  and  $M_0 = X_0/\Pi$ . The Tube Theorem shows that  $X$  is  $\Pi$ -diffeomorphic to the union

$$X = X_0 \cup \bigcup_\alpha Y_\alpha = \Pi/H \times_\Gamma P \cup \bigcup_\alpha M_{\psi_\alpha} \times_{\Omega_\alpha} Q_\alpha$$

with the identification on the overlaps  $B_\alpha \times [\frac{1}{2}, 1]$  induced by a reduction of structural groups  $P|_{B_\alpha} \cong \Gamma \times_{\Omega_\alpha} Q_\alpha$ .

Two special  $\Pi$ -manifolds  $(X_1, \pi_1)$  and  $(X_2, \pi_2)$  over  $M$  are called *isomorphic* when there exists a  $\Pi$ -equivariant diffeomorphism  $f: X_1 \rightarrow X_2$  such that the induced diffeomorphism  $\bar{f}: M \rightarrow M$  is the identity. By the smooth isotopy lifting theorem of G. Schwarz [22, Corollary 2.4] this is equivalent to Jänich's original definition where  $\bar{f}$  was assumed to be only isotopic to the identity, by an isotopy fixing  $\partial M$  pointwise. Let  $\mathcal{S}[H, U_A]$  denote the set of isomorphism classes of special  $\Pi$ -manifolds over  $M$ , with isotropy group system fine equivalent to  $(H, U_A)$  (cf. [13, § 2]).

8.2. THEOREM [13, 3.2]. *Let  $(H, U_A)$  be an admissible isotropy group system over  $M$ , where  $M$  is a smooth, compact connected manifold with boundary. Then*

$$\mathcal{S}[H, U_A] \cong [M, \partial M_A; B\Gamma, B\Omega_A].$$

In order to classify equivariant  $(\Pi, G)$ -bundles  $(E, p)$  over a special  $\Pi$ -manifold  $X$ , called *special  $(\Pi, G)$ -bundles* for short, we will generalize the results of Lashof [16] to describe the bundles over  $\Pi$ -spaces  $X_0$  and  $Y_\alpha$  with one orbit type, and then follow Jänich's method to glue the pieces together.

First some general definitions: if  $\rho: H \rightarrow G$  is a (smooth) homomorphism,  $[\rho]$  denotes the set of homomorphisms  $\rho': H \rightarrow G$  such that  $\rho'(h) = g\rho(h)g^{-1}$  for some  $g \in G$  and all  $h \in H$ . We will say that *the fibre over  $x$  belongs to  $[\rho]$*  if for each  $z \in p^{-1}(x)$  there exists  $\rho' \in [\rho]$  such that  $hz = z \cdot \rho'(h)$  for all  $h \in H$ . Then let

$$X^{[\rho]} = \{x \in X^H \mid \text{the fibre over } x \text{ belongs to } [\rho]\}.$$

Let  $X_0^{[\rho]} = X_0 \cap X^{[\rho]}$  and notice that  $X_0^{[\rho]} \subset P = \{x \in X \mid \Pi_x = H\}$ . By [16, Lemma 1.2], the space

$$E^\rho = \{z \in E \mid hz = z \cdot \rho(h), \forall h \in H\}$$

is a  $Z_\rho$ -bundle over  $X^{[\rho]}$ , where  $Z_\rho$  is the centralizer of  $\rho$  in  $G$ . The group  $\widehat{G} = \Pi \times G$  has a left action on the total space  $E$  given by the formula  $(\gamma, g) \cdot z = \gamma z \cdot g^{-1}$  for any  $(\gamma, g) \in \widehat{G}$  and any  $z \in E$ . Let us set

$$H\langle \rho \rangle := \{(h, \rho(h)) \mid h \in H\} \subset \Pi \times G$$

and

$$\Gamma\langle \rho \rangle := N(H\langle \rho \rangle) / H\langle \rho \rangle.$$

Then  $E^\rho$  is just the fixed set of  $H\langle \rho \rangle$  in  $E$  under this action.

Two special  $(\Pi, G)$ -bundles  $(E_1, p_1)$  and  $(E_2, p_2)$  over  $M$  are called *equivalent* when there exists a  $\Pi$ -equivariant  $G$ -bundle isomorphism  $\phi: E_1 \rightarrow E_2$ , covering a  $\Pi$ -equivariant diffeomorphism  $f: X_1 \rightarrow X_2$ , so that the induced diffeomorphism  $\bar{f}: M \rightarrow M$  is the identity.

We now give the classification of the part of  $(E, p)$  lying over  $M_0$ . After the remarks above, we see that it follows directly from the Slice Theorem [2, II.5.8].

**8.3. THEOREM [16, 1.9].** *Let  $\rho: H \rightarrow G$  be a smooth homomorphism. The equivalence classes of  $(\Pi, G)$ -equivariant bundles  $(E_0, p_0)$  over  $M_0$  with all fibres belonging to  $[\rho]$  are in bijection with the homotopy classes of maps  $[M_0, B\Gamma\langle \rho \rangle]$ .*

We next define an *admissible*  $(\Pi, G)$ -isotropy group system over  $M$  to be a set  $(H, U_A; \rho, \rho_A)$ , where  $H$  and  $U_A$  are as above,  $\rho: H \rightarrow G$  is a homomorphism, and  $\rho_A = \{\rho_\alpha\}_{\alpha \in A}$  is a set of homomorphisms  $\rho_\alpha: U_\alpha \rightarrow G$  such that  $\rho_\alpha|_H = \rho$ . We define

$$\Omega_\alpha\langle \rho_\alpha \rangle = N(U_\alpha\langle \rho_\alpha \rangle) \cap N(H\langle \rho \rangle) / H\langle \rho \rangle$$

and

$$\Omega_A\langle \rho_A \rangle = \{\Omega_\alpha\langle \rho_\alpha \rangle\}_{\alpha \in A}.$$

A special  $(\Pi, G)$ -bundle  $(E, p)$  realizes an admissible  $(\Pi, G)$ -isotropy group system  $(H, U_A; \rho, \rho_A)$  over  $M$  if

- (i) there exist points  $\{y_\alpha \in Y_\alpha\}$  such that  $\Pi_{y_\alpha} = U_\alpha$ ,
- (ii) for each  $y_\alpha$ , the normal space  $V_{y_\alpha}$  to the orbit  $\Pi \cdot y_\alpha$  has a point  $z_\alpha \in V_{y_\alpha}$  with  $\Pi_{z_\alpha} = H$ ,
- (iii) the images  $c_\alpha(t)$  of rays in  $V_{y_\alpha}$  joining  $z_\alpha$  to  $y_\alpha$  have isotropic liftings  $\tilde{c}_\alpha(t)$  to  $E$  such that  $\tilde{c}_\alpha(t) \subset E^\rho$  for  $0 \leq t < 1$  and  $\tilde{c}_\alpha(1) \subset E^{\rho_\alpha}$ .

It is not difficult to check (following [13, § 2]) that every special  $(\Pi, G)$ -bundle  $(E, p)$  over  $M$  realizes some admissible  $(\Pi, G)$ -isotropy group system  $(H, U_A; \rho, \rho_A)$ . This isotropy group system is unique up to a natural notion of equivalence, extending the ‘fine-orbit structure’ of Jänich.

We say that two isotropy group systems  $(H, U_A; \rho, \rho_A)$  and  $(H', U'_A; \rho', \rho'_A)$  are *fine equivalent* if the following conditions hold:

- (i) there exists an element  $\gamma \in \Pi \times G$  such that  $H'\langle\rho'\rangle = \gamma H\langle\rho\rangle \gamma^{-1}$ , and
- (ii) there exist  $n_\alpha \in NH\langle\rho\rangle$  such that  $U'_\alpha\langle\rho'_\alpha\rangle = (\gamma n_\alpha)U_\alpha\langle\rho_\alpha\rangle(\gamma n_\alpha)^{-1}$ .

Let  $\mathcal{S}[H, U_A; \rho, \rho_A]$  denote the set of equivalence classes of special  $(\Pi, G)$ -bundles over  $M$  realizing the given  $(\Pi, G)$ -isotropy group system, up to fine equivalence.

8.4. THEOREM. *Let  $(H, U_A; \rho, \rho_A)$  be an admissible  $(\Pi, G)$ -isotropy group system over  $M$ , where  $M$  is a smooth, compact connected manifold with boundary. Then*

$$\mathcal{S}[H, U_A; \rho, \rho_A] \cong [M, \partial M_A; B\Gamma\langle\rho\rangle, B\Omega_A\langle\rho_A\rangle].$$

*Proof.* Suppose that we are given an admissible  $(\Pi, G)$ -isotropy group system over  $M$ . Let  $(E, p)$  be a special  $(\Pi, G)$ -bundle over  $M$  realizing the given  $(\Pi, G)$ -isotropy group system. By restricting the bundle to  $M_0$ , we get a map  $\omega_0: M_0 \rightarrow B\Gamma\langle\rho\rangle$  classifying the principal  $\Gamma\langle\rho\rangle$ -bundle  $P\langle\rho\rangle$  (which completely determines  $(E_0, p_0)$ ) by Theorem 8.3. We can apply the Tube Theorem 8.1 to the  $\widehat{G} = \Pi \times G$  action on  $p^{-1}(Y_\alpha) \subset E$ , since  $z \in E^\rho$  means that  $\widehat{G}_z = H\langle\rho\rangle$  and similarly for  $z \in E^{\rho_\alpha}$ . This identifies the restriction of our bundle to the part over  $B_\alpha \times [0, 1]$  as  $M_{\varphi_\alpha} \times_{\Omega_\alpha\langle\rho_\alpha\rangle} Q\langle\rho_\alpha\rangle$ , where

$$\varphi_\alpha: \widehat{G}/H\langle\rho\rangle \rightarrow \widehat{G}/U_\alpha\langle\rho_\alpha\rangle$$

is the  $\widehat{G}$ -equivariant projection, and  $Q\langle\rho_\alpha\rangle$  is a principal right  $\Omega_\alpha\langle\rho_\alpha\rangle$ -bundle over  $B_\alpha$ . The classifying map  $\omega_0$  for  $P\langle\rho\rangle$  therefore extends to a map

$$\omega: (M, \partial M_A) \rightarrow (B\Gamma\langle\rho\rangle, B\Omega_A\langle\rho_A\rangle)$$

where the notation means that each boundary component  $B_\alpha$  is mapped into the  $\alpha$ -component of  $B\Omega_A\langle\rho_A\rangle$ . The restriction of  $\omega$  to  $B_\alpha$  classifies  $Q\langle\rho_\alpha\rangle$ . This shows that  $(E, p)$  is determined up to equivalence by  $\omega$ .

Conversely, if we are given a map  $\omega$  as above we can reconstruct a special  $(\Pi, G)$ -equivariant bundle over  $M$  realizing the isotropy group system, up to fine equivalence. It can be checked that this bundle is unique up to equivalence.  $\square$

8.5. REMARK. Note that a special  $\Pi \times G$ -manifold over  $M$  is a special  $(\Pi, G)$ -bundle over  $M$  precisely when the subgroup  $1 \times G$  acts freely on the total space. The isotropy group system for the bundle is just the collection of isotropy groups for the  $\Pi \times G$ -action. This observation shows that Theorem 8.4 follows from Jänich's results.

8.6. COROLLARY. *Let  $\Pi$  and  $G$  be compact Lie groups. The set of special  $(\Pi, G)$ -equivariant bundles over  $M$  is finite, provided that  $\dim M \leq 1$  and the isotropy groups are semi-simple.*

*Proof.* This is proved using [21], as for the finiteness of  $\mathcal{E}(n, G)$ .  $\square$

To conclude this section, we discuss the connection between these results and Theorem B. Let  $X$  be a special  $\Pi$ -manifold over  $M$ , and let  $\mathcal{E}(X, \mathcal{F})$  denote the set of bundle isomorphism classes of principal  $(\Pi, G)$ -bundles over  $X$  with



isotropy group system fine equivalent to  $\mathcal{F} := (H, U_A; \rho, \rho_A)$ . Here a bundle isomorphism is a  $\Pi$ -equivariant  $G$ -bundle isomorphism  $\phi: E_1 \rightarrow E_2$  covering the identity on  $X$ .

8.7. LEMMA. *There is a commutative diagram*

$$\begin{array}{ccccc}
 \mathcal{E}(X, \mathcal{F}) & \longrightarrow & \mathcal{S}[H, U_A; \rho, \rho_A] & \xrightarrow{\lambda} & \mathcal{S}[H, U_A] \\
 \downarrow v & & \downarrow \approx & & \downarrow \approx \\
 [M, \partial M_A; BZ_\rho, BZ_{\rho_A}] & \xrightarrow{u} & [M, \partial M_A; B\Gamma\langle\rho\rangle, B\Omega_A\langle\rho_A\rangle] & \longrightarrow & [M, \partial M_A; B\Gamma, B\Omega_A]
 \end{array}$$

where the horizontal composites have images represented by the element  $[X]$ .

*Proof.* The exact sequences

$$1 \rightarrow Z_\rho \rightarrow \Gamma\langle\rho\rangle \rightarrow \Gamma \quad \text{and} \quad 1 \rightarrow Z_{\rho_A} \rightarrow \Omega_A\langle\rho\rangle \rightarrow \Omega_A$$

of groups induce a map

$$u: [M, \partial M_A; BZ_\rho, BZ_{\rho_A}] \rightarrow [M, \partial M_A; B\Gamma\langle\rho\rangle, B\Omega_A\langle\rho_A\rangle].$$

By Theorem 8.4 and Theorem 8.2, we also have a surjective map

$$v: \mathcal{E}(X, \mathcal{F}) \rightarrow [M, \partial M_A; BZ_\rho, BZ_{\rho_A}]$$

induced by  $u$  and our construction of equivariant bundles. □

8.8. THEOREM. *Let  $X$  be a special  $\Pi$ -manifold over  $M$ , and let  $\mathcal{F} = (H, U_A; \rho, \rho_A)$  be an admissible isotropy group system. Then*

$$v: \mathcal{E}(X, \mathcal{F}) \cong [M, \partial M_A; BZ_\rho, BZ_{\rho_A}].$$

*Proof.* The map  $v$  is given in the diagram above, and we have already observed that it is surjective. Suppose that  $\xi_1, \xi_2 \in \mathcal{E}(X, \mathcal{F})$  with  $v(\xi_1) = v(\xi_2)$ . Then we have a continuous map

$$(M \times I, \partial(M \times I)) \rightarrow (BZ_\rho, BZ_{\rho_A})$$

realizing the homotopy between the classifying maps for  $\xi_1$  and  $\xi_2$ . By the surjectivity of  $v$  for  $(\Pi, G)$ -bundles over  $M \times I$ , we get a bundle  $(E, p)$  over  $X \times I$  which restricts to  $\xi_1$  and  $\xi_2$  at the ends  $X \times \partial I$ . Since  $E \cong E_0 \times I$ , we get  $\xi_1 \cong \xi_2$ . □

Let  $\text{Aut}(X)$  be the group of  $\Pi$ -equivariant isotopy classes of  $\Pi$ -equivariant diffeomorphisms of  $X$  over the identity of  $M$ . The group  $\text{Aut}(X)$  acts on  $\mathcal{E}(X, \mathcal{F})$  by pulling back:  $f \cdot \xi := (f^{-1})^* \xi$ . This action is well defined by the equivariant Covering Homotopy Theorem of Palais [2, II.7.3]. If

$$\lambda: \mathcal{S}[H, U_A; \rho, \rho_A] \rightarrow \mathcal{S}[H, U_A]$$

denotes the natural forgetful map, then there is an induced map

$$\psi: \mathcal{E}(X, \mathcal{F}) \rightarrow \lambda^{-1}(X)$$

given by applying our stronger equivalence relation on bundles (which allows  $\phi$  to cover a self-diffeomorphism of  $X$ ).

8.9. PROPOSITION. *The map  $\psi$  induces a bijection between  $\lambda^{-1}(X)$  and the quotient of  $\mathcal{E}(X, \mathcal{F})$  by the action of  $\text{Aut}(X)$ .*

*Proof.* The map from one set of bundles to the other is defined by regarding a  $(\Pi, G)$ -bundle over  $X$  as an element of  $\lambda^{-1}(X)$ , and this is well defined since the equivalent relation in  $\mathcal{S}[H, U_A; \rho, \rho_A]$  is stronger. Moreover, two bundles with isotropy group system  $\mathcal{F}$  over  $X$  are equivalent if and only if they are in the same orbit of the action of  $\text{Aut}(X)$ ; hence our correspondence is injective. On the other hand, if  $(E', p')$  is a bundle with base space  $X'$  in  $\lambda^{-1}(X)$ , then there exists a  $\Pi$ -equivariant diffeomorphism  $h: X \rightarrow X'$  covering the identity on  $M$ . Then  $E := h^*(E')$  is an equivalent element in  $\lambda^{-1}(X)$ , and is a bundle over  $X$ , so our correspondence is surjective.  $\square$

These results and Theorem 8.4 can sometimes be used for explicit classification of equivariant  $(\Pi, G)$ -bundles over special  $\Pi$ -manifolds. Notice that Bredon in [2, V.7] together with [2, Theorem V.6.4] has determined  $\text{Aut}(X)$  in many cases of interest.

We shall now specialize to special  $\Pi$ -manifolds over  $I := [-1, 1]$ , and extend Theorem B to this setting. Examples include actions with cohomogeneity 1 on spheres, classified in [25, Theorem C, Table II]. The two components  $\{\pm 1\}$  of  $\partial I$  are denoted  $\{\pm\}$ , and the notation  $\mathcal{F} = (H, U_{\pm}; \rho, \rho_{\pm})$  will be used for the admissible  $(\Pi, G)$ -isotropy group systems, as well as  $\Gamma, \Omega_{\pm}$ , etc. The classification of special  $\Pi$ -manifolds over  $I$  takes the following form.

8.10. THEOREM. *Let  $(H, U_{\pm})$  be an admissible isotropy group system over  $I$ . Then  $\mathcal{S}[H, U_{\pm}]$  is in bijection with the double cosets  $\pi_0(\Omega_{-}) \backslash \pi_0(\Gamma) / \pi_0(\Omega_{+})$ .*

*Proof.* This follows directly from Theorem 8.2, since

$$[I, \partial I; B\Gamma, B\Omega_{\pm}] \cong \pi_1(B\Omega_{-}) \backslash \pi_1(B\Gamma) / \pi_1(B\Omega_{+}) \cong \pi_0(\Omega_{-}) \backslash \pi_0(\Gamma) / \pi_0(\Omega_{+}). \quad \square$$

8.11. REMARK. The bijection of Theorem 8.10 can be seen in a more constructive way. Given a special  $\Pi$ -manifold  $X$  over  $I$ , we can choose an  $H$ -meridian  $c: I \rightarrow X$ , that is, a smooth section of  $X \rightarrow I$  so that  $\Pi_{c(t)} = H$  for  $t$  in the interior of  $I$ ; this can be obtained from a smooth section of the (trivial) principal  $\Gamma$ -bundle  $P \rightarrow I$ . Let  $U_{\pm} := \Pi_{c(\pm 1)}$ . We say that  $(H, U_{\pm})$  is an  $H$ -meridian isotropy group system for  $X$ . Choosing another smooth section of  $P$  gives isotropy groups conjugate to  $U_{\pm}$  by elements in the same connected component of  $N(H)$ .

As in §3.7, the special  $\Pi$ -manifold  $X$  can be reconstructed as a quotient of  $I \times \Pi$ :

$$X = (I \times \Pi / H) / \{(\pm 1, g) \sim (\pm 1, gu_{\pm}), \forall u_{\pm} \in U_{\pm}\}.$$

Therefore,  $X$  is determined by the subgroups  $U_{\pm}$  (compare [7, Proposition 1.6]). Moreover, any set  $(H, U'_{\pm})$ , where the  $U'_{\pm}$  are conjugate to  $U_{\pm}$ , occurs as an  $H$ -meridian isotropy group system for some special  $\Pi$ -manifold  $X'$  over  $I$  (proved as in §4.3). In this way,  $\mathcal{S}[H, U_{\pm}]$  is a quotient of  $\pi_0(N(H)) \times \pi_0(N(H))$ . The diagonal group acts trivially, and by carefully examining the relevant equivalence relation one sees that  $\mathcal{S}[H, U_{\pm}]$  is in bijection with  $\pi_0(\Omega_{-}) \backslash \pi_0(\Gamma) / \pi_0(\Omega_{+})$ .

Let  $X$  be a special  $\Pi$ -manifold over  $I$ . We will now compute the set  $\mathcal{E}(X, G)$  of isomorphism classes of  $\Pi$ -equivariant principal  $G$ -bundles over  $X$ . Fix an

$H$ -meridian isotropy group system  $(H, U_{\pm})$  for  $X$  induced by a smooth  $H$ -meridian  $c: I \rightarrow X$ . If  $\xi = (E, \rho)$  is a  $(\Pi, G)$ -bundle over  $X$ , then we can choose an isotropic lift  $\tilde{c}: I \rightarrow E$ , and obtain an admissible isotropy group system  $\mathcal{F} = (H, U_{\pm}; \rho, \rho_{\pm})$  for the bundle.

Since the composition of an isotropic lift with a bundle isomorphism is again isotropic, the conjugacy classes  $[\rho_{\pm}] \in \mathcal{R}(U_{\pm}, G)$  depend only on  $[\xi] \in \mathcal{E}(X, G)$ . This defines a map

$$J: \mathcal{E}(X, G) \rightarrow \mathcal{R}(U_-, G) \times \mathcal{R}(U_+, G).$$

We write  $\mathcal{R}(U_-, G) \times_H \mathcal{R}(U_+, G)$  for the set of pairs

$$([\rho_-], [\rho_+]) \in \mathcal{R}(U_-, G) \times \mathcal{R}(U_+, G)$$

such that  $\text{Res}[\rho_-] = \text{Res}[\rho_+]$  in  $\mathcal{R}(H, G)$ .

Theorem B generalizes to special manifolds over  $I$  as follows.

8.12. THEOREM. *Let  $X$  be a special  $\Pi$ -manifold over  $I$  realizing the isotropy group system  $(H, U_{\pm})$ . The set  $\mathcal{E}(X, G)$  of isomorphism classes of  $(\Pi, G)$ -bundles over  $X$  is determined by the following properties.*

(i) *The image of  $J$  is  $\mathcal{R}(U_-, G) \times_H \mathcal{R}(U_+, G)$ .*

(ii) *Let  $\rho_{\pm}: U_{\pm} \rightarrow G$  be two smooth homomorphisms such that  $\rho_-|_H = \rho_+|_H =: \rho$ . Then there is a bijection between  $J^{-1}([\rho_-], [\rho_+])$  and the set of double cosets  $\pi_0(Z_{\rho_-}) \backslash \pi_0(Z_{\rho}) / \pi_0(Z_{\rho_+})$ .*

*Proof.* Since  $\rho_{\pm}$  comes from an admissible system, the image of  $J$  is contained in  $\mathcal{R}(U_-, G) \times_H \mathcal{R}(U_+, G)$ . On the other hand, let  $\rho_{\pm}: U_{\pm} \rightarrow G$  be two smooth homomorphisms such that  $\rho_-|_H = \rho_+|_H =: \rho$ . The special  $(\Pi \times G)$ -manifold constructed as in Remark 8.11, with isotropy group system  $(H \langle \rho \rangle, U_{\pm} \langle \rho_{\pm} \rangle)$  (associated to an  $H \langle \rho \rangle$ -meridian) is a  $(\Pi, G)$ -bundle  $\xi$  over  $X$  (the special  $\Pi$ -manifold with  $H$ -meridian isotropy group system  $(H, U_{\pm})$ ). One has  $J(\xi) = ([\rho_-], [\rho_+])$ , which proves Part (i).

If  $(E, \rho) \in J^{-1}([\rho_-], [\rho_+])$  then its isotropy group system is fine equivalent to  $\mathcal{F} = (H, U_{\pm}; \rho, \rho_{\pm})$ . In the notation introduced earlier, we have

$$\mathcal{E}(X, \mathcal{F}) = J^{-1}([\rho_-], [\rho_+])$$

and the result now follows from Theorem 8.8. □

We also have a more explicit version of Proposition 8.9. Note that  $J(f \cdot \xi) = J(\xi)$ , for  $f \in \text{Aut}(X)$ ; so we must investigate the action of  $\text{Aut}(X)$  on a pre-image  $\mathcal{E}(X, \mathcal{F}) \cong J^{-1}([\rho_-], [\rho_+])$ . The group  $\text{Aut}(X)$  has a homotopy description: choose a base point  $(\bullet) \in \Omega_- \backslash \Gamma / \Omega_+$  which corresponds to the class of  $X$  in  $\pi_0(\Omega_-) \backslash \pi_0(\Gamma) / \pi_0(\Omega_+)$ .

8.13. PROPOSITION (Bredon [2, V.7.3, VI.6.4]). *There is a group anti-isomorphism*

$$\text{Aut}(X) \cong [I, \partial I; \Gamma, \Omega_{\pm}]_{\bullet}.$$

*Proof.* The pointed maps  $(I, \partial I) \rightarrow (\Gamma, \Omega_{\pm})$  send  $\partial I$  into the component of  $\Omega_- \backslash \Gamma / \Omega_+$  containing the base point  $(\bullet)$ . Let  $f \in \text{Aut}(X)$ . Using our  $H$ -meridian  $c$ , one defines a smooth path  $d: I \rightarrow \Pi/H$  by the formula

$$f(c(t)) = d(t) \cdot c(t).$$

The map  $f$  being  $\Pi$ -equivariant, one has, for all  $h \in H$ ,

$$hd(t) \cdot c(t) = h \cdot f(c(t)) = f(hc(t)) = f(c(t)) = d(t)c(t).$$

This implies, for all  $t \in I$ , that  $d(t) \in N(H)/H = \Gamma$ . For  $t = \pm 1$ , one gets in addition that  $d(\pm 1) \in \Omega_{\pm}$ . We check that this defines an anti-homomorphism from  $\text{Aut}(X)$  to the group  $[I, \partial_- I, \partial_+ I; \Gamma, \Omega_-, \Omega_+]$ . Now, if  $d(t)$  represents a class in the latter, the formula

$$f_d(\alpha c(t)) := \alpha d(t) \cdot c(t), \quad \text{for } \alpha \in \Pi, t \in I,$$

defines an element  $f_d$  of  $\text{Aut}(X)$  and constitutes an inverse to the above anti-homomorphism.  $\square$

Suppose that the homomorphisms  $\Gamma\langle\rho\rangle \rightarrow \Gamma$  and  $\Omega_{\pm}\langle\rho\rangle \rightarrow \Omega_{\pm}$  are surjective. Then we have a fibre bundle

$$(8.14) \quad Z_{\rho_-} \setminus Z_{\rho} / Z_{\rho_+} \rightarrow \Omega_- \langle \rho \rangle \setminus \Gamma \langle \rho \rangle / \Omega_+ \langle \rho \rangle \rightarrow \Omega_- \setminus \Gamma / \Omega_+.$$

Therefore,  $\pi_1(\Omega_- \setminus \Gamma / \Omega_+)$  acts on  $\pi_0(Z_{\rho_-} \setminus Z_{\rho} / Z_{\rho_+}) = \pi_0(Z_{\rho_-}) \setminus \pi_0(Z_{\rho}) / \pi_0(Z_{\rho_+})$ .

**8.15. THEOREM.** *Suppose that the homomorphisms  $\Gamma\langle\rho\rangle \rightarrow \Gamma$  and  $\Omega_{\pm}\langle\rho\rangle \rightarrow \Omega_{\pm}$  are surjective. Let  $\rho_{\pm}: U_{\pm} \rightarrow G$  be two smooth homomorphisms such that  $\rho_-|_H = \rho_+|_H =: \rho$ . Then the quotient of  $J^{-1}([\rho_-], [\rho_+])$  by the action of  $\text{Aut}(X)$  is in bijection with the quotient of  $\pi_0(Z_{\rho_-}) \setminus \pi_0(Z_{\rho}) / \pi_0(Z_{\rho_+})$  by the action of  $\pi_1(\Omega_- \setminus \Gamma / \Omega_+)$ .*

*Proof.* By Theorem 8.8, there is a surjective map  $\psi: J^{-1}([\rho_-], [\rho_+]) \rightarrow \lambda^{-1}([X])$ . Since  $\Gamma\langle\rho\rangle \rightarrow \Gamma$  and  $\Omega_{\pm}\langle\rho\rangle \rightarrow \Omega_{\pm}$  are surjective, we have the fibration (8.14). From the homotopy exact sequence of this fibration, we see that  $\lambda^{-1}([X])$  is in bijection with the quotient of  $\pi_0(Z_{\rho_-} \setminus Z_{\rho} / Z_{\rho_+})$  by the action of  $\pi_1(\Omega_- \setminus \Gamma / \Omega_+)$ . But this is just the action of  $\text{Aut}(X)$  by Proposition 8.13.  $\square$

**8.16. EXAMPLE.** Consider the standard action of  $\Pi = \text{SO}(n)$  on  $S^n$ , with  $H = \text{SO}(n-1)$ . As  $U_{\pm} = \Pi$ , the homomorphisms  $\Gamma\langle\rho\rangle \rightarrow \Gamma$  and  $\Omega_{\pm}\langle\rho\rangle \rightarrow \Omega_{\pm}$  are surjective.

When  $n = 2$ ,  $H$  is the trivial group. If  $G$  is connected, then  $Z_{\rho} = G$  is connected and the map  $J$  is a bijection, as seen in Theorem A. Also, the group of  $\Pi$ -automorphisms of  $S^2$  is equal to  $\Pi$ , so  $\text{Aut}(S^2)$  is trivial.

If  $n \geq 3$ , the group  $\Gamma = \Omega_{\pm}$  has two elements, the non-trivial one being represented by the diagonal matrix  $D := \text{diag}(1, \dots, 1, -1, -1)$ . As the space  $\Omega_- \setminus \Gamma / \Omega_+$  is reduced to a point, it follows from Theorem 8.15 that the group  $\text{Aut}(S^n)$  acts trivially on  $\mathcal{E}(S^n, G)$ . This has the following consequence.

**8.17. PROPOSITION.** *Let  $\rho_{\pm}: \text{SO}(n) \rightarrow G$  be two representations into a compact Lie group  $G$  such that  $\rho_{\pm}|_{\text{SO}(n-1)} = \rho$ . Then, the element  $\rho_-(D)\rho_+(D)^{-1}$  belongs to  $Z_{\rho}$  and represents the trivial element in  $\pi_0(Z_{\rho_-}) \setminus \pi_0(Z_{\rho}) / \pi_0(Z_{\rho_+})$ .*

*Proof.* As  $\rho_{\pm}|_H = \rho$ , one has, for all  $h \in H$ ,

$$\rho_+(D)^{-1}\rho(h)\rho_+(D) = \rho_-(D)^{-1}\rho(h)\rho_-(D).$$

Therefore,  $\rho_-(D)\rho_+(D)^{-1} \in Z_{\rho}$ .

Recall that our  $H$ -meridian for  $S^n$  is  $c(t) = (0, \dots, \cos(\frac{1}{2}\pi t), \sin(\frac{1}{2}\pi t))$ . By Proposition 4.1 and its proof there exists a  $(\Pi, G)$ -bundle  $\xi$  over  $S^n$ , with a  $\rho$ -isotropic lifting  $\tilde{c}: I \rightarrow E(\xi)$  of  $c$  such that the isotropy representations associated to  $\tilde{c}(\pm 1)$  are  $\rho_{\pm}$ . The curve  $\tilde{c}$  is the horizontal lifting of  $c$  for a  $\Pi$ -invariant connection. There is no problem in extending the definitions of  $c(t)$  and  $\tilde{c}(t)$  for  $t \in \mathbb{R}$ . This defines  $\mu \in G$  by

$$\tilde{c}(3) = \tilde{c}(-1) \cdot \mu \quad \text{and} \quad \tilde{c}(5) = \tilde{c}(1) \cdot \mu.$$

For  $t \in [-1, 1]$ , one has  $D \cdot \tilde{c}(t) = \tilde{c}(2-t)\rho_+(D)$  (since this is true for  $t = 1$  and both sides are horizontal). For  $t = -1$ , this gives

$$\tilde{c}(-1)\mu\rho_+(D) = \tilde{c}(3)\rho_+(D) = D \cdot \tilde{c}(-1) = \tilde{c}(-1)\rho_-(D).$$

Therefore,  $\mu = \rho_-(D)\rho_+(D)^{-1}$ .

Now, let  $\xi_1 := \delta^*\xi$ ; one has

$$E(\xi_1) = \{(x, u) \in S^n \times E(\xi) \mid \delta(x) = \pi(u)\}$$

and the  $\Pi$ -action on  $E(\xi_1)$  comes from the diagonal action. A  $\rho$ -isotropic lifting  $\tilde{c}_1: I \rightarrow E(\xi_1)$  of  $c$  is then given by

$$\tilde{c}_1(t) := (c(t), \tilde{c}(t-2)).$$

By Proposition 4.1, one has  $\xi = \xi_1$  in  $\mathcal{E}(S^n, G)$  if and only if  $\tilde{J}_\rho(\xi) = \tilde{J}_\rho(\xi_1)$  in  $\mathcal{R}_\rho(n, G)$ . By § 4.2, using the curves  $\tilde{c}$  and  $\tilde{c}_1$ , one has, in  $\mathcal{R}_\rho(n, G)$ ,

$$\tilde{J}_\rho(\xi) = [\rho_-, \rho_+] \quad \text{and} \quad \tilde{J}_\rho(\xi_1) = [\mu^{-1}\rho_-, \rho_+].$$

This proves that  $\mu$  represents the unit element in  $\pi_0(Z_{\rho_-}) \setminus \pi_0(Z_\rho) / \pi_0(Z_{\rho_+})$ .  $\square$

8.18. EXAMPLE. Consider the action of  $\text{SO}(3)$  on the traceless  $(3 \times 3)$ -symmetric matrices by conjugation. Restricting this action to the unit sphere (under the invariant scalar product  $\text{tr}(AB)$ ) makes  $S^4$  a special  $\text{SO}(3)$ -manifold over  $I$  (see [7, § 3]) for more details on this classical cohomogeneity 1  $\text{SO}(3)$ -action on  $S^4$ ). Here  $H = S(O(1) \times O(1) \times O(1))$ ,  $U_- = S(O(1) \times O(2))$  and  $U_+ = S(O(2) \times O(1))$ . Observe that  $U_-$  is isomorphic to  $O(2)$ , generated by  $\{1\} \times \text{SO}(2)$  and the element  $(-1, -1, 1)$  of  $H$ . Any homomorphism  $\rho_-: U_- \rightarrow G$  killing  $(-1, -1, 1)$  would be trivial, since  $\text{SO}(2)$  is the commutator subgroup of  $O(2)$ . Using the symmetric statement with  $U_+$ , one sees that if  $\rho_\pm: U_\pm \rightarrow G$  is any pair of homomorphisms with  $\rho_-|_H = \rho_+|_H = \rho$ , then the  $\rho_\pm$  are trivial unless  $\rho$  is injective.

Take  $G = \text{SO}(3)$  and use the standard inclusions of  $U_\pm$  into  $\text{SO}(3)$ . Any non-trivial homomorphism  $\rho_-$  is conjugate to  $\tilde{\rho}_-: (\varepsilon, r_\theta) \mapsto (\varepsilon, r_\theta^p)$ , and any non-trivial  $\rho_+$  is conjugate to  $\tilde{\rho}_+: (r_\theta, \varepsilon) \mapsto (r_\theta^q, \varepsilon)$ , where  $p$  and  $q$  are non-negative integers. By what was said above, in order for  $\tilde{\rho}_\pm$  to be non-trivial, the integers  $p$  and  $q$  must be odd and both  $\tilde{\rho}_\pm$  restrict on  $H$  to the identification  $\rho$  of  $H$  with the diagonal matrices of  $\text{SO}(3)$ . We have  $Z_\rho = H$ ,  $Z_{\tilde{\rho}_-} = \{\text{diag}(1, 1, 1), \text{diag}(1, -1, -1)\}$  and  $Z_{\tilde{\rho}_+} = \{\text{diag}(1, 1, 1), \text{diag}(-1, -1, 1)\}$ . Therefore, the set of double cosets  $\pi_0(Z_{\tilde{\rho}_-}) \setminus \pi_0(Z_\rho) / \pi_0(Z_{\tilde{\rho}_+})$  is trivial and  $\mathcal{E}([S^4 \rightarrow I], \text{SO}(3))$  is in bijection, via the map  $J$  of Theorem 8.12, with  $\{0, 0\} \cup \mathbb{N}_{\text{odd}} \times \mathbb{N}_{\text{odd}}$ .

In [7, § 3],  $(\text{SO}(3), \text{SO}(3))$ -bundles  $P_{p,q}$  over  $S^4$  are constructed for  $p, q \in \mathbb{Z}$  with  $p \equiv 3 \pmod{4}$  (these come from  $(S^3 \times S^3)$ -bundles). These bundles satisfy  $J(P_{p,q}) = (|p|, |q|)$ ; so, up to isomorphism, the sign of  $p$  and  $q$  does not matter.

8.19. EXAMPLE. This is the complex analogue of Example 8.18. One considers the action of  $SU(3)$  on the traceless  $(3 \times 3)$ -Hermitian matrices by conjugation and restrict it to the unit sphere. One thus gets a special  $SU(3)$ -manifold over  $I$  diffeomorphic to  $S^7$ . The isotropy groups are  $H = S(U(1) \times U(1) \times U(1))$ ,  $U_- = S(U(1) \times U(2))$  and  $U_+ = S(U(2) \times U(1))$ . As in Example 8.18, if  $\rho_{\pm}: U_{\pm} \rightarrow G$  is any pair of homomorphisms with  $\rho_-|_H = \rho_+|_H = \rho$ , then  $\rho_{\pm}$  are trivial unless  $\rho$  is injective.

For  $G = SU(3)$ , one checks that any non-trivial homomorphism of  $U_{\pm}$  to  $G$  is conjugate to the standard inclusion. As  $Z_{\rho} = H$  in  $SU(3)$ , the map  $J$  is injective and  $\mathcal{E}(S^7, SU(3))$  consists of two elements.

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