



# Some examples of free actions on products of spheres

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## Abstract

If  $G_1$  and  $G_2$  are finite groups with periodic Tate cohomology, then  $G_1 \times G_2$  acts freely and smoothly on some product  $S^n \times S^n$ .

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## 0. Introduction

The study of free finite group actions on products of spheres is a natural continuation of the spherical space form problem [15]. In this paper, we show that certain products of finite groups do act freely and smoothly on a product of spheres, even though the individual factors can't act freely and smoothly (or even topologically) on a single sphere. This verifies a conjecture of Elliott Stein [24]. The method involves a detailed analysis of the product formulas in surgery theory, and a refinement of Dress induction for surgery obstructions.

If a finite group  $G$  acts freely on  $S^n$ , then (i) every abelian subgroup of  $G$  is cyclic, and (ii) every element of order 2 is central. In [15], Madsen, Thomas and Wall proved that these conditions are sufficient to imply the existence of a free topological action on some sphere. Actually, these two conditions have a very different character. By the work of P.A. Smith and R. Swan [25], condition (i) is necessary and sufficient for a free simplicial action of  $G$  on a finite-dimensional simplicial complex, which is homotopy equivalent to a sphere. The finite groups  $G$  satisfying condition (i) are exactly the groups with periodic Tate cohomology, or equivalently those for which every subgroup of order  $p^2$ ,  $p$  prime, is cyclic (the  $p^2$ -conditions). On the other hand, Milnor [18] proved that condition (ii) is necessary

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for a free  $G$ -action by homeomorphisms on any closed, topological *manifold* which has the mod 2 homology of a sphere. The groups with periodic cohomology satisfying condition (ii) are just those which have no dihedral subgroups, or equivalently those for which every subgroup of order  $2p$ ,  $p$  prime, is cyclic (the  $2p$ -conditions). Milnor's result shows for example that the periodic dihedral groups do not act topologically on  $S^n$ , although they do act simplicially on a finite complex homotopy equivalent to  $S^n$ .

For free finite group actions on a product of spheres, the analogue of condition (i) was suggested by P. Conner [5]: if  $G$  acts freely on a  $k$ -fold product of spheres  $(S^n)^k := S^n \times \cdots \times S^n$ , is every abelian subgroup of  $G$  generated by at most  $k$  elements? Conner proved this statement for  $k = 2$ , and a lot of work [19,4,1,3] has been done to determine what additional conditions are necessary to produce free simplicial actions on a finite-dimensional simplicial complex homotopy equivalent to a product of spheres. The picture is now almost completely clarified, at least for elementary abelian groups and spheres of equal dimension: Adem and Browder [1] and Carlsson [4] showed that  $G = (\mathbf{Z}/p)^r$  acts freely on  $(S^n)^k$ , for  $p$  a prime, implies  $r \leq k$  provided that  $n \neq 1, 3, 7$  in the case  $p = 2$  (the restriction  $n \neq 1$  for  $p = 2$  was recently removed in [29]). The same result is conjectured to hold for finite-dimensional  $G$ -CW complexes homotopy equivalent to a product of spheres of possibly *unequal* dimensions (see [2] for some recent progress).

Much less seems to be known at present about the additional conditions needed to produce free actions by homeomorphisms or diffeomorphisms on the closed manifolds  $(S^n)^k$  for  $k > 1$ . Let  $D_q$  denote the dihedral group of order  $2q$ , with  $q$  an odd prime. Elliott Stein [24] proved that, for every  $n = 4j + 3$  and any  $k \geq 2$ , there exist free, orientation-preserving piece-wise linear actions of  $(D_q)^k$  on  $(S^n)^k$ . Many of these actions are smoothable.

These examples show that a direct generalization of Milnor's condition (ii) is *not* necessary for actions on products of spheres. In this paper we verify a conjecture of Stein's:

**Theorem A.** *If  $G_1$  and  $G_2$  are finite groups with periodic Tate cohomology, then  $G_1 \times G_2$  acts freely and smoothly on some product  $S^n \times S^n$ .*

The techniques used to prove this statement also show that any product of periodic groups  $G_1 \times \cdots \times G_k$ , with  $k > 1$ , acts freely and topologically on  $(S^n)^k$  for some  $n$ . These are smooth actions if  $k \neq 3$  (see Theorem 5.2). Of course there are groups  $G$  satisfying Conner's condition which are not the direct product of periodic groups, so these examples are just the simplest case. The surgery techniques need to be developed further to study more general groups.

## 1. Hyperelementary induction

We need a refinement of Dress's fundamental work on induction, following [12, Section 1] and [11]. In [7] and [8] we are given a Mackey functor  $\mathcal{M}$  and a *family*  $\mathcal{H}$  of subgroups of  $G$ , which by definition is a collection of subgroups closed under conjugation and taking further subgroups. An important example is the family of hyperelementary subgroups.

One can then form what Dress calls an Amitsur complex: this is a chain complex

$$\mathcal{M}(G) \xrightarrow{\partial_0} \bigoplus_{H \in \mathcal{H}} \mathcal{M}(H) \xrightarrow{\partial_1} \cdots$$

where the higher terms are explicitly described sums of  $\mathcal{M}$  applied to elements of  $\mathcal{H}$ . The boundary map  $\partial_0$  is the sum of restriction maps and the higher  $\partial_i$  are just sums and differences of restriction maps.

There is a second Amitsur complex defined using induction maps for which the boundary maps go the other direction.

Dress further assumes that some Green ring, say  $\mathcal{G}$ , acts on  $\mathcal{M}$ . Write

$$\delta_{\mathcal{G}}^{\mathcal{H}}: \bigoplus_{H \in \mathcal{H}} \mathcal{G}(H) \rightarrow \mathcal{G}(G)$$

for the sum of the induction maps.

**Theorem 1.1.** *If there exists  $y \in \bigoplus_{H \in \mathcal{H}} \mathcal{G}(H)$  such that  $\delta_{\mathcal{G}}^{\mathcal{H}}(y) = 1 \in \mathcal{G}(G)$ , then both Amitsur complexes for  $\mathcal{M}$  are contractible.*

**Remark 1.2.** The conclusion can be expressed as

$$\mathcal{M}(G) = \varprojlim_{\mathcal{H}} \mathcal{M}(H) \quad \text{or} \quad \mathcal{M}(G) = \varinjlim_{\mathcal{H}} \mathcal{M}(H)$$

where the first limit is made up of restrictions and the second of inductions. In these cases we say that  $\mathcal{M}$  is  $\mathcal{H}$ -detected or  $\mathcal{H}$ -generated. If both Amitsur complexes are contractible we say that  $\mathcal{M}$  is  $\mathcal{H}$ -computable. The result above follows from [8, Proposition 1.2, p. 305] and the remark just above [8, Proposition 1.3, p. 190].

In order to use this strong theorem, it is necessary to find a computable Green ring which acts on a given Mackey functor. The Burnside ring of finite  $G$ -sets is a Green ring which acts on any Mackey functor, via the formula

$$[G/H] \cdot x = \text{Ind}_H^G(\text{Res}_G^H(x)),$$

but it satisfies Dress’s condition on  $\delta_{\mathcal{G}}^{\mathcal{H}}$  if and only if  $G \in \mathcal{H}$ .

**Example 1.3.** The Dress ring  $GU(G, \mathbf{Z})$  is  $\mathcal{H}$ -computable for  $\mathcal{H}$  the hyperelementary family (see [8, Theorem 3]). Since there is a homomorphism given by tensor product from  $GU(G, \mathbf{Z})$  to the automorphisms of  $(RG, \omega)$ -Morita in  $(R, -)$ -Morita (see [10, 1.B.3] for the definitions), it follows that any Mackey functor out of  $(RG, \omega)$ -Morita is also hyperelementary computable. For this paper, the main examples will be the quadratic and symmetric  $L$ -groups  $L_n(\mathbf{Z}G)$  and  $L^n(\mathbf{Z}G)$ , with various torsion decorations, as defined by Ranicki [21]. These are functors out of  $(\mathbf{Z}G, \omega)$ -Morita, and hence are hyperelementary computable (compare [8, Theorem 1] which covers the quadratic  $L$ -groups at least). Since the 2-localization map  $L_n(\mathbf{Z}G) \rightarrow L_n(\mathbf{Z}G) \otimes \mathbf{Z}_{(2)}$  is injective for  $G$  finite, the 2-hyperelementary family suffices for deciding when a surgery obstruction is zero.

For each prime  $p$ , and for each subgroup  $K$ , we denote by  $\mathcal{F}_p(K)$  the set of subgroups of  $H \subset G$  where  $K$  is a normal subgroup of  $H$  and  $H/K$  is a  $p$ -group. A family  $\mathcal{F}$  is called *hyper $_p$ -closed* if  $\mathcal{F}_p(K) \subseteq \mathcal{F}$  for all  $K \in \mathcal{F}$ .

Let  $X$  be a finite  $G$ -set. The minimal representative  $G$ -set for the family of subgroups  $\{\mathcal{F}_p(K) : K \in \text{Iso}(X)\}$  will be denoted *hyper $_p$ - $X$* . This construction is due to Dress [8, p. 307]. The trivial  $G$ -set is denoted  $\bullet$ . One of Dress’s main results is the following:

**Theorem 1.4** ([7, p. 207]). *Let  $\mathcal{G}$  be a Green ring. For any prime  $p$ , and any finite  $G$ -set  $Y$ , let  $\mathcal{K}(Y) := \ker(\mathcal{G}(\bullet) \otimes \mathbf{Z}_{(p)} \rightarrow \mathcal{G}(Y) \otimes \mathbf{Z}_{(p)})$  and let  $\mathcal{I}(Y) := \text{Im}(\mathcal{G}(\text{hyper}_p\text{-}Y) \otimes \mathbf{Z}_{(p)} \rightarrow \mathcal{G}(\bullet) \otimes \mathbf{Z}_{(p)})$ . Then  $\mathcal{G}(\bullet) \otimes \mathbf{Z}_{(p)} = \mathcal{K}(Y) + \mathcal{I}(Y)$ .*

If  $X$  is a finite  $G$ -set, we will use the notation  $\langle X \rangle$  for the equivalence class of  $X$  in the Dress category  $\mathcal{D}(G)$ . One useful consequence is:

**Lemma 1.5** ([11]). *Let  $\mathcal{G}_0$  be a sub-Green ring of  $\mathcal{G}_1$ . For any prime  $p$ , and any finite  $G$ -set  $X$  with  $\langle X \rangle = \langle \text{hyper}_p\text{-}X \rangle$ , then the natural map  $\mathcal{G}_0(X) \otimes \mathbf{Z}_{(p)} \rightarrow \mathcal{G}_0(\bullet) \otimes \mathbf{Z}_{(p)}$  is surjective if and only if  $\mathcal{G}_1(X) \otimes \mathbf{Z}_{(p)} \rightarrow \mathcal{G}_1(\bullet) \otimes \mathbf{Z}_{(p)}$  is surjective.*

**Proof.** For any Green ring  $\mathcal{G}$  and any finite  $G$ -set  $Y$ , the image of  $\mathcal{G}(Y) \otimes \mathbf{Z}_{(p)}$  in  $\mathcal{G}(\bullet)$  is an ideal. Hence either map is onto if and only if  $1_{\mathcal{G}(\bullet)}$  is in the image. Since  $1_{\mathcal{G}_0(\bullet)}$  goes to  $1_{\mathcal{G}_1(\bullet)}$ , this proves the first implication.

For the converse, notice that the assumption  $\mathcal{G}_1(X) \otimes \mathbf{Z}_{(p)} \rightarrow \mathcal{G}_1(\bullet) \otimes \mathbf{Z}_{(p)}$  is surjective implies that the Amitsur complex is contractible for the restriction maps induced by the transformation  $X \rightarrow \bullet$ . In particular,  $\mathcal{G}_1(\bullet) \otimes \mathbf{Z}_{(p)} \rightarrow \mathcal{G}_1(X) \otimes \mathbf{Z}_{(p)}$  is injective. Therefore  $\mathcal{G}_0(\bullet) \otimes \mathbf{Z}_{(p)} \rightarrow \mathcal{G}_0(X) \otimes \mathbf{Z}_{(p)}$  is injective, and from Dress’s Theorem we conclude that  $\mathcal{G}_0(\text{hyper}_p\text{-}X) \otimes \mathbf{Z}_{(p)} \rightarrow \mathcal{G}_0(\bullet) \otimes \mathbf{Z}_{(p)}$  is surjective.  $\square$

Suppose that  $\mathcal{G}$  is a Green ring which acts on a Mackey functor  $\mathcal{M}$ . For many applications of induction theory, the “best” Green ring for  $\mathcal{M}$  is the *Burnside quotient Green ring*  $\mathcal{A}_{\mathcal{G}}$ , defined as the image of the Burnside ring in  $\mathcal{G}$ . This is a Green ring which acts on  $\mathcal{M}$ , and by construction  $\mathcal{A}_{\mathcal{G}}$  is a sub-Green ring of  $\mathcal{G}$ . In particular, the natural map  $\mathcal{A}_{\mathcal{G}} \rightarrow \mathcal{G}$  is an injection.

We say that a finite  $G$ -set  $X$  is a *Dress generating set* for a Green ring  $\mathcal{G}$ , provided that, for each prime  $p$ ,  $\mathcal{G}(\text{hyper}_p\text{-}X) \otimes \mathbf{Z}_{(p)} \rightarrow \mathcal{G}(\bullet) \otimes \mathbf{Z}_{(p)}$  is surjective.

**Theorem 1.6.** *A finite  $G$ -set  $X$  is a Dress generating set for a Green ring  $\mathcal{G}$  if and only if it is a Dress generating set for the Burnside quotient Green ring  $\mathcal{A}_{\mathcal{G}}$ .*

We can translate this into a computability statement as follows:

**Corollary 1.7.** *Let  $p$  be a prime and  $\mathcal{G}$  be a Green ring. Suppose that  $\mathcal{F}$  is a hyper $_p$ -closed family of subgroups of  $G$ . Then  $\mathcal{G} \otimes \mathbf{Z}_{(p)}$  is  $\mathcal{F}$ -computable if and only if  $\mathcal{A}_{\mathcal{G}} \otimes \mathbf{Z}_{(p)}$  is  $\mathcal{F}$ -computable.*

The advantage of  $\mathcal{A}_{\mathcal{G}}$  over  $\mathcal{G}$  is that  $\mathcal{A}_{\mathcal{G}}$  acts on Mackey functors which are subfunctors or quotient functors of  $\mathcal{M}$  but  $\mathcal{G}$  does not in general. For example,  $\mathcal{G}$  never acts on  $\mathcal{A}_{\mathcal{G}}$  unless they are equal. We next point out another good feature of the Burnside quotient Green ring.

**Theorem 1.8.** *Suppose that  $\mathcal{G}$  is a Green ring which acts on a Mackey functor  $\mathcal{M}$ , and  $\mathcal{F}$  is a hyper $_p$ -closed family of subgroups of  $G$ . If  $\mathcal{G} \otimes \mathbf{Z}_{(p)}$  is  $\mathcal{F}$ -computable, then every  $x \in \mathcal{M}(G) \otimes \mathbf{Z}_{(p)}$  can be written as*

$$x = \sum_{H \in \mathcal{F}} a_H \text{Ind}_H^G(\text{Res}_G^H(x))$$

for some coefficients  $a_H \in \mathbf{Z}_{(p)}$ .

**Proof.** Since  $\mathcal{G} \otimes \mathbf{Z}_{(p)}$  is  $\mathcal{F}$ -computable, we know that  $\mathcal{A}_{\mathcal{G}} \otimes \mathbf{Z}_{(p)}$  is also  $\mathcal{F}$ -computable. Therefore, we can write  $1 = \sum_{K \in \mathcal{F}} b_K \text{Ind}_K^G(y_K)$ , for some  $y_K \in \mathcal{A}_{\mathcal{G}}(K) \otimes \mathbf{Z}_{(p)}$  and  $b_K \in \mathbf{Z}_{(p)}$ . For any  $x \in \mathcal{M}(G) \otimes \mathbf{Z}_{(p)}$  we now have the formula

$$x = 1 \cdot x = \sum_{K \in \mathcal{F}} b_K \text{Ind}_K^G(y_K \cdot \text{Res}_G^K(x)).$$

But each  $y_K \in \mathcal{A}_G(K) \otimes \mathbf{Z}_{(p)}$  can be represented by a sum  $\sum c_{KH}[K/H]$ , with  $c_{KH} \in \mathbf{Z}_{(p)}$ , under the surjection  $\mathcal{A}(K) \rightarrow \mathcal{A}_G(K)$ . Therefore

$$\begin{aligned} x &= \sum_{K \in \mathcal{F}} b_K \sum_{H \subseteq K} c_{KH} \text{Ind}_K^G([K/H] \cdot \text{Res}_G^K(x)) \\ &= \sum_{K \in \mathcal{F}} b_K \sum_{H \subseteq K} c_{KH} \text{Ind}_K^G(\text{Ind}_H^K(\text{Res}_K^H(\text{Res}_G^K(x)))) \\ &= \sum_{K \in \mathcal{F}} b_K \sum_{H \subseteq K} c_{KH} \text{Ind}_H^G(\text{Res}_G^K(x)). \end{aligned}$$

We now define  $a_H = \sum_{K \in \mathcal{F}} b_K c_{KH}$ , and the formula becomes

$$x = \sum_{H \in \mathcal{F}} a_H \text{Ind}_H^G(\text{Res}_G^K(x)). \quad \square$$

## 2. Product formulas in surgery

The existence of free actions on products of spheres will be established by evaluating the surgery obstruction of a product of degree one normal maps. We need to recall the product formulas in algebraic surgery theory due to Ranicki [21, Section 8], [22, Section 8].

**Theorem 2.1** ([21, Proposition 8.1]). *Let  $(A, \epsilon)$  and  $(B, \eta)$  be rings with involution. There are natural products in the symmetric and quadratic  $L$ -groups*

$$\begin{aligned} \otimes : L^n(A, \epsilon) \otimes_{\mathbf{Z}} L^m(B, \eta) &\rightarrow L^{m+n}(A \otimes_{\mathbf{Z}} B, \epsilon \otimes \eta) \\ \otimes : L^n(A, \epsilon) \otimes_{\mathbf{Z}} L_m(B, \eta) &\rightarrow L_{m+n}(A \otimes_{\mathbf{Z}} B, \epsilon \otimes \eta) \\ \otimes : L_n(A, \epsilon) \otimes_{\mathbf{Z}} L^m(B, \eta) &\rightarrow L_{m+n}(A \otimes_{\mathbf{Z}} B, \epsilon \otimes \eta) \\ \otimes : L_n(A, \epsilon) \otimes_{\mathbf{Z}} L_m(B, \eta) &\rightarrow L_{m+n}(A \otimes_{\mathbf{Z}} B, \epsilon \otimes \eta). \end{aligned}$$

These product formulas appear in the formula for the surgery obstruction of the product of normal maps. Let  $(f, b): M \rightarrow X$  be a degree one normal map, where  $M$  is a closed smooth or topological manifold of dimension  $n$ ,  $X$  is a finite Poincaré complex of dimension  $n$ , and  $b: \nu_M \rightarrow \xi$  is a vector bundle map covering  $f$ . The symmetric signature  $\sigma^*(X)$  is a cobordism invariant of  $X$  in the symmetric  $L$ -group  $L^n(\mathbf{Z}\pi_1(X))$ , and the quadratic signature  $\sigma_*(f)$  of the normal map is a normal cobordism invariant lying in the quadratic  $L$ -group  $L_n(\mathbf{Z}\pi_1(X))$ . If  $n \geq 5$ , then  $\sigma_*(f, b) = 0$  if and only if the normal map  $(f, b)$  is normally cobordant to a homotopy equivalence [22, Section 7]. These invariants are defined in [22] by enriching the chain complexes  $C(M)$ ,  $C(\tilde{X})$  of the universal covering spaces with additional structures arising from Poincaré duality. We will suppress mention of the reference maps  $c_X: X \rightarrow K(\pi_1(X), 1)$  and the orientation characters, as well as the  $K$ -theory torsion decorations (see [21, Section 9] and [9, Section 6]).

**Theorem 2.2** ([22, Proposition 8.1(ii)]). *Let  $(f, b): M \rightarrow X$  and  $(g, c): N \rightarrow Y$  be degree one normal maps, with  $\dim X = n$  and  $\dim Y = m$ . Then for the product  $(f \times g, b \times c): M \times N \rightarrow X \times Y$  degree*

one normal map, the surgery obstruction

$$\sigma_*(f \times g, b \times c) = \sigma_*(f, b) \otimes \sigma_*(g, c) + \sigma^*(X) \otimes \sigma_*(g, c) + \sigma_*(f, b) \otimes \sigma^*(Y)$$

as an element in  $L_{m+n}(\mathbf{Z}[\pi_1(X \times Y)])$ .

The naturality of the Ranicki product formulas (see [9, Proposition 6.3]) will be used to reduce the computation of surgery obstructions to hyperelementary groups. Since we will be dealing with subgroups and finite coverings, it will be convenient to introduce the following notation. Let  $X(G)$  denote a finite Poincaré complex with fundamental group  $G$ , and let  $X(H)$  denote its covering space with fundamental group  $H \subseteq G$ . We have a similar notation  $(f(H), b(H)): M(H) \rightarrow X(H)$  for coverings of degree one normal maps. There are two basic operations

$$\text{Res}_G^H: L_n(\mathbf{Z}G) \rightarrow L_n(\mathbf{Z}H)$$

and

$$\text{Ind}_H^G: L_n(\mathbf{Z}H) \rightarrow L_n(\mathbf{Z}G)$$

associated to subgroups, where the first is defined only for subgroups of finite index. These give the  $L$ -groups a natural Mackey functor structure (see [10, Section 1B], [9, Section 5]). The symmetric and quadratic signatures behave nicely under induction and restriction. For restriction, we can just consider  $\sigma^*(X(G)) = (C(X(e)), \varphi_X)$  as a symmetric structure over  $\mathbf{Z}H$  by restricting the  $G$ -action to  $H$ . Here  $e \in G$  denotes the identity element, so  $X(e) = \tilde{X}$ .

For induction from subgroups, the induced symmetric structure is defined on the  $G$ -covering  $G \times_H X(e) \rightarrow X(H)$  associated to the composition  $X(H) \rightarrow K(H, 1) \rightarrow K(G, 1)$  of reference maps. This gives a symmetric structure denoted  $\sigma^*(G \times_H X(e))$  on the  $\mathbf{Z}G$ -chain complex  $C(G \times_H X(e))$  (compare [14, p. 196]). Similarly, we have an *induced* quadratic signature denoted

$$\sigma_*(G \times_H (f(e), b(e)): G \times_H M(e) \rightarrow G \times_H X(e))$$

for a degree one normal map associated to a covering  $X(H) \rightarrow X(G)$ . In summary:

**Lemma 2.3.** *Let  $(f(G), b(G)): M(G) \rightarrow X(G)$  be an  $n$ -dimensional degree one normal map, with  $G$  a finite group. For any subgroup  $H \subseteq G$  the following formulas hold:*

- (i)  $\text{Res}_G^H(\sigma^*(X(G))) = \sigma^*(X(H))$ .
- (ii)  $\text{Ind}_H^G(\sigma^*(X(H))) = \sigma^*(G \times_H X(e))$ .
- (iii)  $\text{Res}_G^H(\sigma_*(f(G), b(G))) = \sigma_*(f(H), b(H))$ .
- (iv)  $\text{Ind}_H^G(\sigma_*(f(H), b(H))) = \sigma_*(G \times_H (f(e), b(e)))$ .

We now combine these formulas with [Theorem 1.8](#).

**Theorem 2.4.** *Let  $(f, b): M(G_1) \rightarrow X(G_1)$  and  $(g, c): N(G_2) \rightarrow Y(G_2)$  be degree one normal maps, with finite fundamental groups  $G_1$  and  $G_2$  respectively. Then the quadratic signature  $\sigma_*(f \times g, b \times c) = 0$  if the products:*

- (i)  $\sigma_*(f(H_1), b(H_1)) \otimes \sigma_*(g(H_2), c(H_2)) = 0$
- (ii)  $\sigma_*(f(H_1), b(H_1)) \otimes \sigma^*(Y(H_2)) = 0$
- (iii)  $\sigma^*(X(H_1)) \otimes \sigma_*(g(H_2), c(H_2)) = 0$

for all 2-hyperelementary subgroups  $H_1 \subseteq G_1$  and  $H_2 \subseteq G_2$ .

**Proof.** The  $L$ -groups needed for the product formulas are all 2-hyerelementary computable after 2-localization (we will assume this is done, but not add to the notation). Moreover, Wall proved that 2-localization is an injection for  $L$ -groups of finite groups, so the geometric surgery obstruction is 2-locally detected. Notice that by Lemma 2.3, for any inclusion  $H \subset G$  of finite groups,

$$\mathrm{Ind}_H^G(\mathrm{Res}_G^H \sigma^*(X(G))) = \mathrm{Ind}_H^G \sigma^*(X(H)),$$

and similarly for normal maps. By Theorem 1.8, we may write the symmetric signature of  $X(G)$  as an integral linear combination of the images under induction from 2-hyerelementary subgroups of the symmetric signatures of the coverings  $X(H)$ , and similarly for the quadratic signatures of normal maps.

By substituting these expressions for  $G_1$  and  $G_2$  into each individual term of the product formula given in Theorem 2.2, and using the naturality of the product pairings under induction, we may write the quadratic signature  $\sigma_*(f \times g, b \times c) \in L_{n+m}(\mathbf{Z}[G_1 \times G_2])$  as an integral linear combination of the products listed, for all 2-hyerelementary subgroups  $H_1 \subseteq G_1$  and  $H_2 \subseteq G_2$ .  $\square$

### 3. Periodic groups and normal invariants

The foundational work of Swan [25] provides an  $n$ -dimensional, finitely dominated, Poincaré complex  $X = X(G)$  with fundamental group  $G$ , for every finite group  $G$  with periodic cohomology, and every sufficiently large  $n \geq 3$  such that  $n + 1$  is a multiple of the period of  $G$ . It has the property that the universal covering  $\tilde{X} = X(e) \simeq S^n$ . We will call these Swan complexes. The topological spherical space form problem is to decide which Swan complexes are homotopy equivalent to a closed topological manifold.

In [26, Theorem 2.2] it was shown the equivalence classes of  $(G, n)$ -polarized Swan complexes (i.e. fixing an identification of the fundamental group with  $G$  and a homotopy equivalence of the universal covering with  $S^n$ ) correspond bijectively with generators of  $H^{n+1}(G, \mathbf{Z})$ . There is a slight difference in our notation from that of [15] since our groups are acting on  $S^n$  and not  $S^{n-1}$ . Recall that there is a classification of periodic groups into types I–VI, and that some (but not all) of the periodic groups admit fixed-point free orthogonal representations. The quotients of  $S^n$  by a fixed-point free orthogonal action are called *orthogonal spherical space forms*.

Here are the basic technical results of [15, Sections 2–3] and [16, Theorem 2].

**Theorem 3.1** (Madsen–Thomas–Wall). *Let  $G$  be a finite group with periodic cohomology. There exist finite  $(G, n)$ -polarized Swan complexes  $X = X(G)$  such that the covering spaces  $X(H)$  are homotopy equivalent to closed manifolds, for each  $H \subseteq G$  which has a fixed-point free orthogonal representation. In addition,  $X$  has a smooth normal invariant which restricts to the normal invariant of an orthogonal spherical space form for the 2-Sylow covering  $X(G_2)$ .*

**Proof.** The first statement is Lemma 2.1 of [15], and the existence of smooth normal invariants is established in Theorem 3.1 of [15]. The existence of a smooth normal invariant which restricts to that of an orthogonal spherical space form over the 2-Sylow covering was established in [17, Theorem 3.10]. This result is also contained in the proof of Theorem 2 in [16]: the argument is based on Lemmas 3.2 and 3.3 of [15], which do not assume the  $2p$ -conditions, and the concluding step is on [15, p. 380].  $\square$



We now recall the classification of 2-hyerelementary periodic groups. These are of type I or II, and those of type I are semi-direct products of the form

$$1 \rightarrow \mathbf{Z}/m \rightarrow G \rightarrow \mathbf{Z}/2^k \rightarrow 1,$$

where  $m$  is odd, and  $t: \mathbf{Z}/2^k \rightarrow \text{Aut}(\mathbf{Z}/m)$  is the twisting map. The type II groups are semi-direct products

$$1 \rightarrow \mathbf{Z}/m \rightarrow G \rightarrow Q(2^\ell) \rightarrow 1$$

of odd order cyclic groups by quaternion 2-groups  $Q(2^\ell)$ .

**Lemma 3.2.** *A 2-hyerelementary periodic group  $G$  has a non-central element of order two if and only if  $G$  is type I and  $\ker t = \{1\}$ .*

We will call these 2-hyerelementary type I groups with  $\ker t = \{1\}$  the *generalized dihedral groups* and denote them  $G = D(m, 2^k)$ . The ordinary dihedral groups  $D_m$  of order  $2m$  are listed as  $D(m, 2)$  in this notation. Then a periodic group  $G$  satisfies the  $2p$ -conditions (for all primes  $p$ ) if and only if  $G$  does not contain any (generalized) dihedral subgroups. Note that the generalized dihedral groups are the only 2-hyerelementary periodic groups which do *not* admit a fixed-point free orthogonal representation.

The main result of [15], that a finite group  $G$  can act freely on some sphere  $S^n$  if and only if  $G$  satisfies the  $p^2$  and  $2p$  conditions for all primes  $p$ , follows immediately from the results above. Here is a re-formulation of the final surgery step.

**Corollary 3.3** (Madsen–Thomas–Wall). *Let  $G$  be a finite group with periodic cohomology. There exists a finite  $(G, n)$ -polarized Swan complex  $X$ , with  $n \equiv 3 \pmod{4}$ , and a smooth degree one normal map  $(f, b): M \rightarrow X$  such that the quadratic signature  $\sigma_*(f, b) = 0 \in L_n(\mathbf{Z}G)$  provided that  $G$  contains no (generalized) dihedral subgroups.*

**Proof.** We include a summary of their proof. If  $G$  has no generalized dihedral subgroups, then every 2-hyerelementary subgroup of  $G$  admits a fixed-point free orthogonal representation. One then starts with a finite  $(G, n)$ -polarized Swan complex  $X$  as in Theorem 3.1, and the smooth degree one normal map  $(f, b): M \rightarrow X$  given by the special choice of normal invariant. Since 2-localization gives an injective map on the surgery obstruction groups, Dress induction implies that  $\sigma_*(f, b) = 0 \in L_n(\mathbf{Z}G)$  provided that

$$\sigma_*(f(H), b(H)) = 0 \in L_n(\mathbf{Z}H)$$

for all 2-hyerelementary subgroups  $H \subset G$ . However, each  $X(H)$  is homotopy equivalent to a closed smooth manifold, and surgery obstructions of normal maps between closed manifolds are detected by restriction to the 2-Sylow subgroup  $G_2 \subset G$ . Therefore  $\sigma_*(f, b) = 0$ . This argument is explained in detail in [15, Section 4].  $\square$

**Remark 3.4.** The main result of [15] gives free smooth actions on homotopy spheres, but not necessarily actions on the *standard* sphere. It doesn't use the full strength of the special choice of smooth normal invariant provided by Theorem 3.1. In [17, Theorem A], Madsen showed that every group  $G$  satisfying the  $p^2$  and  $2p$  conditions, for all primes  $p$ , acts freely and smoothly on the standard sphere. The argument follows from the naturality of the surgery exact sequence under the covering  $X(e) \rightarrow X$ , and the surjectivity of the restriction map  $L_{n+1}(\mathbf{Z}G) \rightarrow L_{n+1}(\mathbf{Z})$ . The special choice of normal invariant ensures that the smooth structure on  $X(G)$  is covered by the standard sphere.



### 4. Surgery obstruction groups

We will need some information about the surgery obstruction groups  $L_3(\mathbf{Z}G)$  and  $L_2(\mathbf{Z}G)$ , for  $G$  a generalized dihedral group. Here we must be precise about the  $K$ -theory decorations: since every Swan complex  $X$  is *weakly-simple*, we can evaluate the surgery obstruction  $\sigma_*(f, b)$  of a degree one normal map  $(f, b): M \rightarrow X$  in  $L'_n(\mathbf{Z}G)$ , where  $L'$  denotes the weakly-simple obstruction groups (see [13, Section 1]) with allowable torsions in the subgroup  $SK_1(\mathbf{Z}G) \subset Wh(\mathbf{Z}G)$ . Our Swan complexes will all have dimensions  $n \equiv 3 \pmod{4}$ , so  $\sigma_*(f, b) \in L'_3(\mathbf{Z}G)$ . The obstruction for the product of two such problems will be evaluated in  $L'_2(\mathbf{Z}[G_1 \times G_2])$ , since  $n + m \equiv 2 \pmod{4}$ .

**Lemma 4.1.** *Let  $G$  be a generalized dihedral group, and let  $(f, b): M \rightarrow X$  be a degree one normal map to a finite  $(G, n)$ -polarized Swan complex.*

- (i) *The image of  $\sigma_*(f, b)$  in  $L_3^h(\widehat{\mathbf{Z}}_2G)$  is non-zero, but its image in  $L_3^p(\widehat{\mathbf{Z}}_2G)$  is zero.*
- (ii) *If  $G_1$  and  $G_2$  are generalized dihedral groups, the natural map*

$$L'_2(\mathbf{Z}[G_1 \times G_2]) \rightarrow L_2^p(\widehat{\mathbf{Z}}_2[G_1 \times G_2])$$

*is an injection.*

**Proof.** For the first statement we refer to [14] or [6] for the result that the image of  $\sigma_*(f, b)$  in  $L_3^h(\widehat{\mathbf{Z}}_2G)$  is detected by semi-characteristics, and that these vanish if and only if  $G$  satisfies the  $2p$ -conditions. On the other hand,  $L_3^p(\widehat{\mathbf{Z}}_2G) = 0$  since by reduction modulo the radical we obtain the odd  $L^p$ -groups of a semisimple ring, and these all vanish [20].

For the second part we start with the computation of  $L'_2(\mathbf{Z}G)$ . The case  $G = D(m, 2)$  is done in [13, Section 13] and we follow the outline given there. Recall that there is a natural splitting (see [12, Section 5]):

$$L'_2(\mathbf{Z}G) = \bigoplus_{d|m} L'_2(\mathbf{Z}G)(d)$$

induced by idempotents in the 2-localized Burnside ring. If  $d \mid m$ , then the restriction map  $\text{Res}_G^H$  is an isomorphism on the  $d$ -component, where  $H = N_G(\mathbf{Z}/d)$  is the subgroup of  $G$  whose odd-order part is  $\mathbf{Z}/d$ . The splitting is compatible with change of coefficients in  $L$ -theory.  $\square$

**Lemma 4.2.** *If  $G$  is a generalized dihedral group, then  $L'_*(\mathbf{Z}G)(d)$  is torsion-free for all  $d \mid m$  such that  $-1 \in \text{Aut}(\mathbf{Z}/d)$  is not contained in  $\text{Im } t$ .*

**Proof.** We combine [13, Theorem 8.3] with a calculation  $H^*(\text{Wh}'(\widehat{\mathbf{Z}}_2G)(d)) = 0$ , similar to that of [13, Section 10], to obtain  $L'_*(\widehat{\mathbf{Z}}_2G)(d) = 0$ . The result that  $L'_*(\mathbf{Z}G)(d)$  is torsion-free now follows from [13, Table 14.15].  $\square$

We assume from now on that  $-1 \in \text{Im } t$  for all generalized dihedral groups under consideration, since our 2-torsion surgery obstructions clearly vanish under restriction to the  $d$ -components where  $L'_*(\mathbf{Z}G)(d)$  is torsion-free. Under this assumption, the various summands of  $\mathbf{Q}G$  all have type  $O$ , so from [13, Table 14.12] the natural map  $L'_2(\mathbf{Z}G) \rightarrow L'_2(\widehat{\mathbf{Z}}_2G)$  is an injection. From [13, Section 9] we check that the kernel of the map

$$\psi_2: L'_2(\widehat{\mathbf{Z}}_2G) \rightarrow L'_2(\mathbf{Z}G \rightarrow \widehat{\mathbf{Z}}_2G)$$

injects into  $L_2^h(\widehat{\mathbf{Z}}_2G)$ . The basic point is that  $L_2'(\widehat{\mathbf{Z}}_2G)$  is a direct sum of terms of the form  $H^1(\widehat{A}_2^\times) \oplus g_2 \cdot \mathbf{Z}/2$ , where the summand  $g_2 \cdot \mathbf{Z}/2$  injects into  $L_2^p(\widehat{\mathbf{Z}}_2G)$ , and the summand  $H^1(\widehat{A}_2^\times)$  injects into  $H^1((\widehat{A}_2 \otimes \mathbf{Q})^\times)$  under the map  $\psi_2$ .

We now have a comparison sequence

$$\dots \rightarrow H^0(\tilde{K}_0(\widehat{\mathbf{Z}}_2G)) \rightarrow L_2^h(\widehat{\mathbf{Z}}_2G) \rightarrow L_2^p(\widehat{\mathbf{Z}}_2G) \dots$$

where (by convention) we use the involution  $[P] \mapsto -[P*]$  on  $\tilde{K}_0((\widehat{\mathbf{Z}}_2G))$ . However,  $\tilde{K}_0((\widehat{\mathbf{Z}}_2G))$  is free abelian, with induced involution  $-1$ , so  $H^0(\tilde{K}_0(\widehat{\mathbf{Z}}_2G)) = 0$ .

The steps for computing  $L_2'(\mathbf{Z}[G_1 \times G_2])$ , if  $G_1$  and  $G_2$  are generalized dihedral, are the same. The summands of  $\mathbf{Q}[G_1 \times G_2]$  all have type  $O$ , and we get an injection into  $L_2'(\widehat{\mathbf{Z}}_2[G_1 \times G_2])$ . The same arguments as above show that  $\ker \psi_2$  injects into  $L_2^p(\widehat{\mathbf{Z}}_2[G_1 \times G_2])$ .  $\square$

We also need one computation in the non-oriented case. Let  $G \times \mathbf{Z}/2^-$  denote the group  $G \times \mathbf{Z}/2$  with the non-trivial orientation character  $w: G \times \mathbf{Z}/2 \rightarrow \{\pm 1\}$  obtained by projecting onto the second factor.

**Lemma 4.3.** *Let  $G = D(m, 2^k)$  be a generalized dihedral group. Then the natural map*

$$L_1'(\mathbf{Z}[G \times \mathbf{Z}/2^-]) \rightarrow L_1^h(\widehat{\mathbf{Z}}_2[G \times \mathbf{Z}/2^-])$$

*is an isomorphism.*

**Proof.** The existence of a central element  $T$  of order two with  $w(T) = -1$  implies that there are no involution invariant simple factors of the rational group algebra. In other words, we have type  $GL$  for all factors, so the relative groups  $L_*'(\mathbf{Z}G \rightarrow \widehat{\mathbf{Z}}_2G) = 0$ , and  $L_1'(\mathbf{Z}[G \times \mathbf{Z}/2^-]) \cong L_1'(\widehat{\mathbf{Z}}_2[G \times \mathbf{Z}/2^-])$ . By Morita theory (see [9, Section 8] for references), there is an isomorphism

$$L_1'(\mathbf{Z}[G \times \mathbf{Z}/2^-]) \cong \bigoplus_{d|m} L_1'(A(d)[\mathbf{Z}/2^-])$$

where  $A(d) := \widehat{\mathbf{Z}}_2[\zeta_d]^\sigma$  is an unramified 2-ring (with trivial involution) since  $\zeta_d$  is a primitive odd order root of unity. The subgroup  $\sigma := t^{-1}\langle 2 \rangle \subseteq \mathbf{Z}/2^k$  stabilizes each dyadic prime in  $\widehat{\mathbf{Z}}_2[\zeta_d]$ , so acts as Galois automorphisms on the ring of integers. Now by [27, Proposition 3.2.1], the Tate cohomology groups  $H^*(Wh'(A(d)[\mathbf{Z}/2^-])) = 0$  (note the we are in the *exceptional case* mentioned in the paragraph immediately following Proposition 3.2.1). By the Rothenberg sequence, it follows that  $L_1'(\widehat{\mathbf{Z}}_2[G \times \mathbf{Z}/2^-]) \cong L_1^h(\widehat{\mathbf{Z}}_2[G \times \mathbf{Z}/2^-])$ .  $\square$

In the next section, we will need to compute an  $S^1$ -transfer on  $L$ -groups in one particular situation. If  $S^1 \rightarrow E \rightarrow B$  is an  $S^1$  fibration, let  $\phi: \pi_1(B) \rightarrow \{\pm 1\}$  represent its first Stiefel–Whitney class. For any orientation character  $w: \pi_1(B) \rightarrow \{\pm 1\}$ , we have a transfer map  $\Omega_n(B, w) \rightarrow \Omega_{n+1}(E, w\phi)$  defined on the singular bordism groups of closed manifolds whose orientation character is pulled back from  $w$  or  $w\phi$  respectively. If  $\mathbf{Z} \rightarrow G \rightarrow \pi \rightarrow 1$  is the fundamental group sequence of the bundle, then Wall [28, Theorem 11.6] defined a transfer map  $L_n(\mathbf{Z}\pi, w) \rightarrow L_{n+2}(\mathbf{Z}G \rightarrow \mathbf{Z}\pi, w\phi)$  and embedded it in a long exact sequence with relative groups  $LS_n(\Phi) \equiv LN_n(G \rightarrow \pi)$ . The  $S^1$ -bundle transfer associated to the fibration is the composition

$$L_n(\mathbf{Z}\pi, w) \rightarrow L_{n+2}(\mathbf{Z}G \rightarrow \mathbf{Z}\pi, w\phi) \rightarrow L_{n+1}(\mathbf{Z}G, w\phi).$$

The product formulas are natural with respect to  $S^1$ -bundle transfers.

**Lemma 4.4.** *If  $S^1 \rightarrow E \rightarrow B$  is a fibration with fundamental group sequence  $\mathbf{Z} \rightarrow G_2 \rightarrow \pi \rightarrow 1$ , then for any group  $G_1$  the following diagram commutes*

$$\begin{array}{ccc} L_n(\mathbf{Z}G_1) \otimes \Omega_{m-1}(B, w) & \longrightarrow & L_{m+n-1}(\mathbf{Z}[G_1 \times \pi], 1 \times w) \\ \downarrow & & \downarrow \\ L_n(\mathbf{Z}G_1) \otimes \Omega_m(E, w\phi) & \longrightarrow & L_{m+n}(\mathbf{Z}[G_1 \times G_2], 1 \times w\phi) \end{array}$$

where the vertical maps are  $S^1$ -bundle transfers and the horizontal maps are obtained by products with the symmetric signature.

**Proof.** Let  $(f, b): M \rightarrow X$  be degree one normal map with fundamental group  $G_1$ , and suppose that  $S^1 \rightarrow E \rightarrow B$  is an  $S^1$ -bundle with the given fundamental group and orientation data. Elements of the cobordism group  $\Omega_{m-1}(B, w)$  are represented by pairs  $(Y, h)$ , where  $Y$  is an  $(m - 1)$ -dimensional closed manifold and  $h: Y \rightarrow B$  is a reference map, with  $w_1(Y) = h^*(w)$ . In this case,  $\sigma_*(f, b) \otimes \sigma^*(Y, h) = h_*(\sigma_*(f \times id, b \times id))$ . In other words, the product is represented by the quadratic signature associated to the degree one map  $f \times id_Y: M \times Y \rightarrow X \times Y$ . Then we can define the  $S^1$ -bundle transfer by pulling back the bundle  $p: E \rightarrow B$  to  $p^*E \rightarrow Y$ , and taking the quadratic signature of the  $(m + n)$ -dimensional degree one map  $f \times id: M \times p^*E \rightarrow X \times p^*E$ .  $\square$

In [23, p. 808] the relative groups (in the quadratic case) were given an algebraic description

$$LN_n(G \rightarrow \pi) = L_n(\mathbf{Z}G, \beta, u)$$

where  $(\beta, u)$  is an anti-structure on the group ring  $\mathbf{Z}G$  (see [13, Section 1]). If the base  $B$  is a non-orientable Poincaré space, but the total space  $E$  is orientable, let  $w: \pi \rightarrow \{\pm 1\}$  denote the orientation character for the base, and  $\langle t \rangle \subseteq G$  denote the subgroup of  $\pi_1(E)$  generated by  $\pi_1(S^1)$ . We define  $w: G \rightarrow \{\pm 1\}$  by composition. Under these assumptions,  $\beta(g) = g^{-1}$  if  $w(g) = +1$  and  $\beta(g) = -g^{-1}t$  if  $w(g) = -1$  (see [23, p. 805]).

**Example 4.5.** Let  $Q = Q(2^\ell)$  be a generalized quaternion group. There exists a non-orientable fiber bundle

$$S^1 \rightarrow S^3/Q(2^\ell) \rightarrow RP^2$$

with fundamental group sequence  $\mathbf{Z} \rightarrow Q(2^\ell) \rightarrow \mathbf{Z}/2 \rightarrow 1$ , and orientable total space. The image of  $\pi_1(S^1)$  is a cyclic subgroup  $\mathbf{Z}/2^{\ell-1} \subseteq Q(2^\ell)$ . If  $X = X(G)$  is a Swan complex, then the fibration

$$S^1 \rightarrow X \times S^3/Q(2^\ell) \rightarrow X \times RP^2$$

realizes the fundamental group sequence  $\mathbf{Z} \rightarrow G \times Q \rightarrow G \times \mathbf{Z}/2 \rightarrow 1$ .

**Corollary 4.6.** *Let  $G$  be a generalized dihedral group, and  $Q = Q(2^\ell)$  a generalized quaternion group. The  $S^1$ -bundle transfer*

$$L'_1(\mathbf{Z}[G \times \mathbf{Z}/2^-]) \rightarrow L'_2(\mathbf{Z}[G \times Q])$$

is zero.

**Proof.** The first map  $L'_1(\mathbf{Z}[G \times \mathbf{Z}/2^-]) \rightarrow L'_3(\mathbf{Z}[G \times Q \rightarrow G \times \mathbf{Z}/2])$  in the definition of the  $S^1$ -transfer is already zero. We will check this by computing the previous term in the long exact sequence

$$\cdots \rightarrow LN_1(G \times Q \rightarrow G \times \mathbf{Z}/2) \rightarrow L'_1(\mathbf{Z}[G \times \mathbf{Z}/2^-]) \rightarrow L'_3(\mathbf{Z}[G \times Q \rightarrow G \times \mathbf{Z}/2]).$$

In this case,  $LN_1(G \times Q \rightarrow G \times \mathbf{Z}/2) = L'_1(\mathbf{Z}[G \times Q, \beta, u])$  and we are computing the map induced by projection

$$L'_1(\mathbf{Z}[G \times Q, \beta, u]) \rightarrow L'_1(\mathbf{Z}[G \times \mathbf{Z}/2^-]).$$

But we know from Lemma 4.3 that  $L'_1(\mathbf{Z}[G \times \mathbf{Z}/2^-]) \cong L_1^h(\widehat{\mathbf{Z}}_2[G \times \mathbf{Z}/2^-])$ , so it is enough to show that projection induces an isomorphism on the 2-adic  $L$ -groups. But this is clear since  $Q(2^\ell)$  is a finite 2-group, and reduction modulo the radical induces an isomorphism. Therefore  $L_1^h(\widehat{\mathbf{Z}}_2[G \times Q, \beta, u]) \cong L_1^h(\widehat{\mathbf{Z}}_2[G \times \mathbf{Z}/2^-])$ .  $\square$

## 5. The proof of Theorem A

Let  $G_1$  and  $G_2$  be finite groups with periodic cohomology. We choose finite polarized Swan complexes  $X = X(G_1)$  and  $Y = Y(G_2)$ , as given in Theorem 3.1, of dimensions  $n \equiv 3 \pmod{4}$  and  $m \equiv 3 \pmod{4}$ . In particular, the covering spaces  $X(H_1)$  and  $Y(H_2)$  are homotopy equivalent to closed manifolds, for each  $H_1 \subseteq G_1$  and each  $H_2 \subseteq G_2$  which has a fixed-point free orthogonal representation. By Corollary 3.3, these Poincaré complexes also admit smooth degree one normal maps  $(f, b): M \rightarrow X$  and  $(g, c): N \rightarrow Y$  such that the quadratic signatures  $\sigma_*(f, b) = 0 \in L_n(\mathbf{Z}G_1)$  (respectively  $\sigma_*(g, c) = 0 \in L_m(\mathbf{Z}G_2)$ ) if and only if  $G_1$  (respectively  $G_2$ ) contains no generalized dihedral subgroups. We remark that, by taking joins, it is possible to arrange for  $n = m$ , as required in the statement of Theorem A. Our construction actually gives free actions on  $S^n \times S^m$  for any  $n$  and  $m$  arising from the Swan complexes above.

We will now compute the product formula (Theorem 2.2) to show that the quadratic signature  $\sigma_*(f \times g, b \times c) = 0 \in L_{m+n}(\mathbf{Z}[G_1 \times G_2])$ . It will then follow by surgery theory, that there is a smooth, closed manifold  $W \simeq X \times Y$  such that the universal covering  $\widetilde{W}$  is homotopy equivalent to  $S^n \times S^m$ . Since  $n + m \equiv 2 \pmod{4}$ , the surgery exact sequence [28, Section 10] implies that  $\widetilde{W}$  is diffeomorphic to  $S^n \times S^m$ , and we obtain a free action on a product of standard spheres.

Now we proceed with the calculation of the quadratic signature. According to Theorem 2.4,  $\sigma_*(f \times g, b \times c) = 0$  provided that:

- (i)  $\sigma_*(f(H_1), b(H_1)) \otimes \sigma_*(g(H_2), c(H_2)) = 0$
- (ii)  $\sigma_*(f(H_1), b(H_1)) \otimes \sigma^*(Y(H_2)) = 0$
- (iii)  $\sigma^*(X(H_1)) \otimes \sigma_*(g(H_2), c(H_2)) = 0$

for all 2-hyerelementary subgroups  $H_1 \subseteq G_1$  and  $H_2 \subseteq G_2$ . The calculation is now reduced to evaluating these products in the following three situations:

- A1.  $L'_3(\mathbf{Z}H_1) \otimes L'_3(\mathbf{Z}H_2) \rightarrow L'_2(\mathbf{Z}[H_1 \times H_2])$ , where  $H_1$  and  $H_2$  are both generalized dihedral,
- A2.  $L'_3(\mathbf{Z}H_1) \otimes L^3(\mathbf{Z}H_2) \rightarrow L'_2(\mathbf{Z}[H_1 \times H_2])$ , where  $H_1$  and  $H_2$  are both generalized dihedral,
- A3.  $L'_3(\mathbf{Z}H_1) \otimes L^3(\mathbf{Z}H_2) \rightarrow L'_2(\mathbf{Z}[H_1 \times H_2])$ , where  $H_1$  is generalized dihedral, and  $H_2$  has no dihedral subgroups.

In addition, we may reverse the roles of  $H_1$  and  $H_2$  in A2 and A3 to cover all the possible cases. No new calculations are involved.

Cases A1 and A2

By Lemma 4.1, the image of  $\sigma_*(f, b)$  in  $L_3^h(\widehat{\mathbf{Z}}_2 H_1)$  is non-zero, but its image in  $L_3^p(\widehat{\mathbf{Z}}_2 H_1)$  is zero. On the other hand, the natural map  $L_2'(\mathbf{Z}[H_1 \times H_2]) \rightarrow L_2^p(\widehat{\mathbf{Z}}_2[H_1 \times H_2])$  is an injection. The vanishing of  $\sigma_*(f, b) \otimes \sigma_*(g, c)$  in Case A1 follows from applying these facts to the diagram

$$\begin{array}{ccc} L_3^h(\widehat{\mathbf{Z}}_2 H_1) \otimes L_3^h(\widehat{\mathbf{Z}}_2 H_2) & \longrightarrow & L_2^h(\widehat{\mathbf{Z}}_2[H_1 \times H_2]) \\ \downarrow & & \downarrow \\ L_3^p(\widehat{\mathbf{Z}}_2 H_1) \otimes L_3^p(\widehat{\mathbf{Z}}_2 H_2) & \longrightarrow & L_2^p(\widehat{\mathbf{Z}}_2[H_1 \times H_2]) \end{array}$$

For Case A2 we use a similar diagram with  $L_3^3(\widehat{\mathbf{Z}}_2 H_2)$  instead of  $L_3^h(\widehat{\mathbf{Z}}_2 H_2)$ . We get  $\sigma_*(f, b) \otimes \sigma^*(Y) = 0$ , and similarly  $\sigma^*(X) \otimes \sigma_*(g, c) = 0$  by reversing the roles of  $H_1$  and  $H_2$ .

Case A3

In this case,  $H_2$  has no dihedral subgroups, so  $X(H_2)$  is homotopy equivalent to an orthogonal spherical space form. Therefore  $\sigma^*(Y)$  is the symmetric signature of a closed manifold, and hence lies in the image

$$\Omega_m(BH_2) \rightarrow L^m(\mathbf{Z}H_2).$$

This subgroup has better induction properties: it is computable using the family of 2-Sylow subgroups (recall that we have localized at 2). It follows that  $\sigma^*(Y)$  as an odd integral multiple of  $\text{Ind}_{K_2}^{H_2}(\sigma^*(Y(K_2)))$ , where  $K_2 \subseteq H_2$  is a 2-Sylow subgroup. By naturality of the product formula, we may assume from the beginning that  $H_2 = \mathbf{Z}/2^\ell$  or  $H_2 = Q(2^\ell)$ .

Since  $\Omega_m(BH_2) \otimes \mathbf{Z}_{(2)} \cong H_m(BH_2; \mathbf{Z}_{(2)}) \cong \mathbf{Z}/|H_2|$ , any orthogonal space form  $Y(H_2)$  is bordant to an (odd) multiple of a generator of this homology group. Suppose that  $m = 4k + 3$ . Then we can represent a generator of  $\Omega_m(BH_2) \otimes \mathbf{Z}_{(2)}$  by a standard quotient  $Y_0(H_2) := S^m/H_2$ , constructed as the join of  $k + 1$  copies of a free orthogonal action on  $S^3$ . By linearity of the product formula, it is enough to evaluate the product formula for  $Y_0(H_2)$ .

If  $H_2 = \mathbf{Z}/2^\ell$ , then  $Y_0 = S^m/\mathbf{Z}/2^\ell$  is the total space of an oriented  $S^1$ -bundle:

$$S^1 \rightarrow S^m/\mathbf{Z}/2^\ell \rightarrow \mathbb{C}P^{2k+1}$$

and  $\mathbb{C}P^{2k+1}$  is an oriented boundary. Therefore, the product  $\sigma_*(f, b) \otimes \sigma^*(Y_0)$  is the  $S^1$ -bundle transfer of the product  $\sigma_*(f, b) \otimes \sigma^*(\mathbb{C}P^{2k+1}) = 0$ .

**Remark 5.1.** An alternate argument could be given by computing the group  $L_2'(\mathbf{Z}[H_1 \times H_2])$  using [27, Proposition 3.2.3] as above.

If  $H_2 = Q(2^\ell)$ , then  $Y_0 = S^m/Q(2^\ell)$  is the total space of a non-orientable  $S^1$ -bundle

$$S^1 \rightarrow S^m/Q(2^\ell) \rightarrow \mathbb{C}P^{2k+1}/\mathbf{Z}/2$$

where the base is the orbit space of  $\mathbb{C}P^{2k+1}$  under a free, orientation-reversing  $\mathbf{Z}/2$ -action. In homogeneous co-ordinates the action of the generator is given by

$$[z_0, z_1, \dots, z_{2k}, z_{2k+1}] \mapsto [-\bar{z}_1, \bar{z}_0, \dots, -\bar{z}_{2k+1}, \bar{z}_{2k}]$$

where  $\bar{z}$  denotes complex conjugation. There is a commutative diagram of quotients

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^m & \longrightarrow & \mathbb{C}P^{2k+1} \\ \downarrow & & \downarrow & & \downarrow \\ S^1/\mathbf{Z}/2^{\ell-1} & \longrightarrow & S^m/Q(2^\ell) & \longrightarrow & \mathbb{C}P^{2k+1}/\mathbf{Z}/2 \end{array}$$

arising from embedding the extension  $1 \rightarrow \mathbf{Z}/2^{\ell-1} \rightarrow Q(2^\ell) \rightarrow \mathbf{Z}/2 \rightarrow 1$  in the unit quaternions  $S^3$ , and taking joins to get  $S^{4k+3} = S^3 * \dots * S^3$ . Let  $Z_0 = \mathbb{C}P^{2k+1}/\mathbf{Z}/2$  denote this orbit space.

The product  $\sigma_*(f, b) \otimes \sigma^*(Z_0)$  lies in  $L'_1(\mathbf{Z}[H_1 \times \mathbf{Z}/2^-])$ , where  $H_1$  is a generalized dihedral group. By Corollary 4.6, the  $S^1$ -bundle transfer

$$L'_1(\mathbf{Z}[H_1 \times \mathbf{Z}/2^-]) \rightarrow L'_2(\mathbf{Z}[H_1 \times H_2])$$

is zero, so the product  $\sigma_*(f, b) \otimes \sigma^*(Y_0) = 0$  by Lemma 4.4. This completes the proof of Theorem A.

The multiplicativity  $\sigma^*(X \times Y) = \sigma^*(X) \otimes \sigma^*(Y)$  of the symmetric signature [22, Proposition 8.1(i)], and the vanishing of pair-wise products established above, gives many more examples of the same kind:

**Theorem 5.2.** *For  $k > 1$ ,  $k \neq 3$ , any product  $G_1 \times \dots \times G_k$  of finite groups with periodic cohomology acts freely and smoothly on  $(S^n)^k$ , for some  $n \equiv 3 \pmod{4}$ .*

**Remark 5.3.** If  $k = 3$ , the same argument only shows that a product  $G_1 \times G_2 \times G_3$  of periodic groups acts freely and smoothly either on  $(S^n)^3$ , or possibly on  $(S^n)^3 \sharp \Sigma^{3n}$ , where  $\Sigma^{3n}$  denotes the Kervaire homotopy sphere in dimension  $3n \equiv 1 \pmod{4}$ . If the Kervaire sphere is non-standard (e.g. when  $3n \neq 2^t - 3$ ,  $t \geq 2$ ), then  $(S^n)^3 \sharp \Sigma^{3n}$  is not diffeomorphic to  $(S^n)^3$ . The smallest example is  $n = 3$ , and  $\Sigma^9$  is non-standard.

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