Topology change of vortices
weak and strong solutions

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Outline

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- SDE model for vortex filament interaction

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  valid for nearly parallel vortex filaments with filament separation much greater than width of vortex core.

- Topology change is impossible in this approximation.

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where $\mathbf{X}_j(z,t) = (x_j(z,t), y_j(z,t))$ are the coordinates of the vortex centrelines, Γ_j are their circulations, $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\mathbf{b}_j(z,t)$ are independent Gaussian random variables.

SDE model for vortex filaments

We now consider the case of two filaments:

$$\begin{aligned}
\frac{\partial \psi_1}{\partial t} &= \frac{\partial^2 \psi_1}{\partial z^2} + 2\Gamma \frac{\psi_1 - \psi_2}{|\psi_1 - \psi_2|^2} + \sqrt{2\nu'} b_1 \\
\frac{\partial \psi_2}{\partial t} &= \frac{\partial^2 \psi_2}{\partial z^2} - 2 \frac{\psi_1 - \psi_2}{|\psi_1 - \psi_2|^2} + \sqrt{2\nu'} b_2
\end{aligned}$$

where $\psi_j = x_j(z,t) + i\, y_j(z,t)$, $b_j(z,t) = b_{j1} + i\, b_{j2}$, we have set $\Gamma_1 = 1$, $\Gamma = \Gamma_2/\Gamma_1$, and time has been re-scaled by 4π so $\nu' = 4\pi\nu$.

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$$\frac{\partial \psi_1}{\partial t} = 2\Gamma \frac{\psi_1 - \psi_2}{|\psi_1 - \psi_2|^2} + \sqrt{2\nu'} b_1$$

$$\frac{\partial \psi_2}{\partial t} = -2 \frac{\psi_1 - \psi_2}{|\psi_1 - \psi_2|^2} + \sqrt{2\nu'} b_2$$

- The curvature term is not present in two dimensions.

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- Model is computationally efficient
- Model can be analyzed mathematically (Agullo & Verga have given an exact solution in the special case they considered)

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$\longrightarrow$ Analyze symmetric vortex merging interactions in 2D and symmetric vortex reconnection in 3D.

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\hat{\psi}_1(t+\Delta t) &= \hat{\psi}_1(t)\exp[-i\,\Delta t\,k^2] \\
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4. Repeat for each realization to build up pdf.

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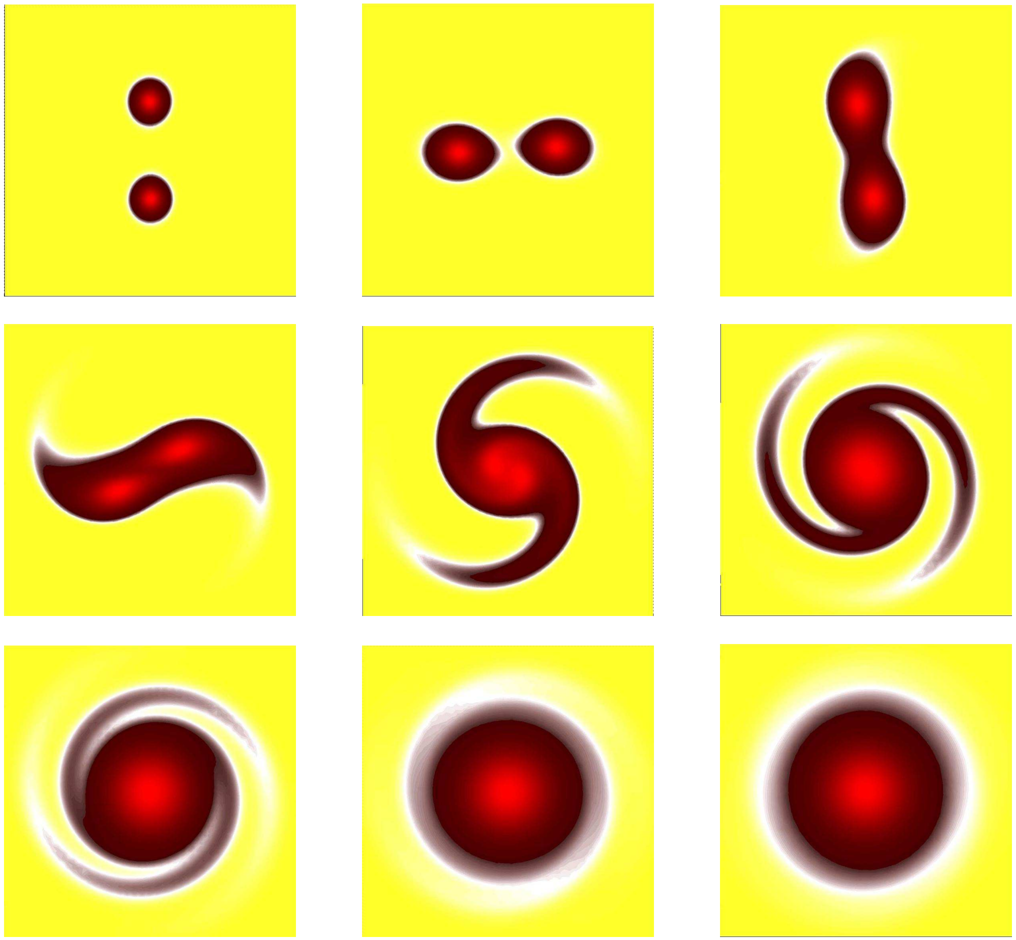
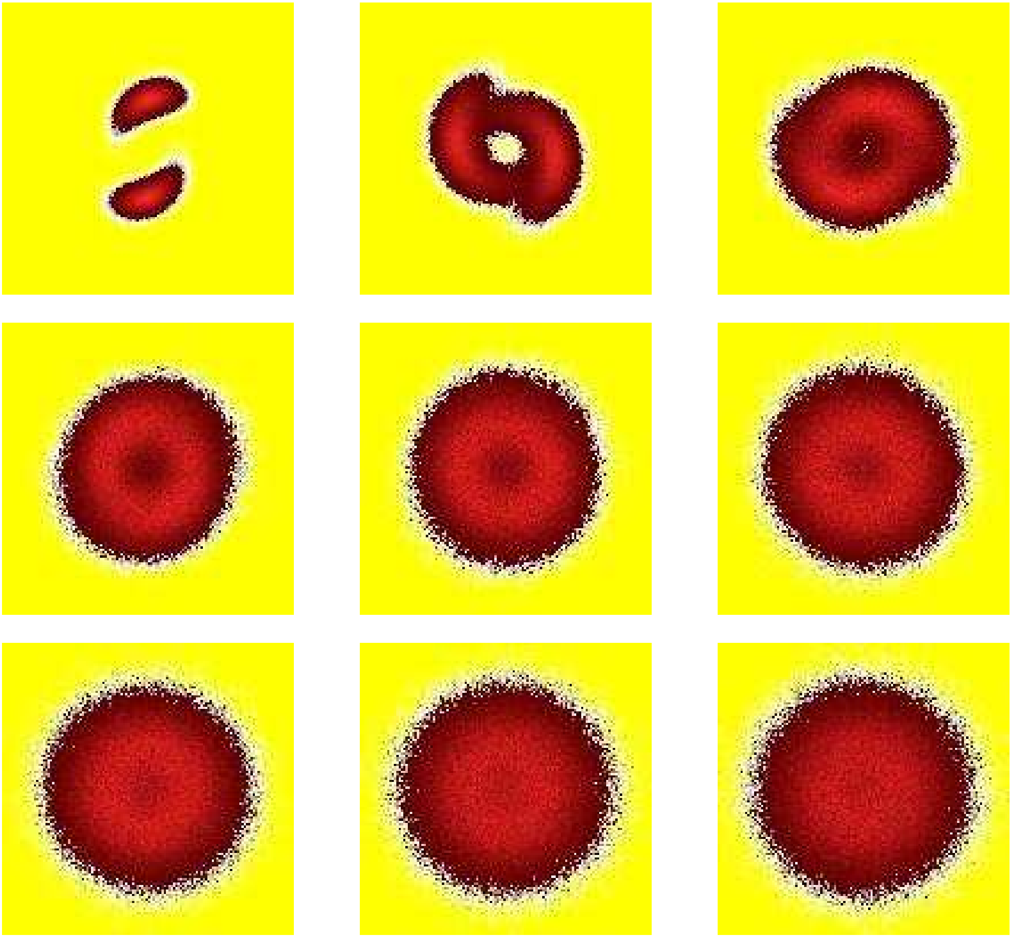
- Compare SDE model with high resolution adaptive wavelet numerical solution of full 2D vorticity equations.

2D vortex merging



Vortex merging at $Re = 1\,000$, full adaptive wavelet solution

2D vortex merging



Vortex merging at $Re = 1\,000$, weak stochastic solution

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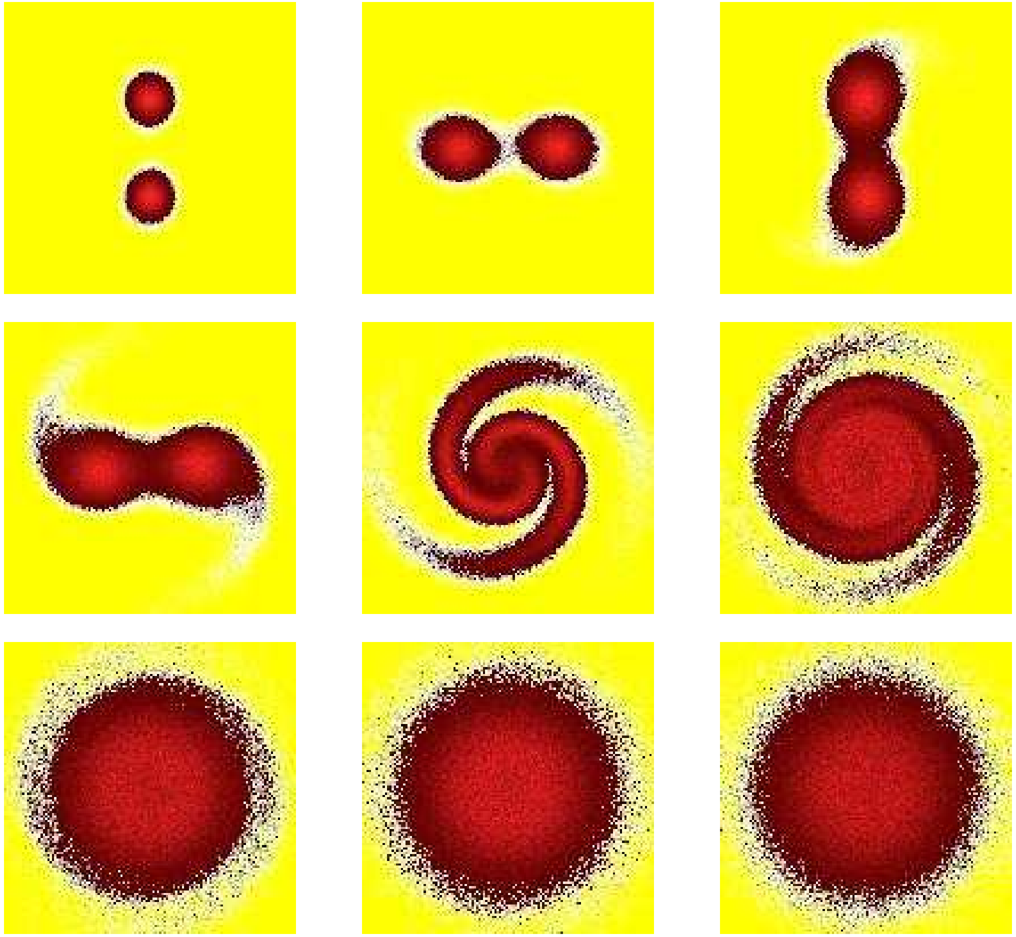
- This correction models the continuous vorticity distribution.

2D vortex merging

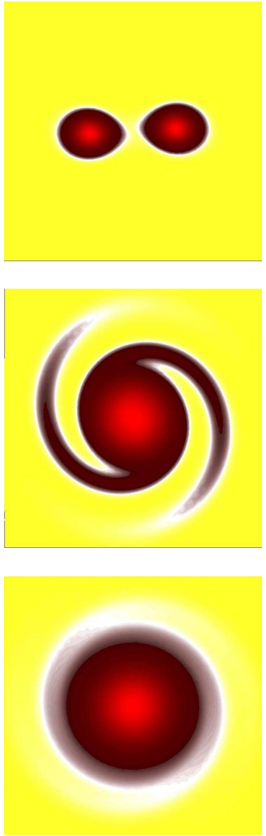


Vortex merging at $Re = 1\,000$, Gaussian velocity field

2D vortex merging

Effect of continuous vorticity on merging: which part of the continuous vorticity field is most important?

2D vortex merging



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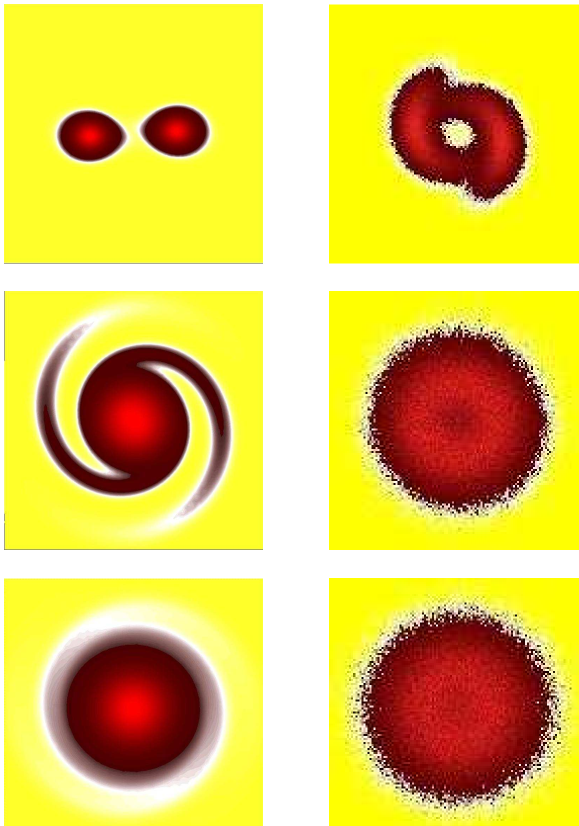

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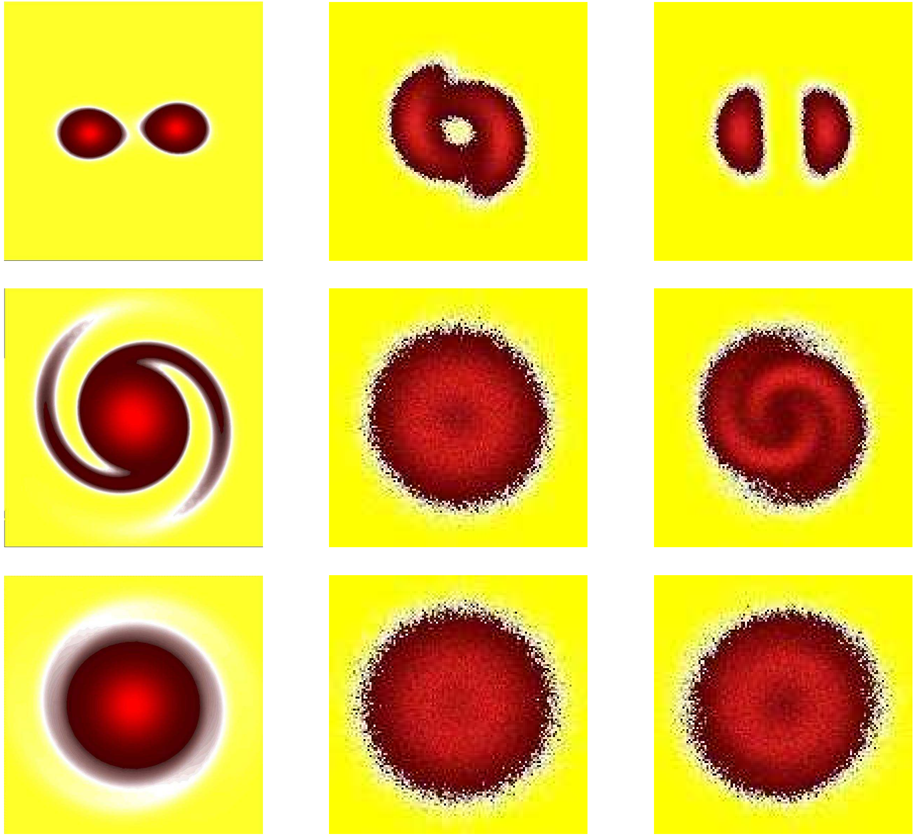

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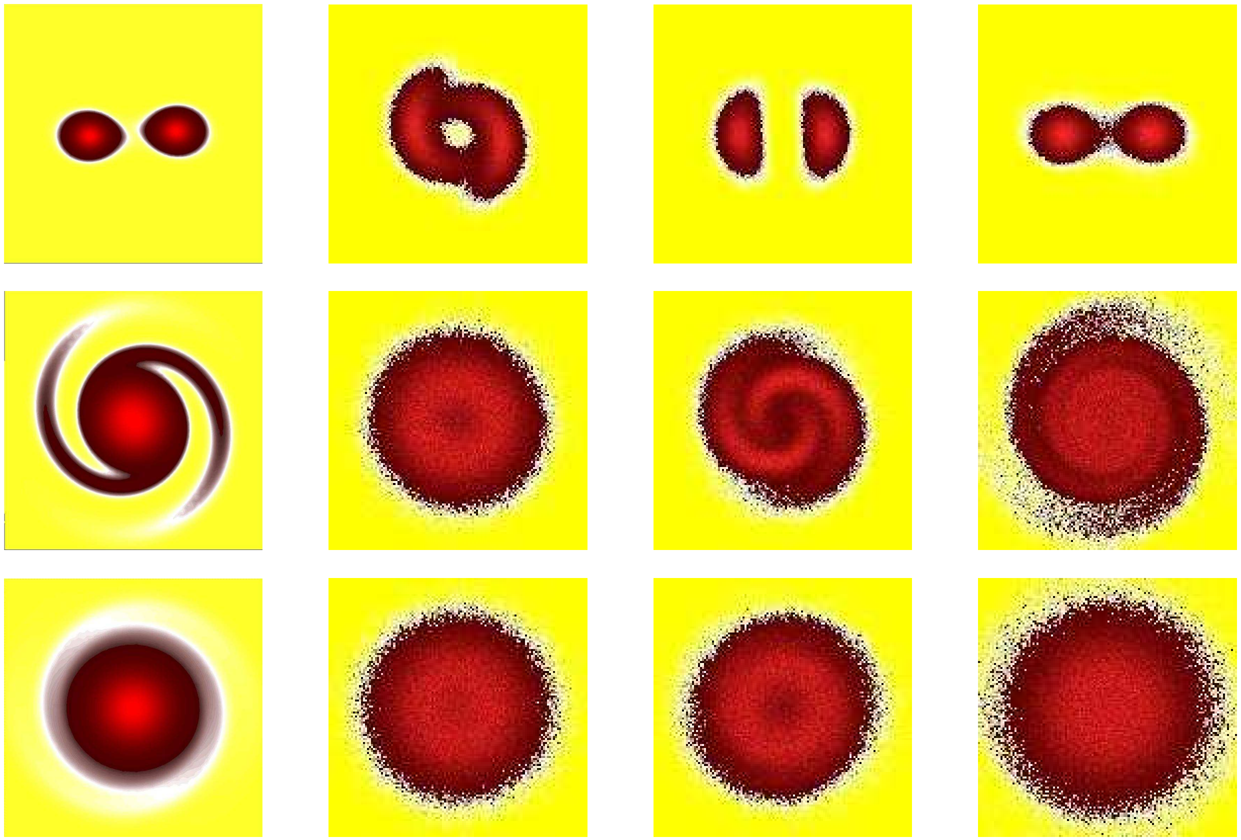

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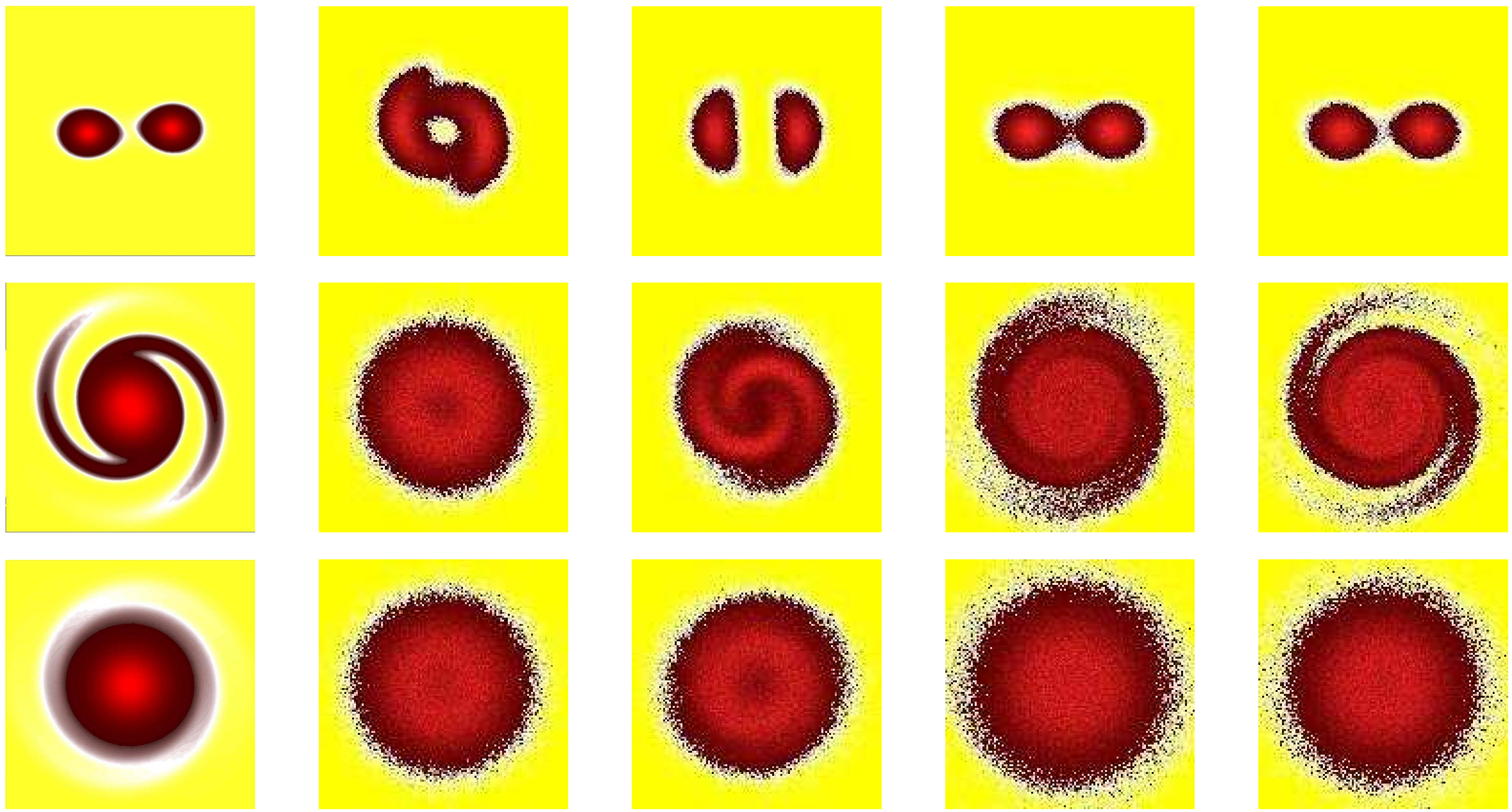

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3D vortex reconnection

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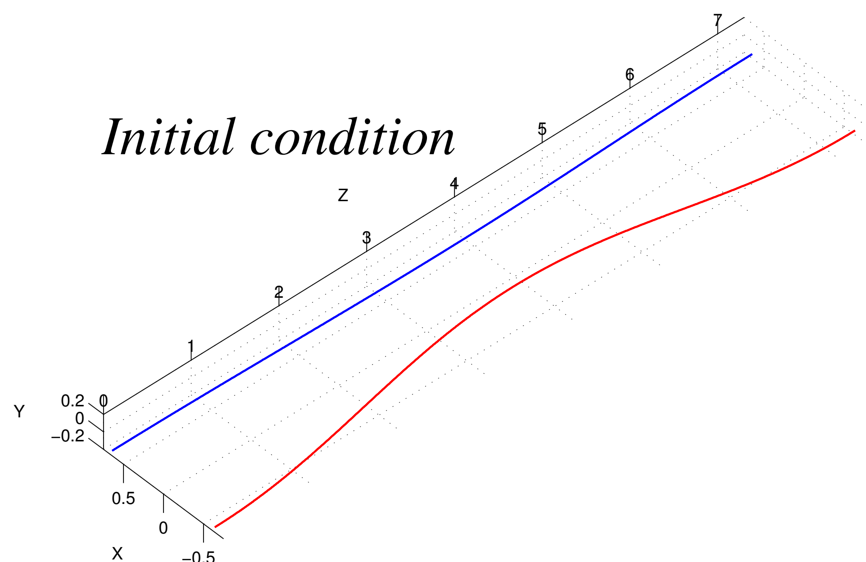
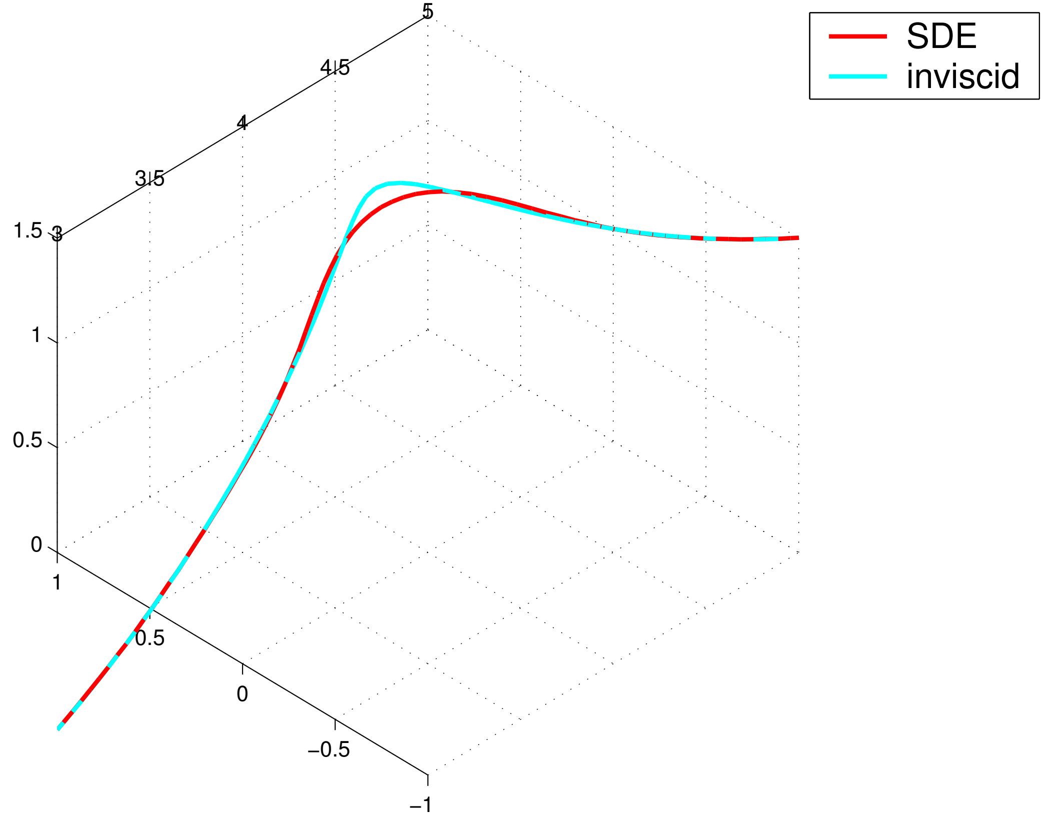


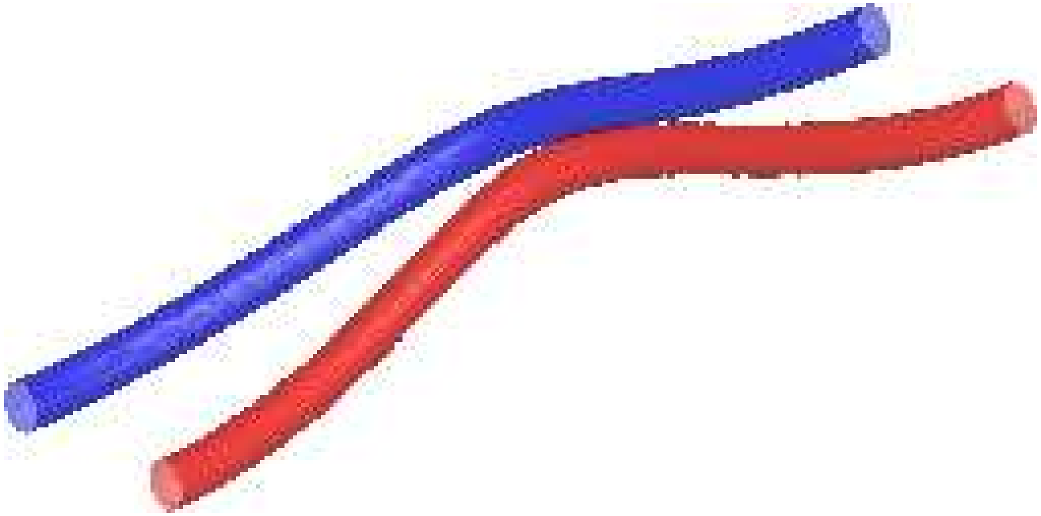
3D vortex reconnection

Comparison of SDE and inviscid models at $t = 0.51$



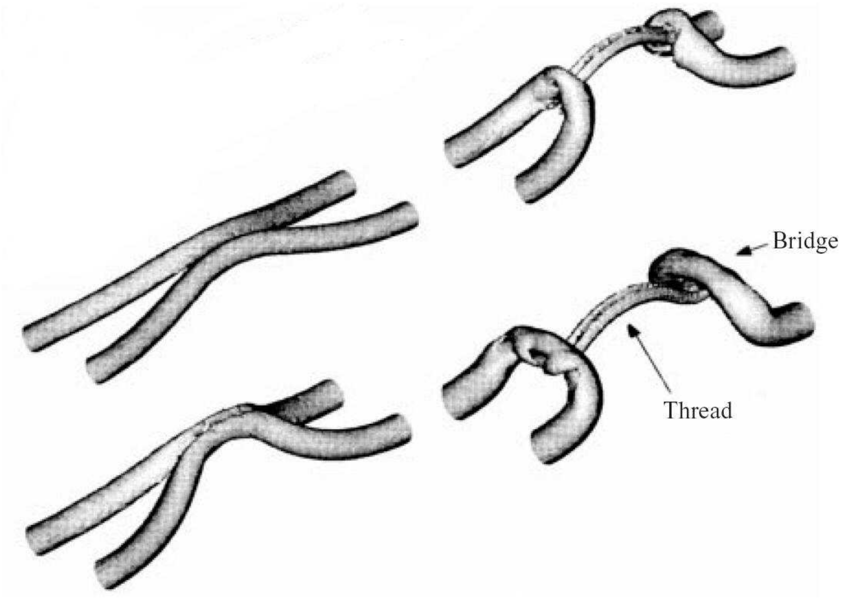
Only positive vortex is shown. Inviscid solution breaks down at $t \approx 0.522$ as vortices develop kinks and touch.

3D vortex reconnection

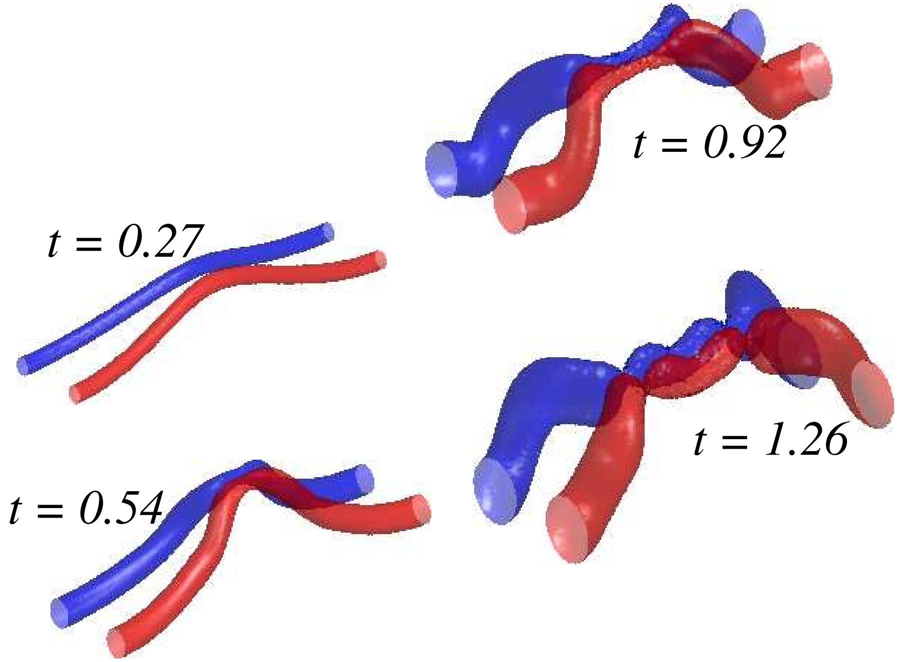


SDE model simulation of vortex reconnection at $Re = 1\,5000$.

3D vortex reconnection



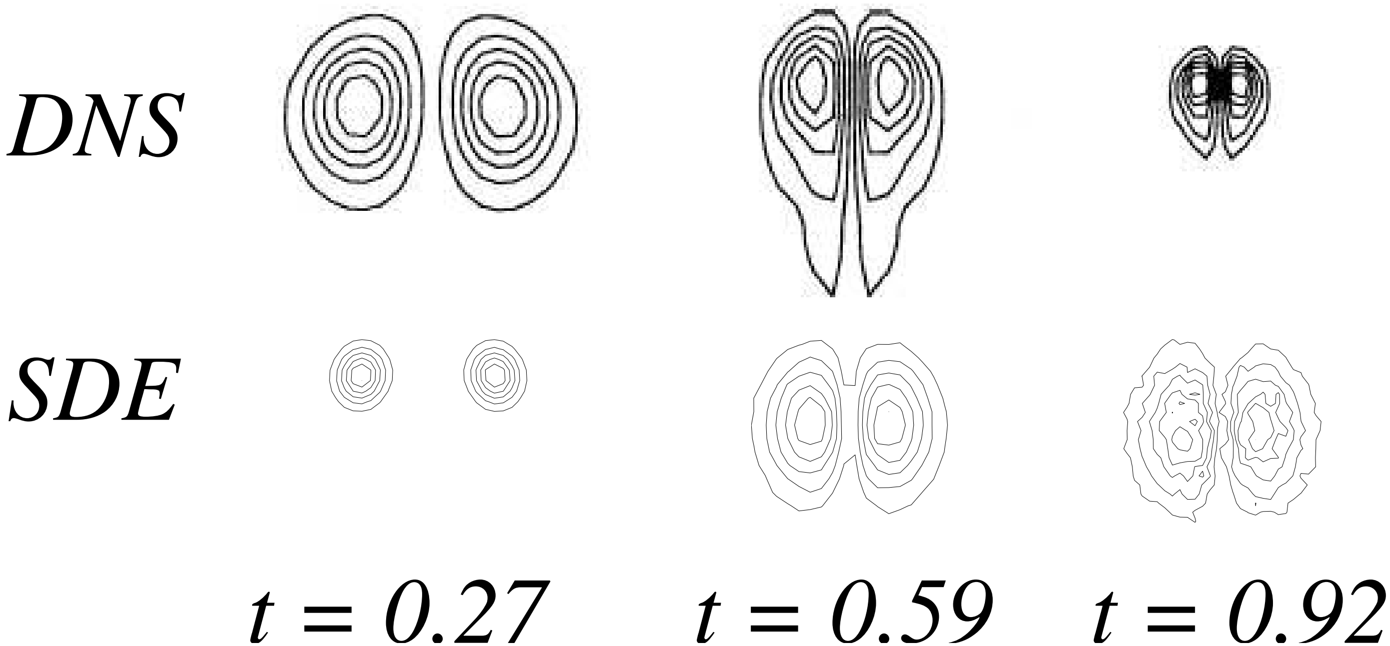
DNS (Marshall et al. 2001)



SDE model

3D vortex reconnection

Vorticity contours in $z = \lambda/2$ plane



(At $t = 0$ the DNS vortices have a finite radius $\sigma_0 = 0.2$.)

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- qualitative agreement is reasonable for times $t \gg t_c \approx 0.522$ where inviscid theory fails
- 3D model is much better than uncorrected 2D