Adaptive wavelet simulation of fluid–structure interaction in 2D and 3D

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Collaborators

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Outline

1. Motivation for adaptive wavelets

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2. Adaptive wavelet collocation method
   - Construction of second generation wavelets
   - Adaptive wavelet collocation method
   - One-dimensional examples: Burgers equation and moving shock

Outline (cont.)

3. Fluid–structure interaction
   - Adaptive wavelet collocation
   - Brinkman penalization
   - Elliptic solver for pressure

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   - Flow past cylinders (2D)
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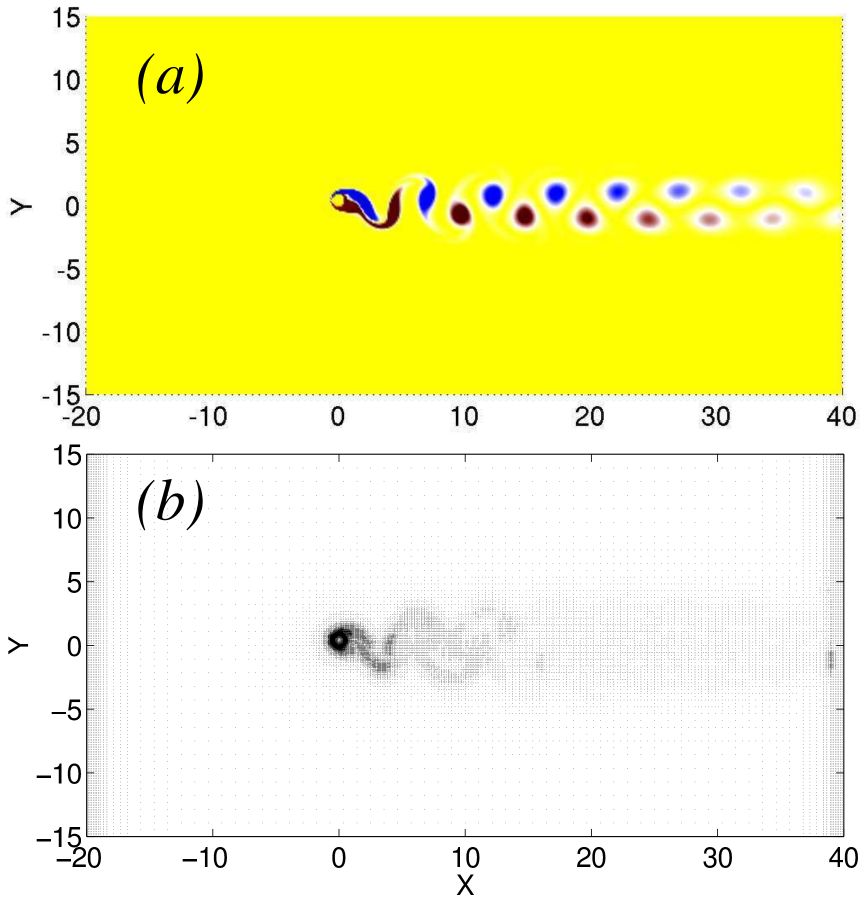
5. Conclusions

Motivation: vortices

Forced isotropic turbulence, $Re_\lambda = 72$, maximum resolution = 128^3, iso-surface of vorticity at $30\% \ ||\vec{\omega}||_\infty$.

Motivation: complex geometry



Moving cylinder at Re = 100, effective grid = $3\ 584 \times 1\ 792$.

Why wavelets?

1. High rate of data compression (e.g. jpeg2 2000 image compression)

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2. Fast $O(\mathcal{N})$ transform
3. Fast signal de-noising (optimal for additive Gaussian noise)
4. Easy to control wavelet properties (e.g. smoothness, boundary conditions)

What are Wavelets?

Basic property:

A set of basis functions that are localized in physical and wavenumber spaces.

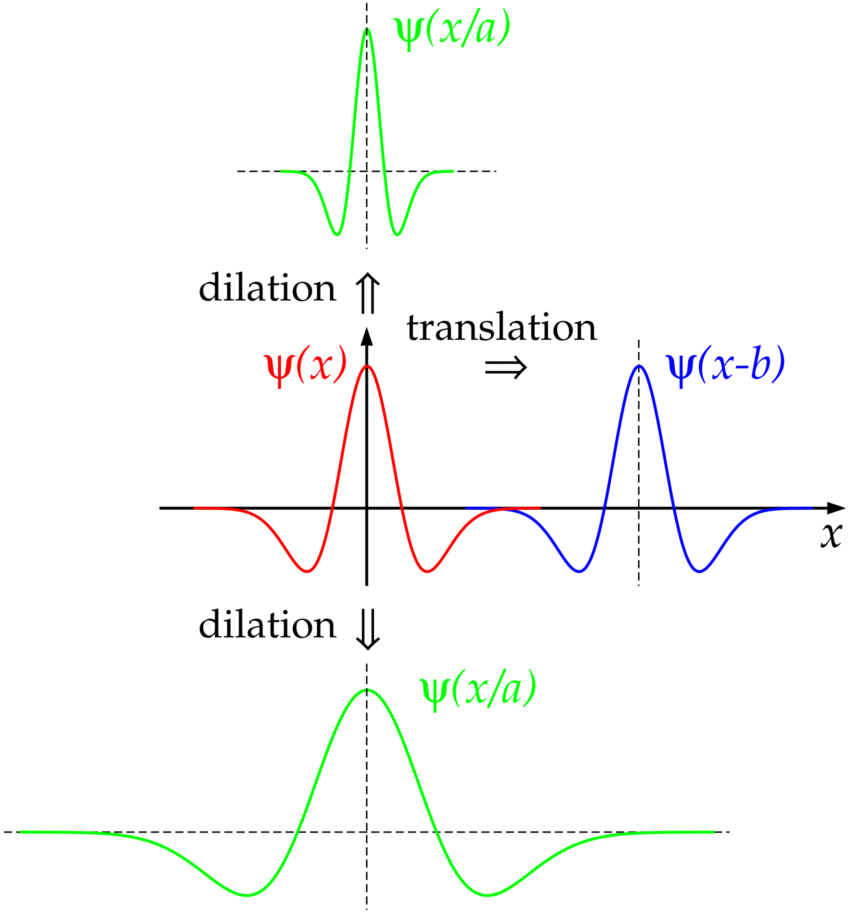
What are Wavelets?

Definition:

A second-generation multi-resolution analysis $\mathbf{M}$ of a function space $\mathbf{L}$ consists of a sequence of closed subspaces $\mathbf{M} = \{\mathcal{V}^j \subset \mathbf{L} \mid j \in \mathcal{J}\}$ such that

1. $\mathcal{V}^j \subset \mathcal{V}^{j+1}$,
2. $\bigcup_{j \in \mathcal{J}} \mathcal{V}^j$ is dense in $\mathbf{L}$, and
3. for each $j \in \mathcal{J}$, $\mathcal{V}^j$ has a Reisz basis given by scaling functions $\{\phi_k^j \mid k \in \mathcal{K}^j\}$.

Construction of wavelet families



Which wavelet family to choose?

- Collocation or Galerkin method?
- Cost of calculating nonlinear terms?
- General boundary conditions?
- Cost of dynamic grid adaptation?
- Cost of calculating spatial operators on an adaptive grid?
- Ease of generalizing to complex geometries?

Second Generation Wavelet*

- Collocation or Galerkin method? *Collocation*
- Cost of calculating nonlinear terms? $O(\mathcal{N})$, *easy*
- General boundary conditions? *Straightforward*
- Cost of dynamic grid adaptation? $O(\mathcal{N})$
- Cost of calculating spatial operators on an adaptive grid? $O(\mathcal{N})$
- Ease of generalizing to complex geometries? *Feasible*

* (Sweldens, 1996)

Second Generation Wavelets

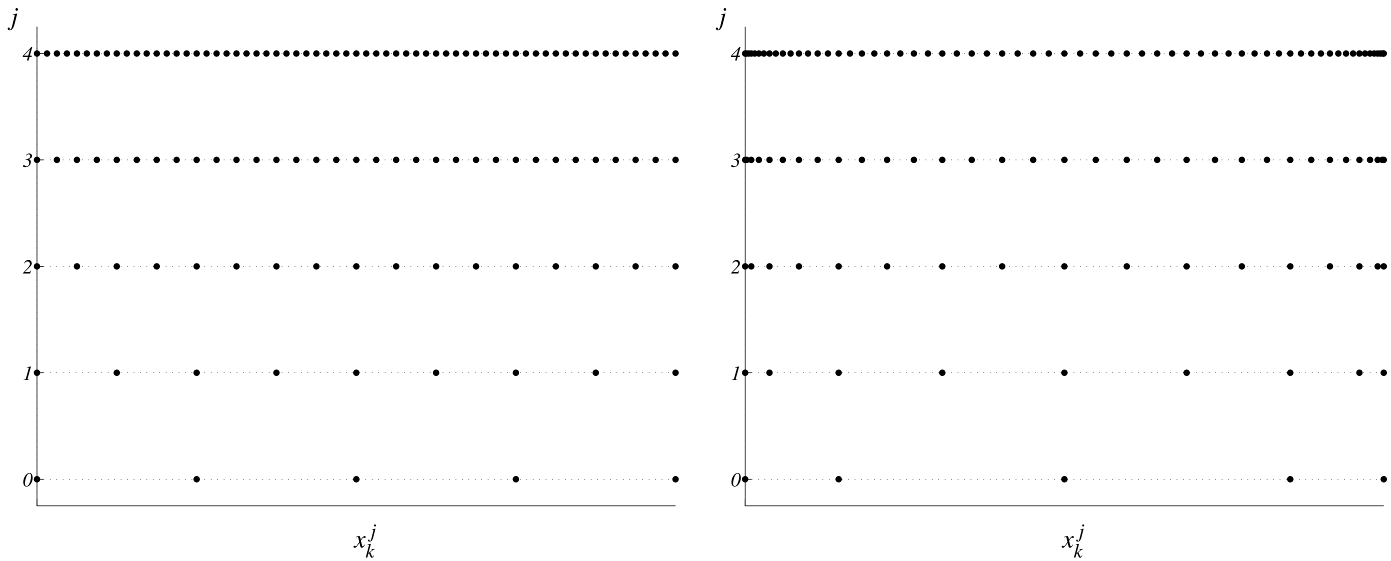
Main properties

- Constructed in spatial domain
- Can be custom designed for complex domains and irregular sampling intervals
- No auxiliary memory is required and the original signal can be replaced with its wavelet transform
- Allows to perform wavelet transform (both forward and inverse) on an adaptive grid

Wavelet Construction

Nested wavelet grids

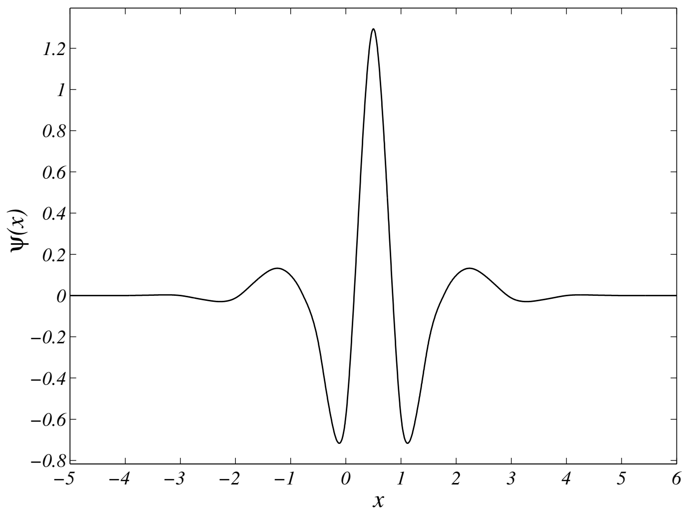
$$\mathcal{G}^j = \left\{x_k^j \in \Omega : \; x_k^j = x_{2k}^{j+1}, \; k \in \mathcal{K}^j\right\}, \; j \in \mathcal{J}$$

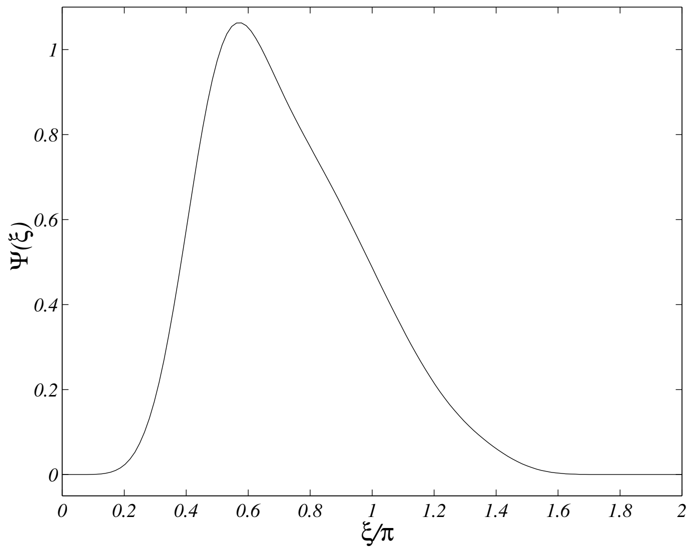


Uniform Grid

Nonuniform Grid

Second Generation Wavelets

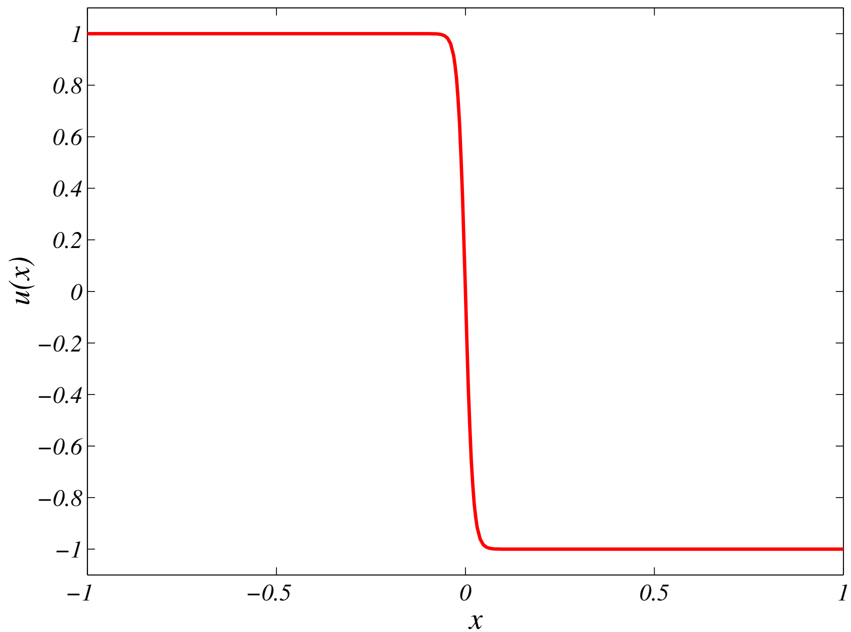




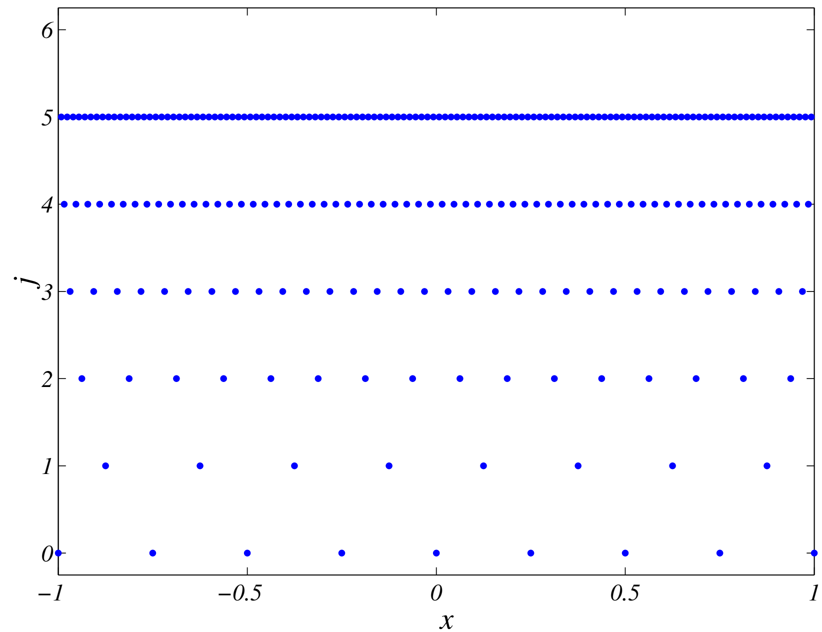
Wavelet Fourier Transform

Wavelet Compression

$$u(\mathbf{x}) = \sum_{j=0}^{+\infty} \sum_{\mathbf{k} \in \mathcal{K}^j} d_{\mathbf{k}}^j \psi_{\mathbf{k}}^j(\mathbf{x})$$



Function $u(x)$



Wavelet locations $x_{\mathbf{k}}^j$

Wavelet Compression

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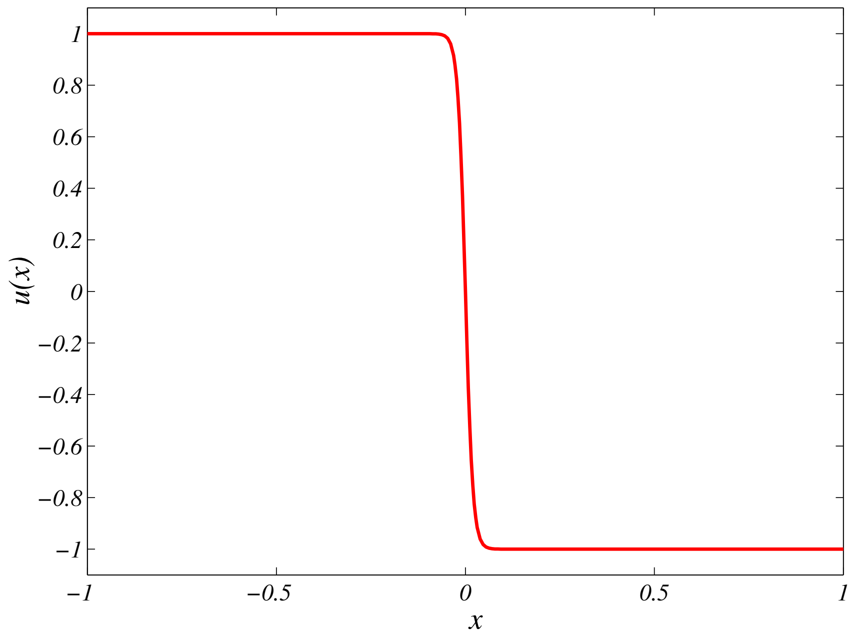
level



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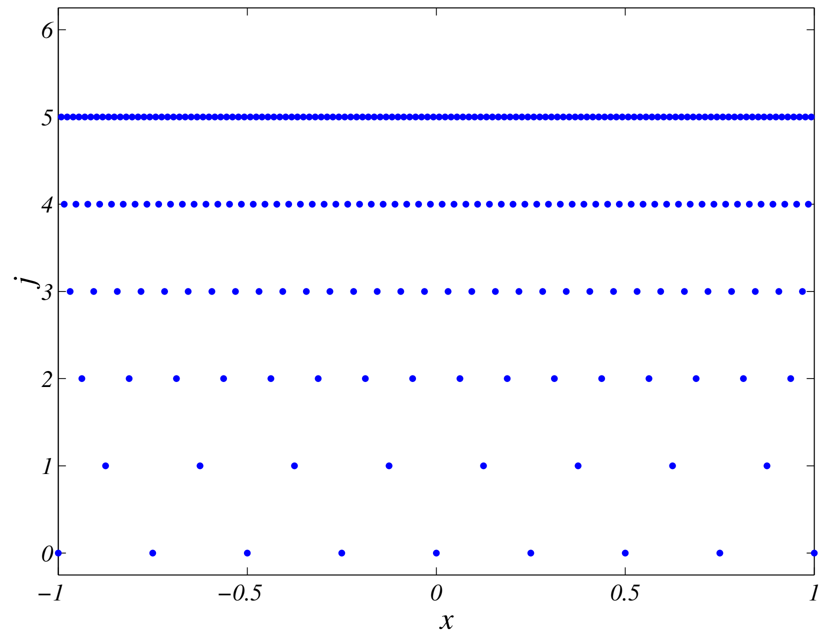

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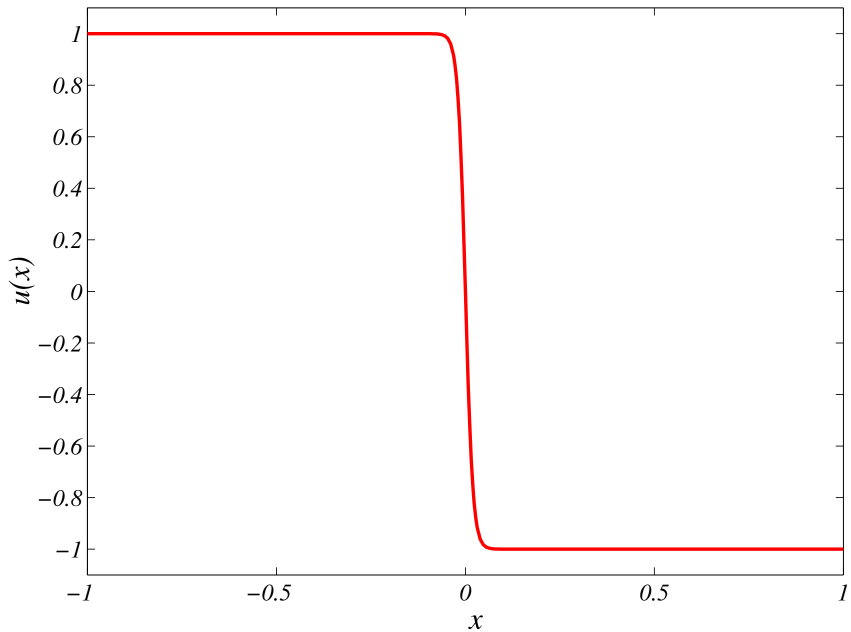

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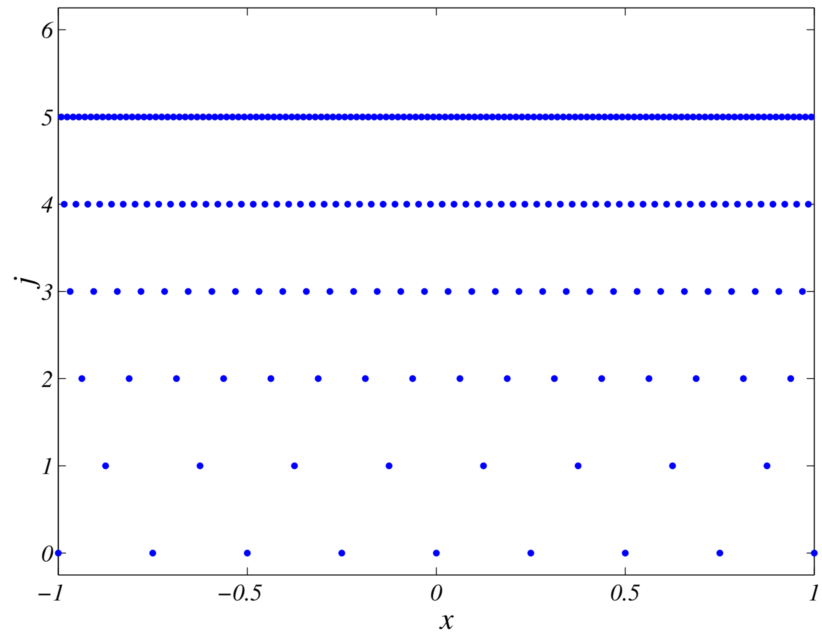

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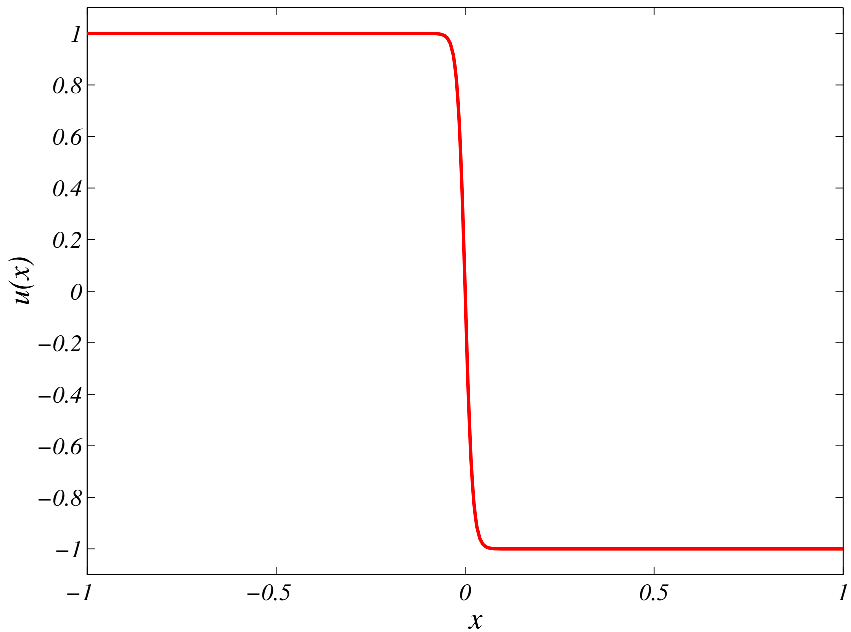

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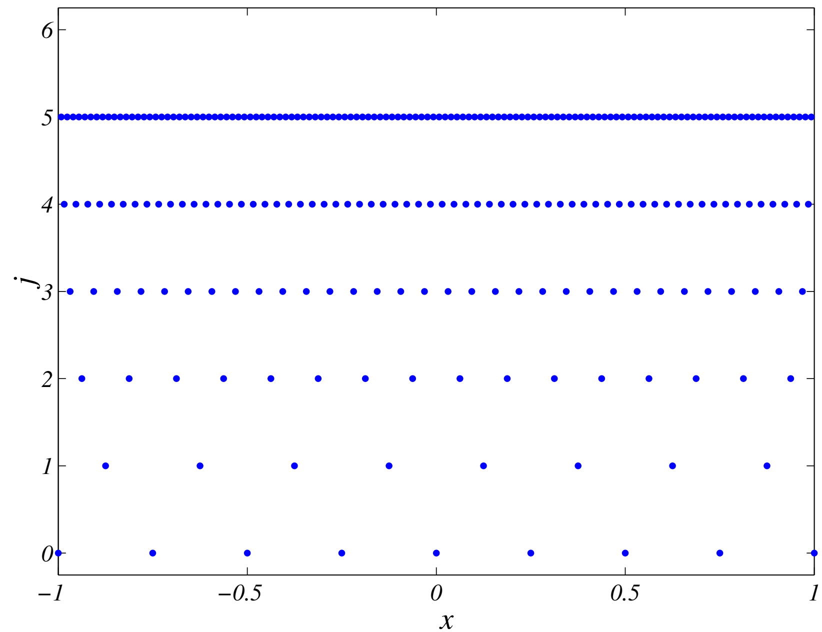
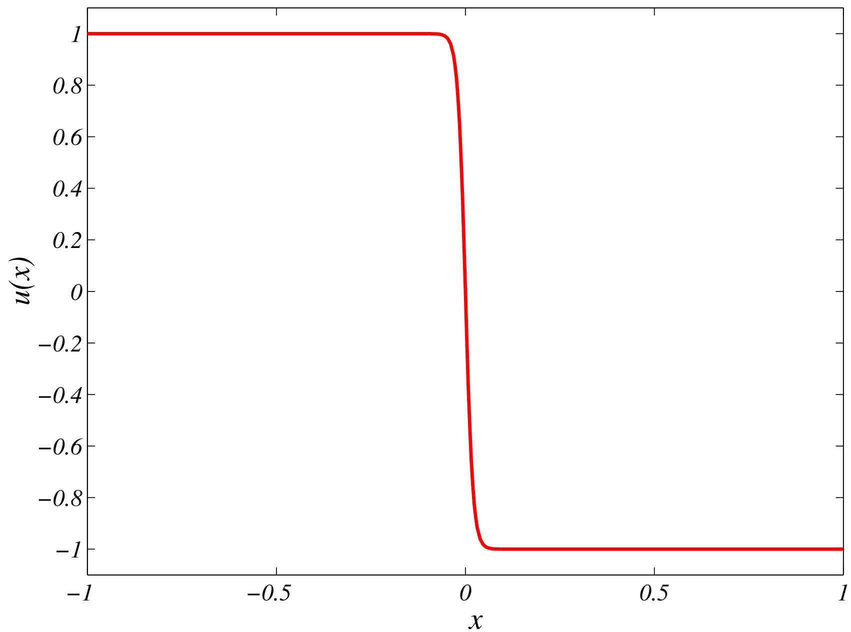


Wavelet locations $x_{\mathbf{k}}^j$

Wavelet Compression

$$u_{\geq}(\mathbf{x}) = \sum_{j=0}^{+\infty} \sum_{\mathbf{k} \in \mathcal{K}^j,\ |d^j_{\mathbf{k}}| \geq \epsilon} d^j_{\mathbf{k}} \psi^j_{\mathbf{k}}(\mathbf{x})$$



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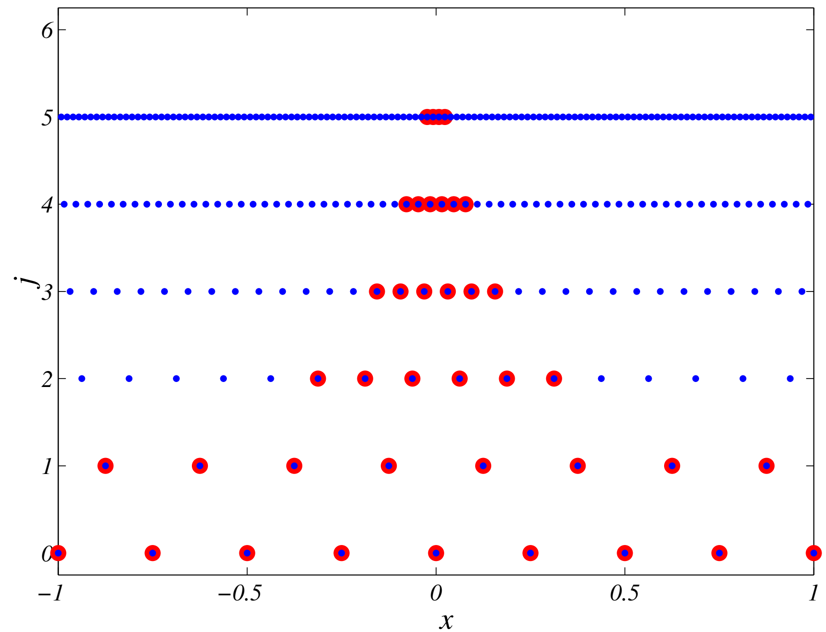
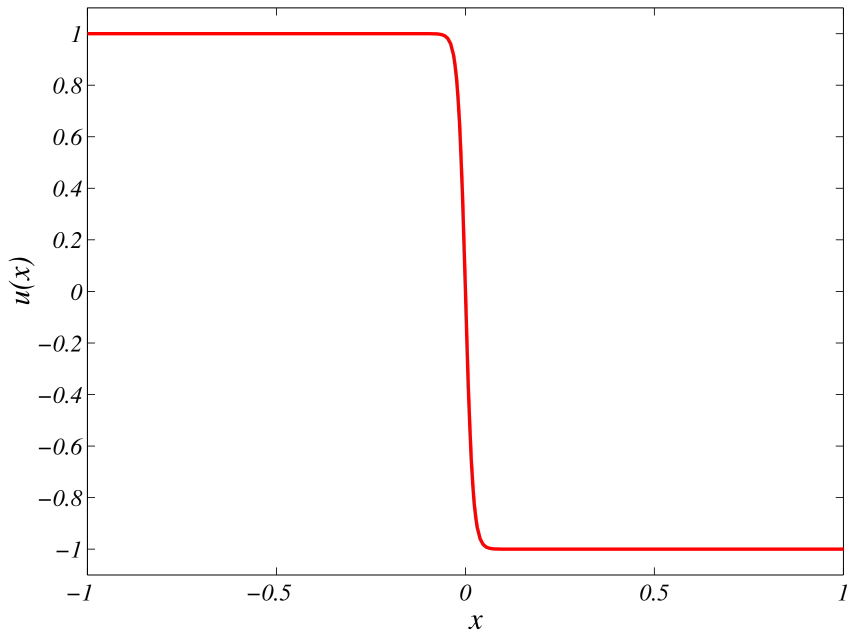


Wavelet locations $x^j_{\mathbf{k}}$ $\epsilon = 10^{-3}$

Wavelet Compression

$$\begin{aligned} |u(\mathbf{x}) - u_{\geq}(\mathbf{x})| &\leq C_1 \epsilon \\ \mathcal{N}^{1/n} &\leq C_2 \epsilon^{-1/p} \\ |u(\mathbf{x}) - u_{\geq}(\mathbf{x})| &\leq C_3 \mathcal{N}^{-p/n} \end{aligned}$$



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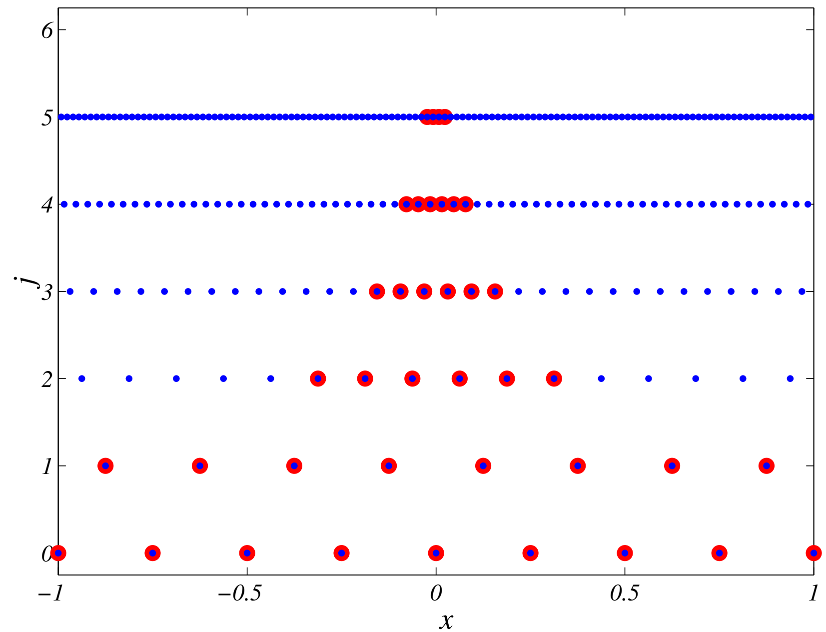

Wavelet locations $x_{\mathbf{k}}^{j}$ $\epsilon = 10^{-3}$

Solving PDEs

$$\boldsymbol{F}\left(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{u}, \nabla \boldsymbol{u}, \boldsymbol{q}, \mathbf{x}, t\right) = 0$$

$$\boldsymbol{\Phi}\left(\boldsymbol{u}, \nabla \boldsymbol{u}, \boldsymbol{q}, \mathbf{x}, t\right) = 0$$

$$u(\mathbf{x}_{\mathbf{k}}^{j}) \implies d_{\mathbf{k}}^{j} \implies \frac{\partial u}{\partial x_i}(\mathbf{x}_{\mathbf{k}}^{j})$$

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$$\overbrace{u(\mathbf{x}^{j}_{\mathbf{k}}) \Longrightarrow d^{j}_{\mathbf{k}} \Longrightarrow \frac{\partial u}{\partial x_i}(\mathbf{x}^{j}_{\mathbf{k}})}^{O(\mathcal{N})}$$

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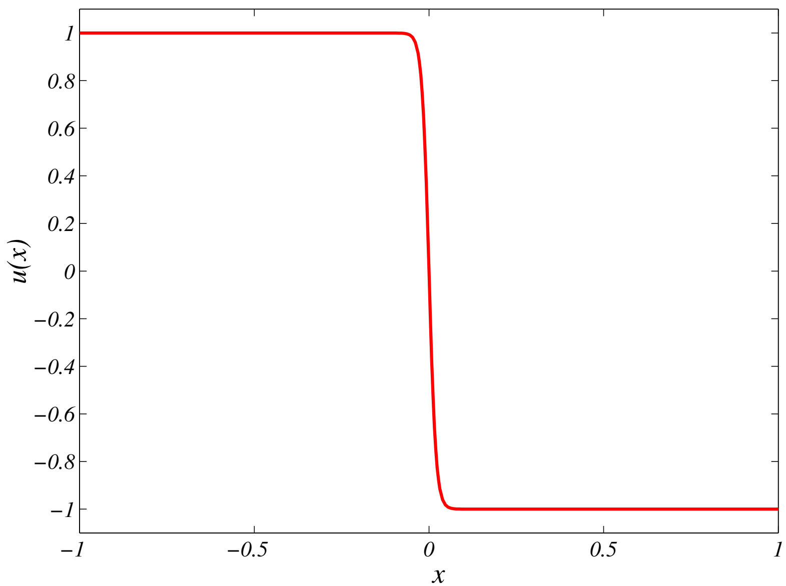

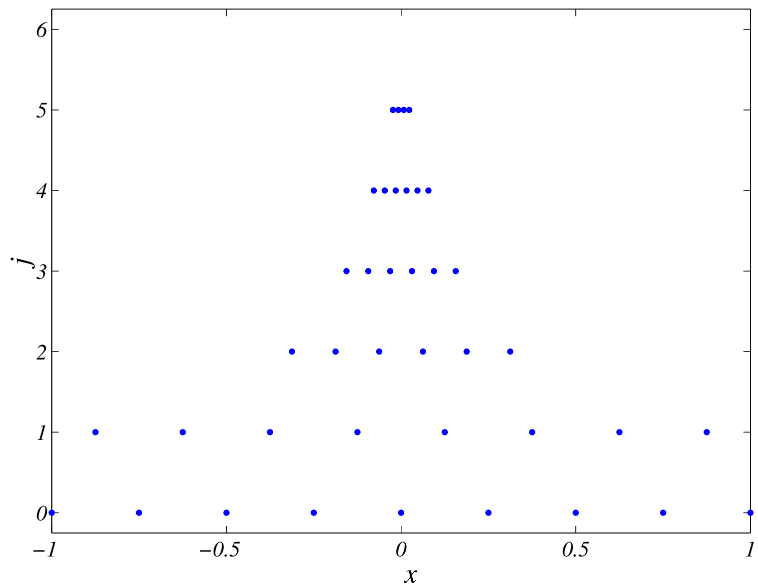

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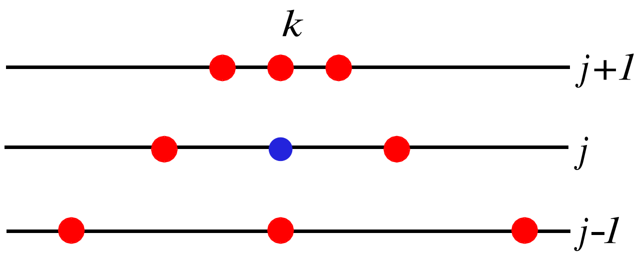
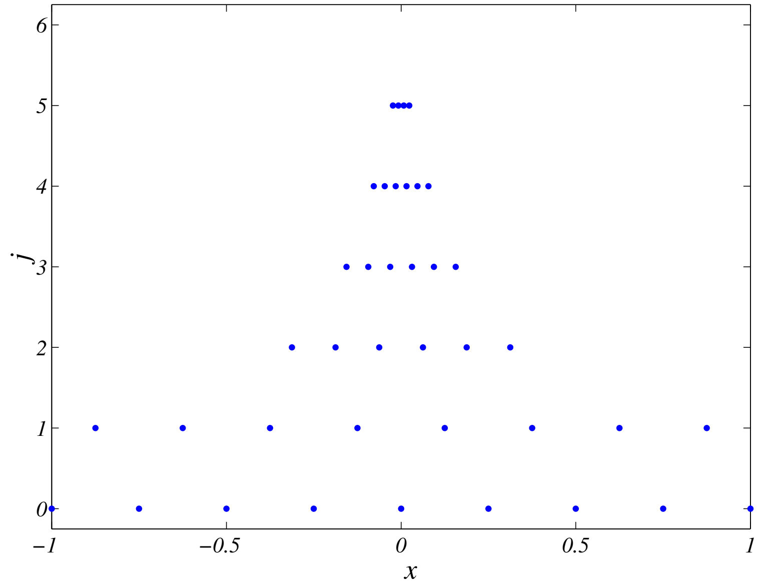
Adjacent zone:





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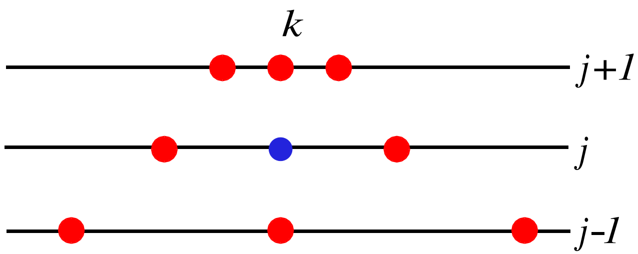
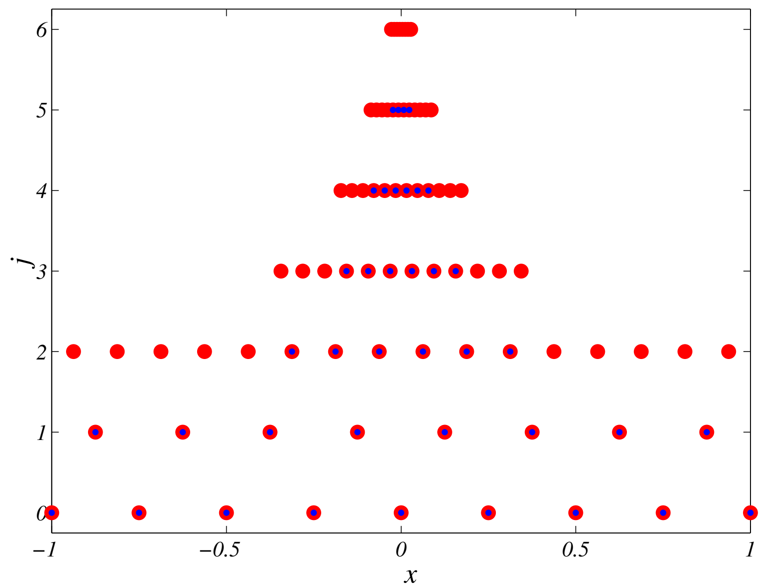




Numerical Algorithm

Evolution problems

1. Perform the wavelet transform of $\mathbf{u_k}(t)$ on $\mathcal{G}^t_{\geq}$
2. Update $\mathcal{G}^{t+\Delta t}_{\geq}$
3. If $\mathcal{G}^{t+\Delta t}_{\geq} = \mathcal{G}^t_{\geq}$, go to step 5
4. Interpolate $\mathbf{u_k}(t)$ to $\mathcal{G}^{t+\Delta t}_{\geq}$
5. Integrate the system of equations to obtain $\mathbf{u_k}(t + \Delta t)$ and go back to step 1

$\mathcal{G}^t_{\geq}$ - computational grid at time t

Test Problem: Burgers Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \quad x \in (-1, 1)\,,\ t > 0$$

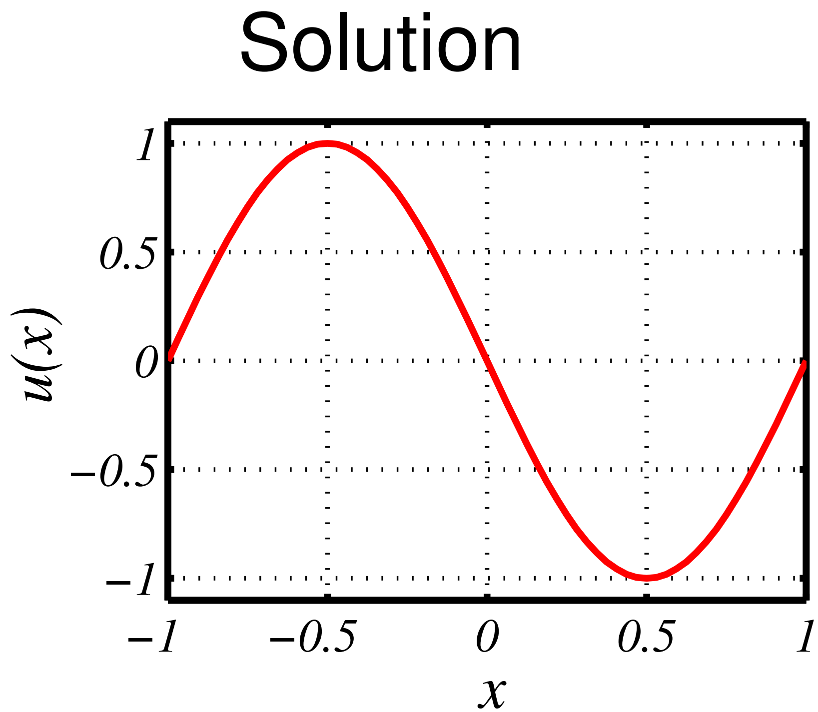
$$u\left(x, 0\right) = -\sin\left(\pi x\right), \quad u\left(\pm 1, t\right) = 0$$

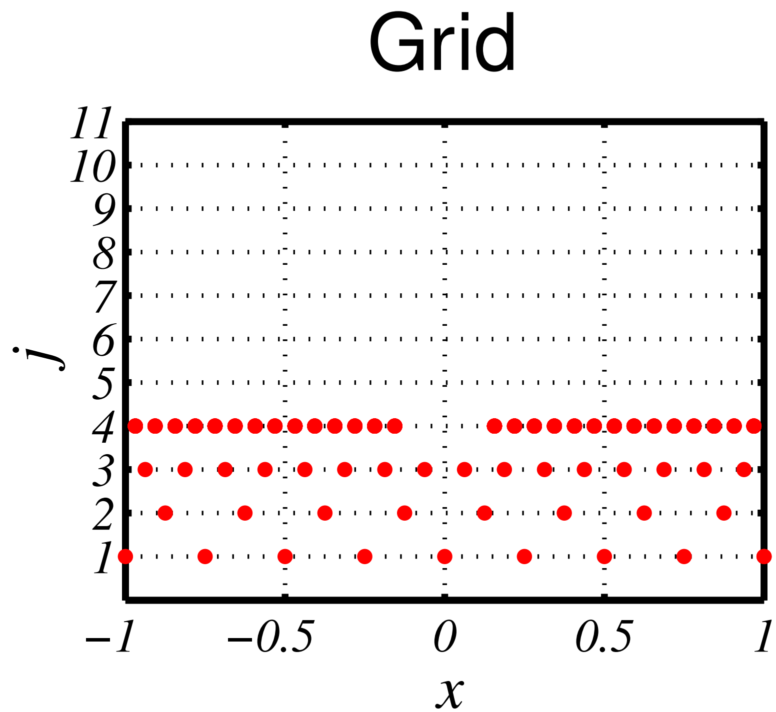
Analytical Solution:

$$u\left(x, t\right) = -\frac{\int_{-\infty}^{+\infty} \sin\left(\pi\left(x-\eta\right)\right)\exp\left(\frac{-\cos(\pi(x-\eta))}{2\pi\nu}\right)\exp\left(-\frac{\eta^2}{4\nu t}\right)d\eta}{\int_{-\infty}^{+\infty}\exp\left(-\frac{\cos(\pi(x-\eta))}{2\pi\nu}\right)\exp\left(\frac{-\eta^2}{4\nu t}\right)d\eta}$$

Parameters: $\nu = 10^{-2}/\pi$, $\epsilon = 10^{-4}$

Test Problem: Burgers Equation





$\epsilon = 10^{-5}$, $N = \widetilde{N} = 3$

Test Problem: Burgers Equation

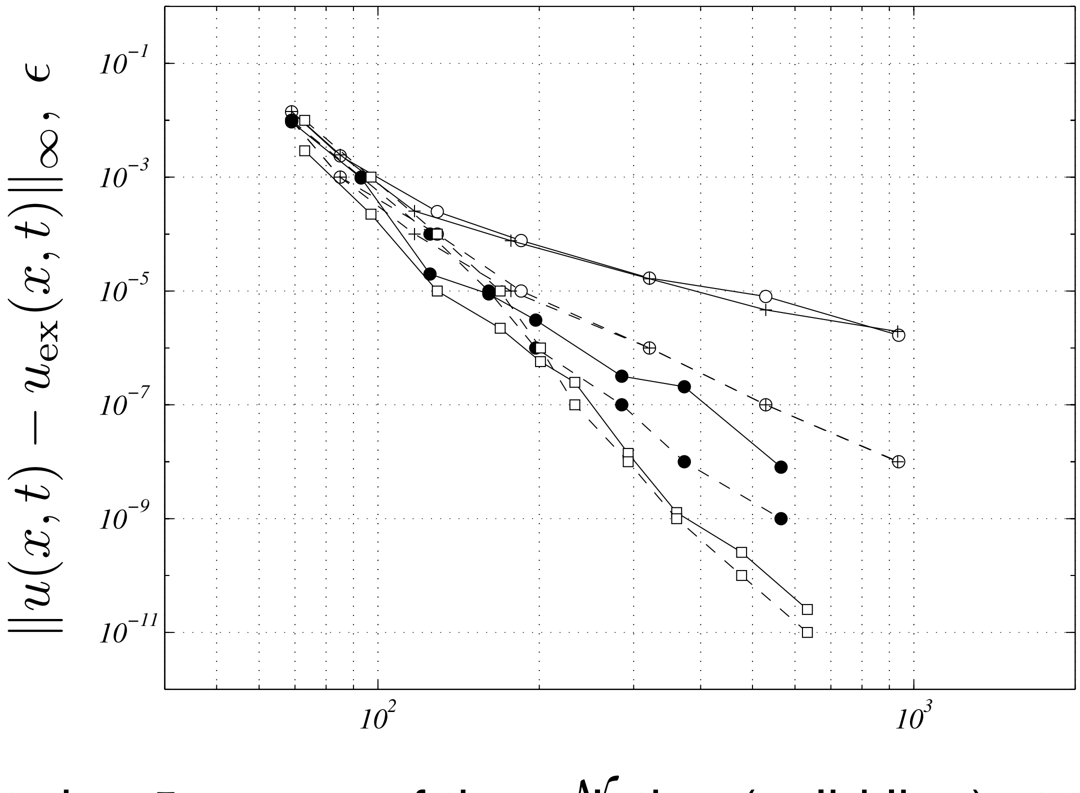


Fig: The pointwise L_∞-error of the solution (solid line) at time $t = 2/\pi$ for different choices of ϵ, N and $\widetilde{N}$: $N = \widetilde{N} = 2$ ($\circ$); $N = 2$, $\widetilde{N} = 0$ ($+$); $N = \widetilde{N} = 3$ ($\bullet$); $N = \widetilde{N} = 4$ ($\Box$). The dashed line shows the value of ϵ as a function of $\mathcal{N}$.

Test Problem: Moving Shock

$$\frac{\partial u}{\partial t} + (v+u)\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad x \in (-\infty, +\infty)\,,\ t > 0$$

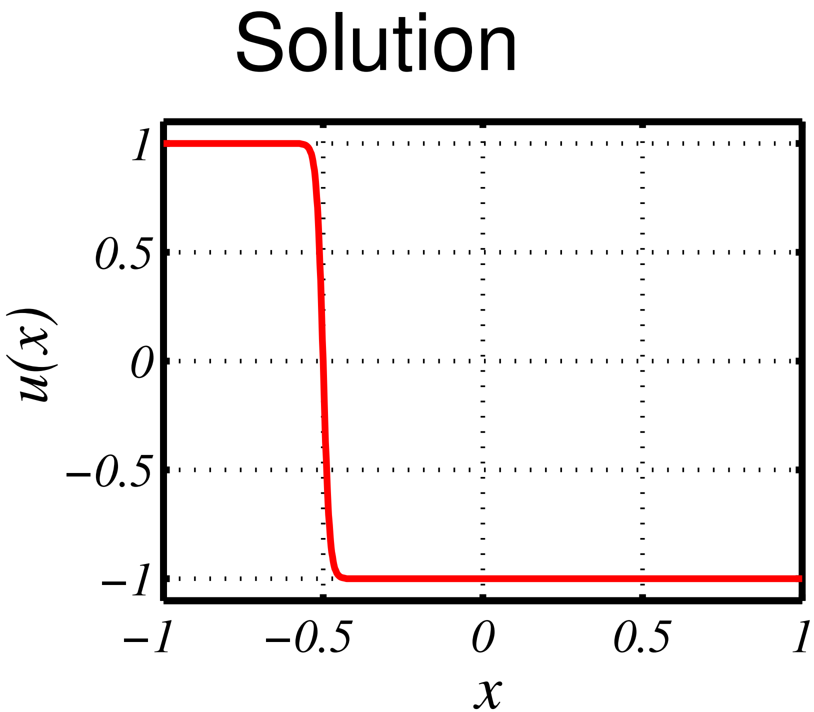
$$u\,(x,0) = -\tanh\left(\frac{x - x_0}{2\nu}\right), \qquad u\,(\pm\infty, t) = \mp 1$$

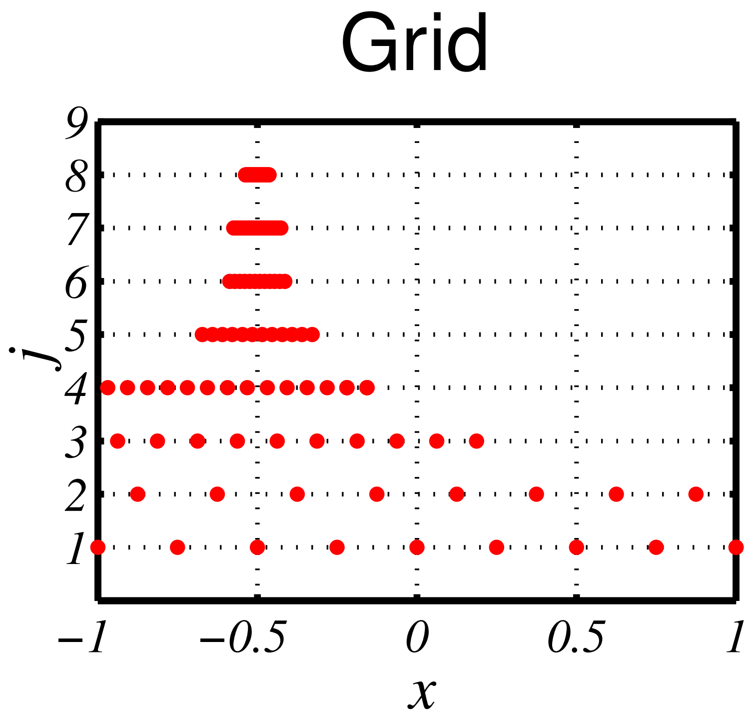
Analytical Solution:

$$u_{1\mathrm{D}}\,(x,t) = -\tanh\left(\frac{x - x_0 - vt}{2\nu}\right)$$

Parameters: $\nu = 10^{-2},\ x_0 = -1/2$, $v = 1$, $\epsilon = 10^{-4}$

Test Problem: Moving Shock





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Fluid–structure interaction

- Moderate to high Reynolds number flow around solid obstacles.

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- Obstacle may be fixed, or may move or deform (e.g. in response to fluid forces).
- Applications: wind engineering of tall buildings, heat exchangers, underwater pipes, aeronautics.

Fluid–structure interaction

Goal

To develop a general code for calculating all kinds of fluid–structure interaction

Fluid–structure interaction

Combine two methods:

Fluid–structure interaction

Combine two methods:

1. *Adaptive wavelet collocation* for grid adaptation and derivatives.

Fluid–structure interaction

Combine two methods:

1. *Adaptive wavelet collocation* for grid adaptation and derivatives.
2. *Brinkman penalization* to impose no-slip boundary conditions at the surface of an obstacle of arbitrary shape.

Fluid–structure interaction

Brinkman penalization of Navier–Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} + \boldsymbol{U}) \cdot \nabla \boldsymbol{u} \ + \ \nabla P = \nu \Delta \boldsymbol{u}$$

$$- \frac{1}{\eta} \chi(\mathbf{x}, t)(\boldsymbol{u} + \boldsymbol{U} - \boldsymbol{U}_o)$$

$$\nabla \cdot \boldsymbol{u} \ = \ 0$$

Fluid–structure interaction

where the solid is defined by

$$\chi(\mathbf{x}, t) = \begin{cases} 1 & \text{if } \mathbf{x} \in \text{solid}, \\ 0 & \text{otherwise.} \end{cases}$$

- The upper bound on the global error of this penalization was shown to be (Angot et al. 1999) $O(\eta^{1/4})$.
- We observe an error of $O(\eta)$.

Fluid–structure interaction

Cylinder response

Cylinder is modelled as a damped harmonic oscillator

$$m\ddot{\mathbf{x}}_o(t) + b\dot{\mathbf{x}}_o(t) + k\mathbf{x}_o = \boldsymbol{F}(t),$$

Fluid–structure interaction

Cylinder response

Cylinder is modelled as a damped harmonic oscillator

$$m\ddot{\mathbf{x}}_o(t) + b\dot{\mathbf{x}}_o(t) + k\mathbf{x}_o = \boldsymbol{F}(t),$$

where the force $\boldsymbol{F}(t)$ is calculated from the penalization

$$\boldsymbol{F}(t) = \frac{1}{\eta}\int \chi(\mathbf{x}, t)(\boldsymbol{u} + \boldsymbol{U} - \boldsymbol{U}_o)\,\mathrm{d}\mathbf{x}.$$

Fluid–structure interaction

Time scheme

- Second order backwards difference
- Semi-implicit discretization of convection term
- Split-step to enforce divergence free velocity

Fluid–structure interaction

Time scheme

- Second order backwards difference
- Semi-implicit discretization of convection term
- Split-step to enforce divergence free velocity
  Poisson equation solved using adaptive wavelet multilevel method

Elliptic Solver: $\mathbf{L}\mathbf{u} = \mathbf{f}$

V-cycle:

$\mathbf{r}^J = \mathbf{f}^J - \mathbf{L}\mathbf{u}^J$

for all levels $j = J \ : \ -1 \ : \ j_{\text{min}} + 1$

do ν_1 steps of **approximate** solver for $\mathbf{L}\mathbf{v}^j = \mathbf{r}^j$

$\mathbf{r}^{j-1} = I_w^{j-1}\left(\mathbf{r}^j - \mathbf{L}\mathbf{v}^j\right)$

enddo

end

Solve for $j = j_{\text{min}}$ level: $\mathbf{L}\mathbf{v}^j = \mathbf{r}^j$

for all levels $j = j_{\text{min}} + 1 \ : \ +1 \ : \ J$

$\mathbf{v}^j = \mathbf{v}^j + \omega_0 I_w^j \mathbf{v}^{j-1}$

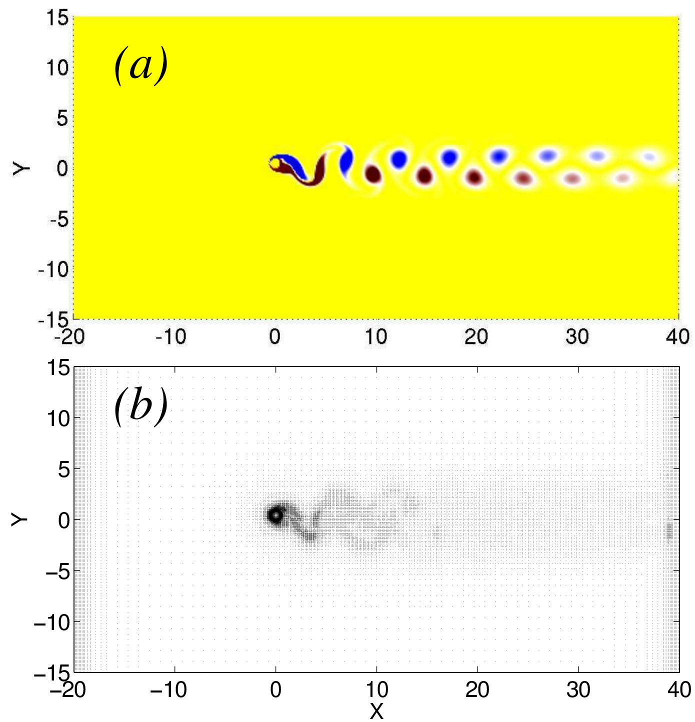
do ν_2 steps of **approximate** solver for $\mathbf{L}\mathbf{v}^j = \mathbf{r}^j$ **enddo**

end

$\mathbf{u}^J = \mathbf{u}^J + \omega_1 \mathbf{v}^J$

do ν_3 steps of **exact** solver for $\mathbf{L}\mathbf{u}^J = \mathbf{f}^J$ **enddo**

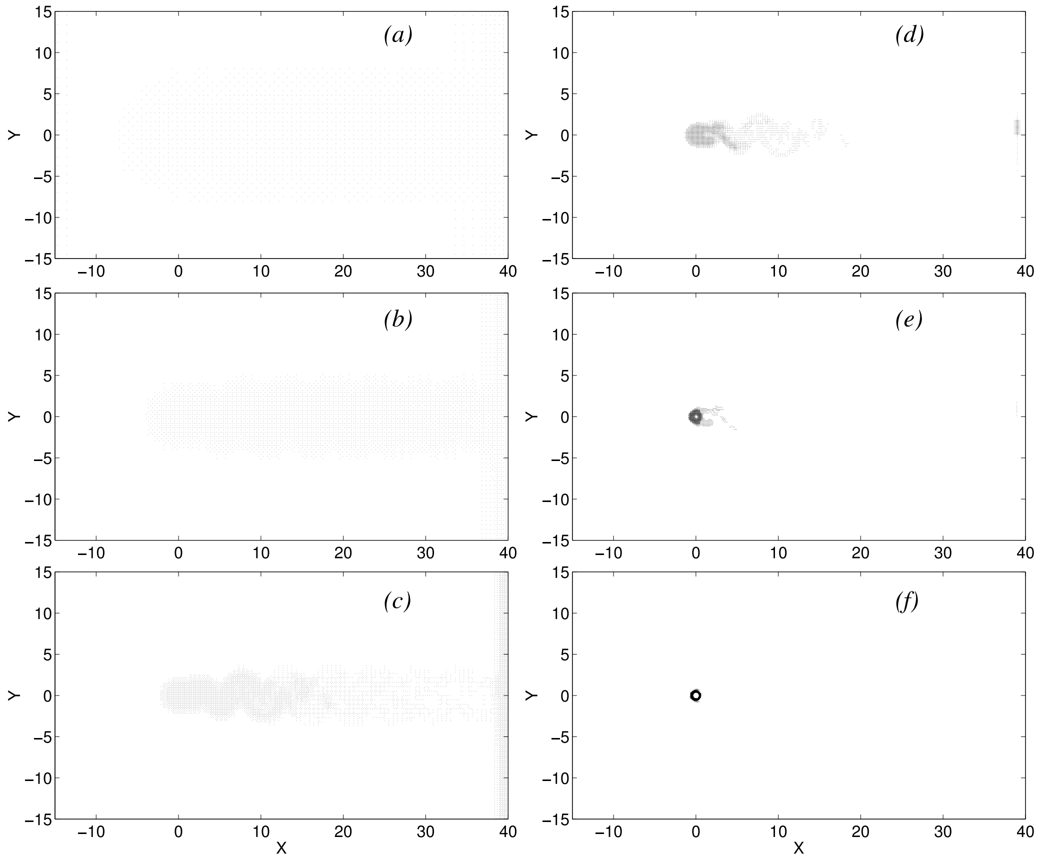
2D Fluid–structure interaction



Moving cylinder at Re = 100.

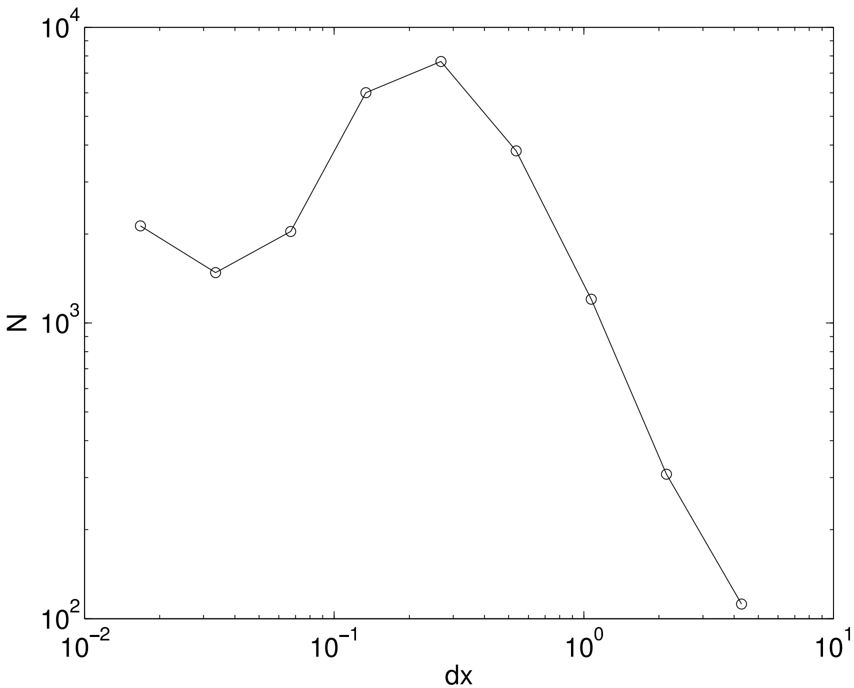
- Full domain $3\ 584 \times 1\ 792$.
- Zoom.

2D Fluid–structure interaction



Grid at scales j = 4 to j = 9.

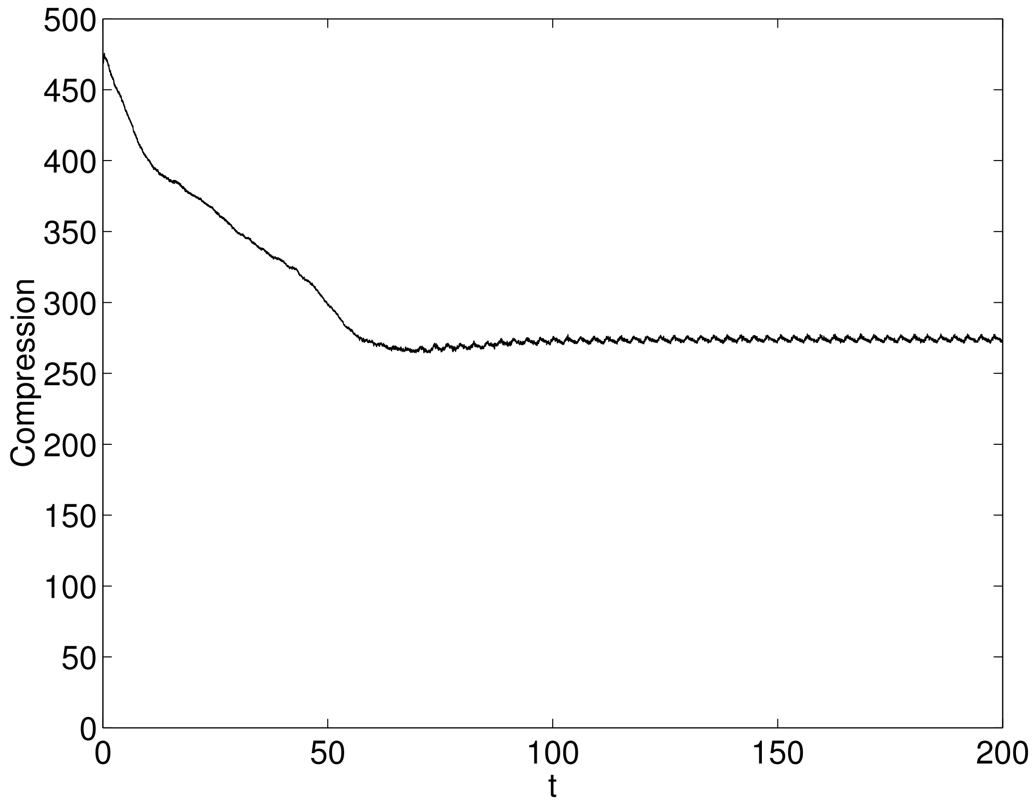
2D Fluid–structure interaction



Number of grid points as a function of grid size for fixed cylinder at $Re = 100$. The grid size $\Delta x = Lx/(14 \times 2^{j-1})$ where j is the scale.

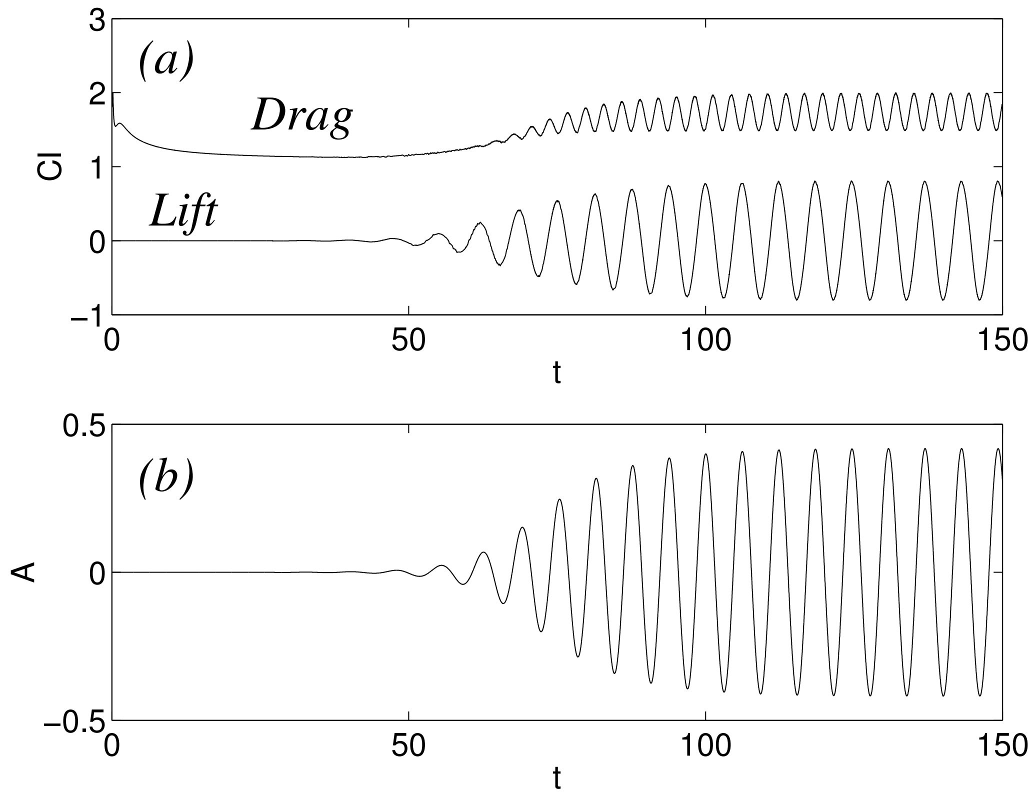
Note that most grid points are near the Taylor scale $Re^{-1/2} = 0.1$.

2D Fluid–structure interaction



Compression for fixed cylinder at $Re = 100$ as a function of time. The average compression ratio is about 270.

2D Fluid–structure interaction



Lift and drag for a fixed cylinder at $Re = 100$. Average drag during the shedding phase is $C_D = 1.35$, Strouhal number is $St = 0.168$.

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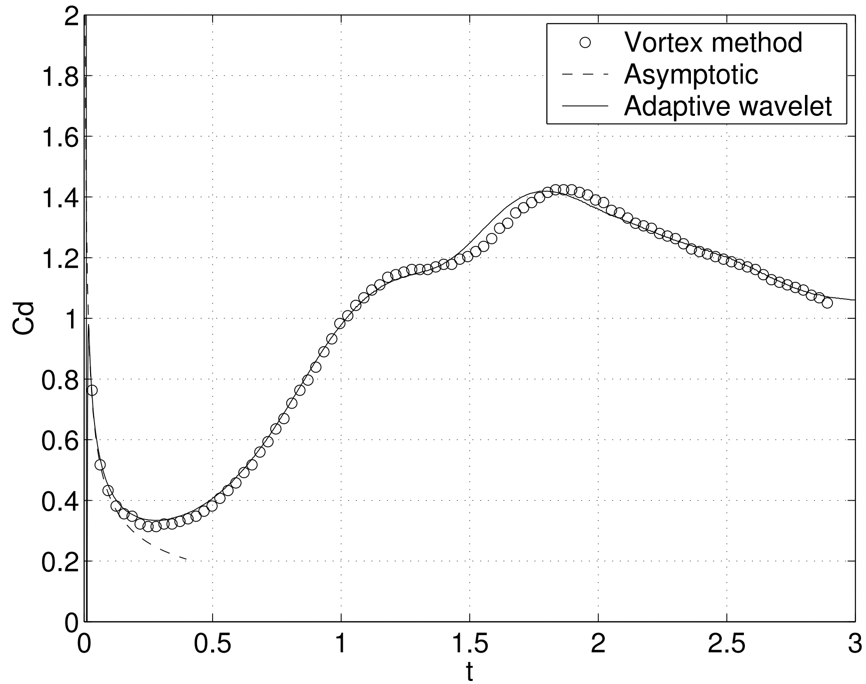

Drag cylinder at $Re = 3\,000$ compared to Bar-Lev & Yang (1975), and the vortex method of Koumoutsakos & Leonard (1995).

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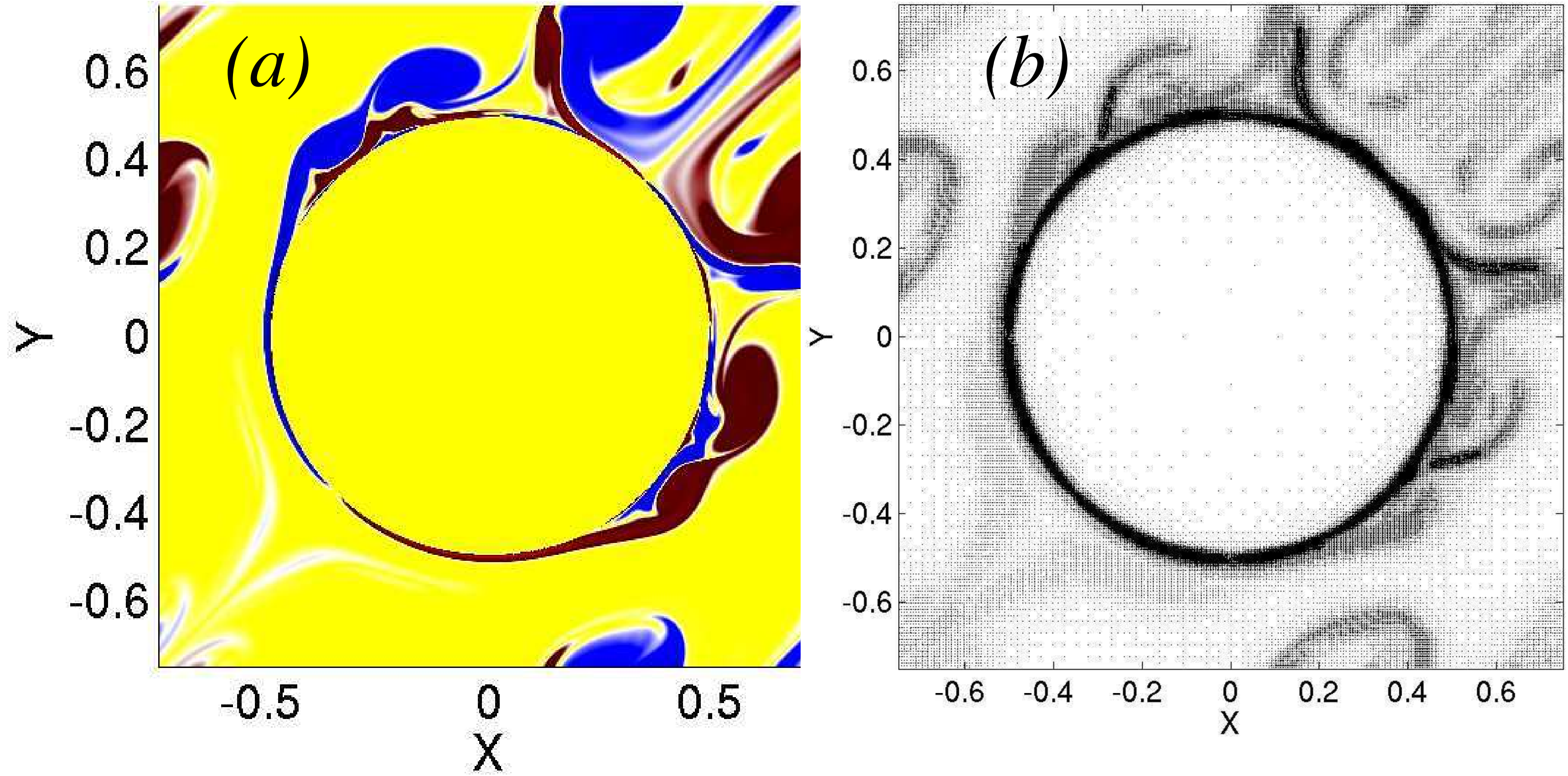
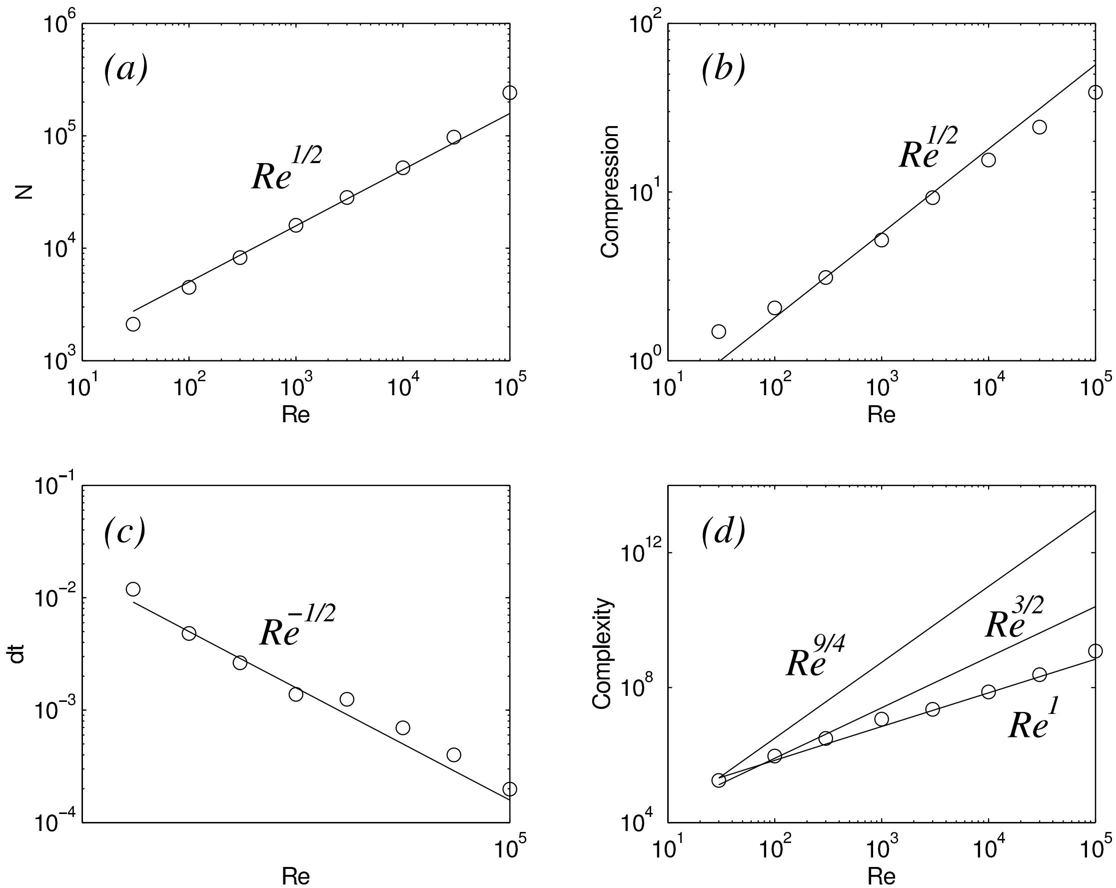


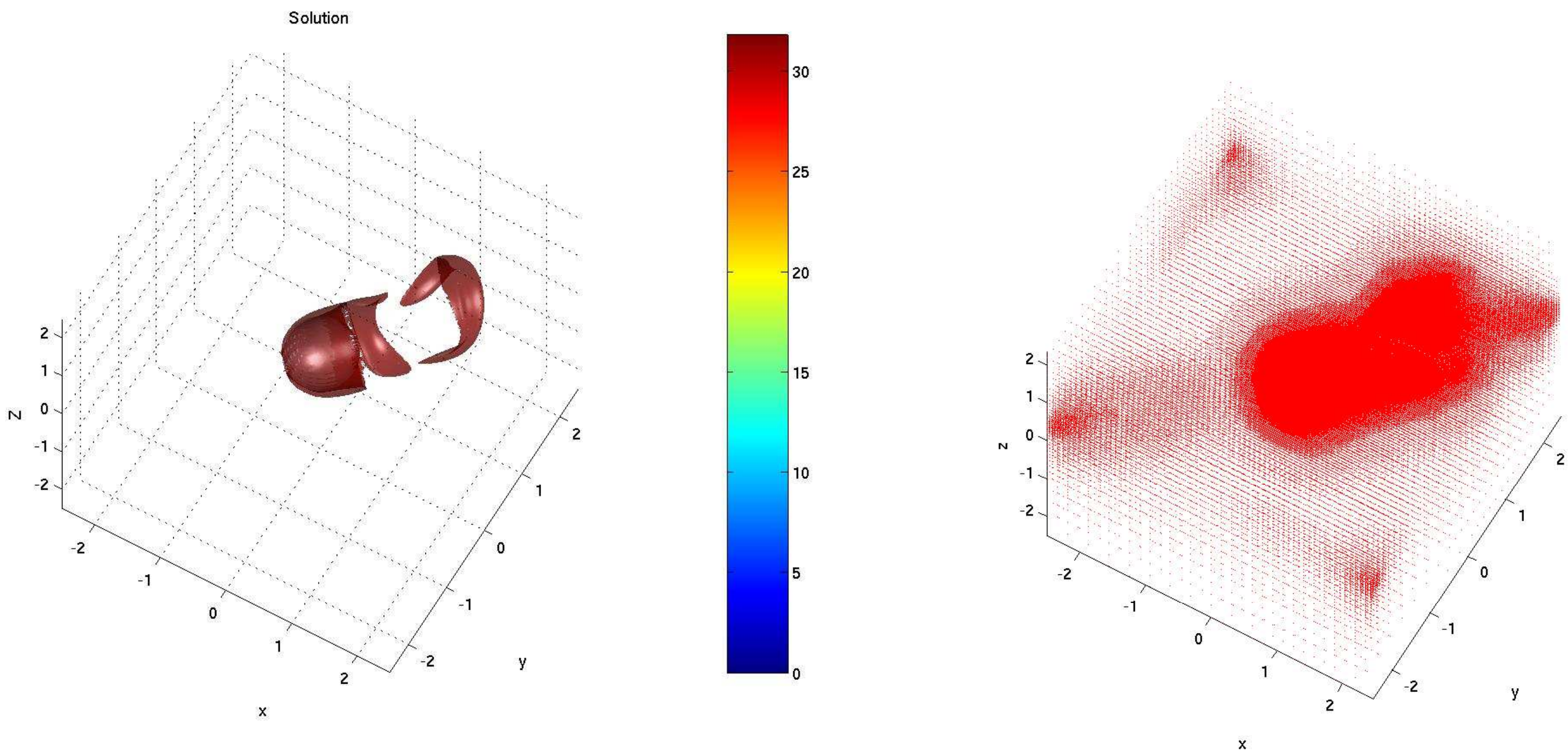
Periodic cylinder array at $Re = 10^4$, $t = 3.5$. (a) Vorticity. (b) Grid.

2D Fluid–structure interaction



Scaling for cylinder array.

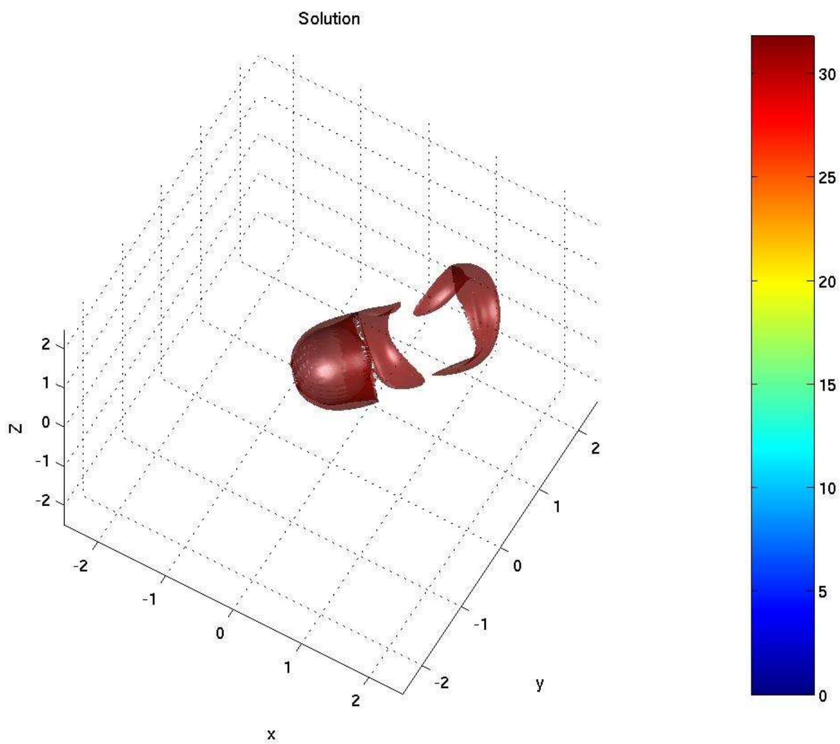
3D Fluid–structure interaction

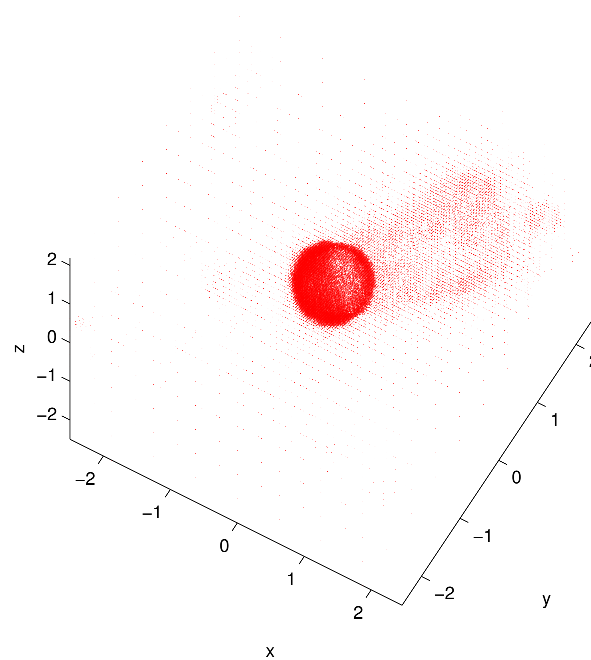


Flow around a sphere at Re=550, max grid 256^3

Vorticity isosurface (30% $||\omega||_\infty$) and grid at $t = 16$.

3D Fluid–structure interaction





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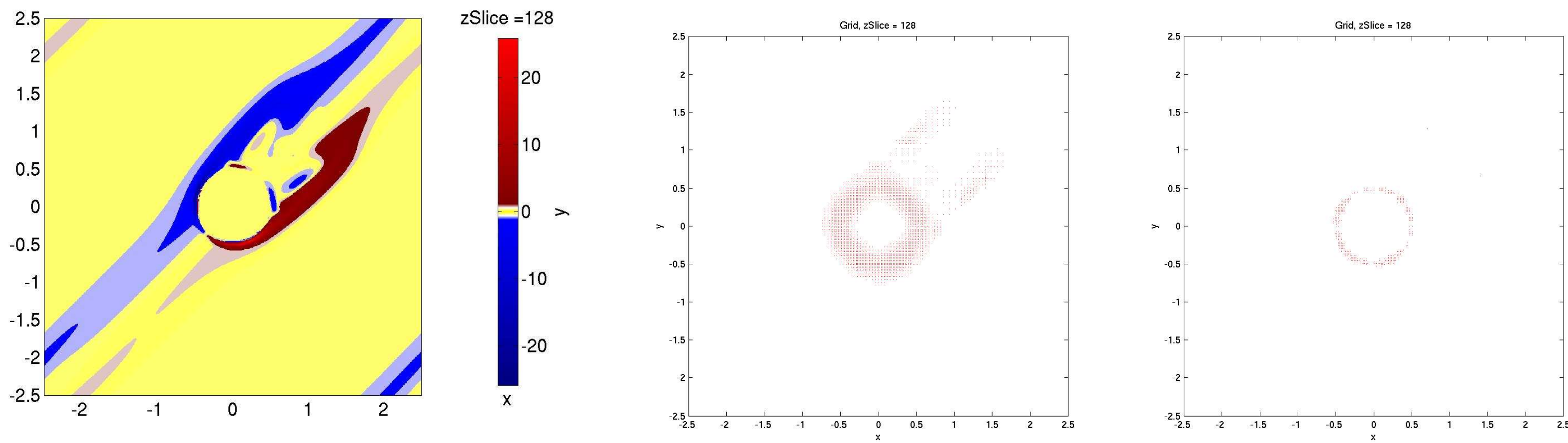
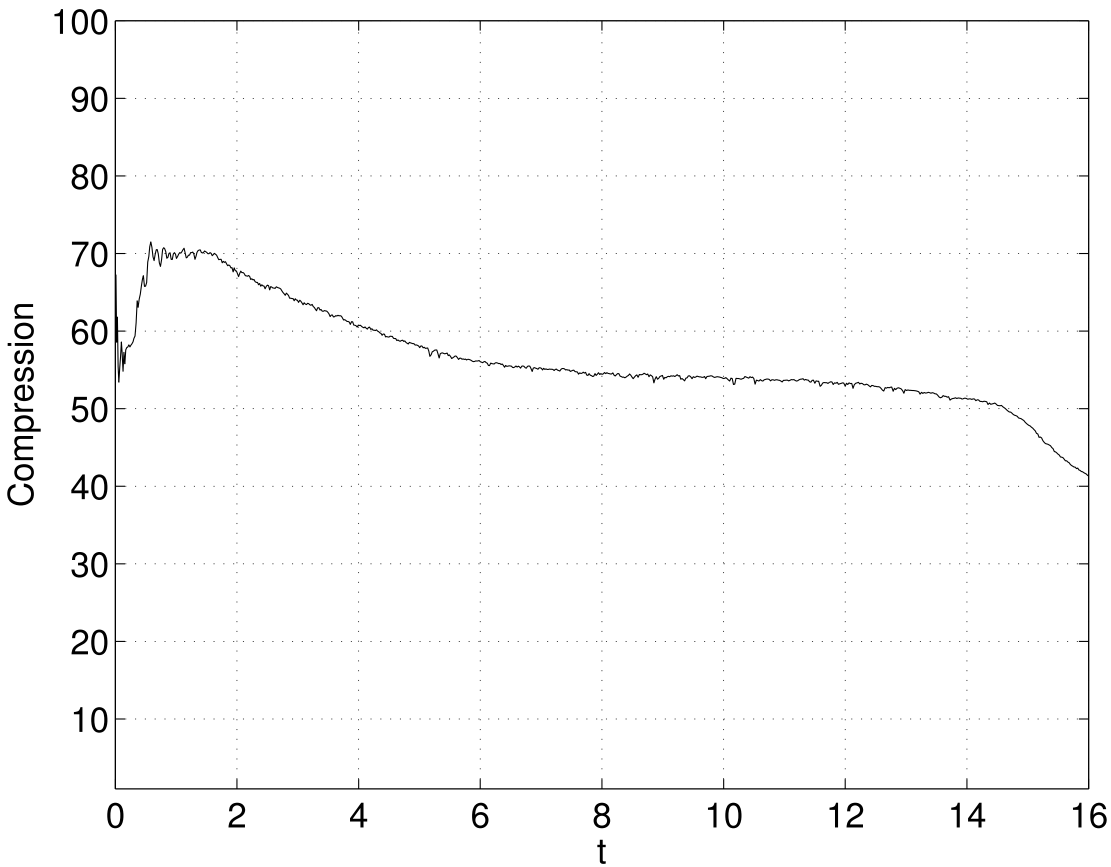


ω_z entire grid grid points $> \epsilon$

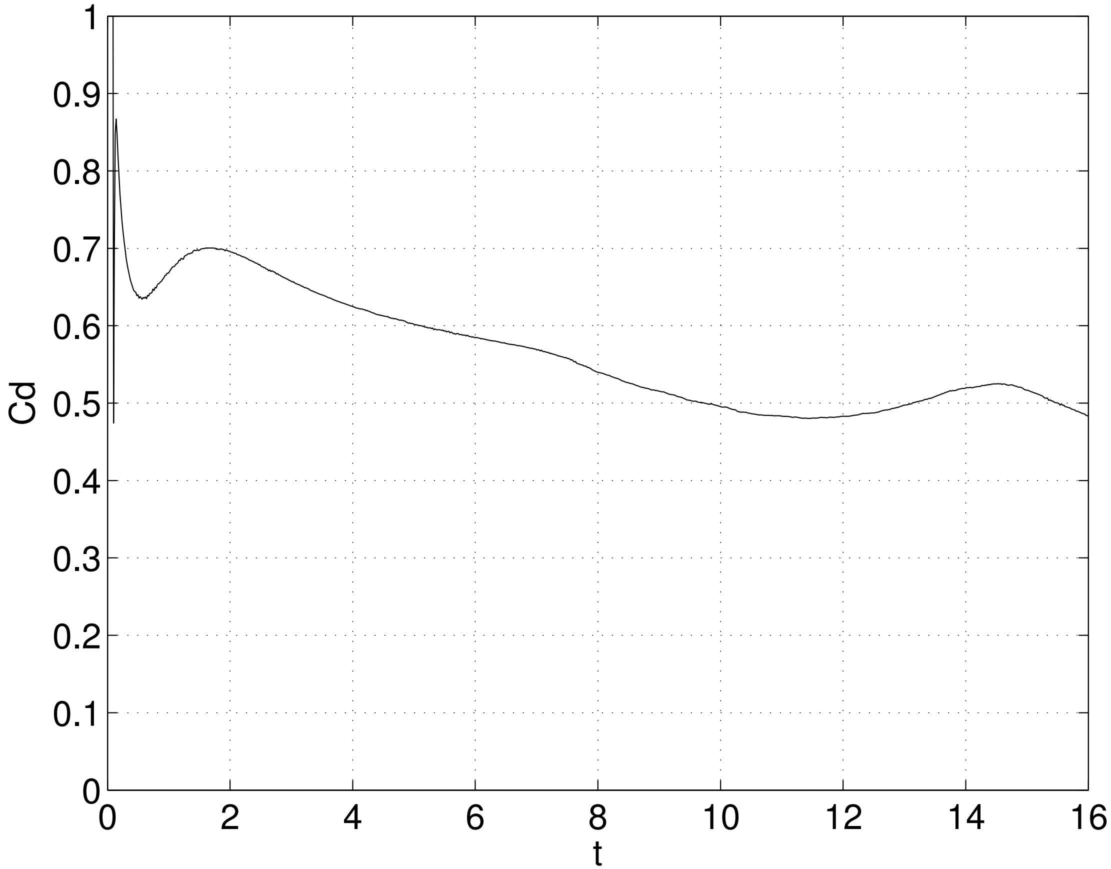
Z-slices through sphere at $t = 16$.

3D Fluid–structure interaction



Wavelet compression for sphere array at $Re = 550$.

3D Fluid–structure interaction



Drag for sphere array at $Re = 550$.

Conclusions

1. Adaptive wavelet collocation method
   - Developed general purpose solver
   - Used for elliptic and time evolution problems
   - Verified accuracy and grid compression on 1D test problems

Conclusions (cont.)

3. 2D fluid–structure interaction
   - Accurate and efficient results
   - Grid compression of $270\times$
   - Works well for moving cylinder
   - Complexity scales like Re

Conclusions (cont.)

3. 2D fluid–structure interaction
   - Accurate and efficient results
   - Grid compression of 270$\times$
   - Works well for moving cylinder
   - Complexity scales like Re
4. 3D fluid–structure interaction
   - Number of grid points scales like $Re^{1/2}S$
   - Drag accurate
   - Compression of 40 to 170$\times$

Future work

1. Parallelize wavelet transform
2. Implement efficient data structure
3. Extend to compressible flows (underway)
4. Measure 3D scaling of number of grid points

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   Dan Goldstein — next talk