CONQUERING CALCULUS

POST-SECONDARY PREPARATION

SOLUTIONS TO ALL EXERCISES

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FOREWORD

This file contains solutions to all exercises from the book Calculus ... *Fear No More* [Review and Reference for First Year College and University Courses], Second Edition, written by Miroslav Lovrić, and published by Nelson.

In general, all details of a solution are provided. Sometimes, related theoretical concepts, methods or formulas are recalled. Keep in mind that – in many cases – there are multiple correct ways of solving a question (and/or simplifying an answer); just because your answer looks different from the one given here does not mean that it's incorrect. Some exercises require that you use a calculator.

Do not read this manual! It is far more beneficial to try to solve an exercise on your own. Start and see how far you can go. If you get stuck, identify the problem first – try to understand why you are having difficulties, and then look up the solution. This way, you will learn not only what the problem is (or what you have problems with), but also how that particular problem has been resolved. If you just read a solution, you might not recognize the hard part(s) - or, even worse, you might miss the whole point of the exercise.

Big Thank you! to Andrijana Burazin for writing solutions to all exercises.

We accept full responsibility for errors and will be grateful to anybody who brings them to our attention. Your comments and suggestions will be greatly appreciated.

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Section 1. Numbers and Operations

1. (a) Note that 360 is a multiple of 72. Thus
$$\frac{360}{72} = \frac{72 \cdot 5}{72} = 5$$
Alternatively, note that both the numerator and the denominator are multiples of 36:
$$\frac{360}{72} = \frac{36 \cdot 10}{36 \cdot 2} = \frac{10}{2} = 5$$
(b) Factor, and then cancel:
$$\frac{3 \cdot 33 \cdot 24}{4 \cdot 5 \cdot 6} = \frac{3 \cdot 33 \cdot 4 \cdot 6}{4 \cdot 5 \cdot 6} = \frac{3 \cdot 33}{5} = \frac{99}{5}$$
Note that it is not necessary to factor all terms (so we did not factor 33), nor to factor as far as possible (i.e., all the way to the prime factors; e.g., $24 = 4 \cdot 6$ suffices, and there was no need to write $24 = 2 \cdot 2 \cdot 2 \cdot 3$).

(c) Factor, and then cancel by 11 and by 13:
$$\frac{121 \cdot 13}{169 \cdot 11} = \frac{11 \cdot 11 \cdot 13}{13 \cdot 13 \cdot 11} = \frac{11}{13}$$
(d) Factor, and then cancel by 7 and by b:
$$\frac{14ab}{7bc} = \frac{7 \cdot 2 \cdot a \cdot b}{7 \cdot b \cdot c} = \frac{2 \cdot a}{c} = \frac{2a}{c}$$
Note that it is not always necessary to use a dot to indicate the product: for instance $ab = a \cdot b$ and $2a = 2 \cdot a$. However, in some cases we must use the dot, as in $14 = 7 \cdot 2$; alternatively, we use brackets: $14 = (7)(2)$.

(e) Both 90 and 33 are multiples of 3:
$$\frac{90z}{33x^2z} = \frac{3 \cdot 30z}{3 \cdot 11x^2z} = \frac{30}{11x^2}$$
(f) Both 36 and 24 are multiples of 12:
$$\frac{36x(x-y)}{24(x-y)} = \frac{12 \cdot 3x(x-y)}{12 \cdot 2(x-y)} = \frac{3x}{2}$$
In the above, we cancelled by 12 and by $x - y$.

2. By "yes" we mean "equal to $\frac{a-b}{c-d}$" and by "no" we mean "not equal to $\frac{a-b}{c-d}$"

(a) $-\dfrac{-a+b}{c-d} = \dfrac{-(-a+b)}{c-d} = \dfrac{a-b}{c-d}$; yes.

(b) $\dfrac{a-b}{d-c} = \dfrac{a-b}{-(-d+c)} = -\dfrac{a-b}{c-d}$; no.

(c) $\dfrac{b-a}{d-c} = \dfrac{-(-b+a)}{-(-d+c)} = \dfrac{-b+a}{-d+c} = \dfrac{a-b}{c-d}$; yes.

(d) $\dfrac{a-b}{d-c} = \dfrac{a-b}{-(c-d)} = -\dfrac{a-b}{c-d}$; no.

(e) $-\dfrac{b-a}{c-d} = \dfrac{-(b-a)}{c-d} = \dfrac{-b+a}{c-d} = \dfrac{a-b}{c-d}$; yes.

(f) $-\dfrac{b-a}{d-c} = -\dfrac{-(a-b)}{-(c-d)} = -\dfrac{a-b}{c-d}$; no.

(g) $-\dfrac{a-b}{d-c} = \dfrac{a-b}{-(d-c)} = \dfrac{a-b}{c-d}$; yes.

(h) $-\dfrac{a-b}{c-d}$; no.

3. By "yes" we mean "equal to $\frac{(x-y)(z-t)}{a-b}$" and by "no" we mean "not equal to $\frac{(x-y)(z-t)}{a-b}$"

(a) $-\dfrac{(x-y)(z-t)}{a-b}$; no.

(b) $-\dfrac{(x-y)(z-t)}{b-a} = \dfrac{(x-y)(z-t)}{-(b-a)} = \dfrac{(x-y)(z-t)}{a-b}$; yes.

(c) $-\dfrac{(y-x)(z-t)}{b-a} = \dfrac{-(y-x)(z-t)}{b-a} = \dfrac{(x-y)(z-t)}{b-a} = \dfrac{(x-y)(z-t)}{-(a-b)} = -\dfrac{(x-y)(z-t)}{a-b}$; no.

(d) $\dfrac{(y-x)(z-t)}{b-a} = \dfrac{-(x-y)(z-t)}{-(a-b)} = \dfrac{(x-y)(z-t)}{a-b}$; yes.

(e) $-\dfrac{(x-y)(t-z)}{a-b} = \dfrac{-(x-y)(t-z)}{a-b} = \dfrac{(x-y)[-(t-z)]}{a-b} = \dfrac{(x-y)(z-t)}{a-b}$; yes.

(f) $\dfrac{(y-x)(t-z)}{a-b} = \dfrac{[-(x-y)][-(z-t)]}{a-b} = \dfrac{(x-y)(z-t)}{a-b}$; yes.

(g) $-\dfrac{(y-x)(t-z)}{a-b} = -\dfrac{[-(x-y)][-(z-t)]}{a-b} = -\dfrac{(x-y)(z-t)}{a-b}$; no.

(h) $\dfrac{(y-x)(t-z)}{b-a} = \dfrac{[-(x-y)][-(z-t)]}{-(a-b)} = \dfrac{(x-y)(z-t)}{-(a-b)} = -\dfrac{(x-y)(z-t)}{a-b}$; no.

4. (a) Since all denominators are equal, we combine the numerators:

$$\frac{5x}{3y} - \frac{7}{3y} + \frac{11x}{3y} = \frac{5x-7+11x}{3y} = \frac{16x-7}{3y}$$

(b) Since all denominators are equal, we combine the numerators:

$$\begin{aligned}\frac{2a+b}{7} - \frac{a}{7} + \frac{b}{7} - \frac{b-a}{7} &= \frac{2a+b-a+b-(b-a)}{7} \\ &= \frac{2a+b-a+b-b+a}{7} = \frac{2a+b}{7}\end{aligned}$$

(c) Since both denominators are equal, we combine the numerators and then simplify:

$$\begin{aligned}\frac{(a+b)^2}{ab} - \frac{(a-b)^2}{ab} &= \frac{(a+b)^2-(a-b)^2}{ab} \\ &= \frac{a^2+2ab+b^2-(a^2-2ab+b^2)}{ab} \\ &= \frac{a^2+2ab+b^2-a^2+2ab-b^2}{ab} = \frac{4ab}{ab} = 4\end{aligned}$$

(d) Since all denominators are equal, we combine the numerators:

$$\frac{c}{c+d} - \frac{d}{c+d} + \frac{3c-d}{c+d} = \frac{c-d+3c-d}{c+d} = \frac{4c-2d}{c+d} = \frac{2(2c-d)}{c+d}$$

(e) Write $4x = \frac{4x}{1}$ as a fraction with denominator 3 and then combine the three fractions:

$$4x + \frac{x+y}{3} - \frac{5y}{3} = \frac{4x\cdot 3}{1\cdot 3} + \frac{x+y}{3} - \frac{5y}{3}$$

$$= \frac{12x}{3} + \frac{x+y}{3} - \frac{5y}{3}$$
$$= \frac{12x + x + y - 5y}{3} = \frac{13x - 4y}{3}$$

(f) The common denominator is 8; thus

$$\frac{3x-y}{4} + \frac{x+y}{2} - \frac{5y}{8} = \frac{(3x-y)(2)}{4(2)} + \frac{(x+y)(4)}{2(4)} - \frac{5y}{8}$$
$$= \frac{6x-2y}{8} + \frac{4x+4y}{8} - \frac{5y}{8}$$
$$= \frac{6x - 2y + 4x + 4y - 5y}{8} = \frac{10x - 3y}{8}$$

(g) The common denominator is $(2b)(3a) = 6ab$; thus

$$\frac{a}{2b} + \frac{2b}{3a} - 4 = \frac{a(3a)}{2b(3a)} + \frac{2b(2b)}{3a(2b)} - \frac{4(2b)(3a)}{1(2b)(3a)}$$
$$= \frac{3a^2}{6ab} + \frac{4b^2}{6ab} - \frac{24ab}{6ab}$$
$$= \frac{3a^2 + 4b^2 - 24ab}{6ab}$$

(h) The common denominator is $30xy$; thus

$$\frac{x-y}{2} - \frac{x+y}{5x} - \frac{1}{3y} + \frac{4}{15x} = \frac{(x-y)(15xy)}{2(15xy)} - \frac{(x+y)(6y)}{5x(6y)} - \frac{1(10x)}{3y(10x)} + \frac{4(2y)}{15x(2y)}$$
$$= \frac{15x^2y - 15y^2x}{30xy} - \frac{6xy + 6y^2}{30xy} - \frac{10x}{30xy} + \frac{8y}{30xy}$$
$$= \frac{15x^2y - 15y^2x - (6xy + 6y^2) - 10x + 8y}{30xy}$$
$$= \frac{15x^2y - 15y^2x - 6xy - 6y^2 - 10x + 8y}{30xy}$$

(i) The common denominator is a^2b^2; thus

$$a + 4b - \frac{a+1}{b} - \frac{2}{ab^2} + \frac{3}{a^2b} = \frac{(a+4b)(a^2b^2)}{a^2b^2} - \frac{(a+1)(a^2b)}{b(a^2b)} - \frac{2(a)}{ab^2(a)} + \frac{3(b)}{a^2b(b)}$$
$$= \frac{a^3b^2 + 4a^2b^3}{a^2b^2} - \frac{a^3b + a^2b}{a^2b^2} - \frac{2a}{a^2b^2} + \frac{3b}{a^2b^2}$$
$$= \frac{a^3b^2 + 4a^2b^3 - (a^3b + a^2b) - 2a + 3b}{a^2b^2}$$
$$= \frac{a^3b^2 + 4a^2b^3 - a^3b - a^2b - 2a + 3b}{a^2b^2}$$

5. (a) Multiply the fractions first, then cancel the resulting fraction and add to $3/c^2$:

$$\frac{a}{bc} \cdot \frac{b}{ac} + \frac{3}{c^2} = \frac{ab}{abc^2} + \frac{3}{c^2} = \frac{1}{c^2} + \frac{3}{c^2} = \frac{4}{c^2}$$

(b) Multiply the fractions and cancel by xy:

$$\frac{x+y+z}{xy} \cdot \frac{xy}{x+y} = \frac{(x+y+z)(xy)}{(xy)(x+y)} = \frac{x+y+z}{x+y}$$

Note: it is not possible to cancel the final answer by $x+y$. However, the fraction can be rewritten in the following way, by viewing $x+y$ as a single term:

$$\frac{x+y+z}{x+y} = \frac{(x+y)+z}{x+y} = \frac{x+y}{x+y} + \frac{z}{x+y} = 1 + \frac{z}{x+y}$$

(c) Multiply and then cancel:

$$3mn \cdot \frac{3}{n} \cdot \frac{m}{27} = \frac{3mn}{1} \cdot \frac{3}{n} \cdot \frac{m}{27} = \frac{(3mn)(3)(m)}{(1)(n)(27)} = \frac{9m^2n}{3 \cdot 9n} = \frac{m^2}{3}$$

(d) First add the terms in parentheses (the common denominator is xyz), and then multiply by xyz:

$$\begin{aligned} xyz\left(\frac{x}{yz} + \frac{y}{xz}\right) &= xyz\left(\frac{x(x)}{yz(x)} + \frac{y(y)}{xz(y)}\right) \\ &= xyz\left(\frac{x^2}{xyz} + \frac{y^2}{xyz}\right) = xyz \cdot \frac{x^2+y^2}{xyz} \\ &= \frac{xyz(x^2+y^2)}{xyz} = x^2 + y^2 \end{aligned}$$

Alternatively, multiply the terms in parentheses by xyz first, and then simplify:

$$\begin{aligned} xyz\left(\frac{x}{yz} + \frac{y}{xz}\right) &= xyz \cdot \frac{x}{yz} + xyz \cdot \frac{y}{xz} \\ &= \frac{x^2yz}{yz} + \frac{xy^2z}{xz} = x^2 + y^2 \end{aligned}$$

(e) Multiply and then cancel by $x+y$:

$$\begin{aligned} 3x(x+y) \cdot \frac{3}{(x+y)(x-2y)} &= \frac{3x(x+y)}{1} \cdot \frac{3}{(x+y)(x-2y)} \\ &= \frac{3x(x+y)(3)}{(1)(x+y)(x-2y)} \\ &= \frac{9x(x+y)}{(x+y)(x-2y)} \\ &= \frac{9x}{x-2y} \end{aligned}$$

Note: whenever needed (as was the case above), we can write an expression A as a fraction: $A = \frac{A}{1}$.

(f) Multiply first, cancel the resulting fractions, and then simplify:

$$\begin{aligned} xy \cdot \frac{3x-2}{x} + x^2y \cdot \frac{4-y}{y} &= \frac{xy}{1} \cdot \frac{3x-2}{x} + \frac{x^2y}{1} \cdot \frac{4-y}{y} \\ &= \frac{xy(3x-2)}{x} + \frac{x^2y(4-y)}{y} \\ &= y(3x-2) + x^2(4-y) \\ &= 3xy - 2y + 4x^2 - x^2y \end{aligned}$$

(g) Multiply and cancel:

$$\frac{a+2b}{3a-b} \cdot \frac{42}{a+2b} \cdot \frac{7(3a-b)}{4} = \frac{(a+2b)(2 \cdot 21)(7(3a-b))}{(3a-b)(a+2b)(2 \cdot 2)} = \frac{21 \cdot 7}{2} = \frac{147}{2}$$

We cancelled by $3a-b$, $a+2b$ and 2.

6. (a) To divide two fractions means to multiply the fraction in the numerator by the reciprocal of the fraction in the denominator:

$$\frac{\frac{11}{17}}{\frac{44}{51}} = \frac{11}{17} \cdot \frac{51}{44} = \frac{11 \cdot 51}{17 \cdot 44} = \frac{11 \cdot 17 \cdot 3}{17 \cdot 11 \cdot 4} = \frac{3}{4}$$

(b) We write 4 as a fraction $\frac{4}{1}$, then divide the two fractions and cancel:

$$\frac{4}{\frac{128}{21}} = \frac{\frac{4}{1}}{\frac{128}{21}} = \frac{4}{1} \cdot \frac{21}{128} = \frac{4 \cdot 21}{128} = \frac{4 \cdot 21}{4 \cdot 32} = \frac{21}{32}$$

(c) We write 5 as a fraction $\frac{5}{1}$, then divide the two fractions and cancel:

$$\frac{\frac{60}{13}}{5} = \frac{\frac{60}{13}}{\frac{5}{1}} = \frac{60}{13} \cdot \frac{1}{5} = \frac{60}{13 \cdot 5} = \frac{12 \cdot 5}{13 \cdot 5} = \frac{12}{13}$$

(d) As in (b) and (c),

$$\begin{aligned}
\frac{\frac{3}{7}}{6} + \frac{2}{\frac{3}{4}} - \frac{\frac{1}{2}}{\frac{3}{4}} &= \frac{\frac{3}{7}}{\frac{6}{1}} + \frac{\frac{2}{1}}{\frac{3}{4}} - \frac{\frac{1}{2}}{\frac{3}{4}} \\
&= \frac{3}{7} \cdot \frac{1}{6} + \frac{2}{1} \cdot \frac{4}{3} - \frac{1}{2} \cdot \frac{4}{3} \\
&= \frac{3}{42} + \frac{8}{3} - \frac{4}{6} \\
&= \frac{3}{42} + \frac{8(14)}{3(14)} - \frac{4(7)}{6(7)} = \frac{3}{42} + \frac{112}{42} - \frac{28}{42} \\
&= \frac{3 + 112 - 28}{42} = \frac{87}{42}
\end{aligned}$$

In combining the three fractions, we used the fact that the common denominator of 42, 3 and 6 is 42. Note that both 87 and 42 are multiples of 3 (How do we know? The sums of digits of the two numbers, $8 + 7 = 15$ and $4 + 2 = 6$, are divisible by 3). Thus, we cancel:

$$\frac{87}{42} = \frac{29 \cdot 3}{14 \cdot 3} = \frac{29}{14}$$

(e) Write $3c = \frac{3c}{1}$, divide the two fractions and simplify:

$$\frac{\frac{3ab}{c^2}}{3c} = \frac{\frac{3ab}{c^2}}{\frac{3c}{1}} = \frac{3ab}{c^2} \cdot \frac{1}{3c} = \frac{3ab}{3c^3} = \frac{ab}{c^3}$$

(g) Simplify the top and the bottom fractions first, and then divide:

$$\frac{x + \frac{y}{z}}{x - \frac{y}{z}} = \frac{\frac{x}{1} + \frac{y}{z}}{\frac{x}{1} - \frac{y}{z}} = \frac{\frac{x(z)}{1(z)} + \frac{y}{z}}{\frac{x(z)}{1(z)} - \frac{y}{z}} = \frac{\frac{xz}{z} + \frac{y}{z}}{\frac{xz}{z} - \frac{y}{z}} = \frac{\frac{xz+y}{z}}{\frac{xz-y}{z}} = \frac{xz + y}{z} \cdot \frac{z}{xz - y} = \frac{xz + y}{xz - y}$$

(h) Simplify the top and the bottom fractions first, and then divide (note that $51 = 17 \cdot 3$):

$$\frac{\frac{a}{17} - b}{\frac{c}{51} - d} = \frac{\frac{a}{17} - \frac{b(17)}{1(17)}}{\frac{c}{51} - \frac{d(51)}{1(51)}} = \frac{\frac{a-17b}{17}}{\frac{c-51d}{51}} = \frac{a - 17b}{17} \cdot \frac{51}{c - 51d} = \frac{51(a - 17b)}{17(c - 51d)} = \frac{3(a - 17b)}{c - 51d}$$

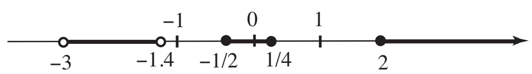
(i) Simplify the expression on the top (the common denominator is ab), and then divide the fractions:

$$\frac{\frac{a}{b} + \frac{b}{a}}{a^2 + b^2} = \frac{\frac{a(a)}{b(a)} + \frac{b(b)}{a(b)}}{a^2 + b^2} = \frac{\frac{a^2}{ab} + \frac{b^2}{ab}}{a^2 + b^2} = \frac{\frac{a^2+b^2}{ab}}{\frac{a^2+b^2}{1}} = \frac{a^2 + b^2}{ab} \cdot \frac{1}{a^2 + b^2} = \frac{1}{ab}$$

(j) Simplify the expression on the top (the common denominator is $2x$), and then divide the fractions:

$$\frac{\frac{1}{x}-\frac{1}{2}}{2-x}=\frac{\frac{1(2)}{x(2)}-\frac{1(x)}{2(x)}}{2-x}=\frac{\frac{2}{2x}-\frac{x}{2x}}{2-x}=\frac{\frac{2-x}{2x}}{\frac{2-x}{1}}=\frac{2-x}{2x}\cdot\frac{1}{2-x}=\frac{1}{2x}$$

7. Remember: we use round bracket or empty dot to say that a point (real number) is not included, and square bracket or filled dot when we wish to include a point (real number).



8. The leftmost interval includes all negative numbers up to and including -2, so we denote it by $(-\infty,-2]$, The middle interval contains all numbers between -1 and 2, not including -1 but including 2; so, we write $(-1,2]$. The rightmost interval is $(3,4)$.

9. (a) $[-4/9,\infty)$. To include $-4/9$ we used the square bracket. Remember to always use round brackets with infinity.

(b) $[-7,-4.4)$. Remember: number included, use square bracket; number not included, use round bracket.

(c) $(-\infty,-8)$ and $(-2,\infty)$

(d) $(0,5.5)$ and $(5.5,\infty)$. Note that 5.5 does not belong to either interval.

(e) $[3,3.7)$ and $(3.7,4]$

(f) $(-\infty,-4)$ and $(-4,0)$. Zero is not a negative number, so we do not include it.

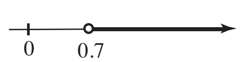
10. First row: interval: $(-1/2,7/3)$; inequality: $-1/2<x<7/3$; verbal: all real numbers between $-1/2$ and $7/3$, not including $-1/2$ and not including $7/3$;

number line: 

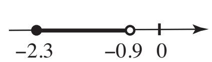
Second row: interval: $(-2,\pi)$; inequality: $-2<x<\pi$; verbal: all real numbers between -2 and π, not including -2 and not including π;

number line: 

Third row: interval: $(0.7,\infty)$; inequality: $x>0.7$; verbal: all real numbers greater than 0.7, not including 0.7;

number line: 

Fourth row: interval: $[-2.3,-0.9)$; inequality: $-2.3\le x<-0.9$; verbal: all real numbers between -2.3 and -0.9, including -2.3 and not including -0.9;

number line: 

Fifth row: interval: $[0, 127.4]$; inequality: $0 \leq x \leq 127.4$; verbal: all real numbers between 0 and 127.4, including both 0 and 127.4;

number line:

11. (a) $\frac{-4}{11} = \frac{4}{-11}$ is correct. Both fractions are equal to $-\frac{4}{11}$.

(b) $\frac{0}{-4} = 0$ is correct. Zero divided by a non-zero number is zero.

(c) $4.7 \geq 4.7$ is correct, because the symbol $\geq$ means greater than *or equal* to.

(d) $4.7 > 4.7$ is incorrect; $A > B$ represents a strict inequality, i.e., it does not allow the two numbers A and B to be equal.

(e) $\dfrac{-4}{0} = 0$ is incorrect. Division by zero does not produce a real number.

(f) $-2.2 \leq -3.1$ is incorrect. On a number line, -2.2 is to the right of -3.7, so the correct relation is $-2.2 > -3.1$.

12. (a) We use the definition, and then simplify:

$$|x+2.9| = \begin{cases} x+2.9 & \text{if } x+2.9 \geq 0 \\ -(x+2.9) & \text{if } x+2.9 < 0 \end{cases} = \begin{cases} x+2.9 & \text{if } x \geq -2.9 \\ -x-2.9 & \text{if } x < -2.9 \end{cases}$$

(b) We use the definition, and then simplify:

$$|4x-7| = \begin{cases} 4x-7 & \text{if } 4x-7 \geq 0 \\ -(4x-7) & \text{if } 4x-7 < 0 \end{cases} = \begin{cases} 4x-7 & \text{if } x \geq \frac{7}{4} \\ -4x+7 & \text{if } x < \frac{7}{4} \end{cases}$$

To solve the inequality $4x - 7 \geq 0$ within the condition, we wrote:

$$\begin{aligned} 4x - 7 &\geq 0 \\ 4x &\geq 7 \\ x &\geq \frac{7}{4} \end{aligned}$$

Replacing $\geq$ with $<$ we obtain the remaining inequality.

(c) As in (a),

$$|3-x| = \begin{cases} 3-x & \text{if } 3-x \geq 0 \\ -(3-x) & \text{if } 3-x < 0 \end{cases} = \begin{cases} 3-x & \text{if } 3 \geq x \\ -3+x & \text{if } 3 < x \end{cases} = \begin{cases} -x+3 & \text{if } x \leq 3 \\ x-3 & \text{if } x > 3 \end{cases}$$

(d) We use the definition, and then simplify:

$$|10-0.2x| = \begin{cases} 10-0.2x & \text{if } 10-0.2x \geq 0 \\ -(10-0.2x) & \text{if } 10-0.2x < 0 \end{cases} = \begin{cases} -0.2x+10 & \text{if } x \leq 50 \\ 0.2x-10 & \text{if } x > 50 \end{cases}$$

To solve the inequality $10 - 0.2x \geq 0$ within the condition, we wrote:

$$\begin{aligned} 10 - 0.2x &\geq 0 \\ 10 &\geq 0.2x \\ \frac{10}{0.2} &\geq x \end{aligned}$$

and therefore $x \leq 50$. Replacing $\geq$ with $<$ we obtain the remaining inequality.

13. (a) Since π is positive, $|\pi| = \pi$.

By definition, $|-1/2| = -(-1/2) = 1/2$, and thus $-|-1/2| = -1/2$.

Since -0.33 is negative, $|-0.33| = -(-0.33) = 0.33$.

Note: of course, we could have written right away that $|-1/2| = 1/2$ or $|-0.33| = 0.33$, since absolute value returns positive numbers. We included intermediate steps to practice the definition.

Why is the definition of absolute value so messy if all it does is to convert a number into a positive number? The reason is that we want to do more than just do calculations with numbers. The definition, as we have seen already, needs to be applied to algebraic expressions as well.

(b) Recall that the distance between two numbers on a number line is computed as the absolute value of their difference; so, the distance between -3 and -4 is $|-3-(-4)| = |-3+4| = |1| = 1$.

(c) Simplify the term involving the absolute value first:

$$\frac{|4-4.5|}{4.5-5} = \frac{|-0.5|}{-0.5} = \frac{0.5}{-0.5} = -1.$$

(d) Start by simplifying the terms involving the absolute value:

$$\frac{2+|3-(-4)|}{|-3-4|} = \frac{2+|7|}{|-7|} = \frac{2+7}{7} = \frac{9}{7}.$$

14. (a) As usual, we write out the definition first, and then simplify:

$$|1-4x| = \begin{cases} 1-4x & \text{if } 1-4x \geq 0 \\ -(1-4x) & \text{if } 1-4x < 0 \end{cases} = \begin{cases} 1-4x & \text{if } 1 \geq 4x \\ 4x-1 & \text{if } 1 < 4x \end{cases} = \begin{cases} -4x+1 & \text{if } x \leq \frac{1}{4} \\ 4x-1 & \text{if } x > \frac{1}{4} \end{cases}$$

(b) As in (a),

$$|6x-5| = \begin{cases} 6x-5 & \text{if } 6x-5 \geq 0 \\ -(6x-5) & \text{if } 6x-5 < 0 \end{cases} = \begin{cases} 6x-5 & \text{if } 6x \geq 5 \\ -6x+5 & \text{if } 6x < 5 \end{cases} = \begin{cases} 6x-5 & \text{if } x \geq \frac{5}{6} \\ -6x+5 & \text{if } x < \frac{5}{6} \end{cases}$$

(c) Draw a number line, start at 5 and walk 12 units to the right (obtaining $5+12 = 17$) and then start at 5 and walk 12 units to the left (obtaining $5-12 = -7$). So the solutions are -7 and 17.

There is a longer alternative – algebraic solution (good for practice!). Note that we are asked to solve the equation $|x-5| = 12$. Recall that

$$|x-5| = \begin{cases} x-5 & \text{if } x-5 \geq 0 \\ -(x-5) & \text{if } x-5 < 0 \end{cases} = \begin{cases} x-5 & \text{if } x \geq 5 \\ -x+5 & \text{if } x < 5 \end{cases}$$

Thus, the equation $|x-5| = 12$ splits into two cases:

$$\begin{cases} x-5 = 12 & \text{if } x \geq 5 \\ -x+5 = 12 & \text{if } x < 5 \end{cases}$$

By simplifying, we obtain two solutions:

$$\begin{cases} x = 17 & \text{if } x \geq 5 \\ x = -7 & \text{if } x < 5 \end{cases}$$

(d) Draw a number line, start at -2 and walk 11 units to the right (obtaining $-2+11=9$) and then start at -2 and walk 11 units to the left (obtaining $-2-11=-13$). So the solutions are -13 and 9.

Alternatively, we can solve $|x-(-2)|=11$ algebraically. The equation $|x+2|=11$ implies $x+2=11$ (and thus $x=9$) or $-(x+2)=11$, and thus $-x=13$ and $x=-13$.

(e) Using the product of absolute values formula $|ab|=|a|\cdot|b|$, we write $|5x|=|5|\cdot|x|=5\,|x|$.

(f) Using the product of absolute values formula $|ab|=|a|\cdot|b|$, we write $|-3x|=|-3|\cdot|x|=3\,|x|$. Thus, $|-3x|=-3|x|$ is not correct.

Alternatively, all we need to do is to find one number for which the given formula does not hold. For instance: if $x=1$, then $|-3x|=|-3(1)|=|-3|=3$ is not equal to $-3|x|=-3\,|1|=-3$.

15. (a) $(-3)^4=(-3)(-3)(-3)(-3)=81$.

(b) Removing the negative number in the exponent, we obtain $4^{-3}=\dfrac{1}{4^3}=\dfrac{1}{64}$.

(c) $0^{-3}=\dfrac{1}{0^3}=\dfrac{1}{0}$ is undefined, since division by zero does not yield a real number.

(d) $(1/2)^{-3}=\dfrac{1}{\left(\frac{1}{2}\right)^3}=\dfrac{1}{\frac{1}{8}}=1\cdot\dfrac{8}{1}=8$. (Remember: it's the top times the reciprocal of the bottom.)

16. (a) Using (the definition) $a^4=a\cdot a\cdot a\cdot a$ we write

$$(-0.1)^4=(-0.1)(-0.1)(-0.1)(-0.1)=0.0001=\frac{1}{10,000}$$

(b) Note that the minus sign, unlike in (a), is not raised to the fourth power. Thus

$$-0.1^4=-[(0.1)(0.1)(0.1)(0.1)]=-0.0001=-\frac{1}{10,000}$$

(c) Remove the negative sign from the exponent and then proceed as in (a):

$$(-0.1)^{-4}=\frac{1}{(-0.1)^4}=\frac{1}{(-0.1)(-0.1)(-0.1)(-0.1)}=\frac{1}{0.0001}=10,000$$

(d) Write $1/10=0.1$ and then simplify:

$$\left(\frac{1}{10}\right)^{-4}=(0.1)^{-4}=\frac{1}{0.1^4}=\frac{1}{(0.1)(0.1)(0.1)(0.1)}=\frac{1}{0.0001}=10,000$$

Alternatively, we could use laws of exponents:

$$\left(\frac{1}{10}\right)^{-4}=(10^{-1})^{-4}=10^{(-1)(-4)}=10^4=10,000$$

or:

$$\left(\frac{1}{10}\right)^{-4}=\frac{1}{\left(\frac{1}{10}\right)^4}=\frac{1}{\frac{1^4}{10^4}}=\frac{1}{\frac{1}{10,000}}=1\cdot\frac{10,000}{1}=10,000$$

(e) Write $1/1000=0.001$ and then simplify:

$$\left(\frac{1}{1000}\right)^{-2}=(0.001)^{-2}=\frac{1}{0.001^2}=\frac{1}{(0.001)(0.001)}=\frac{1}{0.000001}=1,000,000$$

As in (d), there are alternatives, using laws of exponents; for instance:

$$\left(\frac{1}{1000}\right)^{-2} = \left(\frac{1}{10^3}\right)^{-2} = (10^{-3})^{-2} = 10^6 = 1,000,000$$

(f) Remove the negative sign from the exponent first:

$$(-5)^{-2} = \frac{1}{(-5)^2} = \frac{1}{(-5)(-5)} = \frac{1}{25}$$

(g) Remove the negative sign from the exponent, and note that the minus sign in front of 5 is not under the power of -2:

$$-5^{-2} = -\frac{1}{5^2} = -\frac{1}{(5)(5)} = -\frac{1}{25}$$

(h) Remove the negative sign from the exponent, and note that the minus sign in front of 5 is not under the power of -3:

$$-5^{-3} = -\frac{1}{5^3} = -\frac{1}{(5)(5)(5)} = -\frac{1}{125}$$

(i) Remove the negative sign from the exponent:

$$(-5)^{-3} = \frac{1}{(-5)^3} = \frac{1}{(-5)(-5)(-5)} = \frac{1}{-125} = -\frac{1}{125}$$

(j) Remove the negative sign from the exponent:

$$(-0.5)^{-3} = \frac{1}{(-0.5)^3} = \frac{1}{(-0.5)(-0.5)(-0.5)} = \frac{1}{-0.125} = -\frac{1}{0.125} = -8$$

17. (a) To multiply the powers of a, add the exponents:

$$a^4a^{-3}a^2a^{-1} = a^{4+(-3)+2+(-1)} = a^{4-3+2-1} = a^2$$

(b) To divide the powers of a, subtract the exponents:

$$\frac{a^4}{a^{-22}} = a^{4-(-22)} = a^{4+22} = a^{26}$$

(c) By laws of exponents,

$$\frac{x^{12}x^{-2}}{x} = \frac{x^{12+(-2)}}{x} = \frac{x^{10}}{x^1} = x^{10-1} = x^9$$

Note that we used the fact that $x^1 = x$.

(d) Simplify using laws of exponents, and cancel:

$$\frac{2}{y^{-12}} \cdot \frac{y^3}{16} = \frac{2y^3}{16y^{-12}} = \frac{2}{16} \cdot y^{3-(-12)} = \frac{1}{8}y^{15} = \frac{y^{15}}{8}$$

(e) Using laws of exponents,

$$\frac{(x^{15})^3}{(x^{-1})^{-22}} = \frac{x^{(15)(3)}}{x^{(-1)(-22)}} = \frac{x^{45}}{x^{22}} = x^{45-22} = x^{23}$$

(f) Simplify the fraction within the brackets first:

$$\left(\frac{x^{-1}}{x^4}\right)^2 = \left(x^{-1-(4)}\right)^2 = (x^{-5})^2 = x^{(-5)(2)} = x^{-10} = \frac{1}{x^{10}}$$

Alternatively, using $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$:

$$\left(\frac{x^{-1}}{x^4}\right)^2 = \frac{\left(x^{-1}\right)^2}{\left(x^4\right)^2} = \frac{x^{(-1)(2)}}{x^{(4)(2)}} = \frac{x^{-2}}{x^8} = x^{-2-(8)} = x^{-10} = \frac{1}{x^{10}}$$

(g) Simplify the fraction within the brackets first:

$$\left(\frac{x^4}{x^{-1}}\right)^{-2} = \left(x^{4-(-1)}\right)^{-2} = (x^5)^{-2} = x^{-10} = \frac{1}{x^{10}}$$

Alternatively, using $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$:

$$\left(\frac{x^4}{x^{-1}}\right)^{-2} = \frac{\left(x^4\right)^{-2}}{\left(x^{-1}\right)^{-2}} = \frac{x^{(4)(-2)}}{x^{(-1)(-2)}} = \frac{x^{-8}}{x^2} = x^{-8-(2)} = x^{-10} = \frac{1}{x^{10}}$$

(h) As in (f) and (g),

$$\left(\frac{c^{-1}}{c^4}\right)^{-5} = \left(c^{-1-4}\right)^{-5} = \left(c^{-5}\right)^{-5} = c^{25}$$

(i) Simplify the expression inside the brackets first:

$$\left(\frac{a^2b}{ab^{-3}}\right)^{-3} = \left(a^2b \cdot a^{-1}b^{-(-3)}\right)^{-3} = \left(a^{2-1}b^{1+3}\right)^{-3} = \left(ab^4\right)^{-3} = a^{-3}\left(b^4\right)^{-3} = a^{-3}b^{-12} = \frac{1}{a^3b^{12}}$$

(j) Using laws of exponents,

$$\left(\frac{a^{-8}}{b^{-1}}\right)^2\left(\frac{b^{-8}}{a^{-1}}\right)^{-2} = \frac{\left(a^{-8}\right)^2}{\left(b^{-1}\right)^2}\frac{\left(b^{-8}\right)^{-2}}{\left(a^{-1}\right)^{-2}} = \frac{a^{-16}}{b^{-2}} \cdot \frac{b^{16}}{a^2} = a^{-16}b^2 \cdot b^{16}a^{-2} = a^{-18}b^{18} = \frac{b^{18}}{a^{18}} = \left(\frac{b}{a}\right)^{18}$$

(k) Using laws of exponents,

$$\left(\frac{3x-y}{(3x-y)^4}\right)^{-7} = \left(\frac{(3x-y)^1}{(3x-y)^4}\right)^{-7} = \left((3x-y)^{-3}\right)^{-7} = (3x-y)^{21}$$

(l) As in (j),

$$\left(\frac{a}{b^2}\right)^{-4}\left(\frac{a^2}{b}\right)^4 = \frac{a^{-4}}{b^{-8}} \cdot \frac{a^8}{b^4} = \frac{a^4}{b^{-4}} = a^4b^4 = (ab)^4$$

(m) Using laws of exponents,

$$\left(\frac{(a+b-c)^4}{(a+b-c)^{-3}}\right)^{-2} = \left((a+b-c)^7\right)^{-2} = (a+b-c)^{-14} = \frac{1}{(a+b-c)^{14}}$$

(n) Using laws of exponents,

$$\begin{aligned}\left(\frac{xy^3}{x+y}\right)\left(\frac{(x+y)^3}{x^2y^2}\right)^{-2} &= \frac{xy^3}{(x+y)^1} \cdot \frac{(x+y)^{-6}}{x^{-4}y^{-4}} \\ &= xy^3(x+y)^{-1} \cdot (x+y)^{-6}x^4y^4 \\ &= x^5y^7(x+y)^{-7} = \frac{x^5y^7}{(x+y)^7}\end{aligned}$$

18. (a) $\sqrt[4]{-16}$ is not defined since no real number raised to the power of 4 gives a negative number.

(b) Recall that $32 = 2^5$; thus

$$\sqrt[5]{32} = (32)^{\frac{1}{5}} = \left(2^5\right)^{\frac{1}{5}} = 2$$

Note: it is a good idea to remember powers of 2 up to, say, $2^6 = 64$, and the first few powers of 3, say, up to $3^4 = 81$.

(c) We use the fact that $32 = 2^5$ and therefore $-32 = (-2)^5$; thus

$$\sqrt[5]{-32} = (-32)^{\frac{1}{5}} = \left((-2)^5\right)^{\frac{1}{5}} = -2$$

(d) Using laws of exponents, ans recalling that $16 = 2^4$ and $81 = 3^4$, we write

$$\sqrt[4]{\frac{16}{81}} = \left(\frac{16}{81}\right)^{\frac{1}{4}} = \left(\frac{2^4}{3^4}\right)^{\frac{1}{4}} = \frac{\left(2^4\right)^{\frac{1}{4}}}{(3^4)^{\frac{1}{4}}} = \frac{2}{3}$$

(e) Use the fact that $10000 = 10^4$:

$$\sqrt{10000} = (10000)^{\frac{1}{2}} = (10^4)^{\frac{1}{2}} = 10^{4\cdot\frac{1}{2}} = 10^2 = 100$$

(f) Using the law of radicals for fractions, we write

$$\sqrt{\frac{1}{100}} = \frac{\sqrt{1}}{\sqrt{100}} = \frac{1}{10}$$

Alternatively, there is a longer calculation:

$$\sqrt{\frac{1}{100}} = \left(\frac{1}{100}\right)^{\frac{1}{2}} = \left(\frac{1}{10^2}\right)^{\frac{1}{2}} = (10^{-2})^{\frac{1}{2}} = 10^{-1} = \frac{1}{10}$$

(g) $\sqrt[5]{0} = 0$ because $0^5 = 0$.

(h) Recalling the cancellation formula $\sqrt[n]{a^n} = |a|$ when n is even, we write

$$\sqrt[4]{(-2)^4} = |-2| = 2$$

Alternatively,

$$\sqrt[4]{(-2)^4} = \sqrt[4]{16} = 2$$

(i) Recalling the cancellation formula $\sqrt[n]{a^n} = |a|$ when n is even, we write

$$\sqrt[4]{2^4} = |2| = 2$$

Alternatively,

$$\sqrt[4]{2^4} = \sqrt[4]{16} = 2$$

or

$$\sqrt[4]{2^4} = (2^4)^{\frac{1}{4}} = 2^1 = 2$$

(j) Recall the cancellation formula $\sqrt[n]{a^n} = a$ when n is odd. Thus

$$\sqrt[3]{(-2)^3} = -2$$

Alternatively,

$$\sqrt[3]{(-2)^3} = \left((-2)^3\right)^{\frac{1}{3}} = (-2)^1 = -2$$

or

$$\sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2,$$

because $(-2)^3 = 8$.

(k) $\sqrt{-1/10000}$ is not defined, because no real number squared gives a negative number.

(l) Write the square root as a rational exponent:

$$\sqrt{\sqrt{16}} = (\sqrt{16})^{\frac{1}{2}} = (16^{\frac{1}{2}})^{\frac{1}{2}} = 16^{\frac{1}{4}} = (2^4)^{\frac{1}{4}} = 2^1 = 2$$

(m) $\sqrt{-\sqrt{16}}$ is not defined. We can start the calculation by writing

$$\sqrt{-\sqrt{16}} = \sqrt{-4}$$

but this is as far as we can go. No real number squared gives -16.

(n) Using the law of radicals for fractions, we write

$$\sqrt{\frac{100}{121}} = \frac{\sqrt{100}}{\sqrt{121}} = \frac{10}{11}$$

Alternatively,

$$\sqrt{\frac{100}{121}} = \left(\frac{100}{121}\right)^{\frac{1}{2}} = \left(\frac{10^2}{11^2}\right)^{\frac{1}{2}} = \frac{(10^2)^{\frac{1}{2}}}{(11^2)^{\frac{1}{2}}} = \frac{10}{11}$$

(o) Using laws of radicals, we write

$$\sqrt{16}\,\sqrt{\frac{1}{64}} = \sqrt{16\cdot\frac{1}{64}} = \sqrt{\frac{16}{64}} = \sqrt{\frac{1}{4}} = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2}$$

One, of several alternatives:

$$\sqrt{16}\,\sqrt{\frac{1}{64}} = (16)^{\frac{1}{2}}\left(\frac{1}{64}\right)^{\frac{1}{2}} = (4^2)^{\frac{1}{2}}\left(\frac{1}{8^2}\right)^{\frac{1}{2}} = 4\cdot\frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

19. (a) Since $32 = 16\cdot 2$, we get

$$\sqrt{32} = \sqrt{16\cdot 2} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$$

(b) Since $45 = 9\cdot 5$, we write

$$\sqrt{45} = \sqrt{9\cdot 5} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$$

(c) Write $1000 = 100\cdot 10$; then

$$\sqrt{1000} = \sqrt{100\cdot 10} = \sqrt{100}\sqrt{10} = 10\sqrt{10}$$

(d) Since $72 = 36\cdot 2$, we write

$$\sqrt{72} = \sqrt{36\cdot 2} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$$

(e) Write $16 = 8\cdot 2$ (why? because $\sqrt[3]{8} = 2$); then

$$\sqrt[3]{16} = \sqrt[3]{8\cdot 2} = \sqrt[3]{8}\sqrt[3]{2} = 2\sqrt[3]{2}$$

(f) Write $32 = 8\cdot 4$ (why? because $\sqrt[3]{8} = 2$); then

$$\sqrt[3]{32} = \sqrt[3]{8\cdot 4} = \sqrt[3]{8}\sqrt[3]{4} = 2\sqrt[3]{4}$$

(g) Write $81 = 27\cdot 3$ (why? because $\sqrt[3]{27} = 3$); then

$$\sqrt[3]{81} = \sqrt[3]{27\cdot 3} = \sqrt[3]{27}\sqrt[3]{3} = 3\sqrt[3]{3}$$

(h) Write $10000 = 1000\cdot 10$ (why? because $\sqrt[3]{1000} = 10$); then

$$\sqrt[3]{10000} = \sqrt[3]{1000\cdot 10} = \sqrt[3]{1000}\sqrt[3]{10} = 10\sqrt[3]{10}$$

(i) Write $32 = 16 \cdot 2$ (why? because $\sqrt[4]{16} = 2$); then

$$\sqrt[4]{32} = \sqrt[4]{16 \cdot 2} = \sqrt[4]{16}\sqrt[4]{2} = 2\sqrt[4]{2}$$

(j) Write $100000 = 10000 \cdot 10$ (why? because $\sqrt[4]{10000} = 10$); then

$$\sqrt[4]{100000} = \sqrt[4]{10000 \cdot 10} = \sqrt[4]{10000}\sqrt[4]{10} = 10\sqrt[4]{10}$$

20. (a) To multiply, we add exponents:

$$2^{\frac{1}{9}}2^{-\frac{3}{4}} = 2^{\frac{1}{9}-\frac{3}{4}} = 2^{\frac{4}{36}-\frac{27}{36}} = 2^{-\frac{23}{36}}$$

(b) Convert radicals to fractional exponents and simplify:

$$\sqrt[5]{\sqrt{2^{10}}} = \left(\sqrt{2^{10}}\right)^{\frac{1}{5}} = \left((2^{10})^{\frac{1}{2}}\right)^{\frac{1}{5}} = (2^{10})^{\frac{1}{10}} = 2^1 = 2$$

(c) Using the power to another power law of exponents, we write

$$(12^{\frac{1}{4}})^{\frac{3}{5}} = 12^{\frac{3}{20}}$$

(d) Using the power to another power law of exponents, we write

$$\left(33^{-\frac{1}{4}}\right)^{\frac{4}{5}} = 33^{-\frac{1}{4}\cdot\frac{4}{5}} = 33^{-\frac{1}{5}}$$

(e) We write

$$3\sqrt{\frac{14}{9}} = 3 \cdot \frac{\sqrt{14}}{\sqrt{9}} = 3 \cdot \frac{\sqrt{14}}{3} = \sqrt{14}$$

(f) Convert the radical to a fractional exponent and then multiply:

$$x^{\frac{1}{4}}\sqrt[3]{x} = x^{\frac{1}{4}}x^{\frac{1}{3}} = x^{\frac{1}{4}+\frac{1}{3}} = x^{\frac{3}{12}+\frac{4}{12}} = x^{\frac{7}{12}}$$

(g) We can keep radicals

$$(2x - \sqrt{y})\sqrt{y} = 2x\sqrt{y} - \sqrt{y}\sqrt{y} = 2x\sqrt{y} - y$$

(note that $\sqrt{y}\sqrt{y} = (\sqrt{y})^2 = y$), or convert to rational exponents:

$$(2x - \sqrt{y})\sqrt{y} = (2x - y^{\frac{1}{2}})y^{\frac{1}{2}} = 2xy^{\frac{1}{2}} - y^{\frac{1}{2}}y^{\frac{1}{2}} = 2xy^{\frac{1}{2}} - y^1 = 2xy^{\frac{1}{2}} - y$$

(h) Convert to rational exponents and then multiply (when multiplying, keep in mind that $x = x^1$):

$$(2x - \sqrt{y})\sqrt[3]{x} = (2x - y^{\frac{1}{2}})x^{\frac{1}{3}} = 2xx^{\frac{1}{3}} - y^{\frac{1}{2}}x^{\frac{1}{3}} = 2x^{\frac{4}{3}} - y^{\frac{1}{2}}x^{\frac{1}{3}}$$

(i) Convert to rational exponents and then multiply:

$$y^{-\frac{1}{4}}y\sqrt[4]{y} = y^{-\frac{1}{4}}yy^{\frac{1}{4}} = y^{-\frac{1}{4}+1+\frac{1}{4}} = y^1 = y$$

(j) Convert to rational exponents and use laws of exponents:

$$\sqrt[3]{ab^6c^{-12}} = (ab^6c^{-12})^{\frac{1}{3}} = a^{\frac{1}{3}}\left(b^6\right)^{\frac{1}{3}}\left(c^{-12}\right)^{\frac{1}{3}} = a^{\frac{1}{3}}b^2c^{-4} = \frac{a^{\frac{1}{3}}b^2}{c^4}$$

(k) Multiplying each term in the first bracket with each term in the second bracket, we obtain

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = \sqrt{a}\sqrt{a} + \sqrt{b}\sqrt{a} - \sqrt{a}\sqrt{b} - \sqrt{b}\sqrt{b} = (\sqrt{a})^2 - \left(\sqrt{b}\right)^2 = a - b$$

(Note that $\sqrt{b}\sqrt{a} - \sqrt{a}\sqrt{b} = 0$.) Instead of working with radicals, we can use rational exponents:

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = (a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}}) = a^{\frac{1}{2}}a^{\frac{1}{2}} - a^{\frac{1}{2}}b^{\frac{1}{2}} + b^{\frac{1}{2}}a^{\frac{1}{2}} - b^{\frac{1}{2}}b^{\frac{1}{2}} = a^1 - b^1 = a - b$$

Note that we can use the difference of squares formula $(x + y)(x - y) = x^2 - y^2$ (which is reviewed in the next section):

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = \left(\sqrt{a}\right)^2 - \left(\sqrt{b}\right)^2 = a - b$$

(l) Multiplying each term in the first bracket with each term in the second bracket, we obtain

$$\begin{aligned}\left(2\sqrt{a} - \sqrt{b}\right)\left(\sqrt{a} + 5\sqrt{b}\right) &= 2\sqrt{a}\sqrt{a} - \sqrt{b}\sqrt{a} + 10\sqrt{a}\sqrt{b} - 5\sqrt{b}\sqrt{b} \\ &= 2\left(\sqrt{a}\right)^2 - \sqrt{ab} + 10\sqrt{ab} - 5\left(\sqrt{b}\right)^2 \\ &= 2a + 9\sqrt{ab} - 5b\end{aligned}$$

We can use rational exponents as well:

$$\begin{aligned}\left(2\sqrt{a} - \sqrt{b}\right)\left(\sqrt{a} + 5\sqrt{b}\right) &= (2a^{\frac{1}{2}} - b^{\frac{1}{2}})(a^{\frac{1}{2}} + 5b^{\frac{1}{2}}) \\ &= 2a^{\frac{1}{2}}a^{\frac{1}{2}} + 10a^{\frac{1}{2}}b^{\frac{1}{2}} - b^{\frac{1}{2}}a^{\frac{1}{2}} - 5b^{\frac{1}{2}}b^{\frac{1}{2}} \\ &= 2a^1 + 10a^{\frac{1}{2}}b^{\frac{1}{2}} - a^{\frac{1}{2}}b^{\frac{1}{2}} - 5b^1 = 2a + 9b^{\frac{1}{2}}b^{\frac{1}{2}} - 5b\end{aligned}$$

(m) Multiply the bracket by $\sqrt{xy}$ and simplify:

$$\begin{aligned}\sqrt{xy}(\sqrt{x} - 3\sqrt{y}) &= \sqrt{xy}\sqrt{x} - 3\sqrt{xy}\sqrt{y} \\ &= \sqrt{x}\sqrt{y}\sqrt{x} - 3\sqrt{x}\sqrt{y}\sqrt{y} \\ &= \left(\sqrt{x}\right)^2\sqrt{y} - 3\sqrt{x}\left(\sqrt{y}\right)^2 = x\sqrt{y} - 3y\sqrt{x}\end{aligned}$$

Alternatively,

$$\sqrt{xy}(\sqrt{x} - 3\sqrt{y}) = (xy)^{\frac{1}{2}}(x^{\frac{1}{2}} - y^{\frac{1}{2}}) = x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}} - y^{\frac{1}{2}}) = x^{\frac{1}{2}}y^{\frac{1}{2}}x^{\frac{1}{2}} - x^{\frac{1}{2}}y^{\frac{1}{2}}y^{\frac{1}{2}} = xy^{\frac{1}{2}} - 3x^{\frac{1}{2}}y$$

(n) Multiplying each term in the first bracket with each term in the second bracket, we obtain

$$\begin{aligned}(\sqrt[3]{a} + \sqrt[3]{b})(\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2}) &= (a^{\frac{1}{3}} + b^{\frac{1}{3}})(a^{\frac{2}{3}} - (ab)^{\frac{1}{3}} + b^{\frac{2}{3}}) \\ &= (a^{\frac{1}{3}} + b^{\frac{1}{3}})(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}) \\ &= a^{\frac{1}{3}}a^{\frac{2}{3}} - a^{\frac{1}{3}}a^{\frac{1}{3}}b^{\frac{1}{3}} + a^{\frac{1}{3}}ab^{\frac{2}{3}} + b^{\frac{1}{3}}a^{\frac{2}{3}} - b^{\frac{1}{3}}a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{1}{3}}b^{\frac{2}{3}} \\ &= a^1 - a^{\frac{2}{3}}b^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{2}{3}} + b^{\frac{1}{3}}a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}} + b^1 = a + b\end{aligned}$$

(Note: after reviewing the difference of cubes formula, try this question again!)

21. (a) The law of radicals for products implies that $\sqrt{2y} = \sqrt{2}\sqrt{y}$ is correct.

(b) The formula $\sqrt{2 - y} = \sqrt{2} - \sqrt{y}$ is incorrect. For instance, when $y = 1$, then $\sqrt{2 - y} = \sqrt{1} = 1$, whereas $\sqrt{2} - \sqrt{y} = \sqrt{2} - \sqrt{1} = \sqrt{2} - 1 \neq 1$.

Or, when $y = 4$, then $\sqrt{2 - y} = \sqrt{-2}$ is not defined, whereas $\sqrt{2} - \sqrt{y} = \sqrt{2} - \sqrt{4} = \sqrt{2} - 2$.

(c) By the law of radicals for fractions,

$$\sqrt{\frac{16}{x}} = \frac{\sqrt{16}}{\sqrt{x}} = \frac{4}{\sqrt{x}}$$

The formula is correct.

(d) The formula $(x^3)^{\frac{1}{2}} = x^{\frac{7}{2}}$ is incorrect. The correct formula is obtained from the laws of exponents:

$$(x^3)^{\frac{1}{2}} = x^{3\cdot\frac{1}{2}} = x^{\frac{3}{2}}$$

(e) The formula $(x^2)^{\frac{1}{3}} = x^{\frac{2}{3}}$ is correct, since

$$(x^2)^{\frac{1}{3}} = x^{2\cdot\frac{1}{3}} = x^{\frac{2}{3}}$$

(f) The formula $x^3 + x^2 = x^5$ is incorrect. For instance, when $x = 1$, the left side is 2 and the right side is 1.

(g) The formula $\dfrac{a^{\frac{3}{2}}}{a^{\frac{1}{2}}} = a^3$ is incorrect. The correct formula is obtained from the laws of exponents:

$$\frac{a^{\frac{3}{2}}}{a^{\frac{1}{2}}} = a^{\frac{3}{2}-\frac{1}{2}} = a^1 = a$$

(h) The formula $(a^{-1}b)^2 = a^{-2}b^2$ is correct, since

$$(a^{-1}b)^2 = \left(a^{-1}\right)^2 (b)^2 = a^{(-1)2}b^2 = a^{-2}b^2$$

(i) The formula $(a+b)^{-1} = a^{-1} + b^{-1}$ is incorrect. For example, if $a = 1$ and $b = 1$ then $(a+b)^{-1} = 2^{-1} = 1/2$, whereas $a^{-1} + b^{-1} = 1^{-1} + 1^{-1} = 1 + 1 = 2$.

22. (a) $\sqrt[3]{\dfrac{-125}{64}} = \left(\dfrac{-125}{64}\right)^{1/3} = \dfrac{(-125)^{1/3}}{64^{1/3}} = \dfrac{\left((-5)^3\right)^{1/3}}{\left(4^3\right)^{1/3}} = \dfrac{(-5)^{3\cdot(1/3)}}{4^{3\cdot(1/3)}} = \dfrac{(-5)^1}{4^1} = \dfrac{-5}{4}$.

For the record: the formulas that we used in the calculation above, from first to last equals sign: $\sqrt[n]{A} = A^{1/n}$ (with $n = 3$), $(A/B)^n = A^n/B^n$ (with $n = 1/3$); next, we recall that $64 = 4^3$ and $125 = 5^3$, so that $-125 = (-5)^3$; then $(A^m)^n = A^{mn}$ (with $m = 3$ and $n = 1/3$), and finally we simplify exponents and use the fact that $A^1 = A$.

Note: there are other ways to simplify this expression; for instance

$$\sqrt[3]{\frac{-125}{64}} = \sqrt[3]{\frac{(-5)^3}{4^3}} = \sqrt[3]{\left(\frac{-5}{4}\right)^3} = \frac{-5}{4};$$

in the last step we used the formula $\sqrt[n]{A^n} = A$ which holds for all odd exponents n (in our case $n = 3$).

Try to find other ways of correctly simplifying this!

(b) One way to simplify is: $\dfrac{\sqrt{100}}{\sqrt{200}} = \dfrac{10}{\sqrt{100\cdot 2}} = \dfrac{10}{\sqrt{100}\sqrt{2}} = \dfrac{10}{10\sqrt{2}} = \dfrac{1}{\sqrt{2}}$.

If needed, we rationalize the denominator and write

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Alternatively, we could simplify given expression as

$$\frac{\sqrt{100}}{\sqrt{200}} = \sqrt{\frac{100}{200}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

In the first and the last steps we used the formula $\sqrt{A/B} = \sqrt{A}/\sqrt{B}$.

(c) $\left(\dfrac{4}{9}\right)^{-3/2} = \dfrac{4^{-3/2}}{9^{-3/2}} = \dfrac{(2^2)^{-3/2}}{(3^2)^{-3/2}} = \dfrac{2^{2\cdot(-3/2)}}{3^{2\cdot(-3/2)}} = \dfrac{2^{-3}}{3^{-3}} = \dfrac{\frac{1}{2^3}}{\frac{1}{3^3}} = \dfrac{1}{2^3}\dfrac{3^3}{1} = \dfrac{3^3}{2^3} = \dfrac{27}{8}$.

As usual, there are other ways of simplifying; for instance:

$$\left(\frac{4}{9}\right)^{-3/2} = \left(\frac{2^2}{3^2}\right)^{-3/2} = \left(\left(\frac{2}{3}\right)^2\right)^{-3/2} = \left(\frac{2}{3}\right)^{2\cdot(-3/2)} = \left(\frac{2}{3}\right)^{-3} = \frac{1}{\left(\frac{2}{3}\right)^3} = \frac{1}{\left(\frac{2^3}{3^3}\right)} = \frac{1}{\left(\frac{8}{27}\right)} = \frac{27}{8}.$$

(d) Likewise, $16^{-3/2}9^{3/2} = (4^2)^{-3/2}(3^2)^{3/2} = 4^{2\cdot(-3/2)}3^{2\cdot(3/2)} = 4^{-3}3^3 = \frac{3^3}{4^3} = \frac{27}{64}$.

23. (a) Combine the cube roots:

$$\sqrt[3]{20}\sqrt[3]{50} + 7 = \sqrt[3]{20 \cdot 50} + 7 = \sqrt[3]{1000} + 7 = 10 + 7 = 17$$

(b) It helps to write 4 as 2^2:

$$-4^{\frac{5}{2}} + 4^{\frac{3}{2}} - 4^{\frac{1}{2}} = -(2^2)^{\frac{5}{2}} + (2^2)^{\frac{3}{2}} - (2^2)^{\frac{1}{2}} = -2^5 + 2^3 - 2 = -32 + 8 - 2 = -26$$

(c) Because the denominator of the power is 2, we write the numbers involved as squares:

$$\left(\frac{25}{16}\right)^{-\frac{3}{2}} = \left(\frac{5^2}{4^2}\right)^{-\frac{3}{2}} = \frac{5^{-3}}{4^{-3}} = \frac{4^3}{5^3} = \frac{64}{125}$$

(d) Same idea as in (c): write 8 as a cube and 0.01 as a square:

$$8^{\frac{2}{3}} + 0.01^{-\frac{1}{2}} = (2^3)^{\frac{2}{3}} + (0.1^2)^{-\frac{1}{2}} = 2^2 + 0.1^{-1} = 4 + 10 = 14$$

Alternatively,

$$8^{\frac{2}{3}} + 0.01^{-\frac{1}{2}} = (2^3)^{\frac{2}{3}} + (10^{-2})^{-\frac{1}{2}} = 4 + 10 = 14$$

(e) Write both numbers as squares:

$$(0.09)^{\frac{1}{2}} + (0.04)^{-\frac{3}{2}} = \left(0.3^2\right)^{\frac{1}{2}} + \left(0.2^2\right)^{-\frac{3}{2}} = 0.3 + 0.2^{-3} = 0.3 + 125 = 125.3$$

To simplify 0.2^{-3}, we wrote:

$$0.2^{-3} = \frac{1}{0.2^3} = \frac{1}{\left(\frac{1}{5}\right)^3} = \frac{1}{\frac{1}{5^3}} = 1 \cdot \frac{5^3}{1} = 5^3 = 125$$

Alternatively,

$$\begin{aligned}(0.09)^{\frac{1}{2}} + (0.04)^{-\frac{3}{2}} &= \left(\frac{9}{100}\right)^{\frac{1}{2}} + \left(\frac{4}{100}\right)^{-\frac{3}{2}} \\ &= \left(\frac{3^2}{10^2}\right)^{\frac{1}{2}} + \left(\frac{2^2}{10^2}\right)^{-\frac{3}{2}} \\ &= \frac{3}{10} + \frac{2^{-3}}{10^{-3}} \\ &= \frac{3}{10} + \frac{10^3}{2^3} = \frac{3}{10} + \frac{1000}{8} \\ &= \frac{3}{10} + 125 = \frac{1253}{10}\end{aligned}$$

(f) Divide the two roots and rewrite $\sqrt{75}$ using a smaller number under the square root:

$$\frac{\sqrt{200}}{\sqrt{2}}\sqrt{3} + \sqrt{75} = \sqrt{\frac{200}{2}}\sqrt{3} + \sqrt{25 \cdot 3} = \sqrt{100}\sqrt{3} + 5\sqrt{3} = 10\sqrt{3} + 5\sqrt{3} = 15\sqrt{3}$$

(g) Rewrite the second and the third terms using a smaller number under the square root:

$$-7\sqrt{5}+\sqrt{125}-\sqrt{45}=-7\sqrt{5}+\sqrt{25\cdot 5}-\sqrt{9\cdot 5}=-7\sqrt{5}+5\sqrt{5}-3\sqrt{5}=-5\sqrt{5}$$

(h) Multiply and then comine the square roots:

$$\left(\sqrt{8}+7\sqrt{2}\right)\sqrt{2}=\sqrt{8}\sqrt{2}+7\sqrt{2}\sqrt{2}=\sqrt{16}+7\sqrt{4}=4+7(2)=18$$

Alternatively:

$$\left(\sqrt{8}+7\sqrt{2}\right)\sqrt{2}=((2^3)^{\frac{1}{2}}+7\cdot 2^{\frac{1}{2}})2^{\frac{1}{2}}=2^{\frac{3}{2}}2^{\frac{1}{2}}+7\cdot 2^{\frac{1}{2}}2^{\frac{1}{2}}=2^2+7\cdot 2=4+14=18$$

24. (a) $127=\left(\sqrt{127}\right)^2=\left((127)^{1/2}\right)^2$

(b) $a^2+b^2=\left(\sqrt{a^2+b^2}\right)^2=\left((a^2+b^2)^{\frac{1}{2}}\right)^2$

(c) $x^5=\left(\sqrt{x^5}\right)^2=\left(x^{5/2}\right)^2$

(d) $xyz^3=\left(\sqrt{xyz^3}\right)^2=\left((xyz^3)^{\frac{1}{2}}\right)^2=\left(x^{\frac{1}{2}}y^{\frac{1}{2}}z^{\frac{3}{2}}\right)^2$

(e) $\sqrt[3]{x}=\left(\sqrt{\sqrt[3]{x}}\right)^2=\left((x^{\frac{1}{3}})^{\frac{1}{2}}\right)^2=\left(x^{\frac{1}{6}}\right)^2=\left(\sqrt[6]{x}\right)^2$

(f) $\sqrt[4]{x}=\left(\sqrt{\sqrt[4]{x}}\right)^2=\left((x^{\frac{1}{4}})^{\frac{1}{2}}\right)^2=\left(x^{\frac{1}{8}}\right)^2=\left(\sqrt[8]{x}\right)^2$

25. (a) $127=\left(\sqrt[3]{127}\right)^3=\left((127)^{1/3}\right)^3$

(b) $x-y=\left(\sqrt[3]{x-y}\right)^3=\left((x-y)^{1/3}\right)^3$

(c) $x^2=\left(\sqrt[3]{x^2}\right)^3=\left((x^2)^{\frac{1}{3}}\right)^3=\left(x^{\frac{2}{3}}\right)^3$

(d) $a^4b^6c^8=\left(\sqrt[3]{a^4b^6c^8}\right)^3=\left((a^4b^6c^8)^{\frac{1}{3}}\right)^3=\left(a^{\frac{4}{3}}b^2c^{\frac{8}{3}}\right)^3$

(e) $\sqrt{x}=\left(\sqrt[3]{x^{\frac{1}{2}}}\right)^3=\left((x^{\frac{1}{2}})^{\frac{1}{3}}\right)^3=\left(x^{\frac{1}{6}}\right)^3=\left(\sqrt[6]{x}\right)^3$

(f) Using $x=\left(\sqrt[3]{x}\right)^3$, we write $\dfrac{1}{x}=\left(\dfrac{1}{\sqrt[3]{x}}\right)^3$.

26. (a) Using laws of exponents,

$$\begin{aligned}\left(\frac{x^{0.4}x^{4.5}}{x^{-1.3}}\right)^{0.9}&=\left(\frac{x^{0.4+4.5}}{x^{-1.3}}\right)^{0.9}\\&=\left(\frac{x^{4.9}}{x^{-1.3}}\right)^{0.9}=\left(x^{4.9-(-1.3)}\right)^{0.9}\\&=\left(x^{6.2}\right)^{0.9}=x^{6.2\cdot 0.9}=x^{5.58}\end{aligned}$$

First equals sign is due to the formula $x^m\cdot x^n=x^{m+n}$, and the third one due to $x^m/x^n=x^{m-n}$. We computed $\left(x^{6.2}\right)^{0.9}$ using $(x^m)^n=x^{mn}$.

(b) $\left(\dfrac{x^3y^{-2}}{x^{-1}y^4}\right)^{-1/2}\left(x^{-1/2}y^{3/4}\right)^{-1}=\left(x^4y^{-6}\right)^{-1/2}\left(x^{1/2}y^{-3/4}\right)=\left(x^{-2}y^3\right)\left(x^{1/2}y^{-3/4}\right)=x^{-3/2}y^{9/4}$.

In the first step we simplified the fraction

$$\frac{x^3y^{-2}}{x^{-1}y^4} = \frac{x^3}{x^{-1}}\frac{y^{-2}}{y^4} = x^{3-(-1)}y^{-2-4} = x^2y^{-6}.$$

The second equals sign is due to the formulas $(AB)^n = A^nB^n$, here applied with $A = x^4$, $B = y^{-6}$ and $n = -1/2$, and $(x^m)^n = x^{mn}$:

$$\left(x^4y^{-6}\right)^{-1/2} = \left(x^4\right)^{-1/2}\left(y^{-6}\right)^{-1/2} = x^{4\cdot(-1/2)}y^{-6\cdot(-1/2)} = x^{-2}y^3.$$

Finally, in the last step we use the fact that multiplying two powers with the same base results in the addition of their exponents.

(c) We get

$$\frac{(x-a)^{-2}}{(x-a)^{-3}}(x-a)^0 = (x-a)^{-2-(-3)}(x-a)^0 = (x-a)^1(x-a)^0 = x-a,$$

since $(x-a)^1 = x-a$ and $(x-a)^0 = 1$.

27. (a) Using laws of exponents,

$$\left(xy^{0.3}\right)^2 = x^2\left(y^{0.3}\right)^2 = x^2y^{0.6}$$

(b) Using laws of exponents,

$$\begin{aligned}(xy^{-0.3}z^{-0.11})^{0.6} &= x^{0.6}\left(y^{-0.3}\right)^{0.6}\left(z^{-0.11}\right)^{0.6}\\ &= x^{0.6}y^{(-0.3)(0.6)}z^{(-0.11)(0.6)}\\ &= x^{0.6}y^{-0.18}z^{-0.66} = \frac{x^{0.6}}{y^{0.18}z^{0.66}}\end{aligned}$$

(c) Divide inside the brackets first:

$$\left(\frac{y^{0.4}}{y^{-0.7}}\right)^{-3.2} = \left(y^{0.4-(-0.7)}\right)^{-3.2}(y^{1.1})^{-3.2} = y^{(1.1)(-3.2)} = y^{-3.52} = \frac{1}{y^{3.52}}$$

(d) Using laws of exponents,

$$\left(a^{\sqrt{2}}b^2\right)^{3\sqrt{2}} = \left(a^{\sqrt{2}}\right)^{3\sqrt{2}}\left(b^2\right)^{3\sqrt{2}} = a^{\sqrt{2}\cdot3\sqrt{2}}b^{2\cdot3\sqrt{2}} = a^6b^{6\sqrt{2}}$$

(e) Work on each fraction first, and then multiply (it helps to write 4 as 2^2):

$$\begin{aligned}\left(\frac{c}{2d}\right)^{0.6}\left(\frac{4d^{2.4}}{c}\right)^{-0.6} &= \left(\frac{c}{2d}\right)^{0.6}\left(\frac{2^2d^{2.4}}{c}\right)^{-0.6}\\ &= \frac{c^{0.6}}{2^{0.6}d^{0.6}}\cdot\frac{2^{2(-0.6)}d^{(2.4)(-0.6)}}{c^{1(-0.6)}}\\ &= \frac{c^{0.6}}{2^{0.6}d^{0.6}}\cdot\frac{2^{-1.2}d^{-1.44}}{c^{-0.6}}\\ &= c^{0.6}2^{-0.6}d^{-0.6}\cdot2^{-1.2}d^{-1.44}c^{0.6}\\ &= c^{0.6+0.6}d^{-0.6-1.44}2^{-0.6-1.2}\\ &= c^{0.12}d^{-2.04}2^{-1.8}\\ &= \frac{c^{0.12}}{2^{1.8}d^{2.04}}\end{aligned}$$

(f) Simplify the fraction first:

$$\begin{aligned} x^3(x+y)^{-1.3}\left(\frac{x}{x+y}\right)^{7.2} &= x^3(x+y)^{-1.3}\cdot\frac{x^{7.2}}{(x+y)^{7.2}} \\ &= x^{3+7.2}(x+y)^{-1.3-7.2} \\ &= x^{10.2}(x+y)^{-8.5} \\ &= \frac{x^{10.2}}{(x+y)^{8.5}} \end{aligned}$$

28. (a) The formula for the product of fractions $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$ applied from right to left (so we are "undoing" the product) implies that $\dfrac{1}{x^2y}=\dfrac{1}{x^2}\dfrac{1}{y}$ is correct.

(b) The formula $\dfrac{1}{x^2}+\dfrac{1}{y}=\dfrac{1}{x^2+y}$ is incorrect. If $x=1$ and $y=1$, then the left side is 2, whereas the right side is 1/2. Another example: if $x=0$ and $y=3$, then the left side is not defined, whereas the right side is 1/3.

(c) The formula $\sqrt{xy}+\sqrt{y}=\sqrt{xy^2}$ is incorrect. If $x=0$ and $y=1$, then the left side is 1, whereas the right side is 0.

(d) The calculation (recall the way we divide fractions!)

$$\frac{x^2}{\frac{1}{x^2}}=x^2\cdot\frac{x^2}{1}=x^4$$

proves that $\dfrac{x^2}{\frac{1}{x^2}}=1$ is incorrect.

(e) The multiplication law for radicals $\sqrt[n]{a}\,\sqrt[n]{b}=\sqrt[n]{ab}$ implies that

$$\sqrt{xy}\sqrt{y}=\sqrt{xy\cdot y}=\sqrt{xy^2}$$

The given formula is correct.

(f) The formula $\frac{1}{x}-\frac{1}{y}=\frac{x-y}{xy}$ is incorrect. For instance, if $x=1$ and $y=5$, then $\frac{1}{x}-\frac{1}{y}=\frac{1}{1}-\frac{1}{5}=\frac{4}{5}$ whereas $\frac{x-y}{xy}=\frac{1-5}{(1)(5)}=-\frac{4}{5}$.

Alternatively, by computing the common denominator we obtain the correct formula:

$$\frac{1}{x}-\frac{1}{y}=\frac{y}{xy}-\frac{x}{xy}=\frac{y-x}{xy}$$

(g) The formula $\frac{2}{7}-\frac{1}{4}=\frac{1}{3}$ is incorrect; by computing the common denominator we obtain

$$\frac{2}{7}-\frac{1}{4}=\frac{8}{28}-\frac{7}{28}=\frac{1}{28}$$

(h) We do not square a binomial, or a trinomial just by squaring each term. Thus, the formula $(x+y-2)^2=x^2+y^2-4$ is incorrect.

It's easy to find values for x and y to prove this. For instance, if $x=0$ and $y=0$ then $(x+y-2)^2=(-2)^2=4$, whereas $x^2+y^2-4=-4$.

(i) The formula $|3a-4|=|3a|-4$ is incorrect. If $a=0$, then the left side is $|3a-4|=|-4|=4$, whereas the right side is $|3a|-4=-4$.

Note: If $a = 7$, then $|3a - 4| = |3(7) - 4| = 17$ and $|3a| - 4 = |3(7)| - 4 = 17$. This, however, does not prove that the formula is correct (nor would any number of calculations such as this one). The formula needs to hold for all real numbers a. If it does not hold for a single value of a, then the formula is not correct.

(j) The formula $x^k + x^k = x^{2k}$ is incorrect: if $x = 1$, then the left side is 2 and the right side is 1.

(k) Since $x^k \cdot x^{2k} = x^{k+2k} = x^{3k}$, the given formula is correct.

(l) Using $(x^a)^b = x^{ab}$ we compute $(x^k)^k = x^{k^2}$. Thus, $(x^k)^k = x^{2k}$ is incorrect.

Alternatively, Take $x = 2$ and $k = 3$; then $(x^k)^k = (2^3)^3 = 8^3 = 512$, whereas $x^{2k} = 2^{2(3)} = 2^6 = 64$.

Section 2. Basic Algebra

1. (a) $mx + b = 4x - 2.3$. So, $m = 4$ and $b = -2.3$.

(b) $mx + b = -2 - 6x = -6x - 2$. So, $m = -6$ and $b = -2$.

(c) $mx + b = 3x - \frac{2}{7}$. So, $m = 3$ and $b = -\frac{2}{7}$.

(d) $mx + b = \frac{4x}{5} - 12 = \frac{4}{5}x - 12$. So, $m = \frac{4}{5}$ and $b = -12$.

(e) Think of -3 as $-3 = 0 \cdot x + (-3)$. Comparing $mx + b = -3 = 0x + (-3)$ we see that $m = 0$ and $b = -3$.

(f) From $mx + b = x = 1 \cdot x + 0$ we get $m = 1$ and $b = 0$.

(g) From $mx + b = 0 = 0 \cdot x + 0$ we get $m = 0$ and $b = 0$.

(h) $mx + b = \frac{2x-1}{6} = \frac{2x}{6} - \frac{1}{6} = \frac{x}{3} - \frac{1}{6} = \frac{1}{3}x - \frac{1}{6}$. So, $m = \frac{1}{3}$ and $b = -\frac{1}{6}$.

(i) $mx + b = \frac{x}{5} - \frac{3}{11} = \frac{1}{5}x - \frac{3}{11}$. So, $m = \frac{1}{5}$ and $b = -\frac{3}{11}$.

(j) $mx + b = 4(2 - x) = 8 - 4x = -4x + 8$. So, $m = -4$ and $b = 8$.

(k) $mx + b = -2 \cdot \frac{2x-1}{5} = \frac{-4x+2}{5} = -\frac{4x}{5} + \frac{2}{5} = -\frac{4}{5}x + \frac{2}{5}$. So, $m = -\frac{4}{5}$ and $b = \frac{2}{5}$.

2. (a) $ax^2 + bx + c = 4 - x^2 + 9x = -x^2 + 9x + 4$. So, $a = -1$, $b = 9$, and $c = 4$.

(b) $ax^2 + bx + c = 1 - \frac{x}{7} + \frac{x^2}{3} = \frac{x^2}{3} - \frac{x}{7} + 1 = \frac{1}{3}x^2 - \frac{1}{7}x + 1$. So, $a = \frac{1}{3}$, $b = -\frac{1}{7}$, and $c = 1$.

(c) $ax^2 + bx + c = \frac{3-2x+x^2}{4} = \frac{3}{4} - \frac{2x}{4} + \frac{x^2}{4} = \frac{1}{4}x^2 - \frac{1}{2}x + \frac{3}{4}$. So, $a = \frac{1}{4}$, $b = -\frac{1}{2}$, and $c = \frac{3}{4}$.

(d) From $ax^2 + bx + c = -x^2 = (-1)x^2 + 0 \cdot x + 0$ we see that $a = -1$, $b = 0$, $c = 0$

(e) Comparing $ax^2 + bx + c = 1 - x = 0 \cdot x^2 + (-1) \cdot x + 1$ we get $a = 0$, $b = -1$, $c = 1$.

(f) Comparing $ax^2 + bx + c = 0 = 0 \cdot x^2 + 0 \cdot x + 0$ we get $a = 0$, $b = 0$ and $c = 0$.

3. (a) $mx^2 + nx + p = -32 = 0 \cdot x^2 + 0 \cdot x - 32$. So, $m = 0$, $n = 0$, and $p = -32$.

(b) $mx^2 + nx + p = \frac{3x}{16} = 0 \cdot x^2 + \frac{3}{16}x + 0$. So, $m = 0$, $n = \frac{3}{16}$, and $p = 0$.

(c) $mx^2 + nx + p = \frac{x-3x^2}{2} = \frac{x}{2} - \frac{3x^2}{2} = -\frac{3}{2}x^2 + \frac{1}{2}x + 0$. So, $m = -\frac{3}{2}$, $n = \frac{1}{2}$, and $p = 0$.

(d) $mx^2 + nx + p = -3(2x - 14) = -6x + 42 = 0 \cdot x^2 - 6x + 42$. So, $m = 0$, $n = -6$, and $p = 42$.

4. (a) Multiply each with each, to get

$$\begin{aligned}(x^2 - x + 1)(3x - 4) &= (x^2)(3x) - x(3x) + 1(3x) + (x^2)(-4) - x(-4) + 1(-4) \\ &= 3x^3 - 3x^2 + 3x - 4x^2 + 4x - 4 = 3x^3 - 7x^2 + 7x - 4\end{aligned}$$

(b) Using the formula for the square of the difference, we compute

$$\begin{aligned}(x^2 - 0.2)^2 &= (x^2)^2 - 2(x^2)(0.2) + (0.2)^2 \\ &= x^4 - 0.4x^2 + 0.04\end{aligned}$$

(c) We multiply the first two terms first (or the last two – difference of squares!)

$$\begin{aligned}x(x^3 - 1)(x^3 + 1) &= (x^4 - x)(x^3 + 1) \\ &= (x^4)(x^3) - x(x^3) + x^4 - x \\ &= x^7 - x^4 + x^4 - x = x^7 - x\end{aligned}$$

(d) Using the cube of the difference formula, we get

$$\begin{aligned}(2x-1)^3 &= (2x)^3 - 3(2x)^2(1) + 3(2x)(1)^2 - (1)^3 \\ &= 8x^3 - 3(4x^2) + 6x - 1 = 8x^3 - 12x^2 + 6x - 1\end{aligned}$$

(e) Using the formula for the square of the difference, we compute

$$\begin{aligned}\left(3x - \frac{5}{2}\right)^2 &= (3x)^2 - 2(3x)\cdot\frac{5}{2} + \left(\frac{5}{2}\right)^2 \\ &= 9x^2 - 15x + \frac{25}{4}\end{aligned}$$

Alternatively, using EWE,

$$\begin{aligned}\left(3x - \frac{5}{2}\right)^2 &= \left(3x - \frac{5}{2}\right)\left(3x - \frac{5}{2}\right) \\ &= (3x)(3x) - (3x)\left(\frac{5}{2}\right) - \left(\frac{5}{2}\right)(3x) + \left(\frac{5}{2}\right)\left(\frac{5}{2}\right) \\ &= 9x^2 - \frac{15x}{2} - \frac{15x}{2} + \frac{25}{4} \\ &= 9x^2 - 15x + \frac{25}{4}\end{aligned}$$

(f) Simplify and use the square of the sum formula:

$$\begin{aligned}\left(\frac{5+2x}{4}\right)^2 &= \frac{(5+2x)^2}{4^2} \\ &= \frac{5^2 + 2(5)(2x) + (2x)^2}{16} \\ &= \frac{25 + 20x + 4x^2}{16} \\ &= \frac{25}{16} + \frac{20x}{16} + \frac{4x^2}{16} \\ &= \frac{25}{16} + \frac{5x}{4} + \frac{x^2}{4}\end{aligned}$$

(g) Using the formula for the square of the difference, we compute

$$(0.1x - 0.03)^2 = (0.1x)^2 - 2(0.1x)(0.03) + (0.03)^2 = 0.01x^2 - 0.006x + 0.0009$$

Alternatively, using EWE,

$$\begin{aligned}(0.1x - 0.03)^2 &= (0.1x - 0.03)(0.1x - 0.03) \\ &= (0.1x)(0.1x) - (0.1x)(0.03) - (0.03)(0.1x) + (0.03)(0.03) \\ &= 0.01x^2 - 0.003x - 0.003x + 0.0009 \\ &= 0.01x^2 - 0.006x + 0.0009\end{aligned}$$

(h) Using EWE,

$$\begin{aligned}(x^4 - x^2 + 1)(2 - x) &= (x^4)(2) - (x^4)(x) - (x^2)(2) + (x^2)(x) + (1)(2) - (1)(x) \\ &= 2x^4 - x^5 - 2x^2 + x^3 + 2 - x \\ &= -x^5 + 2x^4 + x^3 - 2x^2 - x + 2\end{aligned}$$

(i) Recognizing the expression as a difference of squares, we find

$$(2x - 7y)(2x + 7y) = (2x)^2 - (7y)^2 = 4x^2 - 49y^2$$

Alternatively, using EWE,

$$\begin{aligned}(2x - 7y)(2x + 7y) &= (2x)(2x) + (2x)(7y) - (7y)(2x) - (7y)(7y)\\ &= 4x^2 + 14xy - 14xy - 49y^2\\ &= 4x^2 - 49y^2\end{aligned}$$

(j) Recognizing the expression as a difference of squares, we find

$$\left(x^2 - 11\right)\left(x^2 + 11\right) = \left(x^2\right)^2 - (11)^2 = x^4 - 121$$

(k) Multiply the two brackets (using the difference of squares formula) and then multiply by -2:

$$-2\left(7 + x^4\right)\left(7 - x^4\right) = -2\left[(7)^2 - \left(x^4\right)^2\right] = -2\left[49 - x^8\right] = -98 + 2x^8$$

Alternatively, without using the difference of squares formula:

$$\begin{aligned}-2(7 + x^4)(7 - x^4) &= -2[(7)(7) - (7)(x^4) + (x^4)(7) - (x^4)(x^4)]\\ &= -2[49 - 7x^4 + 7x^4 - x^8]\\ &= -98 + 2x^8 = 2x^8 - 98\end{aligned}$$

(l) Using the difference of squares $(a - b)(a + b) = a^2 - b^2$ with $a = -1$ and $b = x^2$, we obtain

$$\left(-1 - x^2\right)\left(-1 + x^2\right) = (-1)^2 - \left(x^2\right)^2 = 1 - x^4$$

Alternatively, using EWE,

$$\begin{aligned}(-1 - x^2)(-1 + x^2) &= (-1)(-1) - (1)(x^2) + (x^2)(1) - (x^2)(x^2)\\ &= 1 - x^2 + x^2 - x^4\\ &= 1 - x^4 = -x^4 + 1\end{aligned}$$

(m) Using the difference of squares formula, we obtain

$$\left(\frac{x}{3} - 12\right)\left(\frac{x}{3} + 12\right) = \left(\frac{x}{3}\right)^2 - (12)^2 = \frac{x^2}{9} - 144$$

Using EWE,

$$\begin{aligned}\left(\frac{x}{3} - 12\right)\left(\frac{x}{3} + 12\right) &= \left(\frac{x}{3}\right)\left(\frac{x}{3}\right) + \left(\frac{x}{3}\right)(12) - (12)\left(\frac{x}{3}\right) - (12)(12)\\ &= \frac{x^2}{9} - \frac{12x}{3} + \frac{12x}{3} - 144\\ &= \frac{x^2}{9} - 4x + 4x - 144\\ &= \frac{x^2}{9} - 144\end{aligned}$$

(n) Using the cube of the sum formula,

$$\begin{aligned}4x(x + 2)^3 &= 4x\left(x^3 + 3x^2(2) + 3x(2^2) + 2^3\right)\\ &= 4x\left(x^3 + 6x^2 + 12x + 8\right) = 4x^4 + 24x^3 + 48x^2 + 32x\end{aligned}$$

If we cannot recall the formula we need, we can still do it using EWE:

$$\begin{aligned}4x(x+2)^3 = 4x(x+2)(x+2)(x+2) &= 4x(x+2)(x^2+2x+2x+4)\\ &= 4x(x+2)(x^2+4x+4)\\ &= 4x(x^3+4x^2+4x+2x^2+8x+8)\\ &= 4x(x^3+6x^2+12x+8)\\ &= 4x^4+24x^3+48x^2+32x\end{aligned}$$

(o) Using the cube of the difference formula,

$$\begin{aligned}\left(\frac{x}{2}-4\right)^3 &= \left(\frac{x}{2}\right)^3 - 3\left(\frac{x}{2}\right)^2(4) + 3\left(\frac{x}{2}\right)(4^2) - 4^3\\ &= \frac{x^3}{8} - \frac{3x^2}{4}(4) + \frac{3x}{2}(16) - 64\\ &= \frac{x^3}{8} - 3x^2 + 24x - 64\end{aligned}$$

As in (n), if we are not sure about the formula, we use EWE:

$$\begin{aligned}\left(\frac{x}{2}-4\right)^3 = \left(\frac{x}{2}-4\right)\left(\frac{x}{2}-4\right)\left(\frac{x}{2}-4\right) &= \left(\frac{x^2}{4}-2x-2x+16\right)\left(\frac{x}{2}-4\right)\\ &= \left(\frac{x^2}{4}-4x+16\right)\left(\frac{x}{2}-4\right)\\ &= \frac{x^3}{8} - \frac{4x^2}{2} + \frac{16x}{2} - \frac{4x^2}{4} + 16x - 64\\ &= \frac{x^3}{8} - 2x^2 + 8x - x^2 + 16x - 64\\ &= \frac{x^3}{8} - 3x^2 + 24x - 64\end{aligned}$$

5. (a) $ax^2+bx+c = -3x(x-4) = -3x^2+12x$. So, $a=-3$, $b=12$, and $c=0$.

(b) From $ax^2+bx+c = (2x-1)(3-7x) = 6x-3-14x^2+7x = -14x^2+13x-3$ it follows that $a=-14$, $b=13$, and $c=-3$.

(c) From

$$ax^2+bx+c = (x-5)\left(\frac{3}{4}-x\right) = \frac{3x}{4} - \frac{15}{4} - x^2 + 5x = -x^2 + \frac{23x}{4} - \frac{15}{4}$$

we conclude that $a=-1$, $b=23/4$, and $c=-15/4$.

(d) From

$$ax^2+bx+c = (0.4-0.1x)(2.5x+1.5) = x+0.6-0.25x^2-0.15x = -0.25x^2+0.85x+0.6$$

it follows that $a=-0.25$, $b=0.85$, and $c=0.6$.

(e) From

$$ax^2+bx+c = \left(\frac{3}{4}-x\right)\left(\frac{2x}{3}-\frac{1}{6}\right) = \frac{6x}{12} - \frac{3}{24} - \frac{2x^2}{3} + \frac{x}{6} = -\frac{2x^2}{3} + \frac{2x}{3} - \frac{1}{8}$$

it follows that $a=-2/3$, $b=2/3$, and $c=-1/8$.

(f) From

$$ax^2 + bx + c = (2.8x - 1.2)(0.2 - 2.2x) = 0.56x - 6.16x^2 - 0.24 + 2.64x = -6.16x^2 + 3.2x - 0.24$$

we obtain $a = -6.16$, $b = 3.2$, and $c = -0.24$.

6. (a) Using EWE,

$$(x^{-2} + 1)^2 = (x^{-2} + 1)(x^{-2} + 1) = x^{-4} + x^{-2} + x^{-2} + 1 = x^{-4} + 2x^{-2} + 1$$

Using the square of the sum formula,

$$(x^{-2} + 1)^2 = \left(x^{-2}\right)^2 + 2x^{-2}(2) + 1^2 = x^{-4} + 2x^{-2} + 1$$

(b) Using EWE,

$$(x^{-1} + 1)(3x + 2x^{-3}) = 3 + 2x^{-4} + 3x + 2x^{-3} = 3x + 2x^{-3} + 2x^{-4} + 3$$

Note that in the above calculation we used $x^{-1}(3x) = 3x^{-1}x^1 = 3x^{-1+1} = 3x^0 = 3(1) = 3$.

(c) We find

$$\left(\frac{2}{x} - 1\right)\left(4 - \frac{3}{5x}\right) = \frac{8}{x} - \frac{6}{5x^2} - 4 + \frac{3}{5x} = \frac{43}{5x} - \frac{6}{5x^2} - 4$$

(d) Using EWE,

$$(x^{-1} + 3)(x^{-1} - 3) = x^{-2} - 3x^{-1} + 3x^{-1} - 9 = x^{-2} - 9$$

Using the difference of squares formula,

$$(x^{-1} + 3)(x^{-1} - 3) = \left(x^{-1}\right)^2 - (3)^2 = x^{-2} - 9$$

(e) Using EWE,

$$\left(\sqrt{2x} - 3\right)\left(\sqrt{2x} + 3\right) = \left(\sqrt{2x}\right)^2 + 3\sqrt{2x} - 3\sqrt{2x} - 9 = 2x - 9$$

Alternatively, converting to fractional powers,

$$\begin{aligned}\left(\sqrt{2x} - 3\right)\left(\sqrt{2x} + 3\right) &= \left((2x)^{\frac{1}{2}} - 3\right)\left((2x)^{\frac{1}{2}} + 3\right) \\ &= (2x)^{\frac{1}{2}}(2x)^{\frac{1}{2}} + (2x)^{\frac{1}{2}}(3) - (3)(2x)^{\frac{1}{2}} - (3)(3) = 2x - 9\end{aligned}$$

Using the difference of squares formula,

$$\left(\sqrt{2x} - 3\right)\left(\sqrt{2x} + 3\right) = \left(\sqrt{2x}\right)^2 - 3^2 = 2x - 9$$

(f) Using EWE,

$$\left(\sqrt{x} + 4\right)^2 = \left(\sqrt{x} + 4\right)\left(\sqrt{x} + 4\right) = \sqrt{x}\sqrt{x} + 4\sqrt{x} + 4\sqrt{x} + 16 = x + 8\sqrt{x} + 16$$

Using the square of the sum formula,

$$\left(\sqrt{x} + 4\right)^2 = \left(\sqrt{x}\right)^2 + 2\sqrt{x}\,(4) + 4^2 = x + 8\sqrt{x} + 16$$

(g) Using EWE,

$$\left(3\sqrt{x} - 5\right)^2 = \left(3\sqrt{x} - 5\right)\left(3\sqrt{x} - 5\right) = 9\sqrt{x}\sqrt{x} - 15\sqrt{x} - 15\sqrt{x} + 25 = 9x - 30\sqrt{x} + 25$$

Using the square of the sum formula,

$$\left(3\sqrt{x} - 5\right)^2 = \left(3\sqrt{x}\right)^2 - 2\left(3\sqrt{x}\right)(5) + 5^2 = 9x - 30\sqrt{x} + 25$$

(h) Using EWE,

$$\begin{aligned}\left(\sqrt{x+1}-\sqrt{3x}\right)^2 &= \left(\sqrt{x+1}-\sqrt{3x}\right)\left(\sqrt{x+1}-\sqrt{3x}\right)\\ &= (x+1)-\sqrt{3x}\sqrt{x+1}-\sqrt{3x}\sqrt{x+1}+3x\\ &= 4x-2\sqrt{3x}\sqrt{x+1}+1\end{aligned}$$

Alternatively, using the square of the difference formula,

$$\begin{aligned}\left(\sqrt{x+1}-\sqrt{3x}\right)^2 &= \left(\sqrt{x+1}\right)^2-2\sqrt{x+1}\sqrt{3x}+\left(\sqrt{3x}\right)^2\\ &= x+1-2\sqrt{x+1}\sqrt{3x}+3x = 4x+1-2\sqrt{x+1}\sqrt{3x}\end{aligned}$$

(i) Using EWE,

$$\begin{aligned}\left(\frac{1}{x}-2\right)^3 &= (x^{-1}-2)^3\\ &= (x^{-1}-2)(x^{-1}-2)(x^{-1}-2)\\ &= (x^{-2}-4x^{-1}+4)(x^{-1}-2)\\ &= x^{-3}-4x^{-2}+4x^{-1}-2x^{-2}+8x^{-1}-8\\ &= 12x^{-1}-6x^{-2}+x^{-3}-8\end{aligned}$$

Using the formula for the cube of the difference,

$$\begin{aligned}\left(\frac{1}{x}-2\right)^3 &= \left(\frac{1}{x}\right)^3-3\left(\frac{1}{x}\right)^2(2)+3\left(\frac{1}{x}\right)(2^2)-2^3\\ &= \frac{1}{x^3}-\frac{6}{x^2}+\frac{12}{x}-8\end{aligned}$$

(j) Using the formula for the cube of the sum,

$$\begin{aligned}\left(\sqrt{x}+3\right)^3 &= \left(x^{1/2}+3\right)^3\\ &= \left(x^{1/2}\right)^3+3\left(x^{1/2}\right)^2(3)+3\left(x^{1/2}\right)(3^2)+3^3\\ &= x^{3/2}+9x+27x^{1/2}+27\end{aligned}$$

(k) Using the difference of squares formula,

$$\left(\sqrt{x^2+1}-\sqrt{7}\right)\left(\sqrt{x^2+1}+\sqrt{7}\right)=\left(\sqrt{x^2+1}\right)^2-\left(\sqrt{7}\right)^2=x^2+1-7=x^2-6$$

7. Look at the formulas $(a+b)^2=a^2+2ab+b^2$ and $(a-b)^2=a^2-2ab+b^2$. Both expansions have three terms, two of which are squares, and the sign of the third term is determined by the sign within the binomial being squared. In this exercise we think in the opposite direction: we know the three terms which are the result of squaring a binomial, and we need to guess that binomial.

(a) Looking at $x^2+12x+36$ we identify two squares: x^2 and $36=6^2$. The sign of the remaining term is $+$, so our guess is $(x+6)^2$. Check: $(x+6)^2=x^2+2(x)(6)+6^2=x^2+12x+36$. Works.

(b) $x^2-14x+49=(x-7)^2$

(c) In $a^2+4ab+4b^2$ we identify the squares a^2 and $4b^2=(2b)^2$. Thus, we try $(a+2b)^2$. Check: $(a+2b)^2=a^2+2(a)(2b)+(2b)^2=a^2+4ab+4b^2$.

(d) Note that $16x^2 = (4x)^2$. Thus $16x^2 + 8x + 1 = (4x + 1)^2$.

(e) Note that $9x^2 = (3x)^2$ and $4 = 2^2$. Thus $9x^2 - 12x + 4 = (3x - 2)^2$.

(f) Note that $16a^2 = (4a)^2$ and $(25b^2) = (5b)^2$. Thus $16a^2 - 40ab + 25b^2 = (4a - 5b)^2$.

Check:

$$(4a - 5b)^2 = (4a)^2 - 2(4a)(5b) + (5b)^2 = 16a^2 - 40ab + 25b^2.$$

(g) $x^2 + 0.2x + 0.01 = (x + 0.1)^2$

(h) Note that $\frac{1}{4} = \left(\frac{1}{2}\right)^2$. Thus $x^2 + x + \frac{1}{4} = \left(x + \frac{1}{2}\right)^2$. Check:

$$\left(x + \frac{1}{2}\right)^2 = x^2 + 2(x)\frac{1}{2} + \left(\frac{1}{2}\right)^2 = x^2 + x + \frac{1}{4}.$$

(i) Note that $\frac{x^2}{9} = \left(\frac{x}{3}\right)^2$. Thus $\frac{x^2}{9} - 6x + 81 = \left(\frac{x}{3} - 9\right)^2$. Check:

$$\left(\frac{x}{3} - 9\right)^2 = \left(\frac{x}{3}\right)^2 - 2\left(\frac{x}{3}\right)(9) + 9^2 = \frac{x^2}{9} - 6x + 81$$

(j) Note that $\frac{4x^2}{9} = \left(\frac{2x}{3}\right)^2$. Thus

$$\frac{4x^2}{9} - \frac{x}{3} + \frac{1}{16} = \left(\frac{2x}{3} - \frac{1}{4}\right)^2$$

8. (a) We add and subtract $(b/2)^2 = (16/2)^2 = 64$, to get

$$x^2 + 16x = x^2 + 16x + (64 - 64) = (x^2 + 16x + 64) - 64 = (x + 8)^2 - 64$$

Keep in mind that the three terms in the bracket are equal to $\left(x + \frac{b}{2}\right)^2$; i.e., $x^2+16x+64 = (x+8)^2$.

(b) We add and subtract $(b/2)^2 = (16/2)^2 = 64$, to get

$$x^2 + 16x + 9 = x^2 + 16x + (64 - 64) + 9 = (x^2 + 16x + 64) - 64 + 9 = (x + 8)^2 - 55$$

(c) We add and subtract $(b/2)^2 = (7/2)^2 = 49/4$, to get

$$x^2 + 7x = x^2 + 7x + \left(\frac{49}{4} - \frac{49}{4}\right) = \left(x^2 + 7x + \frac{49}{4}\right) - \frac{49}{4} = \left(x + \frac{7}{2}\right)^2 - \frac{49}{4}$$

(d) We add and subtract $(b/2)^2 = (7/2)^2 = 49/4$, to get

$$x^2 + 7x + 4 = \left(x^2 + 7x + \frac{49}{4}\right) - \frac{49}{4} + 4 = \left(x + \frac{7}{2}\right)^2 - \frac{33}{4}$$

(e) Factor 3 out first

$$\begin{aligned} 3x^2 - 12x &= 3\left[x^2 - 4x\right] \\ &= 3\left[(x^2 - 4x + 4) - 4\right] \\ &= 3\left[(x - 2)^2 - 4\right] = 3(x - 2)^2 - 12 \end{aligned}$$

(f) Factor 3 out of the terms involving x and then complete the square within the square brackets:

$$\begin{aligned} 3x^2 - 12x + 5 &= 3\left[x^2 - 4x\right] + 5 = 3\left[(x^2 - 4x + 4) - 4\right] + 5 \\ &= 3\left[(x - 2)^2 - 4\right] + 5 = 3(x - 2)^2 - 7 \end{aligned}$$

(g) We factor out the coefficient of x and then complete the square:

$$\begin{aligned} 4x^2 - 9x &= 4\left[x^2 - \frac{9}{4}x\right] \\ &= 4\left[\left(x^2 - \frac{9}{4}x + \frac{81}{64}\right) - \frac{81}{64}\right] \\ &= 4\left[\left(x - \frac{9}{8}\right)^2 - \frac{81}{64}\right] = 4\left(x - \frac{9}{8}\right)^2 - \frac{81}{16} \end{aligned}$$

(h) We factor out the coefficient of x and then complete the square within the square brackets:

$$\begin{aligned} 3x^2 - 9x + 2 = 3\left[x^2 - 3x\right] + 2 &= 3\left[\left(x^2 - 3x + \frac{9}{4}\right) - \frac{9}{4}\right] + 2 \\ &= 3\left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4}\right] + 2 = 3\left(x - \frac{3}{2}\right)^2 - \frac{27}{4} + 2 = 3\left(x - \frac{3}{2}\right)^2 - \frac{19}{4} \end{aligned}$$

(i) We factor out 1/4 frist:

$$\begin{aligned} \frac{x^2}{4} + 6x - 1 &= \frac{1}{4}\left[x^2 + 24x\right] - 1 = \frac{1}{4}\left[(x^2 + 24x + 144) - 144\right] - 1 \\ &= \frac{1}{4}\left[(x+12)^2 - 144\right] - 1 = \frac{1}{4}(x+12)^2 - 36 - 1 = \frac{1}{4}(x+12)^2 - 37 \end{aligned}$$

(j) The coefficient of x is $b = 1/7$; thus, $b/2 = 1/14$, and $(b/2)^2 = 1/196$:

$$x^2 + \frac{x}{7} - 1 = \left[\left(x^2 + \frac{x}{7} + \frac{1}{196}\right) - \frac{1}{196}\right] - 1 = \left(x + \frac{1}{14}\right)^2 - \frac{197}{196}$$

(k) Factor 1/3 out of the terms involving x:

$$\begin{aligned} \frac{x^2}{3} - \frac{x}{6} + \frac{1}{9} &= \frac{1}{3}\left[x^2 - \frac{x}{2}\right] + \frac{1}{9} = \frac{1}{3}\left[\left(x^2 - \frac{x}{2} + \frac{1}{16}\right) - \frac{1}{16}\right] + \frac{1}{9} \\ &= \frac{1}{3}\left[\left(x - \frac{1}{4}\right)^2 - \frac{1}{16}\right] + \frac{1}{9} \\ &= \frac{1}{3}\left(x - \frac{1}{4}\right)^2 - \frac{1}{48} + \frac{1}{9} = \frac{1}{3}\left(x - \frac{1}{4}\right)^2 + \frac{13}{144} \end{aligned}$$

(l) Factor 0.1 out of the terms involving x:

$$\begin{aligned} 0.1x^2 + 0.2x - 0.5 &= 0.1(x^2 + 2x) - 0.5 = 0.1[(x^2 + 2x + 1) - 1] - 0.5 \\ &= 0.1(x+1)^2 - 0.1 - 0.5 = 0.1(x+1)^2 - 0.6 \end{aligned}$$

9. (a) $3x^4 - 6x^2 + 42x = 3x(x^3 - 2x + 14)$

(b) Group the first two and the last two terms:

$$x^3 + 4x^2 - 3x - 12 = x^2(x+4) - 3(x+4) = (x+4)(x^2 - 3)$$

Alternatively, group the first and the third, and the second and the fourth terms:

$$x^3 + 4x^2 - 3x - 12 = x^3 - 3x + 4x^2 - 12 = x(x^2 - 3) + 4(x^2 - 3) = (x+4)(x^2 - 3)$$

(c) Group the first two and the last two terms (or in some other way, as in (b)):

$$x^3 + xy - x^2y^2 - y^3 = x(x^2 + y) - y^2(x^2 + y) = (x^2 + y)(x - y^2)$$

(d) Group the first two and the last two terms:

$$2a^2 - 6ab + 2ab^2 - 6b^3 = 2a(a-3b) + 2b^2(a-3b) = (a-3b)(2a+2b^2) = 2(a-3b)(a+b^2)$$

(e) Two numbers whose sum is 11 and whose product is 28 are 7 and 4. Thus,

$$x^2 + 11x + 28 = (x+7)(x+4)$$

(f) Two numbers whose sum is 3 and whose product is -28 are 7 and -4. Thus,

$$x^2 + 3x - 28 = (x+7)(x-4)$$

(g) Factor 3 out first, and then proceed by finding numbers whose sum is 1 and whose product is -20. Thus

$$3x^2 + 3x - 60 = 3(x^2 + x - 20) = 3(x+5)(x-4)$$

(h) Factor $5x$ out, and then recall the difference of squares formula:

$$5x^3 - 5x = 5x(x^2 - 1) = 5x(x+1)(x-1)$$

(i) Factor out the common term x^3, and then proceed as usual (sum is 5, product is 4):

$$x^5 + 5x^4 + 4x^3 = x^3(x^2 + 5x + 4) = x^3(x+4)(x+1)$$

10. (a) We write $2x^2 - x - 1 = (2x+a)(x+b)$ and check all combinations of integers a and b so that $ab = -1$. In this case, there are only two possibilities, $a = 1$ and $b = -1$, and $a = -1$ and $b = 1$. By trying $(2x+1)(x-1)$ and $(2x-1)(x+1)$, we see that the first one works. Thus,

$$2x^2 - x - 1 = (2x+1)(x-1)$$

(b) As in (a), we write $2x^2 + x - 1 = (2x+a)(x+b)$ and check the two cases of integers a and b so that $ab = -1$. We find

$$2x^2 + x - 1 = (2x-1)(x+1)$$

(c) Write $5x^2 - 12x + 4 = (5x+a)(x+b)$ and check all pairs of values for a and b such that $ab = 4$: $a = 1$ and $b = 4$ (i.e., $(5x+1)(x+4)$), $a = -1$ and $b = -4$ (i.e., $(5x-1)(x-4)$), $a = 4$ and $b = 1$ (i.e., $(5x+4)(x+1)$), $a = -4$ and $b = -1$ (i.e., $(5x-4)(x-1)$), $a = 2$ and $b = 2$ (i.e., $(5x+2)(x+2)$), $a = -2$ and $b = -2$ (i.e., $(5x-2)(x-2)$). It turns out that

$$5x^2 - 12x + 4 = (5x-2)(x-2)$$

(d) Write $3x^2 - 22x + 7 = (3x+a)(x+b)$ and check the pairs of values for a and b such that $ab = 7$: $a = 1$ and $b = 7$, $a = 7$ and $b = 1$, $a = -1$ and $b = -7$, and $a = -7$ and $b = -1$. It turns out that

$$3x^2 - 22x + 7 = (3x-1)(x-7)$$

(e) There are two cases: $6x^2 + 7x + 2 = (6x+a)(x+b)$ and $6x^2 + 7x + 2 = (3x+a)(2x+b)$. Because $ab = 2$, possible values are $a = 1$ and $b = 2$ (i.e., $(6x+1)(x+2)$ and $(3x+1)(2x+2)$), $a = 2$ and $b = 1$ (i.e., $(6x+2)(x+1)$ and $(3x+2)(2x+1)$), $a = -1$ and $b = -2$ (i.e., $(6x-1)(x-2)$ and $(3x-1)(2x-2)$), and $a = -2$ and $b = -1$ (i.e., $(6x-2)(x-1)$ and $(3x-2)(2x-1)$). It turns out that

$$6x^2 + 7x + 2 = (3x+2)(2x+1)$$

(f) There are two cases: $6x^2 - x - 2 = (6x + a)(x + b)$ and $6x^2 - x - 2 = (3x + a)(2x + b)$. Because $ab = -2$, possible values are $a = 1$ and $b = -2$ (i.e., $(6x + 1)(x - 2)$ and $(3x + 1)(2x - 2)$), $a = -2$ and $b = 1$ (i.e., $(6x - 2)(x + 1)$ and $(3x - 2)(2x + 1)$), $a = -1$ and $b = 2$ (i.e., $(6x - 1)(x + 2)$ and $(3x - 1)(2x + 2)$), and $a = 2$ and $b = -1$ (i.e., $(6x + 2)(x - 1)$ and $(3x + 2)(2x - 1)$). We find that

$$6x^2 - x - 2 = (2x + 1)(3x - 2)$$

(g) There are two cases: $4x^2 - 4x - 3 = (4x + a)(x + b)$ and $4x^2 - 4x - 3 = (2x + a)(2x + b)$. As in (e) and (f), we have check pairs of integers a and b such that $ab = -3$. Sooner or later, we discover that

$$4x^2 - 4x - 3 = (2x + 1)(2x - 3)$$

(h) There are two cases: $8x^2 - 6x + 1 = (8x + a)(x + b)$, $8x^2 - 6x + 1 = (4x + a)(2 + b)$. As in (e) and (f), we have check pairs of integers a and b such that $ab = 1$. We find

$$8x^2 - 6x + 1 = (4x - 1)(2x - 1)$$

11. (a) Write $x^2 - 0.04 = (x)^2 - (0.2)^2$ and use $a^2 - b^2 = (a + b)(a - b)$ with $a = x$ and $b = 0.2$ to get

$$x^2 - 0.04 = (x)^2 - (0.2)^2 = (x - 0.2)(x + 0.2).$$

(b) We write

$$x^2 - \frac{4}{81} = x^2 - \left(\frac{2}{9}\right)^2 = \left(x - \frac{2}{9}\right)\left(x + \frac{2}{9}\right)$$

(c) Even though the numbers might not look like it, they are squares:

$$2x^2 - \frac{3}{5} = \left(\sqrt{2}\,x\right)^2 - \left(\frac{\sqrt{3}}{\sqrt{5}}\right)^2 = \left(\sqrt{2}\,x - \frac{\sqrt{3}}{\sqrt{5}}\right)\left(\sqrt{2}\,x + \frac{\sqrt{3}}{\sqrt{5}}\right)$$

(d) Factor 5 out first:

$$5x^2 - 125 = 5(x^2 - 25) = 5\left((x)^2 - (5)^2\right) = 5(x - 5)(x + 5)$$

(e) Factor 5 out first:

$$5x^2 - 55 = 5(x^2 - 11) = 5\left((x)^2 - (\sqrt{11})^2\right) = 5(x - \sqrt{11})(x + \sqrt{11})$$

(f) $a^2 - b^2c^4 = (a)^2 - \left(bc^2\right)^2 = \left(a - bc^2\right)\left(a + bc^2\right)$

(g) Start with $x^4 - 2 = (x^2)^2 - (\sqrt{2})^2$ to get

$$x^4 - 2 = (x^2 - \sqrt{2})(x^2 + \sqrt{2}).$$

Now $x^2 + \sqrt{2}$ cannot be factored any further, but $x^2 - \sqrt{2}$ can:

$$x^2 - \sqrt{2} = (x)^2 - (\sqrt[4]{2})^2 = (x - \sqrt[4]{2})(x + \sqrt[4]{2}).$$

Thus

$$x^4 - 2 = (x^2 - \sqrt{2})(x^2 + \sqrt{2}) = (x - \sqrt[4]{2})(x + \sqrt[4]{2})(x^2 + \sqrt{2}).$$

(h) Recall that $x = \left(\sqrt{x}\right)^2$. Thus

$$x - 16 = \left(\sqrt{x}\right)^2 - (4)^2 = \left(\sqrt{x} - 4\right)\left(\sqrt{x} + 4\right)$$

(i) Write $2x = \left(\sqrt{2}x\right)^2$ and $5 = \sqrt{5}$. Thus

$$2x - 5 = \left(\sqrt{2x}\right)^2 - \left(\sqrt{5}\right)^2 = \left(\sqrt{2x} + \sqrt{5}\right)\left(\sqrt{2x} - \sqrt{5}\right)$$

(j) Using $a^2 - b^2 = (a+b)(a-b)$ with $a = x+1$ and $b = y-1$, we obtain

$$(x+1)^2 - (y-1)^2 = [(x+1) - (y-1)]\,[(x+1) + (y-1)] = (x-y+2)(x+y)$$

12. (a) Using the sum of the cubes formula $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ with $a = x$ and $b = 3$, we obtain

$$x^3 + 27 = (x)^3 + (3)^3 = (x+3)(x^2 - 3x + 3^2) = (x+3)(x^2 - 3x + 9)$$

(b) Using the sum of the cubes $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ with $a = x$ and $b = 4$, we obtain

$$x^3 + 64 = (x)^3 + (4)^3 = (x+4)(x^2 - 4x + 4^2) = (x+4)(x^2 - 4x + 16)$$

(c) Start with a difference of squares

$$x^6 - 1 = (x^3)^2 - (1)^2 = (x^3+1)(x^3-1)$$

Each binomial can further be factored; thus

$$x^6 - 1 = (x^3+1)(x^3-1) = (x+1)(x^2-x+1)(x-1)(x^2+x+1)$$

Alternatively, if we start with a difference of cubes, we get

$$x^6 - 1 = (x^2)^3 - (1)^3 = (x^2-1)((x^2)^2 + x^2 + 1^2) = (x+1)(x-1)(x^4+x^2+1)$$

It is not obvious how to factor $x^4 + x^2 + 1$. It involves a trick – we add and subtract x^2:

$$\begin{aligned} x^4 + x^2 + 1 &= x^4 + x^2 + 1 + (x^2 - x^2) = (x^4 + x^2 + 1 + x^2) - x^2 \\ &= (x^4 + 2x^2 + 1) - x^2 = (x^2+1)^2 - x^2 \end{aligned}$$

Thus is now the difference of squares, so

$$(x^2+1)^2 - x^2 = (x^2+1+x)(x^2+1-x)$$

(and our answer agrees with the one we obtained earlier).

(d) We write

$$\begin{aligned} x + 1 &= \left(x^{1/3}\right)^3 + (1)^3 \\ &= \left(x^{1/3} + 1\right)\left(\left(x^{1/3}\right)^2 - x^{1/3}(1) + (1)^2\right) = \left(x^{1/3}+1\right)\left(x^{2/3} - x^{1/3} + 1\right) \end{aligned}$$

(e) Write $x^3 + y^3z^6 = (x)^3 + (yz^2)^3$ and use the sum of the cubes formula $a^3+b^3 = (a+b)(a^2-ab+b^2)$ with $a = x$ and $b = yz^2$:

$$x^3 + y^3z^6 = (x)^3 + (yz^2)^3 = (x+yz^2)((x)^2 - (x)(yz^2) + (yz^2)^2) = (x+yz^2)(x^2 - xyz^2 + y^2z^4)$$

(f) Factor out ab^4 first:

$$a^4b^4 - ab^7 = ab^4\left(a^3 - b^3\right) = ab^4(a-b)(a^2+ab+b^2)$$

13. (a) We group the terms and then factor:

$$2x^2y + y^2 - 4x^2 - 2y = (2x^2y + y^2) - (4x^2 + 2y) = y(2x^2+y) - 2(2x^2+y) = (2x^2+y)(y-2).$$

Alternatively, we group the first and the third terms and the second and the fourth terms:

$$2x^2y + y^2 - 4x^2 - 2y = (2x^2y - 4x^2) + (y^2 - 2y) = 2x^2(y-2) + y(y-2) = (y-2)(2x^2 + y).$$

(b) Factor out x^2 and then use the difference of squares:

$$x^4 - x^2 = x^2(x^2 - 1) = x^2(x-1)(x+1).$$

(c) Factor 3 out: $27y^2 - 75x^2 = 3(9y^2 - 25x^2)$ and use the difference of squares:

$$3(9y^2 - 25x^2) = 3\left((3y)^2 - (5x)^2\right) = 3(3y - 5x)(3y + 5x).$$

(d) Using the difference of squares twice, we get

$$x^5 - x = x(x^4 - 1) = x(x^2 - 1)(x^2 + 1) = x(x-1)(x+1)(x^2+1).$$

(e) $8x^2 - 50 = 2(4x^2 - 25) = 2((2x)^2 - (5)^2) = 2(2x-5)(2x+5)$

(f) Again, it's the difference of squares:

$$\frac{x^2}{16} - \frac{y^2}{49} = \left(\frac{x}{4}\right)^2 - \left(\frac{y}{7}\right)^2 = \left(\frac{x}{4} - \frac{y}{7}\right)\left(\frac{x}{4} + \frac{y}{7}\right)$$

(g) $4x^2 - 12 = 4(x^2 - 3) = 4(x - \sqrt{3})(x + \sqrt{3})$.

(h) We factor the given expression in the form $4x^2 - 11x + 6 = (4x + a)(x + b)$ and $4x^2 - 11x + 6 = (2x + a)(2x + b)$ and try combinations of a and b for which $ab = 6$. We get

$$4x^2 - 11x + 6 = (4x - 3)(x - 2).$$

(i) We factor $3x^2 - 8x - 3 = (3x + a)(x + b)$ and check integer values of a and b for which $ab = -3$. We get

$$3x^2 - 8x - 3 = (3x + 1)(x - 3)$$

(j) We hope to factor the given expression in the form $x^2 + 3x - 10 = (x + a)(x + b)$; thus, we need to find two numbers whose sum is 3 and product is -10. We find that 5 and -2 satisfy these requirements, and so

$$x^2 + 3x - 10 = (x - 2)(x + 5).$$

(k) Start with $2x^2 + 7x - 4 = (2x + a)(x + b)$ and try combinations of a and b whose product is -4.

Try $a = 1$, $b = -4$: $(2x+1)(x-4) = 2x^2 - 8x + x - 4 = 2x^2 - 7x - 4$, which does not work. Choosing $a = -1$ and $b = 4$, we get $(2x-1)(x+4) = 2x^2 + 8x - x - 4 = 2x^2 + 7x - 4$ – works!

Had it not worked, we would have tried $a = 2$, $b = -2$ or $a = -2$, $b = 2$, etc.

(l) Start by factoring 2 out: $2x^2 + 2x - 24 = 2(x^2 + x - 12)$. Now, we are looking for two numbers whose sum is 1 and whose product is -12. The numbers 4 and -3 work, and so

$$2x^2 + 2x - 24 = 2(x^2 + x - 12) = 2(x-3)(x+4).$$

(m) Factor x out and then use the same strategy as in (a) or (c):

$$x^3 - 2x^2 - 15x = x(x^2 - 2x - 15) = x(x-3)(x+5).$$

(n) Because $x^4 - 16 = (x^2)^2 - (4)^2$, we use the difference of squares

$$x^4 - 16 = (x^2 - 4)(x^2 + 4).$$

The first factor is the difference of squares again, and thus

$$x^4 - 16 = (x-2)(x+2)(x^2+4).$$

(o) Combine the first two and the last two terms:

$$a^2b - ac + ab^2c - bc^2 = (a^2b - ac) + (ab^2c - bc^2) = a(ab-c) + bc(ab-c) = (a+bc)(ab-c)$$

14. (a) $4x^3y^{\frac{1}{2}} - 2xy^{\frac{3}{2}} = 2xy^{\frac{1}{2}}(2x^2 - y)$

(b) Note that each term contains a^x:

$$a^{x+4} - a^x + a^{x+1} = a^x a^4 - a^x + a^x a^1 = a^x(a^4 - 1 + a)$$

(c) $xyz - 3x^2z^3 + xyz^2 = xz(y - 3xz^2 + yz)$

(d) Factor out $x + 2y$:

$$a(x+2y) - c(x+2y) = (a-c)(x+2y)$$

(e) Rewrite the last two terms to identify a common factor:

$$3x(a-b) - a + b = 3x(a-b) - (a-b) = (3x-1)(a-b)$$

(f) Combine the first two and the last two terms:

$$ac + bc - ad - bd = c(a+b) - d(a+b) = (a+b)(c-d)$$

(g) Combine the first two and the last two terms:

$$x^2y + y - 4x^2 - 4 = y(x^2+1) - 4(x^2+1) = (y-4)(x^2+1)$$

(h) Rewrite $-(x-y)$ in the second term as $y - x$:

$$3a(y-x) - ab(x-y) = 3a(y-x) + ab(y-x) = (3a+ab)(y-x) = a(3+b)(y-x)$$

Alternatively, factor out $a(x-y)$:

$$3a(y-x) - ab(x-y) = a(x-y)[-3-b] = a(x-y)[(-1)(3+b)] = -a(x-y)(3+b)$$

(i) Combine the first two and the last two terms:

$$ab - 2xb - 2xa + 4x^2 = b(a-2x) - 2x(a-2x) = (b-2x)(a-2x)$$

(j) Difference of squares:

$$a^2 - 9c^2 = (a)^2 - (3c)^2 = (a-3c)(a+3c)$$

(k) Difference of squares:

$$1 - 49y^2 = (1)^2 - (7y)^2 = (1-7y)(1+7y)$$

(l) Factor out x^2, then use the difference of squares formula:

$$x^4 - x^2 = x^2(x^2-1) = x^2(x-1)(x+1)$$

(m) Difference of cubes:

$$a^3b^3 - 27 = (ab)^3 - (3)^3 = (ab-3)((ab)^2 + 3ab + 3^2) = (ab-3)(a^2b^2 + 3ab + 9)$$

(n) Difference of squares:

$$1-(x+y)^2 = 1^2-(x+y)^2 = [1-(x+y)][1+(x+y)] = (1-x-y)(1+x+y)$$

(o) Factor out y^2, then use the difference of squares formula:

$$x^4y^2-y^4 = y^2(x^4-y^2) = y^2((x^2)^2-y^2) = y^2(x^2-y)(x^2+y)$$

(p) Group the first two terms using the difference of squares:

$$\begin{aligned} x^2-4-(x+2)^2 &= (x+2)(x-2)-(x+2)(x+2) \\ &= (x+2)(x-2-(x+2)) = (x+2)(-4) = -4(x+2) \end{aligned}$$

(q) Combine the first two and the last two terms:

$$1+x+x^2+x^3 = (1+x)+x^2(1+x) = (1+x)(1+x^2)$$

(r) Factor out a^2b^2 and then use the difference of cubes formula:

$$a^5b^2-a^2b^5 = a^2b^2(a^3-b^3) = a^2b^2(a-b)(a^2+ab+b^2)$$

(s) Difference of squares:

$$(a-b)^2-c^2 = [(a-b)-c][(a-b)+c] = (a-b-c)(a-b+c)$$

(t) Start with the difference of squares formula:

$$\begin{aligned} (2x-y)^2-(x+4y)^2 &= [(2x-y)-(x+4y)]\,[(2x-y)+(x+4y)] \\ &= (2x-y-x-4y)(2x-y+x+4y) \\ &= (x-5y)(3x+3y) \\ &= 3(x-5y)(x+y) \end{aligned}$$

(u) Use the difference of squares and the difference of cubes formulas:

$$\begin{aligned} a^2-b^2-a^3+b^3 &= (a^2-b^2)-(a^3-b^3) \\ &= (a-b)(a+b)-(a-b)(a^2+ab+b^2) \\ &= (a-b)[(a+b)-(b^2+ab+a^2)] \\ &= (a-b)(a+b-b^2-ab-a^2) \end{aligned}$$

15. (a) Cancel by $x+y$:

$$\frac{(2x-y)(x+y)}{4(x+y)} = \frac{2x-y}{4}$$

(b) Cancel by $7x$:

$$\frac{7x}{21x(x^2+y)} = \frac{1}{3(x^2+y)}$$

(c) Cancel by $(2a+b)^2$:

$$\frac{(2a+b)^2}{(2a+b)^3} = \frac{1}{2a+b} = (2a+b)^{-1}$$

(d) The only way to cancel 6 and 3 is to split the fraction:

$$\frac{4x+y+6}{3} = \frac{4x}{3} + \frac{y}{3} + \frac{6}{3} = \frac{4x}{3} + \frac{y}{3} + 2$$

(e) Factor, and then cancel:

$$\frac{x^2-9}{(x-3)^2} = \frac{(x+3)(x-3)}{(x+3)^2} = \frac{x-3}{x+3}$$

(f) Factor, and then cancel:

$$\frac{4-x^2}{x-2} = \frac{(2-x)(2+x)}{x-2} = \frac{-(x-2)(2+x)}{x-2} = -(2+x)$$

Note that we factored out the minus sign: $2 - x = -(-2 + x) = -(x - 2)$.

(g) Factor, and then cancel:

$$\frac{(x+y)^2}{6x+6y} = \frac{(x+y)^2}{6(x+y)} = \frac{x+y}{6}$$

(h) Cancel by $(x + 3y)^3$:

$$\frac{(x-y)^2(x+3y)^3}{(x+3y)^4} = \frac{(x-y)^2}{x+3y}$$

(i) Factor using the techniques we have covered:

$$\frac{x^2-4}{x^2+5x+6} = \frac{(x-2)(x+2)}{(x+2)(x+3)} = \frac{x-2}{x+3}$$

(j) Factor, and then cancel by $x - 2$:

$$\frac{(x^2-4)^2}{x^2-5x+6} = \frac{(x^2-4)(x^2-4)}{(x-3)(x-2)} = \frac{(x-2)(x+2)(x-2)(x+2)}{(x-3)(x-2)} = \frac{(x-2)(x+2)(x+2)}{x-3}$$

(k) Factor and then cancel:

$$\frac{x^2-a^2}{x^3-a^3} = \frac{(x-a)(x+a)}{(x-a)(x^2+xa+a^2)} = \frac{x+a}{x^2+xa+a^2}$$

(l) We can factor the denominator, but there is nothing to cancel:

$$\frac{x^2+a^2}{x^3+a^3} = \frac{x^2+a^2}{(x+a)(x^2-xa+a^2)}$$

(m) Factor out -3 and cancel:

$$\frac{x-y}{3y-3x} = \frac{x-y}{-3(x-y)} = \frac{1}{-3} = -\frac{1}{3}$$

(n) Factor the difference of squares in the denominator and cancel by $c - d$:

$$\frac{(c-d)^2}{d^2-c^2} = \frac{(c-d)^2}{(d-c)(d+c)} = \frac{(c-d)^2}{-(c-d)(d+c)} = \frac{c-d}{-(d+c)} = -\frac{c-d}{c+d}$$

In order to cancel, we factored out the minus sign: $d - c = -(-d + c) = -(c - d)$.

(o) Factor the numerator using the difference of squares, and factor out the minus sign in the denominator:

$$-\frac{x^2-100}{10-x} = -\frac{(x-10)(x+10)}{-(x-10)} = x+10$$

Alternatively,

$$-\frac{x^2-100}{10-x}=\frac{-(x^2-100)}{10-x}=\frac{100-x^2}{10-x}=\frac{(10-x)(10+x)}{10-x}=10+x$$

(p) Factor and cancel:

$$\frac{x^2-x}{x-1}=\frac{x(x-1)}{x-1}=x$$

(q) Factor and cancel:

$$\frac{x^3-x}{(1-x)^2}=\frac{x(x^2-1)}{(1-x)(1-x)}=\frac{x(x-1)(x+1)}{-(x-1)(1-x)}=-\frac{x(x+1)}{1-x}$$

(r) Factor out the minus sign in the denominator (no need to factor anything else):

$$\frac{(a-b)(c^2-d^2)}{d^2-c^2}=\frac{(a-b)(c^2-d^2)}{-(c^2-d^2)}=-(a-b)$$

(s) Rearrange the numerator and then factor:

$$\frac{64-a^3}{a^2-16}=\frac{-(a^3-64)}{a^2-16}=-\frac{a^3-4^3}{a^2-16}=-\frac{(a-4)(a^2+4a+16)}{(a-4)(a+4)}=-\frac{a^2+4a+16}{a+4}$$

(t) Factor out the minus signs and then cancel:

$$\frac{(x-y)(z-t)^3}{(y-x)(t-z)^2}==\frac{(x-y)\left[-(t-z)^3\right]}{\left[-(x-y)\right](t-z)^2}=\frac{-(x-y)(t-z)^3}{-(x-y)(t-z)^2}=(t-z)$$

Note that $(z-t)^3=[-(t-z)]^3=-(t-z)^3$.

16. (a) Factor out $5x$ in the numerator and then cancel:

$$\frac{15xy-55x^2}{20x}=\frac{5x(3y-11x)}{20x}=\frac{3y-11x}{4}$$

(b) Factor and then cancel by x:

$$\frac{x^3y-9xy}{x^3y+3x^3}=\frac{xy(x^2-9)}{x^3(y+3)}=\frac{xy(x-3)(x+3)}{x^3(y+3)}=\frac{y(x-3)(x+3)}{x^2(y+3)}$$

(c) Factor by grouping and then cancel:

$$\frac{ax+bx-ay-by}{ax+bx+ay+by}=\frac{x(a+b)-y(a+b)}{x(a+b)+y(a+b)}=\frac{(x-y)(a+b)}{(x+y)(a+b)}=\frac{x-y}{x+y}$$

(d) $\dfrac{a^3b+ab^2}{a^3b-a^2b^2}=\dfrac{ab(a^2+b)}{a^2b(a-b)}=\dfrac{a^2+b}{a(a-b)}$

(e) Factor $5x$ out of both numerator and denominator:

$$\frac{15xy-55x^2}{20x-15x^3}=\frac{5x(3y-11x)}{5x(4-3x^2)}=\frac{3y-11x}{4-3x^2}$$

(f) Keep in mind that x^2+y^2 cannot be factored:

$$\frac{x^4+y^2x^2}{x^2y+xy^2}=\frac{x^2(x^2+y^2)}{xy(x+y)}=\frac{x(x^2+y^2)}{y(x+y)}$$

(g) $\dfrac{3xy-x^2y}{9y-yx^2}=\dfrac{xy(3-x)}{y(9-x^2)}=\dfrac{xy(3-x)}{y(3-x)(3+x)}=\dfrac{x}{3+x}$

(h) Factor and cancel:

$$\frac{2ax+2bx-ay-by}{4xa^2-2ya^2}=\frac{2x(a+b)-y(a+b)}{2a^2(2x-y)}=\frac{(2x-y)(a+b)}{2a^2(2x-y)}=\frac{a+b}{2a^2}$$

17. (a) Factor and cancel by a:

$$\frac{a^3-a^5+a}{a^2}=\frac{a(a^2-a^4+1)}{a^2}=\frac{a^2-a^4+1}{a}$$

(b) Keep in mind that $1+a^2$ cannot be factored.

$$\frac{a^3+a^6}{a+a^4}=\frac{a^3(1+a^2)}{a(1+a^3)}=\frac{a^2(1+a^2)}{1+a^3}$$

(c) Factor and cancel:

$$\frac{a^2+a^3}{b+ab}=\frac{a^2(1+a)}{b(1+a)}=\frac{a^2}{b}$$

(d) Factor and cancel:

$$\frac{a^2b^3+ab^4}{a^2b^2+a^3b}=\frac{ab^3(a+b)}{a^2b(b+a)}=\frac{b^2}{a}$$

(e) Factor a^x in the numerator:

$$\frac{a^x+a^{x+2}}{a^{x+1}}=\frac{a^x+a^xa^2}{a^xa}=\frac{a^x(1+a^2)}{a^xa}=\frac{1+a^2}{a}$$

(f) To cancel with the denominator, factor out a^{x-1} in the numerator:

$$\frac{a^x+a^{x+2}}{a^{x-1}}=\frac{a^{x-1}(a+a^3)}{a^{x-1}}=a+a^3=a(1+a^2)$$

Alternatively:

$$\frac{a^x+a^{x+2}}{a^{x-1}}=\frac{a^x(1+a^2)}{a^xa^{-1}}=\frac{1+a^2}{a^{-1}}=(1+a^2)a$$

(g) Factor and cancel:

$$\frac{a^x+a^{x+1}}{a^y+a^{y+1}}=\frac{a^x+a^xa}{a^y+a^ya}=\frac{a^x(1+a)}{a^y(1+a)}=\frac{a^x}{a^y}$$

(h) Factor a^x in the denominator:

$$\frac{a^x}{a^x-a^{x+1}}=\frac{a^x}{a^x-a^xa}=\frac{a^x}{a^x(1-a)}=\frac{1}{1-a}$$

18. (a) The fraction is not defined for $x=0$ (that's why we say $x>0$ and not $x\geq 0$):

$$\frac{|x|}{x}=\begin{cases}\dfrac{x}{x} & \text{if } x>0\\ \dfrac{-x}{x} & \text{if } x<0\end{cases}=\begin{cases}1 & \text{if } x>0\\ -1 & \text{if } x<0\end{cases}$$

(b) In the first line, we say $x-2>0$ to exclude $x=2$.

$$\frac{|x-2|}{x-2}=\begin{cases}\dfrac{(x-2)}{x-2} & \text{if } x-2>0\\ \dfrac{-(x-2)}{x-2} & \text{if } x-2<0\end{cases}=\begin{cases}1 & \text{if } x>2\\ -1 & \text{if } x<2\end{cases}$$

(c) We exclude the value $x = 5/3$ for which the denominator is zero. To cancel, rewrite the denominator as $5 - 3x = -(-5 + 3x) = -(3x - 5)$:

$$\frac{|3x-5|}{5-3x} = \begin{cases} \dfrac{3x-5}{5-3x} & \text{if } 3x-5>0 \\ \dfrac{-(3x-5)}{5-3x} & \text{if } 3x-5<0 \end{cases} = \begin{cases} \dfrac{3x-5}{-(3x-5)} = -1 & \text{if } x > \frac{5}{3} \\ \dfrac{-(3x-5)}{-(3x-5)} = 1 & \text{if } x < \frac{5}{3} \end{cases}$$

(d) As in (c),

$$\frac{|1-2x|}{2x-1} = \begin{cases} \dfrac{1-2x}{-(1-2x)} & \text{if } 1-2x>0 \\ \dfrac{-(1-2x)}{-(1-2x)} & \text{if } 1-2x<0 \end{cases} = \begin{cases} -1 & \text{if } x < \frac{1}{2} \\ 1 & \text{if } x > \frac{1}{2} \end{cases}$$

19. (a) The common denominator of $x - 2$ and $x - 4$ is their product; therefore

$$\begin{aligned} \frac{x}{x-2} - \frac{2x-1}{x-4} &= \frac{x(x-4)}{(x-2)(x-4)} - \frac{(2x-1)(x-2)}{(x-2)(x-4)} \\ &= \frac{x^2-4x-(2x^2-x-4x+2)}{(x-2)(x-4)} = \frac{-x^2+x-2}{(x-2)(x-4)} \end{aligned}$$

(b) The least common multiple of x, $x - 1$ and $(x - 1)^2$ is $x(x - 1)^2$. Thus

$$\begin{aligned} \frac{3}{x} - \frac{2x}{x-1} + \frac{4}{(x-1)^2} &= \frac{3(x-1)^2}{x(x-1)^2} - \frac{2x(x)(x-1)}{x(x-1)^2} + \frac{4x}{x(x-1)^2} \\ &= \frac{3(x^2-2x+1)-2x^2(x-1)+4x}{x(x-1)^2} \\ &= \frac{3x^2-6x+3-2x^3+2x^2+4x}{x(x-1)^2} \\ &= \frac{-2x^3+5x^2-2x+3}{x(x-1)^2} \end{aligned}$$

(c) Since $x^2 - 5x + 6 = (x - 2)(x - 3)$ we see that the common denominator is $(x - 2)(x - 3)$:

$$\begin{aligned} \frac{2x+1}{x^2-5x+6} - \frac{1}{x-2} + \frac{x}{x-3} &= \frac{2x+1}{(x-2)(x-3)} - \frac{x-3}{(x-2)(x-3)} + \frac{x(x-2)}{(x-2)(x-3)} \\ &= \frac{2x+1-x+3+x^2-2x}{(x-2)(x-3)} \\ &= \frac{x^2-x+4}{(x-2)(x-3)} \end{aligned}$$

20. (a) The common denominator is $(x - y)(x + y)$:

$$\begin{aligned} \frac{1}{x-y} + \frac{2}{x+y} &= \frac{1(x+y)}{(x-y)(x+y)} + \frac{2(x-y)}{(x+y)(x-y)} \\ &= \frac{(x+y)+2(x-y)}{(x+y)(x-y)} \\ &= \frac{x+y+2x-2y}{(x+y)(x-y)} = \frac{3x-y}{(x+y)(x-y)} \end{aligned}$$

(b) Since $ab + b^2 = b(a+b)$, the common denominator is $b(a+b)$:

$$\begin{aligned}\frac{2a+b}{b} - \frac{1}{a+b} + \frac{b}{ab+b^2} &= \frac{(2a+b)(a+b)}{b(a+b)} - \frac{1(b)}{(a+b)(b)} + \frac{b}{b(a+b)} \\ &= \frac{(2a+b)(a+b) - b + b}{b(a+b)} \\ &= \frac{(2a+b)(a+b)}{b(a+b)} = \frac{2a+b}{b}\end{aligned}$$

(c) Since $a^2 - 4b^2 = (a+2b)(a-2b)$, the common denominator is $(a+2b)(a-2b)$:

$$\begin{aligned}\frac{2}{a+2b} - \frac{3}{a-2b} - \frac{1}{a^2-4b^2} &= \frac{2(a-2b)}{(a+2b)(a-2b)} - \frac{3(a+2b)}{(a-2b)(a+2b)} - \frac{1}{(a-2b)(a+2b)} \\ &= \frac{2(a-2b) - 3(a+2b) - 1}{(a-2b)(a+2b)} \\ &= \frac{2a - 4b - 3a - 6b - 1}{(a-2b)(a+2b)} \\ &= \frac{-a - 10b - 1}{(a-2b)(a+2b)} \\ &= -\frac{a + 10b + 1}{(a-2b)(a+2b)}\end{aligned}$$

(d) The common denominator is xyz:

$$\begin{aligned}\frac{x^2 - y + z}{xy} - \frac{x+1}{yz} + \frac{y}{xz} &= \frac{(x^2 - y + z)z}{xy(z)} - \frac{(x+1)(x)}{yz(x)} + \frac{y(y)}{xz(y)} \\ &= \frac{(x^2 - y + z)z - (x+1)(x) + y(y)}{xyz} \\ &= \frac{x^2z - yz + z^2 - x^2 - x + y^2}{xyz}\end{aligned}$$

(e) The common denominator is $(x-y)(x+y)$:

$$\begin{aligned}\frac{x+y}{x-y} + \frac{x-y}{x+y} &= \frac{(x+y)(x+y)}{(x-y)(x+y)} + \frac{(x-y)(x-y)}{(x+y)(x-y)} \\ &= \frac{(x+y)(x+y) + (x-y)(x-y)}{x^2 - y^2} \\ &= \frac{x^2 + 2xy + y^2 + x^2 - 2xy + y^2}{x^2 - y^2} \\ &= \frac{2x^2 + 2y^2}{x^2 - y^2} = \frac{2(x^2 + y^2)}{x^2 - y^2}\end{aligned}$$

(f) Factor the denominators to figure out the common denominator:

$$\begin{aligned}\frac{2}{x^2-4} - \frac{3}{(x+2)^2} - \frac{1}{2x-4} &= \frac{2}{(x-2)(x+2)} - \frac{3}{(x+2)^2} - \frac{1}{2(x-2)} \\ &= \frac{2(x+2)(2)}{(x-2)(x+2)^2(2)} - \frac{3(x-2)(2)}{(x+2)^2(x-2)(2)} - \frac{1(x+2)^2}{2(x-2)(x+2)^2} \\ &= \frac{4(x+2) - 6(x-2) - (x+2)^2}{2(x-2)(x+2)^2}\end{aligned}$$

$$= \frac{4x+8-6x+12-x^2-4x-4}{2(x-2)(x+2)^2}$$
$$= \frac{-x^2-6x+16}{2(x-2)(x+2)^2} = \frac{-(x^2+6x-16)}{2(x-2)(x+2)^2}$$
$$= \frac{-(x+8)(x-2)}{2(x-2)(x+2)^2} = \frac{-(x+8)}{2(x+2)^2}$$

(g) Because $b^2+2b-8=(b+4)(b-2)$, the common denominator is $(b+4)(b-2)$:

$$\frac{1}{2}+\frac{2}{b-2}-\frac{1}{b+4}+\frac{b}{b^2+2b-8} = \frac{1}{2}+\frac{2}{b-2}-\frac{1}{b+4}+\frac{b}{(b+4)(b-2)}$$
$$= \frac{1(b+4)(b-2)}{2(b+4)(b-2)}+\frac{2(b+4)(2)}{(b-2)(b+4)(2)}-\frac{1(b-2)(2)}{(b+4)(b-2)(2)}+\frac{b(2)}{(b+4)(b-2)(2)}$$
$$= \frac{(b+4)(b-2)+4(b+4)-2(b-2)+2b}{2(b+4)(b-2)}$$
$$= \frac{b^2+2b-8+4b+16-2b+4+2b}{2(b+4)(b-2)}$$
$$= \frac{b^2+6b+12}{2(b+4)(b-2)}$$

(h) Note that because $y-x=-(x-y)$ the first two fractions add up to zero. Thus

$$\frac{1}{y-x}+\frac{1}{x-y}-\frac{1}{(x-y)^2}+\frac{3}{(x-y)^3} = -\frac{1}{x-y}+\frac{1}{x-y}-\frac{1}{(x-y)^2}+\frac{3}{(x-y)^3}$$
$$= -\frac{1}{(x-y)^2}+\frac{3}{(x-y)^3}$$
$$= -\frac{x-y}{(x-y)^3}+\frac{3}{(x-y)^3}$$
$$= \frac{-(x-y)+3}{(x-y)^3} = \frac{-x+y+3}{(x-y)^3}$$

(i) The common denominator is x^m; thus

$$\frac{1}{x}-\frac{1}{x^m} = \frac{x^{m-1}}{x^m}-\frac{1}{x^m} = \frac{x^{m-1}-1}{x^m}$$

(j) The common denominator is a^{x+1}; thus

$$\frac{2}{a^x}+\frac{1}{a^{x+1}} = \frac{2(a)}{a^x(a)}+\frac{1}{a^{x+1}} = \frac{2a}{a^{x+1}}+\frac{1}{a^{x+1}} = \frac{2a+1}{a^{x+1}}$$

(k) The common denominator is a^{x+1}; thus

$$\frac{2}{a^x}-\frac{1}{a^{x+1}}-\frac{1}{a^{x-2}} = \frac{2(a)}{a^x(a)}-\frac{1}{a^{x+1}}-\frac{1(a^3)}{a^{x-2}(a^3)} = \frac{2a}{a^{x+1}}-\frac{1}{a^{x+1}}-\frac{a^3}{a^{x+1}} = \frac{2a-1-a^3}{a^{x+1}}$$

(l) Factoring using the difference of cubes and the difference of squares formulas and then canceling the fraction, we get

$$\frac{x^3-27}{x^2-9} = \frac{(x)^3-(3)^3}{(x)^2-(3)^2} = \frac{(x-3)(x^2+3x+9)}{(x-3)(x+3)} = \frac{x^2+3x+9}{x+3}$$

(m) Factor, and then cancel:

$$\frac{x^4 - x^2}{2x^2 + 3x + 1} = \frac{x^2(x^2 - 1)}{(2x + 1)(x + 1)} = \frac{x^2(x - 1)(x + 1)}{(2x + 1)(x + 1)} = \frac{x^2(x - 1)}{2x + 1}$$

In the denominator, we factored by writing $2x^2 + 3x + 1 = (2x + a)(x + b)$ and then checking integer values of a and b such that $ab = 1$.

(n) Factor and cancel first:

$$(ab + b^2)\frac{a - b}{a + b} + b^2 = b(a + b)\frac{a - b}{a + b} + b^2 = b(a - b) + b^2 = ab$$

Alternatively, compute the common denominator:

$$\begin{aligned}(ab + b^2)\frac{a - b}{a + b} + b^2 &= (ab + b^2)\frac{a - b}{a + b} + b^2\frac{a + b}{a + b} \\ &= \frac{(ab + b^2)(a - b) + b^2(a + b)}{a + b} \\ &= \frac{a^2b + ab^2 - ab^2 - b^3 + ab^2 + b^3}{a + b} \\ &= \frac{a^2b + ab^2}{a + b} = \frac{ab(a + b)}{a + b} = ab\end{aligned}$$

(o) Factor using the difference of squares formula, and then cancel:

$$\frac{x - y}{a^2 - b^2}\,\frac{(a + b)^2}{x^2 - y^2} = \frac{x - y}{(a - b)(a + b)}\,\frac{(a + b)(a + b)}{(x - y)(x + y)} = \frac{a + b}{(a - b)(x + y)}$$

(p) Factor and cancel:

$$\frac{2x - 4}{x^3 + x}\,\frac{x}{3x - 6} = \frac{2(x - 2)}{x(x^2 + 1)}\,\frac{x}{3(x - 2)} = \frac{2}{3(x^2 + 1)}$$

(q) Simplify the terms in the parentheses first:

$$\left(\frac{1}{a} - \frac{1}{b}\right)(a + b) = \frac{b - a}{ab}(a + b) = \frac{b^2 - a^2}{ab}$$

(r) Compute the common denominator, factor, and cancel:

$$\left(\frac{1}{a} + \frac{1}{b}\right)\frac{a^3b^3}{b^2 - a^2} = \frac{b + a}{ab}\,\frac{a^3b^3}{(b - a)(b + a)} = \frac{a^2b^2}{b - a}$$

(s) Recall: the top times the reciprocal of the bottom:

$$\frac{\frac{a^2b}{5}}{\frac{ab^3}{15}} = \frac{a^2b}{5}\,\frac{15}{ab^3} = \frac{3a}{b^2}$$

(t) Simplify the numerator first:

$$\frac{\frac{a}{b} - \frac{b}{a}}{a + b} = \frac{\frac{a^2 - b^2}{ab}}{a + b} = \frac{a^2 - b^2}{ab}\,\frac{1}{a + b} = \frac{(a - b)(a + b)}{ab}\,\frac{1}{a + b} = \frac{a - b}{ab}$$

(u) Simplifying the double fraction,

$$\frac{\frac{a^x}{b^{x-1}}}{\frac{a^{x+1}}{b^x}} = \frac{a^x}{b^{x-1}}\,\frac{b^x}{a^{x+1}} = \frac{a^x}{b^x b^{-1}}\,\frac{b^x}{a^x a} = \frac{1}{b^{-1}a} = \frac{b}{a}$$

(v) Start by calculating common denominators:

$$\frac{1-\frac{3}{x}}{1-\frac{6}{x}+\frac{9}{x^2}} = \frac{\frac{x}{x}-\frac{3}{x}}{\frac{x^2}{x^2}-\frac{6x}{x^2}+\frac{9}{x^2}} = \frac{\frac{x-3}{x}}{\frac{x^2-6x+9}{x^2}} = \frac{x-3}{x}\ \frac{x^2}{x^2-6x+9}$$

In the last step, we simplified using the "top times the reciprocal of the bottom" rule. Next, we cancel the fraction

$$\frac{x-3}{x}\ \frac{x^2}{x^2-6x+9} = \frac{x-3}{x}\ \frac{x^2}{(x-3)^2} = \frac{x}{x-3}$$

(w) Start by calculating common denominators, then simplify the double fraction and cancel:

$$\frac{1+\frac{1}{x-1}}{1-\frac{1}{x-1}} = \frac{\frac{x-1}{x-1}+\frac{1}{x-1}}{\frac{x-1}{x-1}-\frac{1}{x-1}} = \frac{\frac{x}{x-1}}{\frac{x-2}{x-1}} = \frac{x}{x-1}\ \frac{x-1}{x-2} = \frac{x}{x-2}$$

21. (a) How do we factor 6 out of $\frac{3x^4}{4}$ (or any other term)? We need to "generate" 6 first, and then factor it out. Multiply and divide by 6, thus not changing the value of the expression:

$$\frac{3x^4}{4} = \frac{6}{6}\cdot\frac{3x^4}{4} = 6\cdot\frac{3x^4}{6\cdot 4} = 6\cdot\frac{3x^4}{24} = 6\cdot\frac{x^4}{8}$$

We do the same with the remaining terms (some are obvious, such as $-6x^2$):

$$\frac{3x^2}{4} - 6x^2 - \frac{x}{2} - 11 = 6\left(\frac{x^4}{8} - x^2 - \frac{x}{12} - \frac{11}{6}\right)$$

(b) We need to identify x^4 in each term (keep in mind the formula $x^{a+b} = x^a x^b$):

$$3x^4 - 6x^2 - 5x - 12 = 3x^4 - 6x^4x^{-2} - 5x^4x^{-3} - 12x^4x^{-4} = x^4(3 - 6x^{-2} - 5x^{-3} - 12x^{-4})$$

We can check our answer by expanding the right side (by multiplying by x^4).

(c) We need to identify x^2 in each term:

$$-x^3 - x^2 + x + 11 = -x^2x^1 - x^2 + x^2x^{-1} + 11x^2x^{-2} = x^2(-x - 1 + x^{-1} + 11x^{-2})$$

(d) We need to identify $x^{1/2}$ in each term (keep in mind the formula $x^{a+b} = x^a x^b$):

$$\begin{aligned}-x^3 + 3x^2 - x - 12 &= -x^{1/2}x^{5/2} + 3x^{1/2}x^{3/2} - x^{1/2}x^{1/2} - 12x^{1/2}x^{-1/2}\\ &= x^{\frac{1}{2}}\left(-x^{5/2} + 3x^{3/2} - x^{1/2} - 12x^{-1/2}\right)\end{aligned}$$

(e) We need to identify $x^{-1} = \frac{1}{x}$ in each term:

$$\begin{aligned}-x^3 - x^2 + x + 11 &= -x^4x^{-1} - x^3x^{-1} + x^2x^{-1} + 11x^1x^{-1}\\ &= x^{-1}\left(-x^4 - x^3 + x^2 + 11x\right) = \frac{1}{x}\left(-x^4 - x^3 + x^2 + 11x\right)\end{aligned}$$

(f) We need to identify $x^{-3/2}$ in each term:

$$\begin{aligned}-x^3 + 3x^2 - x - 12 &= = -x^{-3/2}x^{9/2} + 3x^{-3/2}x^{7/2} - x^{-3/2}x^{5/2} - 12x^{-3/2}x^{3/2}\\ &= x^{-3/2}(-x^{9/2} + 3x^{7/2} - x^{5/2} - 12x^{3/2})\end{aligned}$$

(g) We write

$$\begin{aligned}-2x^{-2/3}(x-5) + 7x^{1/3}(x-4) &= -2x^{1/3}x^{-1}(x-5) + 7x^{1/3}(x-4)\\ &= x^{1/3}\left[-2x^{-1}(x-5) + 7(x-4)\right]\end{aligned}$$

(h) We write

$$\frac{1}{3}x^{-2/3}(x-5)^2 + \frac{4}{5}x^{1/3}(x-5) = \frac{1}{3}x^{-2/3}(x-5)^2 + \frac{4}{5}x^{-2/3}x^1(x-5)$$
$$= x^{-2/3}(x-5)\left[\frac{1}{3}(x-5) + \frac{4}{5}x\right] = x^{-2/3}(x-5)\left[\frac{17}{15}x - \frac{5}{3}\right]$$

(i) As in (h),

$$\frac{1}{3}x^{-2/3}(x-5)^2 + \frac{4}{3}x^{1/3}(x-5) = \frac{1}{3}x^{-2/3}(x-5)^2 + \frac{4}{3}x^{-2/3}x^1(x-5)$$
$$= \frac{x^{-\frac{2}{3}}(x-5)}{3}\left[(x-5)+4x\right] = \frac{x^{-\frac{2}{3}}(x-5)}{3}\left[5(x-1)\right]$$

(j) We need to identify x^{-1} in each term:

$$\frac{1}{3x^3} + \frac{4}{x^2} - \frac{1}{7x} + 2x - 1 = \frac{1}{3}x^{-3} + 4x^{-2} - \frac{1}{7}x^{-1} + 2x - 1$$
$$= \frac{1}{3}x^{-2}x^{-1} + 4x^{-1}x^{-1} - \frac{1}{7}x^{-1} + 2x^2x^{-1} - x^1x^{-1}$$
$$= x^{-1}\left[\frac{1}{3}x^{-2} + 4x^{-1} - \frac{1}{7} + 2x^2 - x\right]$$

(k) We need to identify $x^{-1/2}$ in each term:

$$\frac{1}{3x^3} + \frac{4}{x^2} - \frac{1}{7x} + 2x - 1 = \frac{1}{3}x^{-3} + 4x^{-2} - \frac{1}{7}x^{-1} + 2x - 1$$
$$= \frac{1}{3}x^{-1/2}x^{-5/2} + 4x^{-1/2}x^{-3/2} - \frac{1}{7}x^{-1/2}x^{-1/2} + 2x^{-1/2}x^{3/2} - x^{-1/2}x^{1/2}$$
$$= x^{-1/2}\left[\frac{1}{3}x^{-5/2} + 4x^{-3/2} - \frac{1}{7}x^{-1/2} + 2x^{3/2} - x^{1/2}\right]$$

(l) We need to identify $\sqrt{x} = x^{1/2}$ in each term:

$$x^2 - 3\sqrt{x} - 5 + \frac{4}{x} - \frac{1}{\sqrt{x}} = x^2 - 3x^{1/2} - 5 + 4x^{-1} - x^{-1/2}$$
$$= x^{1/2}x^{3/2} - 3x^{1/2} - 5x^{1/2}x^{-1/2} + 4x^{1/2}x^{-3/2} - x^{1/2}x^{-1}$$
$$= x^{1/2}\left(x^{3/2} - 3 - 5x^{-1/2} + 4x^{-3/2} - x^{-1}\right)$$
$$= \sqrt{x}\left(x\sqrt{x} - 3 - \frac{5}{\sqrt{x}} + \frac{4}{x\sqrt{x}} - \frac{1}{x}\right)$$

Note that $x^{3/2} = x \cdot x^{1/2} = x\sqrt{x}$.

22. (a)

$$\begin{array}{r|l} & x^2 + 4x + 13 \\ \hline x-4 & x^3 - 3x + 5 \\ & -(x^3 - 4x^2) \\ \hline & 4x^2 - 3x \\ & -(4x^2 - 16x) \\ \hline & 13x + 5 \\ & -(13x - 52) \\ \hline & 57 \end{array}$$

Thus, $\dfrac{x^3 - 3x + 5}{x - 4} = x^2 + 4x + 13 + \dfrac{57}{x-4}$.

(b)

$$\begin{array}{r|l} & x+6 \\ \hline x+1 & x^2+7x-2 \\ & -(x^2+x) \\ \hline & 6x-2 \\ & -(6x+6) \\ \hline & -8 \end{array}$$

Thus, $\dfrac{x^2+7x-2}{x+1} = x+6-\dfrac{8}{x+1}$.

(c)

$$\begin{array}{r|l} & 1 \\ \hline x^2+7 & x^2+2 \\ & -(x^2+7) \\ \hline & -5 \end{array}$$

Thus, $\dfrac{x^2+2}{x^2+7} = 1-\dfrac{5}{x^2+7}$

(d)

$$\begin{array}{r|l} & x^2-1 \\ \hline x^2+1 & x^4 \\ & -(x^4+x^2) \\ \hline & -x^2 \\ & -(-x^2-1) \\ \hline & 1 \end{array}$$

Thus, $\dfrac{x^4}{x^2+1} = x^2-1+\dfrac{1}{x^2+1}$

(e)

$$\begin{array}{r|l} & x+3 \\ \hline x+2 & x^2+5x+7 \\ & -(x^2+2x) \\ \hline & 3x+7 \\ & -(3x+6) \\ \hline & 1 \end{array}$$

Thus, $\dfrac{x^2+5x+7}{x+2} = x+3+\dfrac{1}{x+2}$

(f)

$$\begin{array}{r|l} & 1 \\ \hline x-6 & x \\ & -(x-6) \\ \hline & 6 \end{array}$$

Thus, $\dfrac{x}{x-6} = 1+\dfrac{6}{x-6}$

(g)

$$\begin{array}{r|l} & x^2-7x+49 \\ \hline x+7 & x^3 \\ & -(x^3+7x^2) \\ \hline & -7x^2 \\ & -(-7x^2-49x) \\ \hline & 49x \\ & -(49x+343) \\ \hline & -343 \end{array}$$

Thus, $\dfrac{x^3}{x+7} = x^2-7x+49-\dfrac{343}{x+7}$

(h)

$$\begin{array}{r|l} & \quad 1 \\ \hline x^3+1 & \quad x^3 \\ & -(x^3+1) \\ \hline & \quad\quad -1 \end{array}$$

Thus, $\dfrac{x^3}{x^3+1} = 1 - \dfrac{1}{x^3+1}$

23. (a) Subtract and add 6:

$$\frac{x}{x-6} = \frac{x-6+6}{x-6} = \frac{(x-6)+6}{x-6} = \frac{x-6}{x-6} + \frac{6}{x-6} = 1 + \frac{6}{x-6}$$

Check: see Exercise 22(f).

(b) Subtract and add 4:

$$\frac{x^2}{x^2-4} = \frac{x^2-4+4}{x^2-4} = \frac{(x^2-4)+4}{x^2-4} = \frac{x^2-4}{x^2-4} + \frac{4}{x^2-4} = 1 + \frac{4}{x^2-4}$$

Check:

$$\begin{array}{r|l} & \quad 1 \\ \hline x^2-4 & \quad x^2 \\ & -(x^2-4) \\ \hline & \quad\quad 4 \end{array}$$

(c) Subtract and add 15, to get

$$\frac{6x}{2x-5} = \frac{6x-15+15}{2x-5} = \frac{6x-15}{2x-5} + \frac{15}{2x-5} = \frac{3(2x-5)}{2x-5} + \frac{15}{2x-5} = 3 + \frac{15}{2x-5}$$

Check:

$$\begin{array}{r|l} & \quad 3 \\ \hline 2x-5 & \quad 6x \\ & -(6x-15) \\ \hline & \quad\quad 15 \end{array}$$

(d) Add and subtract 2, thus getting

$$\frac{2x^2}{x^2+1} = \frac{2x^2+2-2}{x^2+1} = \frac{2x^2+2}{x^2+1} + \frac{-2}{x^2+1} = \frac{2(x^2+1)}{x^2+1} - \frac{2}{x^2+1} = 2 - \frac{2}{x^2+1}$$

Check:

$$\begin{array}{r|l} & \quad 2 \\ \hline x^2+1 & \quad 2x^2 \\ & -(2x^2+2) \\ \hline & \quad\quad -2 \end{array}$$

(e) Subtract and add 4:

$$\frac{3x-1}{3x-4} = \frac{3x-4+4-1}{3x-4} = \frac{(3x-4)+3}{3x-4} = \frac{3x-4}{3x-4} + \frac{3}{3x-4} = 1 + \frac{3}{3x-4}$$

Check:

$$\begin{array}{r|l} & \quad 1 \\ \hline 3x-4 & \; 3x-1 \\ & -(3x-4) \\ \hline & \qquad 3 \end{array}$$

(f) Write $11 = 5 + 6$:

$$\frac{4x+11}{4x+5} = \frac{4x+5+6}{4x+5} = \frac{(4x+5)+6}{4x+5} = \frac{4x+5}{4x+5} + \frac{6}{4x+5} = 1 + \frac{6}{4x+5}$$

Check:

$$\begin{array}{r|l} & \quad 1 \\ \hline 4x+5 & \; 4x+11 \\ & -(4x+5) \\ \hline & \qquad 6 \end{array}$$

24. We try to identify a root of $x^3 + 2x^2 - 11x - 12$ by guessing. Recall that, the root must divide the free coefficient -12. Substituting $x = 1$, we get $(1)^3 + 2(1)^2 - 11(1) - 12 = -20$, so 1 is not a root. Substituting $x = -1$, we get $(-1)^3 + 2(-1)^2 - 11(-1) - 12 = 0$! So, $x = 1$ is a root of the given polynomial, i.e., $x + 1$ is its factor.

Thus

$$x^3 + 2x^2 - 11x - 12 = (x+1)q(x)$$

where the unknown polynomial $q(x)$ is identified using long division:

$$\begin{array}{r|l} & x^2 + x - 12 \\ \hline x+1 & x^3 + 2x^2 - 11x - 12 \\ & -(x^3 + x^2) \\ \hline & \qquad x^2 - 11x \\ & \quad -(x^2 + x) \\ \hline & \qquad\quad -12x - 12 \\ & \qquad -(-12x - 12) \\ \hline & \qquad\qquad\qquad 0 \end{array}$$

We write

$$\frac{x^3 + 2x^2 - 11x - 12}{x+1} = x^2 + x - 12$$

i.e.,

$$x^3 + 2x^2 - 11x - 12 = (x+1)(x^2 + x - 12).$$

Now we factor $x^2 + x - 12 = (x+4)(x-3)$ and so

$$x^3 + x^2 - 10x + 8 = (x+1)(x+4)(x-3)$$

Of course, we could have initially guessed some other root ($x = -4$ or $x = 3$). There would be no change, however – we follow the same procedure and arrive at the same factorization.

25. (a) Multiplying and dividing by $\sqrt{7}+3$, we get

$$\frac{4}{\sqrt{7}-3}=\frac{4}{\sqrt{7}-3}\,\frac{\sqrt{7}+3}{\sqrt{7}+3}=\frac{4(\sqrt{7}+3)}{(\sqrt{7})^2-(3)^2}=\frac{4\sqrt{7}+12}{-2}=-2\sqrt{7}-6$$

(b) Multiply and divide by the conjugate of the denominator:

$$\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}=\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}\cdot\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}=\frac{5-\sqrt{5}\sqrt{3}-\sqrt{3}\sqrt{5}+3}{(\sqrt{5})^2-(\sqrt{3})^2}=\frac{8-2\sqrt{15}}{2}=4-\sqrt{15}$$

(c) Multiply and divide by the conjugate of the denominator:

$$\frac{2}{\sqrt{x}-1}=\frac{2}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}=\frac{2(\sqrt{x}+1)}{\left(\sqrt{x}\right)^2-(1)^2}=\frac{2(\sqrt{x}+1)}{x-1}$$

(d) As above,

$$\frac{4-\sqrt{x}}{2+\sqrt{x}}=\frac{4-\sqrt{x}}{2+\sqrt{x}}\,\frac{2-\sqrt{x}}{2-\sqrt{x}}=\frac{(4-\sqrt{x})(2-\sqrt{x})}{(2)^2-(\sqrt{x})^2}=\frac{8-6\sqrt{x}+x}{4-x}$$

(e) Multiply and divide by the conjugate of the denominator:

$$\begin{aligned}\frac{x}{\sqrt{x+1}-1}&=\frac{x}{\sqrt{x+1}-1}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}=\frac{x(\sqrt{x+1}+1)}{\left(\sqrt{x+1}\right)^2-(1)^2}\\&=\frac{x(\sqrt{x+1}+1)}{(x+1)-1}\\&=\frac{x(\sqrt{x+1}+1)}{x}=\sqrt{x+1}+1\end{aligned}$$

(f) Multiply and divide by $2\sqrt{x}+6$:

$$\frac{\sqrt{x}}{2\sqrt{x}-6}=\frac{\sqrt{x}}{2\sqrt{x}-6}\cdot\frac{2\sqrt{x}+6}{2\sqrt{x}+6}=\frac{\sqrt{x}\,(2\sqrt{x}+6)}{(2\sqrt{x})^2-(6)^2}=\frac{2x+6\sqrt{x}}{4x-36}=\frac{x+3\sqrt{x}}{2x-18}$$

(g) We multiply and divide by $\sqrt{x}+\sqrt{x-1}$

$$\begin{aligned}\frac{3}{\sqrt{x}-\sqrt{x-1}}&=\frac{3}{\sqrt{x}-\sqrt{x-1}}\,\frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}}\\&=\frac{3(\sqrt{x}+\sqrt{x-1})}{(\sqrt{x})^2-(\sqrt{x-1})^2}\\&=\frac{3(\sqrt{x}+\sqrt{x-1})}{x-(x-1)}=3(\sqrt{x}+\sqrt{x-1})\end{aligned}$$

(h) Multiply and divide by $\sqrt{x+2}-\sqrt{x-2}$:

$$\begin{aligned}\frac{1}{\sqrt{x+2}+\sqrt{x-2}}&=\frac{1}{\sqrt{x+2}+\sqrt{x-2}}\cdot\frac{\sqrt{x+2}-\sqrt{x-2}}{\sqrt{x+2}-\sqrt{x-2}}=\frac{\sqrt{x+2}-\sqrt{x-2}}{\left(\sqrt{x+2}\right)^2-\left(\sqrt{x-2}\right)^2}\\&=\frac{\sqrt{x+2}-\sqrt{x-2}}{(x+2)-(x-2)}=\frac{\sqrt{x+2}-\sqrt{x-2}}{4}\end{aligned}$$

Section 4. Equations and Inequalities

1. (a) Substituting $x = 1$ into the given equation, we compute
$$\frac{1}{2x+1} - \frac{1}{x+1} = \frac{1}{2(1)+1} - \frac{1}{1+1} = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6} \neq 0.$$
So, $x = 1$ is not a solution.

$x = -2$ is not a solution because
$$\frac{1}{2x+1} - \frac{1}{x+1} = \frac{1}{2(-2)+1} - \frac{1}{-2+1} = -\frac{1}{3} + 1 = \frac{2}{3}$$
is not zero.

Substitute $x = 0$:
$$\frac{1}{2x+1} - \frac{1}{x+1} = \frac{1}{2(0)+1} - \frac{1}{0+1} = 1 - 1 = 0$$

So, $x = 0$ is a solution of the given equation.

(b) We are checking whether or not the left side $x^3 - 5x + 11$ is equal to -1:

$x = 11 : (11)^3 - 5(11) + 11 = 1331 - 55 + 11 = 1287 \neq -1$

$x = 1 : (1)^3 - 5(1) + 11 = 1 - 5 + 11 = 7 \neq -1$

$x = 0 : (0)^3 - 5(0) + 11 = 11 \neq -1$

$x = -3 : (-3)^3 - 5(-3) + 11 = -27 + 15 + 11 = -1$

Thus, $x = -3$ is a solution.

(c) We are checking whether or not the left side $x^4 + 2x^3 - 13x^2 - 14x + 24$ is equal to zero:

$x = -4$: $(-4)^4 + 2(-4)^3 - 13(-4)^2 - 14(-4) + 24 = 256 - 128 - 208 + 56 + 24 = 0$

$x = -2$: $(-2)^4 + 2(-2)^3 - 13(-2)^2 - 14(-2) + 24 = 16 - 16 - 52 + 28 + 24 = 0$

$x = 0$: $(0)^4 + 2(0)^3 - 13(0)^2 - 14(0) + 24 = 24 \neq 0$

$x = 2$: $(2)^4 + 2(2)^3 - 13(2)^2 - 14(2) + 24 = 16 + 16 - 52 - 28 + 24 = -24 \neq 0$

Thus, $x = -4$ and $x = -2$ are the solutions.

2. (a) Simplify
$$4(3x - 0.5) + 11 = -3(x + 2)$$
$$12x - 2 + 11 = -3x - 6$$
gather like terms together
$$12x + 3x = -6 + 2 - 11$$
$$15x = -15$$
Dividing by 15, we get $x = -1$.

(b) Multiplying the equation by 52 ($52 = 4 \cdot 13$ is the lowest common denominator), we get
$$\frac{3}{4}x - \frac{7}{13} = 1$$
$$\frac{3 \cdot 4 \cdot 13}{4}x - \frac{7 \cdot 4 \cdot 13}{13} = 52$$
$$39x - 28 = 52$$

$$\begin{aligned} 39x &= 80 \\ x &= \frac{80}{39} \end{aligned}$$

(c) Simplify and gather like terms together

$$\begin{aligned} 3.4x - 6.2 &= -3.3(2 - 3x) \\ 3.4x - 6.2 &= -6.6 + 9.9x \\ 3.4x - 9.9x &= -6.6 + 6.2 \\ -6.5x &= -0.4 \\ x &= \frac{-0.4}{-6.5} = \frac{4}{65} = 0.061538 \end{aligned}$$

(d) Gather like terms together, divide and simplify

$$\begin{aligned} \sqrt{3}\,x - 3 &= 2\sqrt{3} \\ \sqrt{3}\,x &= 2\sqrt{3} + 3 \\ x &= \frac{2\sqrt{3} + 3}{\sqrt{3}} \\ x &= \frac{2\sqrt{3}}{\sqrt{3}} + \frac{3}{\sqrt{3}} = 2 + \sqrt{3} \end{aligned}$$

In the last step we rationalized the denominator (which might, or might not be required):

$$\frac{3}{\sqrt{3}} = \frac{3}{\sqrt{3}}\,\frac{\sqrt{3}}{\sqrt{3}} = \sqrt{3}$$

Note that we could have canceled the fraction by $\sqrt{3}$ instead.

3. (a) Multiplying the equation by 20 (20 is the lowest common denominator), we get

$$\begin{aligned} 4(4x - 2) - 5(3x - 11) &= 0 \\ 16x - 8 - 15x + 55 &= 0 \\ x &= -47 \end{aligned}$$

(b) The lowest common denominator of $2x-1$ and $x+3$ is their product $(2x-1)(x+3)$. Multiplying the given equation by $(2x-1)(x+3)$, we get

$$\begin{aligned} \frac{-2}{2x-1}(2x-1)(x+3) + \frac{4}{x+3}(2x-1)(x+3) &= 0 \\ -2(x+3) + 4(2x-1) &= 0 \\ 6x &= -10 \\ x &= \frac{-10}{-6} = \frac{5}{3} \end{aligned}$$

(c) We multiply the given equation by $x(x+1)$ (or, cross-multiply!)

$$\begin{aligned} \frac{\sqrt{5}}{x} &= \frac{2\sqrt{5}}{x+1} \\ \frac{\sqrt{5}}{x}x(x+1) &= \frac{2\sqrt{5}}{x+1}x(x+1) \\ \sqrt{5}(x+1) &= 2\sqrt{5}x \end{aligned}$$

$$\begin{aligned}\sqrt{5}x+\sqrt{5}&=2\sqrt{5}x\\ -\sqrt{5}x&=-\sqrt{5}\\ x&=1\end{aligned}$$

4. (a) The lowest common denominator is $6x$. Multiplying the given equation by $6x$ yields

$$v$$

$$\begin{aligned}12-15+8&=3x\\ 3x&=5\end{aligned}$$

Dividing by 3, we get $x = 5/3$.

(b) We cross multiply to get rid of fractions:

$$\begin{aligned}\frac{x}{x-3}&=\frac{x+3}{x+5}\\ x(x+5)&=(x-3)(x+3)\\ x^2+5x&=x^2-9\\ 5x&=-9\\ x&=-\frac{9}{5}\end{aligned}$$

(c) The common denominator is $2x-4=2(x-2)$; thus

$$\begin{aligned}\frac{2}{x-2}+\frac{1}{2}&=\frac{3}{2x-4}\\ \frac{2}{x-2}(2x-4)+\frac{1}{2}(2x-4)&=\frac{3}{2x-4}(2x-4)\\ 4+(x-2)&=3\\ x&=1\end{aligned}$$

(d) Simplify each side first, and then combine the terms:

$$\begin{aligned}0.6(2x-0.1)&=0.35(3.4x+1.5)\\ 1.2x-0.06&=1.19x+0.525\\ 1.2x-1.19x&=0.525+0.06\\ 0.01x&=0.585\\ \frac{0.01x}{0.01}&=\frac{0.585}{0.01}\\ x&=58.5\end{aligned}$$

(e) Multiply both sides by the common denominator (12):

$$\begin{aligned}\frac{3x+1}{12}&=\frac{-4x-1}{3}-1\\ \frac{3x+1}{12}(12)&=\left(\frac{-4x-1}{3}\right)(12)-1(12)\\ 3x+1&=(-4x-1)(4)-12\end{aligned}$$

$$
\begin{aligned}
3x + 1 &= -16x - 4 - 12 \\
3x + 16x &= -4 - 12 - 1 \\
19x &= -17 \\
\frac{19x}{19} &= \frac{-17}{19} \\
x &= -\frac{17}{19}
\end{aligned}
$$

(f) The common denominator is the product of the two denominators; thus, multiplying the equation by $(-2x+1)(3x-1)$, we obtain

$$
\begin{aligned}
\frac{11}{-2x+1} - \frac{3}{3x-1} &= 0 \\
11(3x-1) - 3(-2x+1) &= 0 \\
33x - 11 + 6x - 3 &= 0 \\
33x + 6x &= 11 + 3 \\
39x &= 14 \\
x &= \frac{14}{39}
\end{aligned}
$$

(g) Simplify each side and then combine the terms:

$$
\begin{aligned}
4(-5x+3) - 3(1-x) &= -2(x-1) + 13x \\
-20x + 12 - 3 + 3x &= -2x + 2 + 13x \\
-20x + 3x + 2x - 13x &= 2 - 12 + 3 \\
-28x &= -7 \\
x &= \frac{-7}{-28} = \frac{1}{4}
\end{aligned}
$$

5. (a) Computing the square root of both sides, we obtain

$$
\begin{aligned}
x^2 &= \sqrt{2} \\
x &= \pm\sqrt{\sqrt{2}} \\
x &= \pm\left(2^{\frac{1}{2}}\right)^{\frac{1}{2}} = \pm 2^{\frac{1}{4}} = \pm\sqrt[4]{2}
\end{aligned}
$$

(b) Computing the square root of both sides of $x^2 = \sqrt[3]{2}$ we obtain $x = \pm\sqrt{\sqrt[3]{2}} = \pm\sqrt[6]{2}$.
Note that

$$
\sqrt{\sqrt[3]{2}} = \left(\sqrt[3]{2}\right)^{1/2} = \left(2^{1/3}\right)^{1/2} = 2^{(1/3)(1/2)} = 2^{1/6}
$$

(c) The square of any real number is zero or positive. Thus, the given equation has no solutions.

(d) Computing the square root of both sides, we obtain

$$
\begin{aligned}
(3x+1)^2 &= 4 \\
3x + 1 &= \pm\sqrt{4} \\
3x &= \pm 2 - 1
\end{aligned}
$$

$$x = \frac{\pm 2 - 1}{3}$$

So, the solutions are $x = (2-1)/3 = 1/3$ and $x = (-2-1)/3 = -1$.

(e) Compute the square root of both sides and then solve for x:

$$\begin{aligned} (2x-5)^2 &= 2 \\ 2x - 5 &= \pm\sqrt{2} \\ 2x &= \pm\sqrt{2} + 5 \\ x &= \frac{\pm\sqrt{2}+5}{2} = \frac{5 \pm \sqrt{2}}{2} \end{aligned}$$

6. (a) Since the square of a real number cannot be negative, the given equation has no solutions.

(b) Computing the square root of both sides, we get $x - 6 = \pm\sqrt{12}$; i.e., $x = 6 \pm \sqrt{12}$. Recall that $\sqrt{12}$ can be simplified as follows

$$\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4}\,\sqrt{3} = 2\sqrt{3}$$

and so $x = 6 \pm 2\sqrt{3}$.

(c) We are asked to solve $x^2 + 10x + 24 = 0$. Factoring, we get

$$x^2 + 10x + 24 = (x+4)(x+6) = 0$$

and so $x = -4$ and $x = -6$.

(d) We have to solve $3x^2 - x - 4 = 0$.

To factor, we think of writing $3x^2 - x - 4 = (3x+a)(x+b)$ and try those a and b for which $ab = -4$ ($a = 4$ and $b = -1$, $a = -4$ and $b = 1$, $a = 2$ and $b = -2$, and $a = -2$ and $b = 2$). We find that $3x^2 - x - 4 = (3x-4)(x+1)$ Thus, the solutions are (from $3x - 4 = 0$) $x = 4/3$ and (from $x + 1 = 0$) $x = -1$.

Alternatively, using the quadratic formula, we get

$$x = \frac{1 \pm \sqrt{1+48}}{6} = \frac{1 \pm \sqrt{49}}{6} = \frac{1 \pm 7}{6}$$

Thus, the solutions are $x = 8/6 = 4/3$ and $x = -6/6 = -1$.

7. (a) We add and subtract the square of one half of the coefficient of x, which is $(-1/2)^2 = 1/4$, and write $x^2 - x + 1/4 - 1/4 + 5 = 0$, i.e.,

$$\left(x^2 - x + \frac{1}{4}\right) - \frac{1}{4} + 5 = 0$$

Next, we recall that the square term involves one half of the coefficient of x (i.e., one half of -1)

$$\left(x - \frac{1}{2}\right)^2 = -\frac{19}{4}$$

Since the square of every number is zero or positive, the above equation has no solutions.

(b) First, divide the equation by 2 so that the coefficient of x^2 is 1

$$\begin{aligned} 2x^2 - 2x - 7 &= 0 \\ x^2 - x - \frac{7}{2} &= 0 \end{aligned}$$

Now complete the square: one half of the coefficient of x is $-1/2$; square it (so, it's $1/4$) and add and subtract on the left side

$$x^2 - x + \frac{1}{4} - \frac{1}{4} - \frac{7}{2} = 0$$
$$\left(x^2 - x + \frac{1}{4}\right) - \frac{1}{4} - \frac{7}{2} = 0$$
$$\left(x - \frac{1}{2}\right)^2 = \frac{15}{4}$$

Compute the square root of both sides

$$x - \frac{1}{2} = \pm\frac{\sqrt{15}}{2}$$

Thus,

$$x = \frac{1}{2} \pm \frac{\sqrt{15}}{2}.$$

(c) Divide the equation by 3 so that the coefficient of x^2 is 1

$$3x^2 - x - 5 = 0$$
$$x^2 - \frac{x}{3} - \frac{5}{3} = 0$$

Complete the square: one half of the coefficient of x is $-1/6$; square it (so, it's $1/36$) and add to and subtract from the expression on the left side

$$x^2 - \frac{x}{3} + \frac{1}{36} - \frac{1}{36} - \frac{5}{3} = 0$$
$$\left(x^2 - \frac{x}{3} + \frac{1}{36}\right) - \frac{1}{36} - \frac{5}{3} = 0$$
$$\left(x - \frac{1}{6}\right)^2 = \frac{61}{36}$$

Compute the square root of both sides

$$x - \frac{1}{6} = \pm\frac{\sqrt{61}}{6}$$

Thus,

$$x = \frac{1}{6} \pm \frac{\sqrt{61}}{6}.$$

8. (a) Since $x^2 - 18x + 81 = (x-9)^2 = 0$, we conclude that $x - 9 = 0$, or $x = 9$. Alternatively, we could have used the quadratic equation.

(b) Multiplying the equation by the common denominator $(x-3)(x+3) = x^2 - 9$, we get

$$\frac{2}{x^2-9} = \frac{1}{x+3} + \frac{3}{x-3}$$
$$\frac{2}{x^2-9}(x^2-9) = \frac{1}{x+3}(x-3)(x+3) + \frac{3}{x-3}(x-3)(x+3)$$
$$2 = x - 3 + 3(x+3)$$
$$4x = -4$$

and $x = -1$. Note that all expressions involved are defined at -1, so $x = -1$ is indeed as solution.

(c) Move 16 to the right side and compute the square root:

$$\begin{aligned}(3x-14)^2-16&=0\\(3x-14)^2&=16\\(3x-14)&=\pm\sqrt{16}\\3x&=\pm4+14\\x&=\frac{\pm4+14}{3}\end{aligned}$$

So, $x=(4+14)/3=18/3=6$ and $x=(-4+14)/3=10/3$.

(d) Squaring both sides, we get

$$\begin{aligned}(\sqrt{2x+4})^2&=(\sqrt{6x+1}-1)^2\\2x+4&=6x+1-2\sqrt{6x+1}+1\\2\sqrt{6x+1}&=4x-2\end{aligned}$$

Divide by 2 and square again

$$\begin{aligned}\sqrt{6x+1}&=2x-1\\6x+1&=4x^2-4x+1\\4x^2-10x&=0\end{aligned}$$

Factoring, we get

$$4x^2-10x=2x(2x-5)=0,$$

and thus $x=0$ or $2x-5=0$, i.e., $x=5/2$.

Note that $x=0$ is not a solution: the left side of the given equation is $\sqrt{2x+4}=\sqrt{4}=2$, whereas the right side is $\sqrt{6x+1}-1=\sqrt{1}-1=0$.

Substituting $x=5/2$ into the left side, we get

$$\sqrt{2x+4}=\sqrt{2\,\frac{5}{2}+4}=\sqrt{9}=3.$$

The right side is

$$\sqrt{6x+1}-1=\sqrt{6\,\frac{5}{2}+1}-1=\sqrt{16}-1=3.$$

So, $x=5/2$ is the only solution of the given equation.

(e) Multiplying the equation by the common denominator $(x-1)(x+1)=x^2-1$, we get

$$\begin{aligned}\frac{3}{x+1}-\frac{2}{x-1}&=-1\\\frac{3}{x+1}(x-1)(x+1)-\frac{2}{x-1}(x-1)(x+1)&=-1(x^2-1)\\3(x-1)-2(x+1)&=-x^2+1\\x^2+x-6&=0\end{aligned}$$

Factoring $x^2+x-6=(x+3)(x-2)=0$ we get that $x=-3$ and $x=2$. Since the denominators $x-1$ and $x+2$ are not zero when $x=-3$ and $x=2$, the two values of x are indeed solutions.

(f) Remove the denominator first:

$$\begin{aligned}5x^2 + \frac{11x}{2} - 3 &= 0\\ 5x^2(2) + 11x - 3(2) &= 0\\ 10x^2 + 11x - 6 &= 0\end{aligned}$$

Using the quadratic formula,

$$\begin{aligned}x &= \frac{-11 \pm \sqrt{(11)^2 - 4(10)(-6)}}{2(10)}\\ x &= \frac{-11 \pm \sqrt{121 + 240}}{20}\\ x &= \frac{-11 \pm \sqrt{361}}{20} = \frac{-11 \pm 19}{20}\end{aligned}$$

So, $x = (-11 + 19)/20 = 8/20 = 2/5$ and $x = (-11 - 19)/20 = -30/20 = -3/2$.

(g) Using the quadratic formula,

$$\begin{aligned}x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(5)}}{2(2)}\\ x &= \frac{6 \pm \sqrt{36 - 40}}{4}\\ x &= \frac{6 \pm \sqrt{-4}}{4}\end{aligned}$$

We conclude that the given equation has no real number solutions.

(h) Rewrite $2x^2 - 3x = 5$ as $2x^2 - 3x - 5 = 0$. Using the quadratic formula,

$$\begin{aligned}x &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2(2)}\\ x &= \frac{3 \pm \sqrt{9 + 40}}{4}\\ x &= \frac{3 \pm \sqrt{49}}{4} = \frac{3 \pm 7}{4}\end{aligned}$$

So, $x = 10/4 = 5/2$ and $x = -4/4 = -1$.

(i) Multiply $x^2 - \frac{5x}{4} + \frac{1}{4} = 0$ by 4, to obtain $4x^2 - 5x + 1 = 0$. Using the quadratic formula,

$$\begin{aligned}x &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(4)(1)}}{2(4)}\\ x &= \frac{5 \pm \sqrt{25 - 16}}{8}\\ x &= \frac{5 \pm \sqrt{9}}{8} = \frac{5 \pm 3}{8}\end{aligned}$$

So, $x = 8/8 = 1$ and $x = 2/8 = 1/4$.

(j) Simplify first to get rid of denominators:

$$\begin{aligned}2x^2 + \frac{5x}{2} &= \frac{3}{4}\\ 2x^2 + \frac{5x}{2} - \frac{3}{4} &= 0\end{aligned}$$

$$2x^2(4) + \frac{5x}{2}(4) - \frac{3}{4}(4) = 0$$
$$8x^2 + 10x - 3 = 0$$

Using the quadratic formula,

$$x = \frac{-10 \pm \sqrt{(10)^2 - 4(8)(-3)}}{2(8)}$$
$$x = \frac{-10 \pm \sqrt{100 + 96}}{16}$$
$$x = \frac{-10 \pm \sqrt{196}}{16} = \frac{-10 \pm 14}{16} = \frac{-5 \pm 7}{8}$$

So, $x = 2/8 = 1/4$ and $x = -12/8 = -3/2$.

9. (a) Using the quadratic formula, we solve $x^2 + x - 1 = 0$ for x:

$$x = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

Thus,

$$x^2 + x - 1 = \left(x - \frac{-1+\sqrt{5}}{2}\right)\left(x - \frac{-1-\sqrt{5}}{2}\right) = \left(x + \frac{1-\sqrt{5}}{2}\right)\left(x + \frac{1+\sqrt{5}}{2}\right)$$

(b) Instead of writing $4x^2 - 4x - 15 = (4x + a)(x + b)$ and $4x^2 - 4x - 15 = (2x + a)(2x + b)$ and checking which values for a and b work, we solve the quadratic equation $4x^2 - 4x - 15 = 0$:

$$x = \frac{4 \pm \sqrt{256}}{8} = \frac{4 \pm 16}{8}.$$

So, $x = 20/8 = 5/2$ and $x = -12/8 = -3/2$ are the solutions, and we factor

$$4x^2 - 4x - 15 = 4\left(x - \frac{5}{2}\right)\left(x - \frac{-3}{2}\right) = 2\left(x - \frac{5}{2}\right)2\left(x + \frac{3}{2}\right) = (2x-5)(2x+3).$$

(c) We solve $2x^2 + 2x - 3 = 0$ for x:

$$x = \frac{-2 \pm \sqrt{28}}{4} = \frac{-2 \pm 2\sqrt{7}}{4} = \frac{-1 \pm \sqrt{7}}{2}.$$

Note that we simplified $\sqrt{28} = \sqrt{4 \cdot 7} = \sqrt{4}\sqrt{7} = 2\sqrt{7}$. Thus,

$$2x^2 + 2x - 3 = 2\left(x - \frac{-1+\sqrt{7}}{2}\right)\left(x - \frac{-1-\sqrt{7}}{2}\right) = 2\left(x + \frac{1-\sqrt{7}}{2}\right)\left(x + \frac{1+\sqrt{7}}{2}\right)$$

(d) Using the quadratic formula,

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(15)}}{2(1)}$$
$$x = \frac{-2 \pm \sqrt{4 - 60}}{2} = \frac{-2 \pm \sqrt{-56}}{2}$$

There are no solutions for x, and thus the given trinomial cannot be factored.

(e) Using the quadratic formula,

$$x = \frac{-\left(-\frac{5}{4}\right) \pm \sqrt{\left(-\frac{5}{4}\right)^2 - 4(1)\left(\frac{1}{4}\right)}}{2(1)}$$

$$x = \frac{\frac{5}{4} \pm \sqrt{\frac{25}{16} - 1}}{2}$$

$$x = \frac{\frac{5}{4} \pm \sqrt{\frac{9}{16}}}{2} = \frac{\frac{5}{4} \pm \frac{3}{4}}{2} = \frac{5}{8} \pm \frac{3}{8}$$

So, $x = 8/8 = 1$ and $x = 2/8 = 1/4$. We write

$$x^2 - \frac{5}{4}x + \frac{1}{4} = (x-1)\left(x - \frac{1}{4}\right)$$

(f) Factor x out:

$$5x^2 + \frac{11}{2}x = x\left(5x + \frac{11}{2}\right)$$

(g) Using the quadratic formula,

$$3x^2 - 4x - 1 = 0$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-1)}}{2(3)}$$

$$x = \frac{4 \pm \sqrt{16+12}}{6} = \frac{4 \pm \sqrt{28}}{6} = \frac{4 \pm 2\sqrt{7}}{6} = \frac{2 \pm \sqrt{7}}{3}$$

Note that we wrote $\sqrt{28} = \sqrt{4 \cdot 7} = \sqrt{4}\sqrt{7} = 2\sqrt{7}$. Thus,

$$3x^2 - 4x - 1 = 3\left(x - \frac{2+\sqrt{7}}{3}\right)\left(x - \frac{2-\sqrt{7}}{3}\right)$$

(h) Write $x^2 + 2x - 15 = (x+a)(x+b)$ and look for integers a and b such that $a + b = 2$ and $ab = -15$. We find $a = 5$ and $b = -3$. Thus, $x^2 + 2x - 15 = (x+5)(x-3)$.

10. (a) We rearrange the terms

$$13x - 17 > 4x + 1$$

$$9x > 18$$

$$x > 2$$

In interval notation, the solution is $(2, \infty)$.

(b) We break the inequality into two inequalities.

Solving $1 - x \geq 3 - 2x$, we get $x \geq 2$.

Solving $3 - 2x \geq x - 6$, we get $-3x \geq -9$ and (after dividing by -3) $x \leq 3$.

The solution x must satisfy both $x \geq 2$ and $x \leq 3$. Thus, $2 \leq x \leq 3$; using interval notation, we write the solution as $[2, 3]$.

(c) Because only one term involves x, we can work with both inequalities at the same time. Adding 2 to all sides and dividing by 3, we get

$$3 \leq 3x - 2 \leq 4$$

$$5 \leq 3x \leq 6$$

$$\frac{5}{3} \leq x \leq 2$$

Thus, the solution is the interval $[5/3, 2]$.

(d) Multiply the inequality by the common denominator (30):

$$\frac{5x}{2} - \frac{1}{3} < -2x + \frac{2}{5}$$
$$\frac{5x}{2}(30) - \frac{1}{3}(30) < -2x(30) + \frac{2}{5}(30)$$
$$75x - 10 < -60x + 12$$
$$75x + 60x < 12 + 10$$
$$135x < 22$$

Thus, $x < 22/135$. In interval notation, the solution is $(-\infty, 22/135)$.

(e) Because only one term involves x, we can work with both inequalities at the same time. Adding 11 to all sides and then dividing by -2, we get

$$-3 < -2x - 11 < -2$$
$$8 < -2x < 9$$
$$\frac{8}{-2} > x > \frac{9}{-2}$$
$$-4 > x > -\frac{9}{2}$$

Using interval notation, we write the solution as $(-9/2, -4)$.

(f) We need to break into two inequalities.

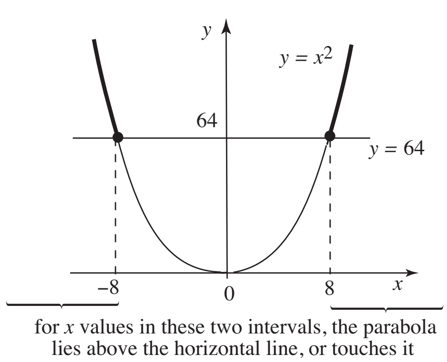
$$2 - 3x \geq x + 2$$
$$2 - 2 \geq x + 3x$$
$$4x \leq 0$$

and thus $x \leq 0$. The second inequality:

$$x + 2 \geq 4x - 5$$
$$2 + 5 \geq 4x - x$$
$$7 \geq 3x$$

and $x \leq 7/3$. The solution for x must satisfy both $x \leq 0$ and $x \leq 7/3$. Thus, $x \leq 0$.

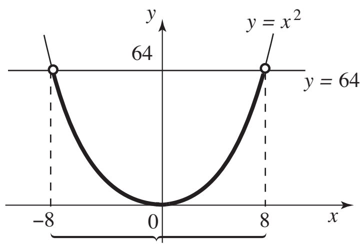
11. (a) We solve $x^2 \geq 64$ geometrically: we identify those parts of the parabola $y = x^2$ that lie above the horizontal line $y = 64$ or touch it; see figure below.



The corresponding values of x lie in the two intervals $(-\infty, -8]$ and $[8, \infty)$.

Of course, we could have solved the inequality algebraically: starting from $x^2 \geq 64$ we write $x^2 - 64 \geq 0$, and then analyze the factors in the inequality $(x-8)(x+8) \geq 0$.

(b) This time, we are looking for x values for which the points on the parabola $y = x^2$ lie below the horizontal line $y = 64$; see figure below. The solution is $(-8, 8)$.



(c) The linear functions involved change the sign at 2 and -2. Thus, we need to analyze intervals $(-\infty, -2)$, $(-2, 2)$ and $(2, \infty)$.

We organize the calculations in the table below.

	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
$x-2$	$-$	$-$	$+$
$x+2$	$-$	$+$	$+$
$\frac{x+2}{x-2}$	$+$	$-$	$+$

The fraction $\dfrac{x+2}{x-2}$ is negative on the interval $(-2, 2)$. Note that we also need to include the case

$$\frac{x+2}{x-2} = 0$$

which happens if $x + 2 = 0$, i.e., if $x = -2$. So, the solution of the given inequality is $[-2, 2)$.

(d) We simplify using a common denominator and then factor:

$$\begin{aligned} \frac{x+2}{x-2} &\leq 1 \\ \frac{x+2}{x-2} - 1 &\leq 0 \\ \frac{x+2}{x-2} - \frac{x-2}{x-2} &\leq 0 \\ \frac{4}{x-2} &\leq 0. \end{aligned}$$

Since the numerator is positive, the denominator has to be negative, so $x - 2 < 0$ and $x < 2$. Note that the fraction is never equal to zero (since its numerator is not zero). Thus, the solution is $(-\infty, 2)$.

(e) Move all terms to the left side and factor:

$$\begin{aligned} x^2 &> 21 - 4x \\ x^2 + 4x - 21 &> 0 \\ (x+7)(x-3) &> 0 \end{aligned}$$

The solutions of the equation $(x+7)(x-3)=0$ are $x=-7$ and $x=3$. The numbers -7 and 3 divide the number line into three intervals:

$$(-\infty,-7),(-7,3) \text{ and } (3,\infty)$$

We check the sign of each factor on each interval and record it in the table below.

	$(-\infty,-7)$	$(-7,3)$	$(3,\infty)$
$x-3$	$-$	$-$	$+$
$x+7$	$-$	$+$	$+$
$(x-3)(x+7)$	$+$	$-$	$+$

The solution (the product has to be positive!) consists of two intervals, $(-\infty,-7)$ and $(3,\infty)$.

(f) Move 2 to the left side and compute the common denominator:

$$\frac{6}{x-5} \leq 2$$
$$\frac{6}{x-5} - 2 \leq 0$$
$$\frac{6}{x-5} - \frac{2(x-5)}{x-5} \leq 0$$
$$\frac{-2x+16}{x-5} \leq 0$$
$$\frac{-2(x-8)}{x-5} \leq 0$$

The linear terms involved change the sign at 5 and 8, so we check the intervals $(-\infty,5)$, $(5,8)$ and $(8,\infty)$.

	$(-\infty,5)$	$(5,8)$	$(8,\infty)$
$x-8$	$-$	$-$	$+$
$x-5$	$-$	$+$	$+$
$\frac{-2(x-8)}{x-5}$	$-$	$+$	$-$

Note that the signs in the last row of the table include the factor of -2. From the table we conclude that the intervals $(-\infty,5)$, and $(8,\infty)$ are part of the solution.

Since equality is allowed, the numerator $-2(x-8)$ can be zero, i.e., x can be equal to 8. Thus, the solution is $(-\infty,5)$ and $[8,\infty)$.

(g) Since $x^3-x=x(x^2-1)=x(x-1)(x+1)$, we need to solve $x(x-1)(x+1)\leq 0$. The factors change their sign at -1, 0 and 1.

	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
$x-1$	$-$	$-$	$-$	$+$
x	$-$	$-$	$+$	$+$
$x+1$	$-$	$+$	$+$	$+$
$x(x-1)(x+1)$	$-$	$+$	$-$	$+$

So, the intervals $(-\infty,-1)$ and $(0,1)$ are part of the solution. Since the inequality allows x^3-x to be equal to zero, we must include $x=0$, $x=-1$ and $x=1$. Thus, the solution to the given inequality is $(-\infty,-1]$ and $[0,1]$.

(h) Rewrite the given inequality as $x^2-x-2\le 0$ and factor

$$(x-2)(x+1)\le 0$$

	$(-\infty,-1)$	$(-1,2)$	$(2,\infty)$
$x-2$	$-$	$-$	$+$
$x+1$	$-$	$+$	$+$
$(x-2)(x+1)$	$+$	$-$	$+$

So, the solution is the interval $[-1,2]$.

Alternatively – let is use test values and checked the whole product at once. To test the interval $(-\infty,-1)$ pick $x=-10$; then $(x-2)(x+1)=(-12)(-9)=108>0$, and so $(-\infty,-1)$ is not part of the solution.

To test the interval $(-1,2)$, we use $x=0$: $(x-2)(x+1)=(-2)(1)=-2<0$, so the interval $(-1,2)$ is part of the solution.

To test the interval $(2,\infty)$ we use $x=5$: $(x-2)(x+1)=(3)(6)=18>0$, so the interval $(2,\infty)$ does not belong to the solution.

So, we have the interval $(-1,2)$. At the endpoints, the product $(x-2)(x+1)$ is zero, and so the solution of $(x-2)(x+1)\le 0$ is $[-1,2]$.

12. (a) Write $|x+2|>3$ as $|x-(-2)|>3$. We are looking for all x whose distance to -2 is greater than 3. The solution is $(-\infty,-5)$ and $(1,\infty)$.

(b) Write $|x+2|\le 3$ as $|x-(-2)|\le 3$. We are looking for all x whose distance from -2 is smaller than or equal to 3. All such x lie in the interval $[-5,1]$.

(c) To solve $|x-1|\ge 12$ means to find all x whose distance from 1 is larger than or equal to 12. Thus, $x\ge 13$ or $x\le -11$. Using interval notation, we write the solution as $(-\infty,-11]$ and $[13,\infty)$.

(d) Write $|x+1|>12$ as $|x-(-1)|>12$. The distance between x and -1 must be greater than 12. Thus, the solution consists of two intervals, $(-\infty,-13)$ and $(11,\infty)$.

13. Keep in mind that $|A-B|$ is the distance between two numbers A and B on the number line. Thus, $|A|=|A-0|$ is the distance between the number A and the origin.

(a) The interval $[-8,8]$ contains all numbers whose distance from the origin is 8 or less; thus, the desired inequality is $|x|\le 8$.

(b) The midpoint of the interval $(3,4)$ is 3.5; we see that $(3,4)$ contains all numbers whose distance from 3.5 is strictly less than 0.5. So, the desired equation is $|x-3.5|<0.5$.

(c) The two intervals contain numbers whose distance from the origin is 1 or more than 1; thus, the equation is $|x|\ge 1$.

(d) The half-way between 0 and 4 is 2. We see that the points in the given interval have the property that their distance from 2 is equal to 2 or larger than 2. Thus, the desired inequality is $|x-2| \geq 2$.

14. (a) Using the definition of absolute value, we write

$$|2x-3| = \begin{cases} 2x-3 & \text{if } \ 2x-3 \geq 0 \\ \\ -(2x-3) & \text{if } \ 2x-3 < 0 \end{cases} = \begin{cases} 2x-3 & \text{if } \ x \geq 3/2 \\ \\ -2x+3 & \text{if } \ x < 3/2 \end{cases}$$

Thus, the equation $|2x-3| = 7$ breaks up into two equations:

If $x \geq 3/2$, it reads $2x-3=7$ (and the solution is $x=5$).

If $x < 3/2$, it reads $-2x+3=7$ (and the solution is $x=-2$).

Thus, there are two solutions, $x=5$ and $x=-2$.

Geometric approach: rewrite $|2x-3| = 7$ as $2|x-\frac{3}{2}| = 7$ i.e.,

$$\left|x-\frac{3}{2}\right| = \frac{7}{2}.$$

Thus, we are looking for x whose distance from $3/2$ is $7/2$. There are two solutions, $x = 3/2+7/2 = 5$ and $x = 3/2-7/2 = -2$.

(b) As in (a),

$$\left|\frac{x}{2}-4\right| = \begin{cases} x/2-4 & \text{if } \ x/2-4 \geq 0 \\ \\ -(x/2-4) & \text{if } \ x/2-4 < 0 \end{cases} = \begin{cases} x/2-4 & \text{if } \ x \geq 8 \\ \\ -x/2+4 & \text{if } \ x < 8 \end{cases}$$

Thus, the equation $\left|\frac{x}{2}-4\right| = 3$ breaks up into two equations:

If $x \geq 8$, it reads $x/2-4=3$ (and the solution is $x=14$).

If $x < 8$, it reads $-x/2+4=3$ (and the solution is $x=2$).

Thus, there are two solutions, $x=2$ and $x=14$.

To interpret $\left|\frac{x}{2}-4\right| = 3$ geometrically, we need to have the left side in the form $|x-$ some number$|$. So, multiply by 2 to get

$$|x-8| = 6.$$

The solutions are those numbers on the number line whose distance from 8 is 6; thus $x = 8-6 = 2$ and $x = 8+6 = 14$.

(c) Using the definition of the absolute value, we write

$$\left|\frac{2x}{3}-1\right| = \begin{cases} \dfrac{2x}{3}-1 & \text{if } \frac{2x}{3}-1 \geq 0 \\ -\left(\dfrac{2x}{3}-1\right) & \text{if } \frac{2x}{3}-1 < 0 \end{cases} = \begin{cases} \dfrac{2x}{3}-1 & \text{if } x \geq \frac{3}{2} \\ 1-\dfrac{2x}{3} & \text{if } x < \frac{3}{2} \end{cases}$$

Note that if $\frac{2x}{3}-1 \geq 0$ then $\frac{2x}{3} \geq 1$, and after multiplying both sides by $\frac{3}{2}$, we obtain $x \geq \frac{3}{2}$. This is how we obtained the inequality in the top line. The bottom line inequality is simplified in the same way.

Thus, the equation $\left|\frac{2x}{3}-1\right|=2$ breaks into two cases. When $x \geq \frac{3}{2}$, then

$$\begin{aligned}\frac{2x}{3}-1&=2\\ \frac{2x}{3}&=3\\ x&=\frac{9}{2}\end{aligned}$$

When $x < \frac{3}{2}$, then

$$\begin{aligned}1-\frac{2x}{3}&=2\\ 3-2x&=6\\ -3&=2x\\ x&=-\frac{3}{2}\end{aligned}$$

To interpret $\left|\frac{2x}{3}-1\right|=2$ geometrically, we need to have the left side in the form $|x-$ some number$|$. We write:

$$\begin{aligned}\left|\frac{2x}{3}-1\right|&=2\\ \left|\frac{2}{3}\left(x-\frac{3}{2}\right)\right|&=2\\ \frac{2}{3}\left|x-\frac{3}{2}\right|&=2\\ \left|x-\frac{3}{2}\right|&=3\end{aligned}$$

(In the last step we multiplied the inequality by 3/2.) So we are looking for x whose distance from 3/2 is 3. Thus, $x=3/2+3=9/2$ and $x=3/2-3=-3/2$.

(d) Using the definition of the absolute value, we write

$$\left|2x-\frac{5}{3}\right|=\begin{cases}2x-\dfrac{5}{3} & \text{if } 2x-\frac{5}{3}\geq 0\\ -\left(2x-\dfrac{5}{3}\right) & \text{if } 2x-\frac{5}{3}<0\end{cases}=\begin{cases}2x-\dfrac{5}{3} & \text{if } x\geq \frac{5}{6}\\ \dfrac{5}{3}-2x & \text{if } x<\frac{5}{6}\end{cases}$$

Note that if $2x-\frac{5}{3}\geq 0$ then $2x\geq\frac{5}{3}$, and after multiplying both sides by $\frac{1}{2}$, we obtain $x\geq\frac{5}{6}$. This is how we obtained the inequality in the top line. The bottom line inequality is simplified in the same way.

Thus, the equation $\left|2x-\frac{5}{3}\right|=7$ breaks into two cases. When $x\geq\frac{5}{6}$, then

$$\begin{aligned}2x-\frac{5}{3}&=7\\ 6x-5&=21\\ 6x&=26\\ x&=\frac{13}{3}\end{aligned}$$

When $x < \frac{5}{6}$, then

$$\begin{aligned} \frac{5}{3} - 2x &= 7 \\ 5 - 6x &= 21 \\ -16 &= 6x \\ -\frac{8}{3} &= x \end{aligned}$$

To interpret $\left|2x - \frac{5}{3}\right| = 7$ geometrically, we need to have the left side in the form $|x-$ some number$|$. We write:

$$\begin{aligned} \left|2x - \frac{5}{3}\right| &= 7 \\ \left|2\left(x - \frac{5}{6}\right)\right| &= 7 \\ 2\left|x - \frac{5}{6}\right| &= 7 \\ \left|x - \frac{5}{6}\right| &= \frac{7}{2} \end{aligned}$$

So we are looking for x whose distance from $5/6$ is $7/2$. Thus, $x = 5/6 + 7/2 = 5/6 + 21/6 = 26/6 = 13/3$ and $x = 5/6 - 7/2 = 5/6 - 21/6 = -16/6 = -8/3$.

15. All exercises (a)-(f) can be solved either algebraically or geometrically.

(a) Recall that $|A| < a$ if and only if $-a < A < a$; thus $|2x - 3| < 2$ ie equivalent to the pair of inequalities $-2 < 2x - 3 < 2$ that we solve simultaneously

$$\begin{aligned} -2 < 2x - 3 &< 2 \\ 1 < 2x &< 5 \\ 1/2 < x &< 5/2 \end{aligned}$$

Now the geometric approach: to interpret the absolute value in $|2x - 3| < 2$ as distance, we need to get rid of 2 in front of x. So, we factor it out: $2|x - \frac{3}{2}| < 2$, then divide the equation by 2 to get $|x - \frac{3}{2}| < 1$. We are looking for all x whose distance from $\frac{3}{2}$ is strictly less than 1; all such x belong to the interval $(\frac{1}{2}, \frac{5}{2})$.

(b) Divide $|3x + 4| \geq 4$ by 3, to get $\left|x + \frac{4}{3}\right| \geq \frac{4}{3}$, i.e.,

$$\left|x - \left(-\frac{4}{3}\right)\right| \geq \frac{4}{3}$$

The solution consists of all x whose distance from $-4/3$ is greater than or equal to $4/3$; thus $x \geq -4/3 + 4/3 = 0$ and $x \leq -4/3 - 4/3 = -8/3$. In other words, the solutions belong to one of the intervals $(-\infty, -8/3]$ and $[0, \infty)$.

(c) The inequality $|3x + 4| < 4$ is equivalent to

$$\begin{aligned} -4 < 3x + 4 &< 4 \\ -8 < 3x &< 0 \\ -8/3 < x &< 0 \end{aligned}$$

Using interval notation, the solution is $(-8/3, 0)$.

(d) Simplify $3|x-5|+2<7$ first, by moving 2 over and dividing by 3; we get $|x-5|<5/3$. We are looking for all x whose distance from 5 is smaller than $5/3$. Thus, $x<5+5/3=20/3$ and $x>5-5/3=10/3$. The solution is $(10/3, 20/3)$.

(e) Multiply $\frac{2}{3}|x-5|<1$ by $\frac{3}{2}$ to get $|x-5|<\frac{3}{2}$. Thus

$$-\frac{3}{2}<x-5<\frac{3}{2}$$
$$5-\frac{3}{2}<x<5+\frac{3}{2}$$
$$\frac{7}{2}<x<\frac{13}{2}$$

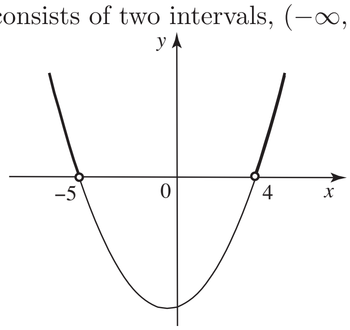
In interval notation, the solution is $(7/2, 13/2)$.

(f) Move 4 over and then multiply by 4, to get $|x+3|<32$. Writing it as

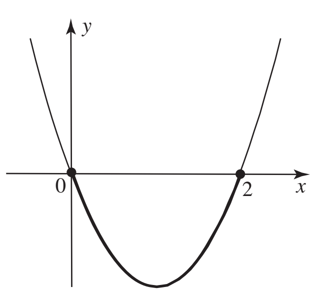
$$|x-(-3)|<32$$

we realize that we are looking for those x whose distance from -3 is less than 32. Thus, $x<-3+32=29$ and $x>-3-32=-35$. The solution is $(-35, 29)$.

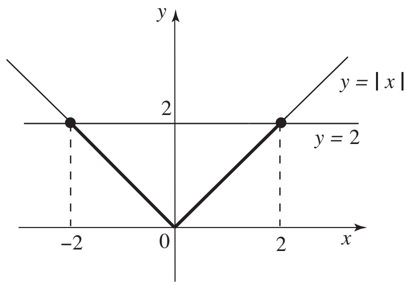
16. (a) The x-intercepts of the parabola $y=x^2+x-20=(x+5)(x-4)$ are -5 and 4. We are looking for those x for which the points on the parabola lie above the x-axis (see figure below). The answer consists of two intervals, $(-\infty, -5)$ and $(4, \infty)$.



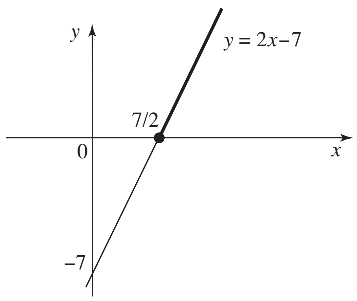
(b) The x-intercepts of the parabola $y=x^2-2x$ are $x=0$ and $x=2$. We are looking for the values of x for which the parabola lies below the x-axis, or touches it. From the figure below we see that the solution is the interval $[0, 2]$.



(c) We are looking for all x such that the graph of the absolute value $y=|x|$ lies below, or touches, the horizontal line $y=2$. See the figure below. The solution is $[-2, 2]$.

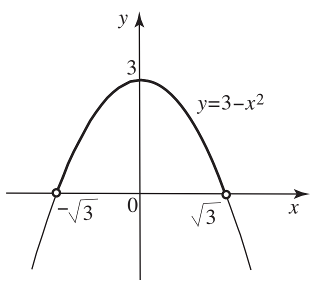


(d) We are looking for all x where the graph of the line $y = 2x - 7$ lies above the x-axes, or touches (crosses) it. See the figure below. The solution is $[7/2, \infty)$.

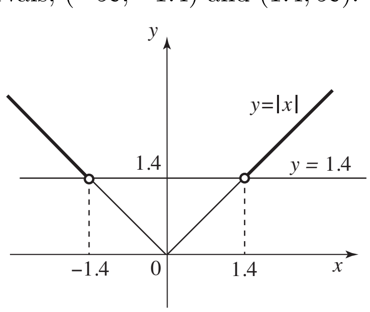


(e) The graph of $y = 3 - x^2$ is a parabola which opens downward, whose x-intercepts are given by $x^2 = 3$, i.e., $x = \pm\sqrt{3}$. Its y-intercept is $y = 3$.

We are looking for all x where this parabola lies above the x-axis. See the figure below. The solution is the interval $(-\sqrt{3}, \sqrt{3})$.



(f) Write $|x| - 1.4 > 0$ as $|x| > 1.4$. We are looking for all x such that the graph of the absolute value $y = |x|$ lies above the horizontal line $y = 1.4$. See the figure below. The solution consists of two intervals, $(-\infty, -1.4)$ and $(1.4, \infty)$.



17. (a) From $x+2y=-11$ we get that $x=-11-2y$. Substituting into the remaining equation, we obtain

$$\begin{aligned} 2x-y&=13\\ 2(-11-2y)-y&=13\\ -22-4y-y&=13\\ -5y&=35\\ y&=-7 \end{aligned}$$

Thus, $x=-11-2y=-11-2(-7)=3$. The solution of the system is $x=3$, $y=-7$.

(b) We compute y from $x-y+4=0$ and substitute into the remaining equation. Thus, $y=x+4$ and so

$$\begin{aligned} y&=x^2+2x+2\\ x+4&=x^2+2x+2\\ 0&=x^2+x-2\\ 0&=(x+2)(x-1) \end{aligned}$$

So, there are two solutions, $x=-2$ and $x=1$. If $x=-2$, then $y=x+4=-2+4=2$. When $x=1$, then $y=x+4=1+4=5$.

Thus, the solutions of the given system are $x=1$, $y=5$ and $x=-2$, $y=2$.

(c) From $x+3y=\frac{5}{2}$ we obtain $x=\frac{5}{2}-3y$. Substituting into the remaining equation, we obtain

$$\begin{aligned} -5\left(\frac{5}{2}-3y\right)-6y&=1\\ -\frac{25}{2}+15y-6y&=1\\ 9y&=\frac{25}{2}+1\\ 9y&=\frac{27}{2}\\ y&=\frac{3}{2} \end{aligned}$$

Thus, $x=\frac{5}{2}-3\left(\frac{3}{2}\right)=-\frac{4}{2}=-2$ and so this system has a single solution: $x=-2$ and $y=3/2$.

(d) From $-x+y-9=0$ we find $x=y-9$. Substituting into the remaining equation, we obtain

$$\begin{aligned} 2(y-9)+10y&=6\\ 12y&=24\\ y&=2 \end{aligned}$$

It follows that $x=2-9=-7$. This system has a single solution: $x=-7$ and $y=2$.

(e) From $x+y=2$ we find $x=2-y$. Substituting into the remaining equation, we obtain

$$\begin{aligned} (2-y)^2+y^2&=10\\ 4-4y+y^2+y^2&=10\\ 2y^2-4y-6&=0 \end{aligned}$$

$$y^2 - 2y - 3 = 0$$
$$(y+1)(y-3) = 0$$

So, $y = -1$ and $y = 3$. When $y = -1$, then $x = 2 - (-1) = 3$. When $y = 3$, then $x = 2 - 3 = -1$.

(f) From $2x + y = 0$ we find $y = -2x$. Substituting into the remaining equation, we obtain

$$\begin{aligned} 4x + y^2 - 2 &= 0 \\ 4x + (-2x)^2 - 2 &= 0 \\ 4x + 4x^2 - 2 &= 0 \\ 2x^2 + 2x - 1 &= 0 \end{aligned}$$
$$x = \frac{-2 \pm \sqrt{12}}{4} = \frac{-2 \pm 2\sqrt{3}}{4} = \frac{-1 \pm \sqrt{3}}{2}$$

Thus, there are two solutions:

$$x = \frac{-1+\sqrt{3}}{2}, \quad y = -2x = -\left(-1+\sqrt{3}\right) = 1 - \sqrt{3}$$

and

$$x = \frac{-1-\sqrt{3}}{2}, \quad y = -2x = -\left(-1-\sqrt{3}\right) = 1 + \sqrt{3}$$

Section 5. Elements of Analytic Geometry

1. (a) The distance between the points $(-3, 4)$ and $(4, -2)$ is

$$\sqrt{(-3-4)^2 + (4-(-2))^2} = \sqrt{49+36} = \sqrt{85}$$

(b) Distance between $(-4, 2)$ and $(0, 5)$ is $\sqrt{(-4-(0))^2 + (2-(5))^2} = \sqrt{16+9} = \sqrt{25} = 5$.
Distance between $(0, 1)$ and $(0, 5)$ is $\sqrt{(0-(0))^2 + (1-(5))^2} = \sqrt{16} = 4$.
Distance between $(3, 1)$ and $(0, 5)$ is $\sqrt{(3-(0))^2 + (1-(5))^2} = \sqrt{9+16} = \sqrt{25} = 5$.
Thus, of the three points, the point $(0, 1)$ is closest to $(0, 5)$.

2. The slope of the line through the points $(1, 3)$ and $(2, -4)$ is

$$m = \frac{-4-3}{2-1} = 7.$$

Using the point-slope form of the equation of the line, we get

$$y - 3 = -7(x-1)$$

i.e., $y = -7x + 10$. We used the point $(1, 3)$ to calculate the equation. If we use $(2, -4)$, we get

$$y - (-4) = -7(x-2)$$

which, of course, gives the same equation, $y = -7x + 10$.

3. Whenever possible, we transform the given equation to the slope-intercept form $y = mx + b$, from which we read the slope (m) and the y-intercept (b).

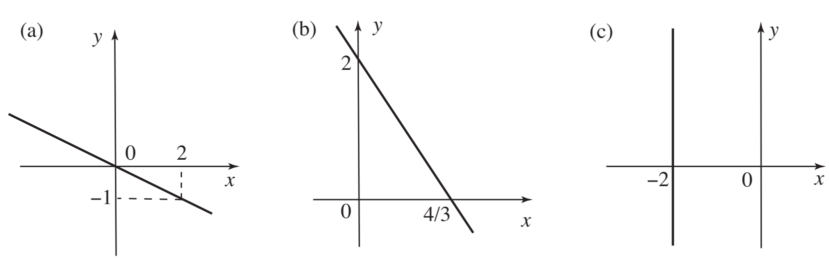
(a) From $-x - 2y = 0$ we get $y = -x/2$; i.e., $m = -1/2$ and $b = 0$. Thus, the graph is a line through the origin of slope $-1/2$; see figure below, (a).

(b) From $3x + 2y - 4 = 0$ we get $2y = -3x + 4$ and $y = -\frac{3}{2}x + 2$. So, it's a line of slope $-3/2$ and y-intercept 2; see figure below, (b). Note that the slope determined by $(0, 2)$ and $(4/3, 0)$ is equal to

$$m = \frac{0-2}{\frac{4}{3}-0} = -\frac{2}{\frac{4}{3}} = -2\frac{3}{4} = -\frac{3}{2},$$

as it should.

(c) This time, we cannot convert to the point-slope form – instead, we write the given equation as $x = -2$. It represents the vertical line through $(-2, 0)$; figure below, (c).



4. (a) Recall that parallel lines have equal slopes.

From $x + 2y + 4 = 0$ we get $2y = -x - 4$ and $y = -\frac{x}{2} - 2$. Thus, the slope is $-1/2$. Since the slope of the line $y = -3x + 4$ is -3, the two lines are not parallel.

(b) Recall that the lines with slopes m_1 and m_2 are perpendicular if and only if $m_2 = -1/m_1$.

From $2x + y - 4 = 0$ we get that $y = -2x + 4$; i.e., the slope of the given line is -2. We conclude that the slope of a line perpendicular to it is $-\frac{1}{-2} = \frac{1}{2}$. We now use the point-slope form to get the desired equation:

$$y - (-2) = \frac{1}{2}(x - 1); \text{ i.e., } y = \frac{1}{2}x - \frac{5}{2}$$

(c) Recall that the condition $m_2 = -1/m_1$ (perpendicular lines) can be written as $m_1 m_2 = -1$.

From $3x - 2y = 6$ we get $2y = 3x - 6$ and $y = \frac{3}{2}x - 3$; so, $m_1 = \frac{3}{2}$.

From $2x + 3y - 12 = 0$ we compute $3y = -2x + 12$ and $y = -\frac{2}{3}x + 4$; so $m_2 = -\frac{2}{3}$.

Since $m_1 m_2 = \frac{3}{2}\left(-\frac{2}{3}\right) - 1$, the two lines are perpendicular.

5. (a) In general, a line parallel to the x-axis has the equation $y = b$, where b represents the y-intercept. So, the desired equation is $y = -4$.

(b) A line parallel to the y-axis has the equation $x = a$, where a represents the x-intercept. So, the desired equation is $x = 3$.

(c) Solving $x - 3y - 7 = 0$ for y, we get $y = \frac{1}{3}x - \frac{7}{3}$. Thus the slope of the given line (and, consequently, the slope of the line we are looking for) is $1/3$. Using the point-slope equation, we get $y - (-4) = \frac{1}{3}(x - 3)$; i.e., $y = \frac{1}{3}x - 5$.

(d) Because the slope of the given line is $1/3$ (from (c)), the slope of a line perpendicular to it is

$$-\frac{1}{1/3} = -3.$$

We now use the point-slope form to get the desired equation:

$$y - (-4) = (-3)(x - 3); \text{ i.e., } y = -3x + 5$$

6. (a) Rewrite $x + y = 0$ as $y = -x$ and compare with $y = mx + b$. Thus, the slope is -1 and the intercept is 0.

(b) Solve $2x + 3 = 0$ for x, to get $x = -3/2$. It's a vertical line, and its slope is not defined (i.e., it is not a real number). There is no y-intercept.

(c) $y + 4 = 0$ represents the horizontal line ($y = -4$) whose y-intercept is -4. As any horizontal line, its slope is zero.

(d) Solve $3x - 5y - 1 = 0$ for y: $5y = 3x - 1$ and $y = \frac{3}{5}x - \frac{1}{5}$. Thus, the slope is $3/5$ and the y-intercept is $-1/5$.

7. (a) The distance is

$$d = \sqrt{((-1) - (-3))^2 + ((-2) - 4)^2} = \sqrt{(2)^2 + (-6)^2} = \sqrt{40} = \sqrt{4 \cdot 10} = 2\sqrt{10}$$

(b) The length of the side joining $(-1, 2)$ and $(3, 8)$ is:

$$d_1 = \sqrt{(3 - (-1))^2 + (8 - 2)^2} = \sqrt{4^2 + 6^2} = \sqrt{52} = \sqrt{4 \cdot 13} = 2\sqrt{13}$$

The length of the side joining $(3, 8)$ and $(-5, -2)$ is:

$$d_2 = \sqrt{(-5-3)^2 + (-2-8)^2} = \sqrt{(-8)^2 + (-10)^2} = \sqrt{164} = \sqrt{4 \cdot 41} = 2\sqrt{41}$$

The length of the side joining $(-5, -2)$ and $(-1, 2)$ is:

$$d_3 = \sqrt{((-1) - (-5))^2 + (2 - (-2))^2} = \sqrt{4^2 + 4^2} = \sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$$

(c) Denoting the origin by O, we find the distances:

$$\begin{aligned} d(A, O) &= \sqrt{(0)^2 + (-3.3)^2} = 3.3 \\ d(B, O) &= \sqrt{(2)^2 + (3.2)^2} = \sqrt{14.24} \approx 3.774 \\ d(C, O) &= \sqrt{(-2)^2 + (2.8)^2} = \sqrt{11.84} \approx 3.441 \\ d(D, O) &= \sqrt{(-2.8)^2 + (-2.3)^2} = \sqrt{13.13} \approx 3.624 \end{aligned}$$

The four points arranged from the closest to the farthest from the origin: A, C, D, B.

(d) The midpoint is

$$M = \left(\frac{(-3.4) + (-12)}{2}, \frac{1 + 4.6}{2} \right) = \left(\frac{-15.4}{2}, \frac{5.6}{2} \right) = (-7.7, 2.8)$$

(e) The midpoint of the line segment from $(4, -3)$ to (a, b) is $\left(\frac{4+a}{2}, \frac{-3+b}{2}\right)$. From

$$\left(\frac{4+a}{2}, \frac{-3+b}{2} \right) = (6, 11)$$

we conclude that $\frac{4+a}{2} = 6$, i.e., $4 + a = 12$ and $a = 8$, and $\frac{-3+b}{2} = 11$, i.e., $-3 + b = 22$ and $b = 25$. So, $(a, b) = (8, 25)$.

(f) From

$$\left(\tfrac{x-4}{2}, \tfrac{9+y}{2}\right) = (-2, -1)$$

we get $\frac{x-4}{2} = -2$ and $\frac{9+y}{2} = -1$ Thus, $x - 4 = -4$, i.e., $x = 0$ and $9 + y = -2$, i.e., $y = -11$.

8. (a) Rewrite $-x - y - 3.4 = 0$ as $y = -x - 3.4$; the slope is -1 and the y intercept is -3.4.

(b) The equation reads $y = 9/7$. This is a horizontal line, so its slope is zero and the y-intercept is $y = 9/7$.

(c) The equation reads $x = -14/3$. This is a vertical line, so its slope is not defined and there is no y-intercept.

(d) Simplify the equation (start by multiplying by 10):

$$\begin{aligned} \frac{x}{2} - \frac{y}{5} &= 7 \\ 5x - 2y &= 70 \\ -2y &= -5x + 70 \\ y &= \frac{5}{2}x - 35 \end{aligned}$$

So, the slope is $5/2$ and the y intercept is -35.

(e) The equation reads $y = 7.6/4$, i.e., $y = 1.9$. This is a horizontal line, so its slope is zero and the y-intercept is $y = 1.9$.

(f) Simplifying (by multiplying by 3) we obtain $x = 12/7$. This is a vertical line; slope is not defined; there is no y-intercept.

(g) We could proceed as usual (by simplifying the equation), or recall the fact that in the linear equation $\frac{x}{a} + \frac{y}{b} = 1$ the numbers a and b are the x- and the y- intercepts (respectively). Thus, the y-intercept is 6, and since the line goes through $(4, 0)$ and $(0, 6)$, its slope is $\frac{6-0}{0-4} = -\frac{6}{4} = -\frac{3}{2}$.

For the record - by simplifying, we obtain (start by multiplying by 24):

$$\frac{x}{4} + \frac{y}{6} = 1$$
$$6x + 4y = 24$$
$$4y = 24 - 6x$$
$$y = 6 - \frac{6}{4}x = 6 - \frac{3}{2}x$$

(h) Simplify to obtain the slope-intercept equation:

$$\frac{3x}{2} - \frac{2y}{5} = 5$$
$$15x - 4y = 50$$
$$y = \frac{15}{4}x - \frac{50}{4}$$

Thus, the slope is $15/4$ and the y-intercept is $y = -50/4 = -25/2$.

9. (a) Rewrite $3x - y + 4 = 0$ as $y = 3x + 4$. The slope is 3.

(b) Rewrite $3x + y - 4 = 0$ as $y = -3x + 4$. The slope is -3.

(c) Rewrite $3y + 4 = 0$ as $y = -4/3$. It is a horizontal line, so its slope is zero.

(d) From

$$3x - 2y + 5 = 0$$
$$2y = 3x + 5$$
$$y = \frac{3}{2}x + \frac{5}{2}$$

we see that the slope is $3/2$.

Slopes ordered, from largest to smallest: (a), (d), (c), and (b).

10. (a) The slope is $m = \frac{1-2.3}{2-3.3} = \frac{-1.3}{-1.3} = 1$. Thus, the desired equation is (we are using the point $(2, 1)$)

$$y - 1 = 1 \cdot (x - 2)$$

i.e., $y = x - 1$.

(b) The given line goes through $(3, 0)$ and $(0, -2)$. Its slope is $m = \frac{0-(-2)}{3-0} = \frac{2}{3}$. So, the equation is $y = \frac{2}{3}x + b = \frac{2}{3}x - 2$.

Alternatively (recall that in the linear equation $\frac{x}{a} + \frac{y}{b} = 1$ the numbers a and b are the x- and the y- intercepts, respectively) we write

$$\frac{x}{3} + \frac{y}{-2} = 1$$

Multiplying by 6, and then simplifying, we get

$$\begin{aligned} 2x - 3y &= 6 \\ 3y &= 2x - 6 \\ y &= \frac{2}{3}x - 2 \end{aligned}$$

(c) The slope is $m = \frac{-5-0}{-2-0} = \frac{5}{2}$. Thus, the line has the equation $y = \frac{5}{2}x$.

(d) Since the x-coordinates of the two points are equal, the line is vertical. Its equation is $x = -4$.

(e) Since the y-coordinates of the two points are equal, the line is horizontal. Its equation is $y = -2$.

(f) Rewrite $x - y - 3 = 0$ in the slope-intercept form: $y = x - 3$. Thus, the slope of the given line (as well as the slope of the line we are looking for) is 1. Using the point-slope equation, we obtain the desired equation: $y - 2 = (1)(x - (-8))$, i.e., $y = x + 10$.

(g) A line parallel to the y-axis contains all points of the form $(x_0, \text{any real number})$ for some real number x_0; its equation is $x = x_0$. In our case, the line goes through $(-2.7, 4.5)$, so its equation is $x = -2.7$.

(h) Rewriting $x - y - 3 = 0$ as $y = x - 3$, we learn that the given line has the slope equal to 1. Thus, the slope of the line perpendicular to it is -1. Using the point-slope equation, we get $y - 1 = (-1)(x - (-4))$, i.e., $y = -x - 3$.

(i) The line $y = m$ is a horizontal line, containing all points of the form $(\text{any real number}, m)$. The line we are looking for is also horizontal, and since it goes through the point (x_0, y_0), its equation is $y = y_0$.

(j) The line $x = n$ is a vertical line, containing all points of the form $(n, \text{any real number})$. The line we are looking for is also vertical, and since it goes through the point (x_0, y_0), its equation is $x = x_0$.

(k) Rewriting $ax + by + c = 0$ as $by = -ax - c$ and

$$y = -\frac{a}{b}x - \frac{c}{b}$$

we see that the slope of the given line is $m = -a/b$. Using the point-slope equation, we get

$$\begin{aligned} y - y_0 &= -\frac{a}{b}(x - x_0) \\ y &= -\frac{a}{b}x + \frac{a}{b}x_0 + y_0 \\ y &= -\frac{a}{b}x + \frac{ax_0 + by_0}{b} \end{aligned}$$

(l) Rewriting $ax + by + c = 0$ as $by = -ax - c$ and

$$y = -\frac{a}{b}x - \frac{c}{b}$$

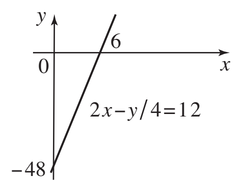
we see that the slope of the given line is $-a/b$. Thus the slope of the line we are looking for is $m = b/a$ (recall that the product of the slopes must be -1). Using the point-slope equation, we get

$$\begin{aligned} y - y_0 &= \frac{b}{a}(x - x_0) \\ y &= \frac{b}{a}x - \frac{b}{a}x_0 + y_0 \\ y &= \frac{b}{a}x + \frac{-bx_0 + ay_0}{a} \end{aligned}$$

11. (a) This is a linear equation, so its graph is a line. Simplify (first multiply by 4):

$$2x - \frac{y}{4} = 12$$
$$8x - y = 48$$
$$y = 8x - 48$$

So, the slope is 8 and the y intercept is -48. See below.



Alternatively, we could have computed two points (for instance: when $x = 0$, then $-y/4 = 12$ and $y = -48$; when $y = 0$, then $2x = 12$ and $x = 6$) and joined them with a straight line.

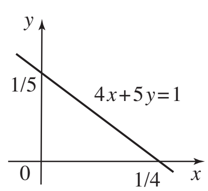
(b) Substituting $x = 0$ into $4x + 5y = 1$ we get $5y = 1$ and $y = 1/5$; so, the y-intercept is $(0, 1/5)$. Substituting $y = 0$ into $4x + 5y = 1$ we get $4x = 1$ and $x = 1/4$; so, the x-intercept is $(1/4, 0)$. See the figure below.

Alternatively, recall that a and b in $x/a + y/b = 1$ are the x- and the y-intercepts. So, if we write

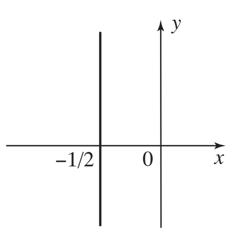
$$4x + 5y = 1 = \frac{x}{1/4} + \frac{y}{1/5} = 1,$$

we can read off the intercepts immediately.

A third way of doing this consists of writing the equation in the slope-intercept form, and using the slope and the y-intercept to graph the line.

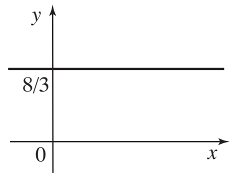


(c) Write $4x = -2$ as $x = -1/2$. This equation represents the vertical line which crosses the x-axis at $(-1/2, 0)$. See below.



(d) By cross-multiplying, we obtain $3y = 8$ and $y = 8/3$.

This is a horizontal line, which crosses the y-axis at $8/3$. See below.



12. In this exercise we need to use the technique of completing the square. If you need to review, read about it in Section 2.

(a) Start with $x^2 + y^2 - 5x + 1 = 0$ and group the x terms and the y terms together

$$x^2 - 5x + y^2 = -1$$

and then complete the square:

$$x^2 - 5x + \frac{25}{4} - \frac{25}{4} + y^2 = -1$$
$$\left(x^2 - 5x + \frac{25}{4}\right) + y^2 = -1 + \frac{25}{4}$$
$$\left(x - \frac{5}{2}\right)^2 + y^2 = \frac{21}{4}$$

Thus, the given equation represents the circle of radius $\sqrt{21}/2$ centered at $(5/2, 0)$.

(b) Again, start with the given equation $x^2 + y^2 - 2x + 8y = 4$, group the x terms and the y terms together

$$x^2 - 2x + y^2 + 8y = 4$$

and complete the square:

$$(x^2 - 2x + 1 - 1) + (y^2 + 8y + 16 - 16) = 4$$
$$\left(x^2 - 2x + 1\right) + \left(y^2 + 8y + 16\right) = 4 + 1 + 16$$
$$(x - 1)^2 + (y + 4)^2 = 21$$

We conclude that the given equation represents the circle of radius $\sqrt{21}$ centered at $(1, -4)$.

(c) Divide the equation by 2, getting $x^2 + y^2 - 3 = 0$ and $x^2 + y^2 = 3$. Thus, it represents the circle centred at the origin of radius $\sqrt{3}$.

13. (a) Divide the equation by 9

$$\frac{x^2}{9} + \frac{y^2}{3} = 1$$

Comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we see that $a^2 = 9$ and $b^2 = 3$.

The given equation represents an ellipse with semi-axes $a = \sqrt{9} = 3$ and $b = \sqrt{3}$, centred at the origin. Its x-intercepts are $\pm a = \pm 3$, and its y-intercepts are $\pm b = \pm\sqrt{3}$.

(b) Complete the square

$$x^2 + y^2 - 2x - 6y = 0$$
$$x^2 - 2x + y^2 - 6y = 0$$
$$(x^2 - 2x + 1 - 1) + (y^2 - 6y + 9 - 9) = 0$$
$$\left(x^2 - 2x + 1\right) + \left(y^2 - 6y + 9\right) = 1 + 9$$

$$(x-1)^2 + (y-3)^2 = 10$$

The equation represents the circle centered at $(1, 3)$, of radius $\sqrt{10}$.

(c) Divide $2x^2 + 3y^2 = 4$ by 4

$$\frac{x^2}{2} + \frac{3y^2}{4} = 1 \qquad \text{i.e.,} \qquad \frac{x^2}{2} + \frac{y^2}{4/3} = 1$$

(thus $a^2 = 2$ and $b^2 = 4/3$).

The equation represents the ellipse with semi-axes $\sqrt{2}$ and $2/\sqrt{3}$, centred at the origin. Its x-intercepts are $\pm\sqrt{2}$, and its y-intercepts are $\pm 2/\sqrt{3}$.

(d) Because the coefficients of x^2 and y^2 are equal, we know that the equation represents a circle.

Complete the square

$$x^2 + y^2 - y = 0$$
$$x^2 + (y^2 - y + \frac{1}{4} - \frac{1}{4}) = 0$$
$$x^2 + \left(y^2 - y + \frac{1}{4}\right) = \frac{1}{4}$$
$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

The equation represents the circle centered at $(0, 1/2)$ of radius $1/2$.

14. (a) Since $a = 1$, the parabola $y = x^2 + 6x + 3$ opens upward. We solve for x-intercepts:

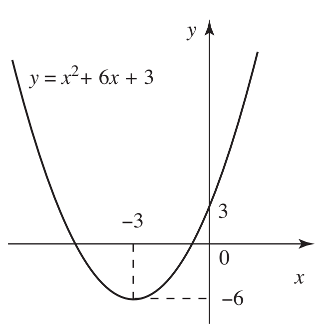
$$x = \frac{-6 \pm \sqrt{24}}{2} = \frac{-6 \pm 2\sqrt{6}}{2} = -3 \pm \sqrt{6}.$$

(Note that $\sqrt{24} = \sqrt{4 \cdot 6} = \sqrt{4}\sqrt{6} = 2\sqrt{6}$.) The x coordinate of the vertex is half-way between the x-intercepts $-3-\sqrt{6}$ and $-3+\sqrt{6}$, i.e., it is equal to -3. (Alternatively, we can use $x = -6/2(1) = -3$.)

The y-coordinate of the vertex is obtained by substituting $x = -3$ into the equation $y = x^2 + 6x + 3$ of the parabola; thus, $y = (-3)^2 + 6(-3) + 3 = -6$.

The y-intercept (substitute $x = 0$ into $y = x^2 + 6x + 3$) is $y = 3$.

So we need to draw a parabola that opens upward, whose vertex is at $(-3, -6)$, that crosses the x-axis at $-3 - \sqrt{6}$ and $-3 + \sqrt{6}$, and that crosses the y-axis at 3; see the figure below.



Alternatively, we use transformations of graphs (explained in Section 6). Complete the square first:

$$y = x^2 + 6x + 3 = (x+3)^2 - 9 + 3 = (x+3)^2 - 6$$

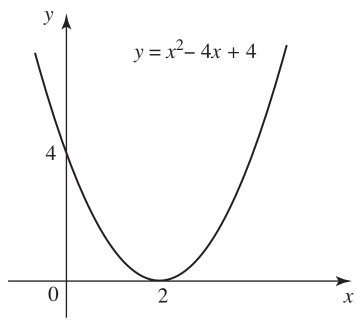
So move the graph of $y = x^2$ to the left for 3 units (to obtain $y = (x+3)^2$) and then move it 6 units down.

(b) The parabola $y = x^2 - 4x + 4$ opens upward since its x^2 coefficient is positive. Note that $y = (x-2)^2$; thus, it has one x-intercept, namely $x = 2$.

The x-coordinate of the vertex is $x = -b/2a = 2$; the corresponding y-coordinate is 0. Thus, the vertex is at $(2, 0)$.

The y-intercept is $y = 4$.

The graph is a parabola that opens upward, has vertex at $(2, 0)$, y-intercept at 4 and $x = 2$ is its only x-intercept. See figure below.

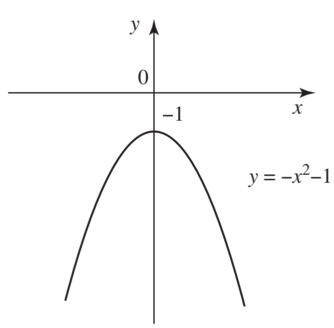


Alternatively, we use transformations of graphs (explained in Section 6): start with the graph of $y = x^2$, and move it 2 units to the right.

(c) The coefficient of x^2 is -1, and thus the parabola $y = -x^2 - 1$ opens downward. From $-x^2 - 1 = 0$ we get the equation $x^2 = -1$ which has no solutions. Thus, there are no x-intercepts.

The x-coordinate of the vertex is $x = -b/2a = 0$; the corresponding y-coordinate is -1; so the vertex is at $(0, -1)$.

The graph is a parabola that opens downward and lies entirely below the x-axis; its vertex is at $(0, -1)$. See figure below.



Alternatively, we use transformations of graphs (explained in Section 6): start with the graph of $y = x^2$, reflect it across the x-axis (to obtain $y = -x^2$) and then move it one unit down.

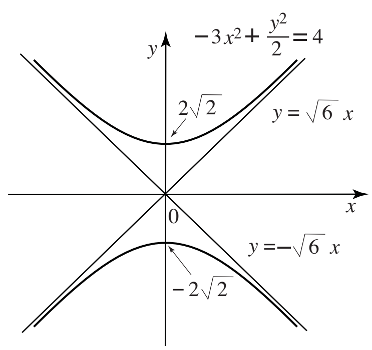
15. (a) Dividing both sides of the equation $-3x^2 + y^2/2 = 4$ by 4, we get

$$-\frac{3x^2}{4} + \frac{y^2}{8} = 1, \qquad \text{i.e.,} \qquad -\frac{x^2}{4/3} + \frac{y^2}{8} = 1$$

Because one of the signs is a minus, the equation represents a hyperbola. From $a^2 = 4/3$ we get $a = 2/\sqrt{3}$; from $b^2 = 8$ we get $b = \sqrt{8} = 2\sqrt{2}$. It follows that the asymptotes are given by

$$y = \pm\frac{b}{a}x = \pm\frac{2\sqrt{2}}{\frac{2}{\sqrt{3}}} = \pm\sqrt{6}x.$$

Substituting $y = 0$ into the given equation (we are looking for x-intercepts) we get $-\frac{3x^2}{4} = 1$, i.e., $x^2 = -\frac{4}{3}$. Since this equation has no solutions, there are no x-intercepts. Substituting $x = 0$, we get $\frac{y^2}{8} = 1$, i.e., $y^2 = 8$, and $y = \pm 2\sqrt{2}$. So, there are two y-intercepts, $2\sqrt{2}$ and $-2\sqrt{2}$. The hyperbola lies in the regions above and below its asymptotes; see the figure below.



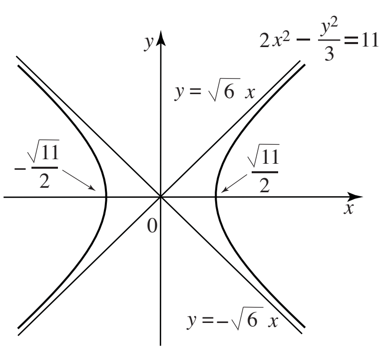
(b) We divide both sides of the equation $2x^2 - y^2/3 = 11$ by 11, to get

$$\frac{2x^2}{11} - \frac{y^2}{33} = 1, \qquad \text{i.e.,} \qquad \frac{x^2}{11/2} - \frac{y^2}{33} = 1$$

As in (a), one of the signs is a minus, and thus the equation represents a hyperbola. From $a^2 = 11/2$ we get $a = \sqrt{11/2}$; from $b^2 = 33$ we get $b = \sqrt{33}$. It follows that the asymptotes are given by

$$y = \pm\frac{b}{a}x = \pm\frac{\sqrt{33}}{\frac{\sqrt{11}}{\sqrt{2}}} = \pm\frac{\sqrt{3}\sqrt{11}}{\frac{\sqrt{11}}{\sqrt{2}}} = \pm\sqrt{6}x.$$

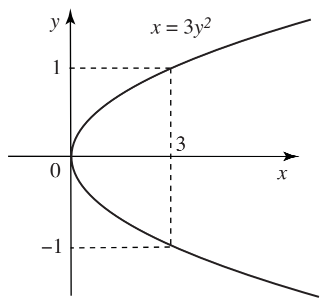
Substituting $y = 0$ into the given equation (we are looking for x-intercepts) we get $2x^2 = 11$, i.e., $x^2 = 11/2$. So, there are two x-intercepts, $x = \pm\sqrt{11/2}$. Substituting $x = 0$, we get $-y^2/3 = 11$, i.e., $y^2 = -33$.Thus, there are no y-intercepts. The hyperbola lies in the regions to the right and to the left of its asymptotes; see the figure below.



(c) Switching x and y we get $y = 3x^2$, which is a parabola that opens upward, with its vertex at the origin. Drawing the mirror image of it with respect to $y = x$, we obtain the graph of $x = 3y^2$; see figure below.

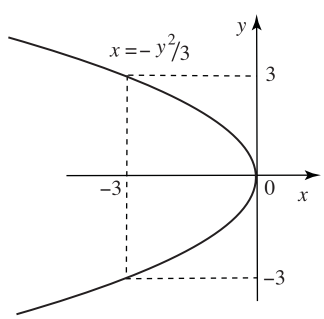
Alternatively, solving $x = 3y^2$ for y, we get $y^2 = x/3$ and $y = \pm\sqrt{x/3}$. Thus, to draw the graph of $x = 3y^2$, we sketch the graph of $y = \sqrt{x/3}$, and then add to it its symmetric image (with respect to x-axis)

It is useful to remember that the equation $x = ay^2$ for $a > 0$ represents a parabola that opens towards the x-axis.



(d) We can use general fact – the equation $x = ay^2$ for $a < 0$ represents a parabola that opens towards the negative x-axis, and its vertex is at the origin.

Alternatively, we proceed as in (c); Switching x and y we get $y = -x^2/3$, which is a parabola that opens downward, with its vertex at the origin. Drawing the mirror image of it with respect to $y = x$, we obtain the graph of $x = -y^2/3$; see figure below.



16. (a) Complete the square for both x and y terms:

$$\begin{aligned} x^2 + 8x + y^2 - 8y &= 4 \\ (x+4)^2 - 16 + (y-4)^2 - 16 &= 4 \\ (x+4)^2 + (y-4)^2 &= 36 \end{aligned}$$

The centre is at $(-4, 4)$ and the radius is 6.

(b) Complete the square:

$$\begin{aligned} x^2 + x + y^2 &= 0 \\ \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + y^2 &= 0 \end{aligned}$$

$$\left(x+\frac{1}{2}\right)^2+y^2=\frac{1}{4}$$

The centre is at $(-1/2, 0)$ and the radius is $1/2$.

(c) Divide by 2 and then complete the square:

$$\begin{aligned}
2x^2+2y^2-7y&=5\\
x^2+y^2-\frac{7}{2}y&=\frac{5}{2}\\
x^2+\left(y-\frac{7}{4}\right)^2-\frac{49}{16}&=\frac{5}{2}\\
x^2+\left(y-\frac{7}{4}\right)^2&=\frac{40}{16}+\frac{49}{16}=\frac{89}{16}
\end{aligned}$$

The centre is at $(0, 7/4)$ and the radius is $\sqrt{89}/4$.

(d) Gather x and y terms together and then complete the square:

$$\begin{aligned}
x^2-\frac{2x}{5}+y^2+\frac{3y}{2}&=1\\
\left(x-\frac{1}{5}\right)^2-\frac{1}{25}+\left(y+\frac{3}{4}\right)^2-\frac{9}{16}&=1\\
\left(x-\frac{1}{5}\right)^2+\left(y+\frac{3}{4}\right)^2&=1+\frac{1}{25}+\frac{9}{16}\\
\left(x-\frac{1}{5}\right)^2+\left(y+\frac{3}{4}\right)^2&=\frac{641}{400}
\end{aligned}$$

The centre is at $(1/5, -3/4)$ and the radius is $\sqrt{641}/20$.

(e) Simplify and complete the square:

$$\begin{aligned}
\frac{x^2+y^2}{4}-x-3y&=1\\
x^2+y^2-4x-12y&=4\\
(x-2)^2-4+(y-6)^2-36&=4\\
(x-2)^2+(y-6)^2&=44
\end{aligned}$$

The centre is at $(2, 6)$, and the radius is $\sqrt{44}=2\sqrt{11}$.

17. (a) Complete the square and simplify to the form given in the text of the exercise:

$$\begin{aligned}
x^2+3x+4y^2-6y&=2\\
x^2+3x+4\left(y^2-\frac{3}{2}y\right)&=2\\
\left(x+\frac{3}{2}\right)^2-\frac{9}{4}+4\left[\left(y-\frac{3}{4}\right)^2-\frac{9}{16}\right]&=2\\
\left(x+\frac{3}{2}\right)^2+4\left(y-\frac{3}{4}\right)^2&=2+\frac{9}{4}+4\cdot\frac{9}{16}\\
\left(x+\frac{3}{2}\right)^2+4\left(y-\frac{3}{4}\right)^2&=\frac{13}{2}
\end{aligned}$$

$$\frac{\left(x+\frac{3}{2}\right)^2}{\frac{13}{2}} + 4\frac{\left(y-\frac{3}{4}\right)^2}{\frac{13}{2}} = 1$$
$$\frac{\left(x+\frac{3}{2}\right)^2}{\frac{13}{2}} + \frac{\left(y-\frac{3}{4}\right)^2}{\frac{1}{4}\cdot\frac{13}{2}} = 1$$

So, the centre of the ellipse is at $(-3/2, 3/4)$, and the semi axes are ($a^2 = 13/2$ and thus) $a = \sqrt{13/2}$ and ($b^2 = 13/8$ and thus) $b = \sqrt{13/8}$.

(b) Complete the square and simplify to the form given in the text of the exercise:

$$2x^2 + y^2 + x = 1$$
$$2\left(x^2 + \frac{x}{2}\right) + y^2 = 1$$
$$2\left[\left(x+\frac{1}{4}\right)^2 - \frac{1}{16}\right] + y^2 = 1$$
$$2\left(x+\frac{1}{4}\right)^2 - \frac{1}{8} + y^2 = 1$$
$$2\left(x+\frac{1}{4}\right)^2 + y^2 = \frac{9}{8}$$
$$2\,\frac{\left(x+\frac{1}{4}\right)^2}{\frac{9}{8}} + \frac{y^2}{\frac{9}{8}} = 1$$
$$\frac{\left(x-\left(-\frac{1}{4}\right)\right)^2}{\frac{9}{16}} + \frac{(y-0)^2}{\frac{9}{8}} = 1$$

We see that the centre of the ellipse is at $(-1/4, 0)$. From $a^2 = 9/16$ and $b^2 = 9/8$ we obtain the semi-axes $a = 3/4$ and $b = 3/\sqrt{8}$.

(c) Complete the square and simplify to the form given in the text of the exercise:

$$x^2 - 2x + \frac{y^2}{5} = 1$$
$$(x-1)^2 - 1 + \frac{y^2}{5} = 1$$
$$(x-1)^2 + \frac{y^2}{5} = 2$$
$$\frac{(x-1)^2}{2} + \frac{y^2}{10} = 1$$

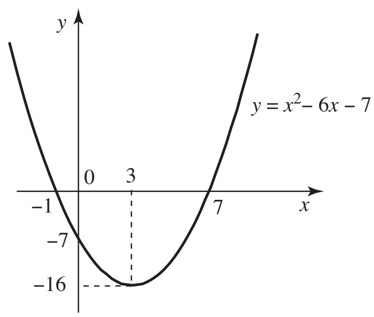
We see that the centre of the ellipse is at $(1, 0)$. From $a^2 = 2$ and $b^2 = 10$ we obtain the semi-axes $a = \sqrt{2}$ and $b = \sqrt{10}$.

18. (a) Since the coefficient of x^2 is positive, the parabola $y = x^2 - 6x - 7$ opens upward. We solve for x-intercepts by factoring: from $y = x^2 - 6x - 7 = (x-7)(x+1)$. we obtain $x = 7$ and $x = -1$. The x coordinate of the vertex is half-way between the x-intercepts, i.e., it is equal to 3. (Alternatively, we can use $x = -b/2a = -(-6)/2 = 3$.)

The y-coordinate of the vertex is $y = (3)^2 - 6(3) - 7 = -16$.

The y-intercept (substitute $x = 0$ into $y = x^2 - 6x - 7$) is $y = -7$.

So we draw a parabola that opens upward, whose vertex is at $(3, -16)$, that crosses the x-axis at -1 and 7, and that crosses the y-axis at -7; see the figure below.



Alternatively, we use transformations of graphs (explained in Section 6). Complete the square first:

$$y = x^2 - 6x - 7 = (x-3)^2 - 9 - 7 = (x-3)^2 - 16$$

So move the graph of $y = x^2$ to the right for 3 units (to obtain $y = (x+3)^2$) and then move it 16 units down.

(b) Since the coefficient of x^2 is positive, the parabola $y = 2x^2 - 3x + 6$ opens upward. From $y = 2x^2 - 3x + 6 = 0$, we find

$$x = \frac{3 \pm \sqrt{3^2 - 4(2)(6)}}{2(2)} = \frac{3 \pm \sqrt{-39}}{4}$$

and conclude that there are no x-intercepts.

Vertex: the x-coordinate is $x = -b/2a = -(-3)/2(2) = 3/4$, and the y-coordinate is

$$y = 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) + 6 = \frac{18}{16} - \frac{9}{4} + 6 = \frac{39}{8}$$

The y-intercept is $y = 6$.

We draw a parabola that opens upward, with vertex at $(3/4, 39/8)$, which crosses the y-axis at $y = 6$.

(c) Since the coefficient of x^2 is negative, the parabola $y = -\frac{x^2}{2} - x + 1$ opens downward. Solving $y = -\frac{x^2}{2} - x + 1 = 0$, i.e., $x^2 + 2x - 2 = 0$,

$$x = \frac{-2 \pm \sqrt{4+8}}{2} = \frac{-2 \pm \sqrt{12}}{2} = \frac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$$

we find the x-intercepts: $x = -1 + \sqrt{3}$ and $x = -1 - \sqrt{3}$.

Vertex: the x-coordinate is $x = -b/2a = 1/(-1) = -1$, and the y-coordinate is $y = -1/2 + 1 + 1 = 3/2$.

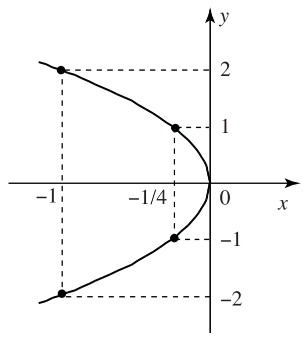
The y-intercept is $y = 1$.

We draw a parabola that opens downward, whose vertex is at $(-1, 3/2)$, that crosses the x-axis at $x = -1 + \sqrt{3}$ and $x = -1 - \sqrt{3}$ and crosses the y-axis at $y = 1$.

(d) Recall that $x = ay^2$ represents the parabola with its vertex at the origin, which opens toward the positive x-axis (if $a > 0$) and negative x-axis (if $a < 0$).

In our case, $a = -1/4$, so the parabola opens toward the negative x-axis. By plotting a few points, we get a more accurate graph; for instance, when $y = \pm 1$, then $x = -1/4$, (so $(-1/4, 1)$ and $(-1/4, -1)$

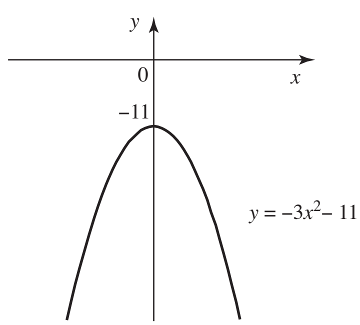
are on the parabola); when $y = \pm 2$, then $x = -1$ (so $(-1, 2)$ and $(-1, -2)$ are on the parabola), and so on. See below.



(e) The coefficient of x^2 is -3, and thus the parabola $y = -3x^2 - 11$ opens downward. From $-3x^2 - 11 = 0$ we get the equation $x^2 = -11/3$, which has no solutions. Thus, there are no x-intercepts.

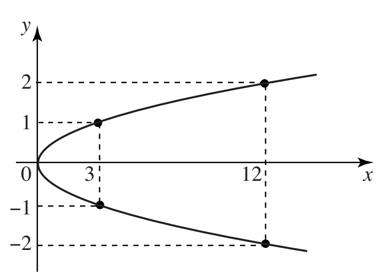
The x-coordinate of the vertex is $x = -b/2a = 0/(-6) = 0$; the corresponding y-coordinate is -11; so the vertex is at $(0, -11)$.

The graph is a parabola that opens downward and lies entirely below the x-axis; its vertex is at $(0, -11)$. See figure below.



(f) Recall that $x = ay^2$ represents the parabola with its vertex at the origin, which opens toward the positive x-axis (if $a > 0$) and negative x-axis (if $a < 0$).

In our case, $a = 3$, so the parabola opens toward the positive x-axis. By plotting a few points, we get a more accurate graph; for instance, when $y = \pm 1$, then $x = 3$ (so $(3, -1)$ and $(3, 1)$ are on the parabola); when $y = \pm 2$, then $x = 12$ (so $(12, 2)$ and $(12, -2)$ are on the parabola), and so on. See below.



Alternatively, make a sketch of $y = 3x^2$ and reflect it across the line $y = x$.

19. (a) Divide the equation $-x^2 + 5y^2 = 5$ by 5, to obtain

$$-\frac{x^2}{5} + \frac{y^2}{1} = 1$$

Thus $a^2 = 5$ and $a = \sqrt{5}$ and $b^2 = 1$ and $b = 1$. The asymptotes are

$$y = \pm\frac{b}{a}x = \pm\frac{1}{\sqrt{5}}x$$

Substituting $x = 0$ into $-x^2 + 5y^2 = 5$ we obtain $5y^2 = 5$ and $y^2 = 1$. We conclude that the y-intercepts are $y = \pm 1$. Substituting $y = 0$ into $-x^2 + 5y^2 = 5$ we obtain $-x^2 = 5$; thus, there are no x-intercepts. The hyperbola lies in the regions above and below its asymptotes.

(b) We divide $x^2 - 5y^2 = 2$ by 2 to obtain 1 on the right side, and then remove the coefficient 5 in front of y^2:

$$\frac{x^2}{2} - \frac{y^2}{2/5} = 1$$

Thus $a^2 = 2$ and $a = \sqrt{2}$ and $b^2 = 2/5$ and $b = \sqrt{2}/\sqrt{5}$. The asymptotes are

$$y = \pm\frac{b}{a}x = \pm\frac{\frac{\sqrt{2}}{\sqrt{5}}}{\sqrt{2}}x = \pm\frac{1}{\sqrt{5}}x$$

Substituting $x = 0$ into $x^2 - 5y^2 = 2$ we obtain $-5y^2 = 2$ and $y^2 = -2/5$ (which has no solutions). So, there are no y-intercepts. Substituting $y = 0$ into $x^2 - 5y^2 = 2$ we obtain $x^2 = 2$; thus, $x = \pm\sqrt{2}$ are the two x-intercepts. We conclude that the hyperbola lies in the regions to the left and to the right of its asymptotes.

(c) Divide $3x^2 - 4y^2 = 12$ by 12 to obtain 1 on the right side:

$$\frac{x^2}{4} - \frac{y^2}{3} = 1$$

Thus $a^2 = 4$ and $a = 2$ and $b^2 = 3$ and $b = \sqrt{3}$. The asymptotes are

$$y = \pm\frac{b}{a}x = \pm\frac{\sqrt{3}}{2}x$$

Substituting $x = 0$ into $3x^2 - 4y^2 = 12$ we obtain $-4y^2 = 12$ and $y^2 = -3$ (which has no solutions). So, there are no y-intercepts. Substituting $y = 0$ into $3x^2 - 4y^2 = 12$ we obtain $3x^2 = 12$ and $x^2 = 4$; thus, $x = \pm 2$ are the two x-intercepts. We conclude that the hyperbola lies in the regions to the left and to the right of its asymptotes.

(d) Rewrite the equation $-3x^2 + 4y^2 = 1$ as

$$-\frac{x^2}{1/3} + \frac{y^2}{1/4} = 1$$

Thus $a^2 = 1/3$ and $a = 1/\sqrt{3}$ and $b^2 = 1/4$ and $b = 1/2$. The asymptotes are

$$y = \pm\frac{b}{a}x = \pm\frac{\frac{1}{2}}{\frac{1}{\sqrt{3}}}x = \pm\frac{\sqrt{3}}{2}x$$

Substituting $x = 0$ into $-3x^2 + 4y^2 = 1$ we obtain $4y^2 = 1$ and $y^2 = 1/4$. We conclude that the y-intercepts are $y = \pm 1/2$. Substituting $y = 0$ into $-3x^2 + 4y^2 = 1$ we obtain $-3x^2 = 1$; thus, there are no x-intercepts. The hyperbola lies in the regions above and below its asymptotes.

(e) Rewrite $\frac{x^2}{12} - 2y^2 = 1$ as

$$\frac{x^2}{12} - \frac{y^2}{1/2} = 1$$

Thus $a^2 = 12$ and $a = \sqrt{12} = 2\sqrt{3}$ and $b^2 = 1/2$ and $b = \sqrt{1/2} = 1/\sqrt{2}$. The asymptotes are

$$y = \pm\frac{b}{a}x = \pm\frac{\frac{1}{\sqrt{2}}}{2\sqrt{3}}x = \pm\frac{1}{2\sqrt{2}\sqrt{3}}x == \pm\frac{1}{2\sqrt{6}}x$$

Substituting $x = 0$ into $\frac{x^2}{12} - 2y^2 = 1$ we obtain $-2y^2 = 1$ and $y^2 = -1/2$ (which has no solutions). Thus, there are no y-intercepts. Substituting $y = 0$ into $\frac{x^2}{12} - 2y^2 = 1$ we obtain $\frac{x^2}{12} = 1$ and $x^2 = 12$; thus, $x = \pm\sqrt{12} = \pm 2\sqrt{3}$ are the two x-intercepts. We conclude that the hyperbola lies in the regions to the left and to the right of its asymptotes.

(f) Divide the equation by 7, to obtain

$$-\frac{x^2}{21} + \frac{y^2}{49} = 1$$

Thus $a^2 = 21$ and $a = \sqrt{21}$ and $b^2 = 49$ and $b = 7$. The asymptotes are

$$y = \pm\frac{b}{a}x = \pm\frac{7}{\sqrt{21}}x$$

Substituting $x = 0$ into $-\frac{x^2}{3}+\frac{y^2}{7} = 7$ we obtain $y^2 = 49$ and the y-intercepts are $y = \pm 7$. Substituting $y = 0$ into $-\frac{x^2}{3} + \frac{y^2}{7} = 7$ we obtain $-x^2 = 21$; thus, no x-intercepts. We conclude that the hyperbola lies in the regions above and below its asymptotes.

20. (a) The coefficients of x^2 and y^2 are positive and not equal; thus, this is an ellipse.

Divide the equation $6x^2 + y^2 = 3$ by 3 to obtain

$$2x^2 + \frac{y^2}{3} = 1, \text{ i.e., } \frac{x^2}{1/2} + \frac{y^2}{3} = 1$$

Comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we see that $a^2 = 1/2$ and $b^2 = 3$.

The given equation represents an ellipse with semi-axes $a = \sqrt{1/2} = 1/\sqrt{2}$ and $b = \sqrt{3}$, centred at the origin. Its x-intercepts are $\pm a = \pm 1/\sqrt{2}$, and its y-intercepts are $\pm b = \pm\sqrt{3}$.

(b) The parabola $y = x^2 + \frac{3x}{2} - 1$ opens upward since its x^2 coefficient is positive. Solving for intercepts, we find

$$\begin{aligned} x^2 + \frac{3x}{2} - 1 &= 0 \\ 2x^2 + 3x - 2 &= 0 \\ x &= \frac{-3 \pm \sqrt{9+16}}{4} = \frac{-3 \pm 5}{4} \end{aligned}$$

Thus, the x-intercepts are $x = 1/2$ and $x = -2$.

The x-coordinate of the vertex is $x = -\frac{b}{2a} = -\frac{3/2}{2} = -\frac{3}{4}$; the corresponding y-coordinate is

$$y = \left(-\frac{3}{4}\right)^2 + \frac{3\left(-\frac{3}{4}\right)}{2} - 1 = \frac{9}{16} - \frac{9}{8} - 1 = -\frac{25}{16}$$

Thus, the vertex is at $(-3/4, -25/16)$.

The y-intercept is $y = -1$.

The graph is a parabola that opens upward, has its vertex at $(-3/4, -25/16)$, and the y-intercept at -1. The x-intercepts are $x = 1/2$ and $x = -2$.

(c) Divide the equation by 6, to obtain $x^2 + y^2 = 1/2$. This is a circle centred at the origin of radius $r = \sqrt{1/2} = 1/\sqrt{2}$.

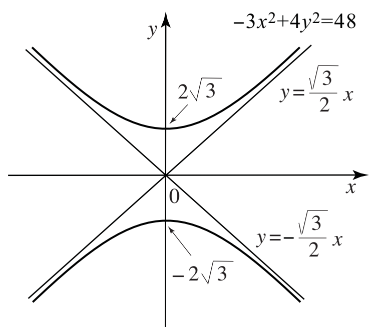
(d) Dividing both sides of the equation $-3x^2 + 4y^2 = 48$ by 48, we obtain

$$-\frac{x^2}{16} + \frac{y^2}{12} = 1$$

Because one of the signs is a minus, the equation represents a hyperbola. From $a^2 = 16$ we get $a = 4$; from $b^2 = 12$ we get $b = \sqrt{12} = 2\sqrt{3}$. It follows that the asymptotes are given by

$$y = \pm\frac{b}{a}x = \pm\frac{2\sqrt{3}}{4} = \pm\frac{\sqrt{3}}{2}x.$$

Substituting $y = 0$ into the given equation (we are looking for x-intercepts) we get $-3x^2 = 48$, i.e., $x^2 = -16$. Since this equation has no solutions, there are no x-intercepts. Substituting $x = 0$, we get $4y^2 = 48$, i.e., $y^2 = 12$, and $y = \pm 2\sqrt{3}$. So, there are two y-intercepts, $2\sqrt{3}$ and $-2\sqrt{3}$. The hyperbola lies in the regions above and below its asymptotes; see the figure below.



(e) Because the coefficients of x^2 and y^2 are equal, this is a circle. We complete the square to find the centre and the radius. Start by multiplying by 3 and gathering the x terms together:

$$\frac{x^2}{3} + \frac{y^2}{3} - 3x = 1$$
$$x^2 - 9x + y^2 = 3$$
$$\left(x - \frac{9}{2}\right)^2 - \frac{81}{4} + y^2 = 3$$
$$\left(x - \frac{9}{2}\right)^2 + y^2 = 3 + \frac{81}{4}$$
$$\left(x - \frac{9}{2}\right)^2 + y^2 = \frac{93}{4}$$

This is a circle centred at $(9/2, 0)$ of radius equal to $\sqrt{93}/2$.

(f) Since the coefficients of x^2 and y^2 are equal, this equation represents a circle. Complete the square to identify its centre and radius:

$$x^2 + 5x + y^2 - y = 0$$

$$\left(x+\frac{5}{2}\right)^2-\frac{25}{4}+\left(y-\frac{1}{2}\right)^2-\frac{1}{4}=0$$
$$\left(x+\frac{5}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{13}{2}$$

Thus, this equation represents the circle centered at $(-5/2, 1/2)$, of radius $\sqrt{13/2}$.

(g) We can use general fact – the equation $x = ay^2$ for $a < 0$ represents a parabola that opens towards the negative x-axis, and its vertex is at the origin.

Alternatively, we proceed as follows: switching x and y we get $y = -x^2$, which is a parabola that opens downward, with its vertex at the origin. Drawing the mirror image of it with respect to $y = x$, we obtain the graph of $x = -y^2$.

(h) Switching x and y, we obtain $y = -x^2 + 1$. The coefficient of x^2 is -1, and so this parabola opens downward. From $-x^2 + 1 = 0$ we get $x^2 = 1$; the x-intercepts are $x = \pm 1$.

The x-coordinate of the vertex is $x = -b/2a = 0/(-2) = 0$; the corresponding y-coordinate is 1; so the vertex is at $(0, 1)$.

Thus, the graph of $y = -x^2 + 1$ is a parabola that opens downward; its vertex is at $(0, 1)$, and it crosses the x-axis at -1 and 1.

Now switching back (i.e., drawing a picture symmetric with respect to the line $y = x$): the graph of $x = -y^2 + 1$ is a parabola that opens to the left, i.e., toward the negative x-axis; its vertex is at $(1, 0)$, and it crosses the y-axis at -1 and 1.

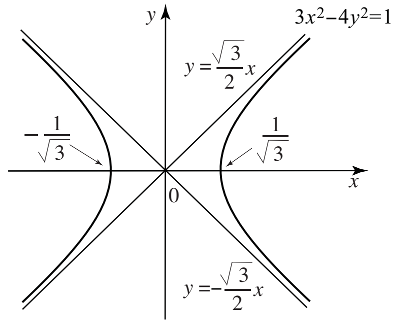
(i) We rewrite $3x^2 - 4y^2 = 1$ as

$$\frac{x^2}{1/3}-\frac{y^2}{1/4}=1$$

One of the signs is a minus, the other is a plus, and so this a hyperbola. From $a^2 = 1/3$ we get $a = 1/\sqrt{3}$; from $b^2 = 1/4$ we get $b = 1/2$. It follows that the asymptotes are given by

$$y=\pm\frac{b}{a}x=\pm\frac{1/2}{1/\sqrt{3}}=\pm\frac{\sqrt{3}}{2}x.$$

Substituting $y = 0$ into the given equation (we are looking for x-intercepts) we get $3x^2 = 1$, i.e., $x^2 = 1/3$. So, there are two x-intercepts, $x = \pm 1/\sqrt{3}$. Substituting $x = 0$, we get $-4y^2 = 1$, i.e., $y^2 = -1/4$.Thus, there are no y-intercepts. The hyperbola lies in the regions to the right and to the left of its asymptotes; see the figure below.



(j) The coefficients of x^2 and y^2 have the same sign, and are not equal. Thus, this is an ellipse. We complete the square and reduce the equation to the form given in the text of Exercise 17:

$$\begin{aligned} x^2 + 6x + 8y^2 &= 0 \\ (x+3)^2 - 9 + 8y^2 &= 0 \\ (x+3)^2 + 8y^2 &= 9 \\ \frac{(x+3)^2}{9} + \frac{y^2}{\frac{9}{8}} &= 1 \end{aligned}$$

This is an ellipse centred at $(-3, 0)$ with semi axes $a = \sqrt{9} = 3$ and $b = \sqrt{9/8} = 3/\sqrt{8}$.

(k) The coefficients of x^2 and y^2 have the same sign, and are not equal. Thus, this is an ellipse. We complete the square and reduce the equation to the form given in the text of Exercise 17:

$$\begin{aligned} x^2 + 5x + 4y^2 - y &= 0 \\ \left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + 4\left(y^2 - \frac{y}{4}\right) &= 0 \\ \left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + 4\left[\left(y - \frac{1}{8}\right)^2 - \frac{1}{64}\right] &= 0 \\ \left(x + \frac{5}{2}\right)^2 + 4\left(y - \frac{1}{8}\right)^2 &= \frac{25}{4} + \frac{4}{64} = \frac{101}{16} \\ \frac{\left(x + \frac{5}{2}\right)^2}{1} + \frac{\left(y - \frac{1}{8}\right)^2}{\frac{1}{4}} &= \frac{101}{16} \\ \frac{\left(x + \frac{5}{2}\right)^2}{\frac{101}{16}} + \frac{\left(y - \frac{1}{8}\right)^2}{\frac{1}{4} \cdot \frac{101}{16}} &= 1 \\ \frac{\left(x + \frac{5}{2}\right)^2}{\frac{101}{16}} + \frac{\left(y - \frac{1}{8}\right)^2}{\frac{101}{64}} &= 1 \end{aligned}$$

This equation represents the ellipse centred at $(-5/2, 1/8)$ with semi axes $a = \sqrt{101/16} = \sqrt{101}/4$ and $b = \sqrt{101/64} = \sqrt{101}/8$.

21. (a) Testing $(1, 1)$: $x^2 + y^2 - 2x + 4y - 4 = (1)^2 + (1)^2 - 2(1) + 4(1) - 4 = 1 + 1 - 2 + 4 - 4 = 0$, so it lies on the curve.

Testing $(0, 1)$: $x^2 + y^2 - 2x + 4y - 4 = (0)^2 + (1)^2 - 2(0) + 4(1) - 4 = 0 + 1 - 0 + 4 - 4 = 1 \neq 0$, so it does not lie on the curve.

Testing $(2, 0)$: $x^2 + y^2 - 2x + 4y - 4 = (2)^2 + (0)^2 - 2(2) + 4(0) - 4 = 4 + 0 - 4 + 0 - 4 = -4 \neq 0$, so it does not lie on the curve.

(b) Testing $(0, 0)$: $-x^2 + y^3 - 2x - 11y = -(0)^2 + (0)^3 - 2(0) - 11(0) = 0$, so it lies on the curve.

Testing $(2, 1)$: $-x^2 + y^3 - 2x - 11y = -(2)^2 + (1)^3 - 2(2) - 11(1) = -4 + 1 - 4 - 11 = -18 \neq 0$, so it does not lie on the curve.

Testing $(-1, -1)$: $-x^2 + y^3 - 2x - 11y = -(-1)^2 + (-1)^3 - 2(-1) - 11(-1) = -1 - 1 + 2 + 11 = 11 \neq 0$, so it does not lie on the curve.

(c) Testing $(1, 0)$: $y = x^3 - x + 12 = 1^3 - 1 + 12 = 12 \neq 0$, so it does not lie on the curve.

Testing $(1, 12)$: $y = x^3 - x + 12 = 1^3 - 1 + 12 = 12$, so it lies on the curve.

Testing $(-1, 12)$: $y = x^3 - x + 12 = (-1)^3 - (-1) + 12 = 12$, so it lies on the curve.

22. (a) The two equations represent lines. Substitute $x = 2y - 4$ obtained from the first equation into the second equation:

$$\begin{aligned} -4(2y-4) - y &= 12 \\ -8y + 16 - y &= 12 \\ -9y &= -4 \\ y &= \frac{4}{9} \end{aligned}$$

When $y = \frac{4}{9}$,

$$x = 2\left(\frac{4}{9}\right) - 4 = \frac{8}{9} - \frac{36}{9} = -\frac{28}{9}$$

Thus, the two lines intersect at the point $(-28/9, 4/9)$.

(b) The two equations represent lines. Writing the second equation as $y = -3x + 6$, we see that both lines have the same slope $m = -3$. Thus, they are parallel (and not identical, since the y-intercepts are not equal), and have no points in common.

(c) The two equations represent lines. Computing x from the first equation ($x = \frac{y}{2} - 1$) and substituting into the first equation, we obtain

$$\begin{aligned} 2y - 4\left(\frac{y}{2} - 1\right) &= 4 \\ 2y - 2y + 4 &= 4 \\ 0 \cdot y &= 0 \end{aligned}$$

Thus, any real number y is a solution. It follows that the two lines have all their points in common, i.e., the two equations actually represent the same line.

An alternative way to do this is to multiply the first equation by 4 and realize that it is identical to the second equation.

(d) The two equations represent lines. Getting rid of fractions, we obtain

$$5x + 3y = 0 \quad \text{and} \quad 3x - 2y = 12.$$

Substituting $y = -\frac{5x}{3}$, obtained from the first equation, into the second equation, we obtain

$$\begin{aligned} 3x - 2\left(-\frac{5x}{3}\right) &= 12 \\ 3x + \frac{10x}{3} &= 12 \\ 9x + 10x &= 36 \\ x &= \frac{36}{19} \end{aligned}$$

Thus

$$y = -\frac{5x}{3} = -\frac{5 \cdot \frac{36}{19}}{3} = -\frac{180}{19} \cdot \frac{1}{3} = -\frac{60}{19}$$

The two lines intersect at $(36/19, -60/19)$.

(e) The two equations represent lines. By adding them, we obtain $2y = 6$ and $y = 3$. By using either equation, we obtain $x = 0$. Thus the two lines intersect at $(0, 3)$.

(f) The two equations represent parallel lines (note that both are of slope -6). Because their y-intercepts are not the same, the two lines have no points in common. (If the two interceptes were equal, then the two lines would be lying on top of each other, i.e., it would be the same line.)

(g) The equation $x^2 + y^2 = 14$ represents the circle centred at the origin of radius $\sqrt{14}$. The equation $y = -x$ gives a line of slope -1 through the origin. Substituting $y = -x$ into $x^2 + y^2 = 14$ we obtain

$$\begin{aligned} x^2 + (-x)^2 &= 14 \\ 2x^2 &= 14 \\ x^2 &= 7 \\ x &= \pm\sqrt{7} \end{aligned}$$

When $x = \sqrt{7}$, then $y = -x = -\sqrt{7}$. When $x = -\sqrt{7}$, then $y = -x = \sqrt{7}$. The given curves have two points in common, $(\sqrt{7}, -\sqrt{7})$ and $(-\sqrt{7}, \sqrt{7})$.

(h) The equation $x^2 + y^2 = 25$ represents the circle centred at the origin of radius 5. The equation $x = -4$ represents the vertical line which crosses the x-axis at $x = -4$. The two curves intersect when

$$\begin{aligned} (-4)^2 + y^2 &= 25 \\ 16 + y^2 &= 25 \\ y^2 &= 9 \\ y &= \pm 3 \end{aligned}$$

The points in common are $(-4, 3)$ and $(-4, -3)$.

(i) The equation $x^2 - y^2 = 1$ represents the hyperbola (think of it as $\frac{x^2}{1} - \frac{y^2}{1} = 1$) with asymptotes $y = \pm 1$ and x-intercepts $x = \pm 1$. It lies to the left and to the right of its asymptotes. The equation $x - 2y = 0$, or $y = x/2$, represents the line through the origin of slope $1/2$.

Substituting $y = x/2$ into $x^2 - y^2 = 1$ we obtain

$$\begin{aligned} x^2 - \frac{x^2}{4} &= 1 \\ \frac{3x^2}{4} &= 1 \\ x^2 &= \frac{4}{3} \\ x &= \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}} \end{aligned}$$

Substituting $x = \pm 2/\sqrt{3}$ into $y = x/2$ we obtain $y = \pm 1/\sqrt{3}$. There are two points of intersection, $(2/\sqrt{3}, 1/\sqrt{3})$ and $(-2/\sqrt{3}, -1/\sqrt{3})$.

(j) The equation $-x^2 + 2y^2 = 4$ represents the hyperbola (think of it as $-\frac{x^2}{4} + \frac{y^2}{2} = 1$) with $a = 2$ and $b = \sqrt{2}$. Its asymptotes are $y = \pm\frac{\sqrt{2}}{2}x = \frac{1}{\sqrt{2}}x$ and the y-intercepts are $y = \pm\sqrt{2}$. It lies above and below its asymptotes. The equation $y = x + 1$ represents the line through $(0, -1)$ of slope 1. Combining the two equations,

$$\begin{aligned} -x^2 + 2(x+1)^2 &= 4 \\ -x^2 + 2(x^2 + 2x + 1) &= 4 \\ x^2 + 4x - 2 &= 0 \end{aligned}$$

Using the quadratic formula,

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{(4)^2 - 4(1)(-2)}}{2(1)} \\ x &= \frac{-4 \pm \sqrt{24}}{2} \\ x &= \frac{-4 \pm 2\sqrt{6}}{2} \\ x &= -2 \pm \sqrt{6} \end{aligned}$$

When $x = -2 + \sqrt{6}$, then $y = (-2 + \sqrt{6}) + 1 = \sqrt{6} - 1$.

When $x = -2 - \sqrt{6}$, then $y = (-2 - \sqrt{6}) + 1 = -\sqrt{6} - 1$.

The points in common are $(-2 + \sqrt{6}, \sqrt{6} - 1)$ and $(-2 - \sqrt{6}, -\sqrt{6} - 1)$.

(k) Writing $2x^2 + y^2 = 3$ as $\frac{2x^2}{3} + \frac{y^2}{3} = 1$, i.e.,

$$\frac{x^2}{3/2} + \frac{y^2}{3} = 1$$

we identify the curve as the ellipse centred at the origin with semi-axes $a = \sqrt{3/2}$ and $b = \sqrt{3}$. Since the highest point on the ellipse has the y-coordinate equal to $\sqrt{3} < 5$, the ellipse has no points in common with the horizontal line $y = 5$.

Alternatively, combining the two equations,

$$\begin{aligned} 2x^2 + (5)^2 &= 3 \\ 2x^2 &= -22 \\ x^2 &= -11 \end{aligned}$$

which has no solutions.

(l) Writing $2x^2 + y^2 = 3$ as $\frac{2x^2}{3} + \frac{y^2}{3} = 1$, i.e.,

$$\frac{x^2}{3/2} + \frac{y^2}{3} = 1$$

we identify the curve as the ellipse centred at the origin with semi-axes $a = \sqrt{3/2}$ and $b = \sqrt{3}$. We are looking for the points of intersection with the horizontal line $y = 1$.

Combining the two equations,

$$\begin{aligned} 2x^2 + (1)^2 &= 3 \\ 2x^2 &= 2 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

Thus, the points in common are $(1, 1)$ and $(-1, 1)$.

(m) Writing $4x^2 + 3y^2 = 1$ as

$$\frac{x^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{3}} = 1$$

we recognize the ellipse centred at the origin, with semi-axes $a = 1/2$ and $b = 1/\sqrt{3}$.

Rewrite $x + y = 1$ as $y = -x + 1$ to recognize the line of slope -1 and y-intercept 1.

Substituting $y = -x + 1$ into $4x^2 + 3y^2 = 1$ we obtain

$$\begin{aligned} 4x^2 + 3(-x + 1)^2 &= 1 \\ 4x^2 + 3x^2 - 6x + 3 &= 1 \\ 7x^2 - 6x + 2 &= 0 \\ x &= \frac{6 \pm \sqrt{36 - 56}}{14} \\ x &= \frac{6 \pm \sqrt{-20}}{14} \end{aligned}$$

Since the expression under the square root (the discriminant) is negative, this quadratic equation has no real solutions. Consequently, the two curves do not intersect.

(n) The equation $2x^2 - y^2 = 10$ represents the hyperbola (think of it as $\frac{x^2}{5} - \frac{y^2}{10} = 1$) with $a = \sqrt{5}$ and $b = \sqrt{10}$. Its asymptotes are $y = \pm\frac{\sqrt{10}}{\sqrt{5}}x = \pm\sqrt{2}x$ and the x-intercepts are ($2x^2 = 10$, $x^2 = 5$ and thus) $x = \pm\sqrt{5}$. It lies to the left and to the right of its asymptotes. The equation $y = 0$ represents the x-axis. Thus, we are looking for the x-intercepts, which we found already: $(\pm\sqrt{5}, 0)$.

23. (a) Recall that in the PQ-space, P is the independent variable (and is plotted on the horizontal axis) and Q is the dependent variable (and is plotted on the vertical axis). Solving $2P - 3Q - 4 = 0$ for Q in terms of P, we obtain $3Q = 2P - 4$ and

$$Q = \frac{2}{3}P - \frac{4}{3}$$

This is a line with slope $2/3$ whose vertical i.e., Q-intercept is equal to $-3/4$.

(b) "P vs. t" means that P is on the vertical axis, and t is on the horizontal axis. (Thus, the curve $P = 2t^2 + 1.4$ is like $y = 2x^2 + 1.4$ in the usual xy-coordinate system.) This equation represents the parabola; there are no t intercepts, since $2t^2 + 1.4 = 0$ gives the equation $t^2 = -1.4/2$, which has no solutions. Setting $t = 0$, we obtain the P-intercept at $(0, 1.4)$. The t-coordinate of the vertex is $t = -0/4 = 0$, and the corresponding P value is 1.4. Thus, the vertex is at $(0, 1.4)$.

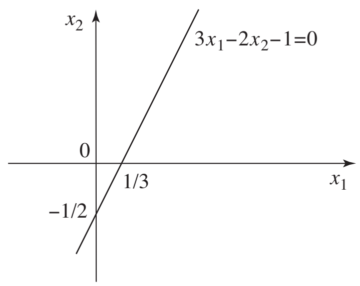
In conclusion, the graph of $P = 2t^2 + 1.4$ is a parabola which opens upward, with its vertex at $(0, 1.4)$. It has no t-intercepts, so it obviously lies above the t-axis.

Alternatively, we can use transformations of graphs (explained in Section 6): we start with the parabola $y = t^2$, expand it vertically by a factor of 2 and then shift up by 1.4 units.

(c) Recall that "x_2 vs. x_1" means that x_1 is on the horizontal, and x_2 on the vertical axis. Solving for x_2, we obtain $2x_2 = 3x_1 - 1$ and

$$x_2 = \frac{3}{2}x_1 - \frac{1}{2}$$

So, the graph is a line of slope $3/2$, with the vertical intercept (x_2-intercept) of $-1/2$. See below.



(d) The equation $aM + bN + c = 0$ is linear in M and N, and so it represents a line. The phrase "MN-coordinate system" suggests that M is the independent variable (plotted on a horizontal axis) and N is the dependent variable (plotted on a vertical axis). Solving $aM + bN + c = 0$ for N, we obtain $bN = -aM - c$, and

$$N = -\frac{a}{b}M - \frac{c}{b}$$

This is a line of slope $-a/b$ and N-intercept $-c/b$.

(e) Rewrite $z_2 - 2z_1 = 0$ as $z_2 = 2z_1$ (so, it is like $y = 2x$). This is a line of slope 2, going through the origin in the z_1z_2-coordinate system.

(f) Substituting $P = 2Q + 4$ obtained from the second equation into the first equation, we find

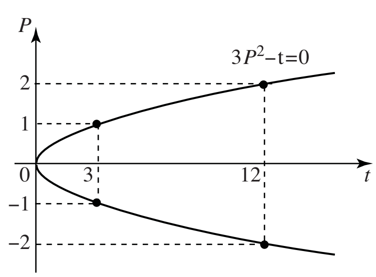
$$\begin{aligned} 3(2Q+4) - \frac{1}{2}Q &= 0 \\ 6Q - \frac{1}{2}Q &= -12 \\ \frac{11}{2}Q &= -12 \\ Q &= -12 \cdot \frac{2}{11} = -\frac{24}{11} \end{aligned}$$

Then $P = 2Q + 4 = 2\left(-\frac{24}{11}\right) + 4 = -\frac{4}{11}$. We conclude that the two lines intersect at the point with coordinates $(P = -4/11, Q = -24/11)$.

(g) The context defines P as dependent, and t as independent variable.

Translated into the usual xy-coordinate system, the equation reads $3y^2 - x = 0$ or $x = 3y^2$. We recognize this equation as representing a parabola which opens toward the positive x-axis, with the vertex at the origin.

Thus, $3P^2 - t = 0$, i.e., $t = 3P^2$, is a parabola which opens toward positive t-axis, with the vertex at the origin. To make a more precise sketch, we plot points: when $P = \pm 1$, then $t = 3$, when $P = \pm 2$, then $t = 12$, and so on. See below.



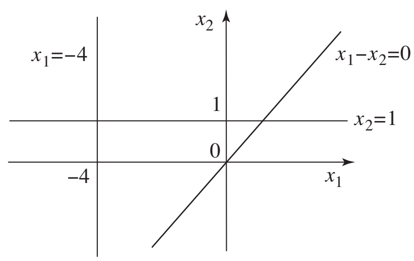
(h) Recall that "x_1x_2-space" means that x_1 is on the horizontal, and x_2 on the vertical axis.

From $x_1 - x_2 = 0$ we obtain $x_2 = x_1$; so this is a line of slope 1 through the origin.

Equation $x_1 = -4$ describes all points (x_1, x_2) in the x_1x_2-coordinate system whose coordinates are of the form $(-4, \text{any real number})$. This is a vertical line, crossing the x_1-axis at -4.

Equation $x_2 = 1$ describes all points (x_1, x_2) whose coordinates are $(\text{any real number}, 1)$. This is a horizontal line, crossing the x_2-axis at 1.

See below.



(i) Rewrite $P^2 + 4Q^2 = 5$ as

$$\frac{P^2}{5} + \frac{Q^2}{\frac{5}{4}} = 1$$

Because Q is on the vertical axis, Q is like y; the coordinate P is like x.

So this is the ellipse centred at the origin of the PQ-coordinate system, with the horizontal semi-axis equal to $\sqrt{5}$ and the vertical semi-axis equal to $\sqrt{5}/2$.

(j) Rewrite $P^2 + 4Q^2 = 5$ as

$$\frac{P^2}{5} + \frac{Q^2}{\frac{5}{4}} = 1$$

Because P is on the vertical axis, P is like y; the coordinate Q is like x, so it makes sense to rewrite the equation as

$$\frac{Q^2}{\frac{5}{4}} + \frac{P^2}{5} = 1$$

So this is the ellipse centred at the origin of the QP-coordinate system, with the horizontal semi-axis equal to $\sqrt{5}/2$ and the vertical semi-axis equal to $\sqrt{5}$.

(k) Multiplying by a, we obtain

$$X^2 + Z^2 = a$$

This is a circle centred at the origin (of the XZ-coordinate system) of radius $\sqrt{a}$.

(l) Recall that "q vs. p" means that p is on the horizontal, and q on the vertical axis (so, p is like x and q is like y).

Because the coefficients of p^2 and q^2 are of opposite signs, this is a hyperbola; we read $a^2 = 2$ and so $a = \sqrt{2}$, and $b^2 = 3$ and so $b = \sqrt{3}$.

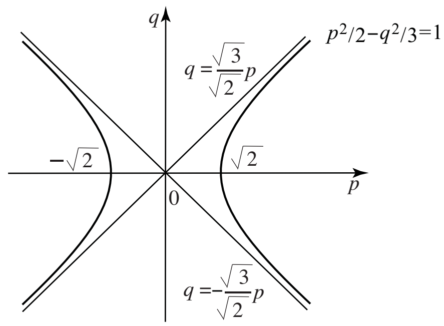
The asymptotes are

$$q = \pm\frac{b}{a}p = \pm\frac{\sqrt{3}}{\sqrt{2}}p$$

Substituting $p = 0$, we obtain $-\frac{q^2}{3} = 1$ and $q^2 = -3$; no solutions, so no q-intercepts.

Substituting $q = 0$, we obtain $\frac{p^2}{2} = 1$ and $p^2 = 2$; p-intercepts are $(\pm\sqrt{2}, 0)$.

Thus, the hyperbola lies to the left and to the right of its asymptotes. See below.



Section 6. Functions

1. (a) $f(x)$ is a polynomial, and so its domain is the set of all real numbers. Using set notation, we write $D = \{x \,|\, x \in R\}$.

(b) The denominator $x - 2$ is zero when $x = 2$. The domain of y consists of all real numbers x except $x = 2$. Using set notation, we write $D = \{x \,|\, x \neq 2\}$.

(c) Since $x^2 + 1 > 0$ no matter what x is, the denominator in this rational function is never 0. Thus, the domain consists of all real numbers.

(d) For $\sqrt{x}$ to be defined, x has to satisfy $x \geq 0$. But since $\sqrt{x}$ is in the denominator, we cannot allow $x = 0$ (since $\sqrt{0} = 0$). Thus, the domain consists of all x such that $x > 0$.

(e) Recall that $\sqrt[3]{x}$ is defined for all real numbers x, and $\sqrt[3]{0} = 0$. Thus, the only restriction on the domain comes from forcing the denominator to be non-zero. We conclude that the domain consists of all x except $x = 0$. In set notation, $D = \{x \,|\, x \neq 0\}$.

(f) Because of the fraction involved, all x in the domain must satisfy $x^2 + 1 \neq 0$. Because of the square root,

$$\frac{x+5}{x^2+1} \geq 0$$

Note that $x^2+1 > 0$ for all x (x^2 is zero or positive, and then 1 is added to it). Thus the denominator of y is not zero, no matter what x is. Since the fraction $\frac{x+5}{x^2+1}$ must be positive or zero, and the denominator is positive, we conclude that $x + 5 \geq 0$, and thus $x \geq -5$. Using interval notation we write $D = [-5, \infty)$.

(g) Since the fourth root is defined for positive numbers and zero, we need $x - 2 \geq 0$; moreover, $x - 2 \neq 0$ guarantees that the denominator is not zero. So, x is in the domain of the given function if $x - 2 > 0$, i.e., $x > 2$. Using set notation, $D = \{x \mid x > 2\}$.

(h) The domain consists of all real numbers except those for which $x^3 - x = 0$. Factoring

$$x^3 - x = x(x^2 - 1) = x(x-1)(x+1) = 0$$

we see that $x = 0$, $x = 1$ or $x = -1$. Therefore, $D = \{x \mid x \neq 0, 1, -1\}$.

(i) Because of $1/x$, we require that $x \neq 0$. On top of that, to make sure that we can calculate the sixth root (recall that $A^{1/6} = \sqrt[6]{A}$) we need

$$1 - \frac{1}{x} \geq 0.$$

To solve this inequality, we calculate common denominator

$$1 - \frac{1}{x} = \frac{x-1}{x} \geq 0$$

and then analyze the signs of the numerator and the denominator:

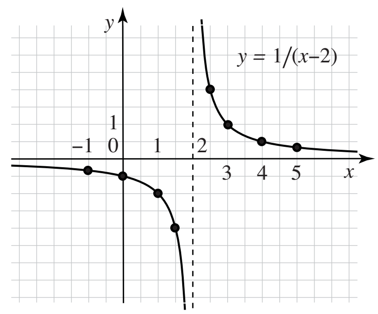
	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$x - 1$	$-$	$-$	$+$
x	$-$	$+$	$+$
$\frac{x-1}{x}$	$+$	$-$	$+$

From the table, we conclude that $1-\frac{1}{x} \geq 0$ when x is in $(-\infty, 0]$ or $[1, \infty)$. Remembering that $x \neq 0$, we see that the domain is $D = \{x \mid x < 0 \text{ or } x \geq 1\} = (-\infty, 0) \cup [1, \infty)$.

2. (a) The domain of the function $y = \frac{1}{x-2}$ is $x \neq 2$. We calculate the points (the more the better), plot them in the coordinate system and join with a curve.

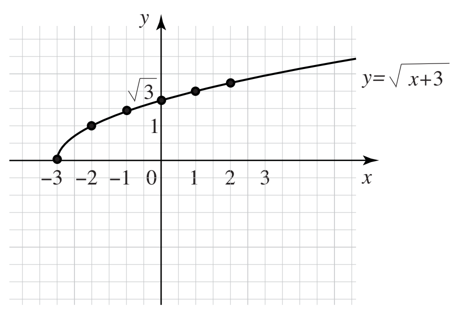
x	$y = \frac{1}{x-2}$
-1	$-1/3$
0	$-1/2$
1	-1
3	1
4	$1/2$
5	$1/3$

x	$y = \frac{1}{x-2}$
1.5	-2
1.8	-5
1.9	-10
2.1	10
2.2	5
2.5	2



(b) The domain of $y = \sqrt{x+3}$ is $x + 3 \geq 0$, i.e., $x \geq -3$.

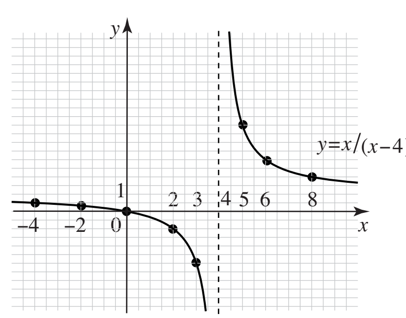
x	$y = \sqrt{x+3}$
-3	0
-2	1
-1	$\sqrt{2}$
0	$\sqrt{3}$
1	2
2	$\sqrt{5}$
6	3



(c) The domain of $y = \frac{x}{x-4}$ is $x \neq 4$.

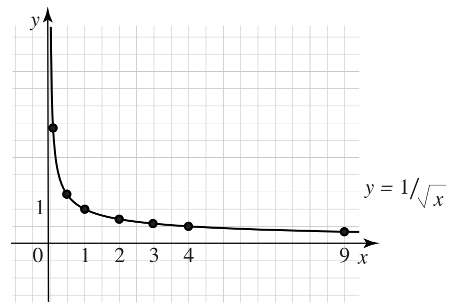
x	$y = \frac{x}{x-4}$
-4	$1/2$
-2	$1/3$
0	0
2	-1
3	-3
3.5	-7

x	$y = \frac{x}{x-4}$
4.2	21
4.5	9
5	5
6	3
8	2
12	1.5
$\dots$	$\dots$



(d) Real numbers x in the domain of the function $y = \frac{1}{\sqrt{x}}$ must satisfy $x \geq 0$ and $x \neq 0$; thus, the domain of y is the set $x > 0$.

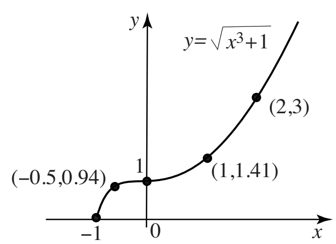
x	$y = \frac{1}{\sqrt{x}}$
0.1	$\sqrt{10} \approx 3.16$
0.5	$\sqrt{2} \approx 1.41$
1	1
2	$1/\sqrt{2} \approx 0.71$
3	$1/\sqrt{3} \approx 0.58$
4	$1/2$
9	$1/3$



3. The domain of f is determined from the requirement $x^3 + 1 \geq 0$, i.e., $x^3 \geq -1$. Looking at the graph of x^3 (or computing the third root of both sides) we see that $x \geq -1$.

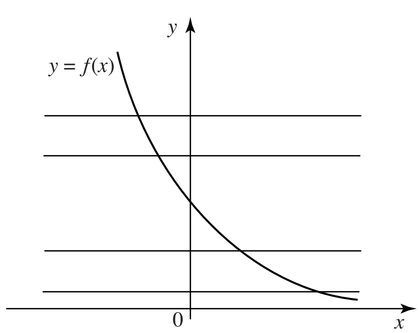
To sketch the graph, we calculate the table of values (below).

x	$\sqrt{x^3+1}$
-1	0
-0.5	0.94
0	1
1	1.41
2	3
2.5	4.08
...	...

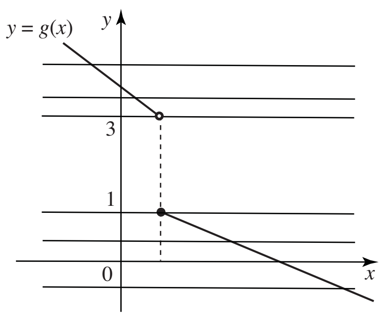


4. Recall that a number c (on the y-axis) is in the range of a function if the horizontal line through c intersects the graph of the function.

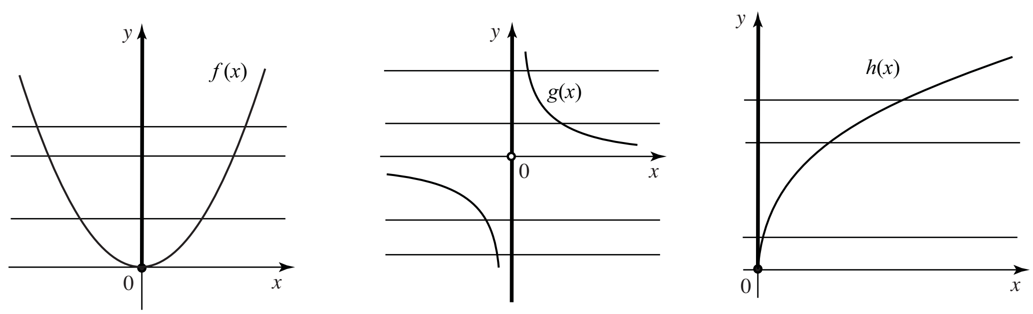
(a) Any horizontal line that crosses the y-axis at some positive number intersects the graph. On the other hand, the x-axis and the horizontal lines that lie below it do not cross the graph. Thus, the range consists of all positive numbers; i.e., $R = (0, \infty)$. See below.



(b) Horizontal lines that cross the y-axis between 1 and 3 do not intersect the graph. The line $y = 1$ intersects the graph, but $y = 3$ does not. All horizontal lines below $y = 1$ and above $y = 3$ intersect the graph. Thus, the range consists of all real numbers smaller than or equal to 1, or greater than 3. in symbols, $R = (-\infty, 1] \cup (3, \infty)$. See figure below.



5. The graphs are shown in figure below; we use horizontal lines (as in previous exercise) to test for the range. The range is shown in thicker line on the y-axis.



6. (a) $f(x) = x + 6$ is a polynomial (more specifically, a linear function). It is defined for all real numbers. By drawing the graph of $f(x)$ (line of slope 1, with y-intercept 6) we can convince ourselves that the range consists of all real numbers. In symbols, $D = \mathbb{R}$, $R = \mathbb{R}$.

Algebraic evidence for the range: we pick any real number c, and have to find x such that $f(x) = x + 6 = c$. Solving, we obtain $x = c - 6$. Thus, no matter which real number c we take, we can make $f(x)$ equal to that value c; in particular, when $x = c - 6$, then $f(x) = x + 6 = (c - 6) + 6 = c$.

(b) $y = -2x$ is a linear function, and thus defined for all real numbers. By drawing the graph of y (line of slope -2 through the origin) we see that the range consists of all real numbers. In symbols, $D = \mathbb{R}$, $R = \mathbb{R}$.

Algebraic evidence for the range: we pick any real number c, and have to find x such that $y = -2x = c$. Solving, we obtain $x = -c/2$. Thus, no matter which real number c we take, we can make y equal to that value c; in particular, when $x = -c/2$, then $y = -2x = -2(-c/2) = c$.

(c) $g(x) = 3x - 14$ is a linear function, and thus its domain is $\mathbb{R}$. By drawing the graph of $g(x)$ (line of slope 3, with y-intercept -14) we can convince ourselves that the range consists of all real numbers.

Algebraic evidence for the range: we pick any real number c, and have to find x such that $g(x) = 3x - 14 = c$. Solving, we obtain $x = (c + 14)/3$. Thus, no matter which real number c we take, we can make $g(x)$ equal to that value c; in particular, when $x = (c+14)/3$, then

$$g(x) = 3x - 14 = 3 \cdot \frac{c+14}{3} - 14 = (c+14) - 14 = c$$

(d) Since we can calculate the square of any number and add 4, the domain is the set of all real numbers.

Since $x^2 \geq 0$, we conclude (by adding 4 to both sides) that $g(x) = x^2 + 4 \geq 4$. Alternatively, sketch the parabola $y = x^2$, shift it up 4 units, and convince yourself (by drawing horizonal lines, as usual) that the range consists of all numbers larger than or equal to 4.

Algebraic evidence: we pick a real number $c \geq 4$ and have to find x such that $g(x) = x^2 + 4 = c$. Solving, we obtain $x^2 = c - 4$ and $x = \pm\sqrt{c-4}$. Thus, no matter which $c \geq 4$ we take, the function $g(x)$ assumes that value c at $x = \pm\sqrt{c-4}$.

In symbols, $D = \mathbb{R}$, $R = [4, \infty)$.

(e) Since we can calculate the square of any number and multiply by -1, the domain is the set of all real numbers.

Multiplying $x^2 \geq 0$ by -1 we obtain $-x^2 \leq 0$. Thus, the range is the set $y \leq 0$. Alternatively, sketch the parabola $y = x^2$, reflect it across the x-axis (to obtain $y = -x^2$), and convince yourself (by drawing horizonal lines) that the range consists of all numbers smaller than or equal to 0.

Algebraic evidence: we pick a real number $c \leq 0$ and have to find x such that $y(x) = -x^2 = c$. Solving, we obtain $x^2 = -c$ and $x = \pm\sqrt{-c}$ (note that $\sqrt{-c}$ is defined because $c \leq 0$). Thus, no matter which $c \leq 0$ we take, the function y assumes that value c at $x = \pm\sqrt{-c}$. Check:

$$y(x) = -x^2 = -\left(\pm\sqrt{-c}\right)^2 = -(-c) = c.$$

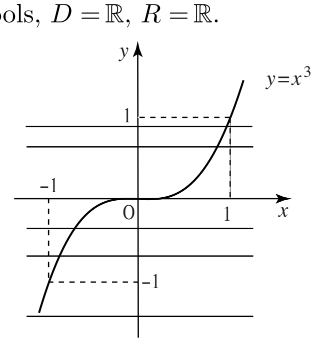
In symbols, $D = \mathbb{R}$, $R = (-\infty, 0]$.

(f) Since we can calculate the cube of any number, the domain is the set of all real numbers.

From the graph of $f(x) = x^3$ (see figure below) we see that the range consists of all real numbers.

Alternatively, we pick a real number c and try to find x such that $f(x) = x^3 = c$. Clearly, $x = \sqrt[3]{c}$; check: $f(\sqrt[3]{c}) = (\sqrt[3]{c})^3 = c$.

In symbols, $D = \mathbb{R}$, $R = \mathbb{R}$.



(g) Since we can calculate the cube of any number and subtract 7 from it, the domain of $f(x) = x^3 - 7$ is the set of all real numbers.

To obtain the graph of $f(x) = x^3 - 7$ shift the graph of $y = x^3$ (see the figure in part (f)) down for 7 units. We see that the range of $f(x) = x^3 - 7$ consists of all real numbers.

Algebraic proof: we pick a real number c and try to find x such that $f(x) = x^3 - 7 = c$. Solving for x, we find $x^3 = c + 7$ and $x = \sqrt[3]{c+7}$. Thus, no matter which real number c we take, the function $f(x)$ assumes that value c at $x = \sqrt[3]{c+7}$. Check:

$$f(x) = x^3 - 7 = \left(\sqrt[3]{c+7}\right)^3 - 7 = (c+7) - 7 = c.$$

In symbols, $D = \mathbb{R}$, $R = \mathbb{R}$.

(h) Since division by zero does not give a real number, the domain of $g(x)$ consists of all numbers $x \neq 3$.

Recall that the only way to make a fraction equal to zero is to make its numerator equal to zero. Since the numerator of $g(x)$ is 2, we conclude that $g(x) \neq 0$ (and thus the range of $g(x)$ does not contain zero).

Now we prove that any non-zero number is in the range of $g(x)$: we pick a real number $c \neq 0$ and find x such that $g(x) = \frac{2}{x-3} = c$. Solving for x, we find

$$\begin{aligned} \frac{2}{x-3} &= c \\ c(x-3) &= 2 \\ cx - 3c &= 2 \\ x &= \frac{2+3c}{c} \end{aligned}$$

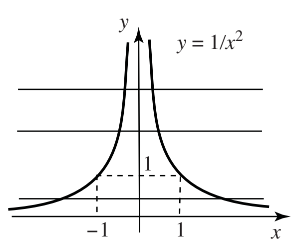
Note that x is defined, since $c \neq 0$. Check: pick any $c \neq 0$, and take $x = \frac{2+3c}{c}$; then

$$g(x) = \frac{2}{x-3} = \frac{2}{\frac{2+3c}{c} - 3} = \frac{2}{\frac{2+3c}{c} - \frac{3c}{c}} = \frac{2}{\frac{2}{c}} = 2 \cdot \frac{c}{2} = c$$

Thus, $D = \{x \,|\, x \in R,\, x \neq 3\}$ and $R = \{y \,|\, y \in R,\, y \neq 0\}$.

(i) To avoid division by zero, we need to remove $x = 0$ from the domain. Thus, $D = \{x \mid x \neq 0\}$.

Again, looking at the graph (below) we see that the range consists of all positive numbers.



To prove this claim algebraically, we show that no matter what $c > 0$ we take, we can find a value x such that $g(x) = c$. From $g(x) = \frac{1}{x^2} = c$ we get

$$x^2 = \frac{1}{c} \quad \text{and thus} \quad x = \pm\sqrt{\frac{1}{c}}.$$

In symbols, $D = \{x \mid x \neq 0\}$, $R = \{y \mid y > 0\}$.

(j) By definition, the domain of $h(x) = |x|$ is the set of all real numbers (i.e., we can calculate the absolute value of any real number).

As well, the definition states that the absolute value is a positive number, or zero. Thus, $D = \mathbb{R}$, $R = \{y \mid y \geq 0\}$.

Alternatively, we sketch the graph of $h(x) = |x|$ and determine the range from it.

(k) The domain of $h(x) = |x|$ is the set of all real numbers. Thus, the domain of $y = |x| - 3$ is $\mathbb{R}$.

By definition, $|x| \geq 0$. Subtracting 3 from both sides, we obtain $|x| - 3 \geq -3$. Thus, the range of y is the set $D = \{y \mid y \in \mathbb{R},\ y \geq -3\}$, or briefly, $y \geq -3$.

Alternatively, we sketch the graph of y by moving the graph of $|x|$ three units down. From the graph (by testing using horizontal lines) we see that the range is $y \geq -3$.

(l) We can calculate absolute value of any real number. Thus, the domain of $f(x) = |x+2|$ is $\mathbb{R}$.

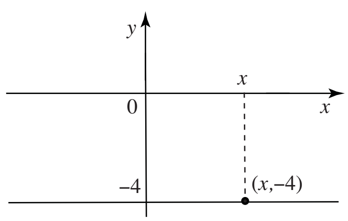
To sketch the graph of $f(x) = |x+2|$ we move the graph of $|x|$ two units to the left (or plot points). From the graph (by testing using horizontal lines) we see that the range is $y \geq 0$.

To prove this claim algebraically, we show that no matter what $c \geq 0$ we take, we can find a value x such that $f(x) = c$. From $f(x) = |x+2| = c$ we get $x + 2 = \pm c$ and $x = \pm c - 2$. Check: given $c \geq 0$, we compute $x = \pm c - 2$. For this value of x,

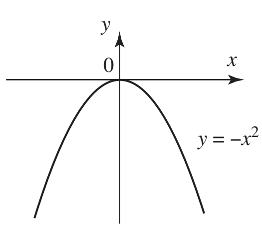
$$f(x) = |x+2| = |(\pm c - 2) + 2| = |\pm c| = |c| = c$$

Details for the last two equals signs: if we take the plus sign, then clearly $|+c| = |c|$ by just dropping the irrelevant plus sign; next, $|c| = c$ because it is assumed that $c \geq 0$. If we take the minus sign, then $|-c| = |c|$ by one of the properties of the absolute value, and $|c| = c$ because $c \geq 0$.

7. (a) The formula $f(x) = -4$ states that, no matter what the x value is, the value of the function $f(x)$ is -4. Thus, the graph of $f(x)$ contains all points of the form $(x, -4)$, where x is a real number. It is a horizontal line that crosses the y-axis at $y = -4$. See below.



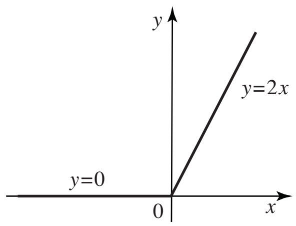
(b) By plotting points, or by reflecting the graph of $y = x^2$ with respect to the x-axis, we obtain the graph below.



(c) Using the definition of absolute value, we get

$$f(x) = |x| + x = \begin{cases} x + x & \text{if } x \geq 0 \\ -x + x & \text{if } x < 0 \end{cases} = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

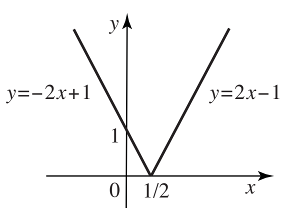
Thus, the graph of this piecewise defined function consists of the line of slope 2 through the origin (for $x \geq 0$) and the x-axis (for $x < 0$). See below.



(d) Using the definition of absolute value, we write

$$|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \geq 0 \\ -(2x-1) & \text{if } 2x-1 < 0 \end{cases} = \begin{cases} 2x-1 & \text{if } x \geq 1/2 \\ -2x+1 & \text{if } x < 1/2 \end{cases}$$

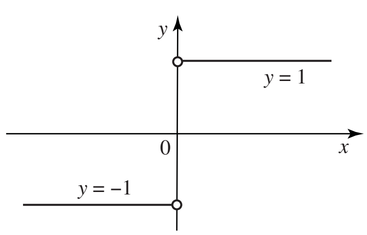
Thus, the graph of this piecewise defined function consists of two lines of slopes 2 and -2 which meet when $x = 1/2$. See below.



(e) The function is not defined at $x = 0$. Using the definition of absolute value, we get

$$f(x) = \frac{|x|}{x} = \begin{cases} x/x & \text{if } x > 0 \\ -x/x & \text{if } x < 0 \end{cases} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Thus, the graph of this piecewise defined function consists of the horizontal line $y = 1$ (for $x > 0$) and the horizontal line $y = -1$ (for $x < 0$). See below.

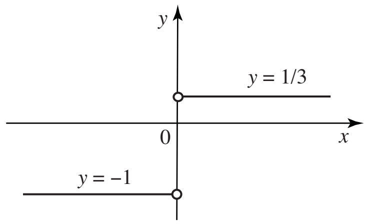


Note how the fact that the function is not defined at $x = 0$ is shown in the graph (no filled dots corresponding to $x = 0$).

(f) The function is not defined at $x = 0$. Using the definition of absolute value, we get

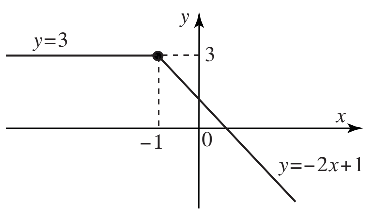
$$g(x) = \frac{2|x|-x}{3x} = \begin{cases} \dfrac{2x-x}{3x} & \text{if } x > 0 \\ \dfrac{-2x-x}{3x} & \text{if } x < 0 \end{cases} = \begin{cases} 1/3 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Thus, the graph of this piecewise defined function consists of the horizontal line $y = 1/3$ (for $x > 0$) and the horizontal line $y = -1$ (for $x < 0$). See below.



Note how the fact that the function is not defined at $x = 0$ is shown in the graph (no filled dots corresponding to $x = 0$).

(g) One piece of $f(x)$ is a horizontal line, and the other is a line of slope -2 and y-intercept 1. Note that when $x = -1$, then $y = -2x + 1 = (-2)(-1) + 1 = 3$. Thus, the two pieces meet at $(-1, 3)$. See below.



8. (a) Using the definition of composition,

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x}{2}\right) = 3\left(\frac{x}{2}\right) + 4 = \frac{3x}{2} + 4$$

(b) $(f \circ f)(x) = f(f(x)) = f(3x + 4) = 3(3x + 4) + 4 = 9x + 16$.

(c) $(g \circ f)(x) = g(f(x)) = g(3x + 4) = \dfrac{3x + 4}{2}$.

9. (a) $(f \circ g)(x) = f(g(x)) = f(x^2 + x + 1) = (x^2 + x + 1)^5$.

(b) $(f \circ f)(x) = f(f(x)) = f(x^5) = (x^5)^5 = x^{25}$.

(c) $(g \circ f)(x) = g(f(x)) = g(x^5) = (x^5)^2 + x^5 + 1 = x^{10} + x^5 + 1$.

10. (a) Replace $f(x)$ in $f(x) = \frac{2}{x-4}$ by y and solve for x

$$\begin{aligned} y &= \frac{2}{x-4} \\ \frac{1}{y} &= \frac{x-4}{2} \\ \frac{2}{y} &= x - 4 \\ x &= \frac{2}{y} + 4 \end{aligned}$$

Thus,

$$f^{-1}(y) = \frac{2}{y} + 4$$

In many situations we prefer to use x as independent variable; so, we write

$$f^{-1}(x) = \frac{2}{x} + 4.$$

(b) We start with $f(x) = 4\sqrt[3]{2-x} - 1$, switch x and y to get $x = 4\sqrt[3]{2-y} - 1$ and solve for y:

$$\begin{aligned} x + 1 &= 4\sqrt[3]{2-y} \\ \frac{x+1}{4} &= \sqrt[3]{2-y} \\ \left(\frac{x+1}{4}\right)^3 &= 2 - y \\ y &= 2 - \left(\frac{x+1}{4}\right)^3 \end{aligned}$$

Thus,

$$f^{-1}(x) = 2 - \left(\frac{x+1}{4}\right)^3$$

(c) We solve $y = 1 - x^3$ for x :

$$\begin{aligned} y &= 1 - x^3 \\ x^3 &= 1 - y \\ x &= \sqrt[3]{1-y} \end{aligned}$$

Thus,

$$f^{-1}(y) = \sqrt[3]{1-y}, \quad \text{or} \quad f^{-1}(x) = \sqrt[3]{1-x}$$

11. (a) $(f \circ g)(x) = f(g(x)) = f(4x - 12) = -(4x - 12) + 4 = -4x + 16$

$(g \circ f)(x) = g(f(x)) = g(-x + 4) = 4(-x + 4) - 12 = -4x + 4$

$(f \circ f)(x) = f(f(x)) = f(-x + 4) = -(-x + 4) + 4 = x$

(b) $(f \circ g)(x) = f(g(x)) = f(4 - 3x) = -5$

$(g \circ f)(x) = g(f(x)) = g(-5) = 4 - 3(-5) = 19$

$(f \circ f)(x) = f(f(x)) = f(-5) = -5$

(c) $(f \circ g)(x) = f(g(x)) = f\left(\frac{x}{2}\right) = \frac{1}{\frac{x}{2}} = \frac{2}{x}$

$(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x}\right) = \frac{\frac{1}{x}}{2} = \frac{1}{2x}$

$(f \circ f)(x) = f(f(x)) = f\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}} = x$

(d) $(f \circ g)(x) = f(g(x)) = f\left(\frac{5}{2x^2}\right) = \sqrt{\frac{5}{2x^2} + 1}$

$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x+1}\right) = \dfrac{5}{2(\sqrt{x+1})^2} = \dfrac{5}{2(x+1)}$

$(f \circ f)(x) = f(f(x)) = f\left(\sqrt{x+1}\right) = \sqrt{\sqrt{x+1}+1}$

(e) $(f \circ g)(x) = f(g(x)) = f(|x+13|) = \dfrac{6}{|x+13|}$

$(g \circ f)(x) = g(f(x)) = g\left(\dfrac{6}{x}\right) = \left|\dfrac{6}{x} + 13\right|$

$(f \circ f)(x) = f(f(x)) = f\left(\dfrac{6}{x}\right) = \dfrac{6}{\frac{6}{x}} = 6 \cdot \dfrac{x}{6} = x$

(f) $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{1}{x}\right) = \dfrac{\frac{1}{x}+1}{\frac{1}{x}-1} = \dfrac{\frac{1+x}{x}}{\frac{1-x}{x}} = \dfrac{1+x}{x} \cdot \dfrac{x}{1-x} = \dfrac{x+1}{1-x}$

$(g \circ f)(x) = g(f(x)) = g\left(\dfrac{x+1}{x-1}\right) = \dfrac{1}{\frac{x+1}{x-1}} = \dfrac{x-1}{x+1}$

$(f \circ f)(x) = f(f(x)) = f\left(\dfrac{x+1}{x-1}\right) = \dfrac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} = \dfrac{\frac{x+1}{x-1}+\frac{x-1}{x-1}}{\frac{x+1}{x-1}-\frac{x-1}{x-1}} = \dfrac{\frac{2x}{x-1}}{\frac{2}{x-1}} = \dfrac{2x}{x-1} \cdot \dfrac{x-1}{2} = x$

(g) $(f \circ g)(x) = f(g(x)) = f\left(x^3 - 12\right) = \sqrt[3]{(x^3-12)+12} = \sqrt[3]{x^3} = x$

$(g \circ f)(x) = g(f(x)) = g\left(\sqrt[3]{x+12}\right) = \left(\sqrt[3]{x+12}\right)^3 - 12 = x + 12 - 12 = x$

$(f \circ f)(x) = f(f(x)) = f\left(\sqrt[3]{x+12}\right) = \sqrt[3]{\sqrt[3]{x+12}+12}$

(h) $(f \circ g)(x) = f(g(x)) = f(x^2) = \sqrt{(x^2)^2+1} = \sqrt{x^4+1}$

$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x^2+1}\right) = \left(\sqrt{x^2+1}\right)^2 = x^2 + 1$

$(f \circ f)(x) = f(f(x)) = f\left(\sqrt{x^2+1}\right) = \sqrt{\left(\sqrt{x^2+1}\right)^2 + 1} = \sqrt{x^2+2}$

12. We find $f(g(x)) = f(x^7) = (x^7)^4 = x^{28}$ and $g(f(x)) = f(x^4) = (x^4)^7 = x^{28}$

13. (a) Replace $f(x)$ by y and solve for x:

$$\begin{aligned} f(x) &= 11 - \frac{x}{4} \\ y &= 11 - \frac{x}{4} \\ 4y &= 44 - x \\ x &= 44 - 4y \end{aligned}$$

Thus, $f^{-1}(x) = 44 - 4x$.

(b) Replace $f(x)$ by y and solve for x:

$$\begin{aligned} f(x) &= -13x \\ y &= -13x \\ -\frac{y}{13} &= x \end{aligned}$$

So, $f^{-1}(x) = -x/13$.

(c) The graph of $f(x) = 4$ is a horizontal line, so function $f(x)$ does not satisfy the horizontal line test. Consequently, it has no inverse function.

(d) Write $g(x) = \frac{2}{3x}$ as $y = \frac{2}{3x}$ and solve for x:

$$\begin{aligned} y &= \frac{2}{3x} \\ 3xy &= 2 \\ x &= \frac{2}{3y} \end{aligned}$$

So, $g^{-1}(x) = \dfrac{2}{3x}$.

(e) Write $f(x) = \sqrt[5]{2x-9}$ as $y = \sqrt[5]{2x-9}$ and solve for x:

$$\begin{aligned} y &= \sqrt[5]{2x-9} \\ y^5 &= 2x - 9 \\ \frac{y^5+9}{2} &= x \end{aligned}$$

So, $f^{-1}(x) = \dfrac{x^5+9}{2}$.

(f) We try the usual calculation:

$$\begin{aligned} y &= -x^4 \\ x^4 &= -y \\ x &= \pm\sqrt[4]{-y} \end{aligned}$$

We do not get a unique function, but a pair of functions. Thus, the given function has no inverse. An alternative is to sketch the graph (x^4 looks like x^2); reflect it across the x-axis to obtain $-x^4$ and convince ourselves that the horizontal line test is not satisfied.

(g) Solve for x:

$$\begin{aligned} y &= \frac{7x-1}{1-x} \\ y(1-x) &= 7x - 1 \\ y - xy &= 7x - 1 \\ y + 1 &= 7x + xy \\ y + 1 &= x(7+y) \\ x &= \frac{y+1}{7+y} \end{aligned}$$

So, $y^{-1}(x) = \dfrac{x+1}{7+x}$.

(h) Start with $y = \sqrt[4]{x-4} - 11$ and solve for x:

$$\begin{aligned} y + 11 &= \sqrt[4]{x-4} \\ (y+11)^4 &= x - 4 \\ (y+11)^4 + 4 &= x \end{aligned}$$

Thus, $f^{-1}(x) = (x+11)^4 + 4$.

(i) Write $f(x) = x^3 - 14$ as $y = x^3 - 14$ and solve for x:

$$\begin{aligned} y &= x^3 - 14 \\ x^3 &= y + 14 \\ x &= \sqrt[3]{y+14} \end{aligned}$$

So, $f^{-1}(x) = \sqrt[3]{x+14}$.

(j) The graph of $h(x) = |x| + 3$ is V-shaped (think of the absolute value graph shifted 3 units up). Since it does not satisfy the horizontal line test, $h(x)$ no inverse function.

(k) Write $f(x) = 4x^{-3}$ as $y = 4x^{-3}$ and solve for x:

$$\begin{aligned} y &= \frac{4}{x^3} \\ x^3 y &= 4 \\ x^3 &= \frac{4}{y} \\ x &= \sqrt[3]{\frac{4}{y}} \end{aligned}$$

So, $f^{-1}(x) = \sqrt[3]{\frac{4}{x}}$.

(l) The graph of the function $y = x^{-2} + 6$ is the graph of $y = x^{-2} = 1/x^2$ moved 6 units up (see *List of Important Functions* in this section for the graph). The horizontal line test implies that y has no inverse.

(m) Start with $y = x^2$ and solve for x to obtain $x = \pm\sqrt{y}$. Since we are given that $x < 0$, there is no ambiguity–we must choose the minus sign. Thus,

$$f^{-1}(x) = -\sqrt{x}.$$

Note that the domain of $f^{-1}(x)$ consists of all numbers $x \geq 0$, and the range is the set $\{y \mid y \leq 0\}$.

(n) Start with $y = x^6$ and solve for x to obtain $x = \pm\sqrt[6]{y}$. Since we are given that $x \geq 0$, there is no ambiguity–we must choose the plus sign. Thus,

$$f^{-1}(x) = \sqrt[6]{x}$$

14. We find

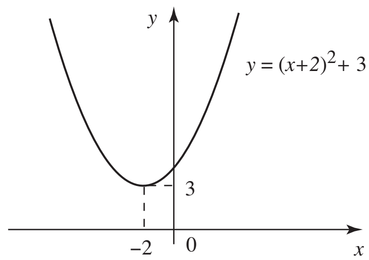
$$f(g(x)) = f\left(\frac{3x}{2-x}\right) = \frac{2\frac{3x}{2-x}}{\frac{3x}{2-x} + 3} = \frac{2\frac{3x}{2-x}}{\frac{3x}{2-x} + \frac{3(2-x)}{2-x}} = \frac{\frac{6x}{2-x}}{\frac{6}{2-x}} = \frac{6x}{2-x} \cdot \frac{2-x}{6} = x$$

and

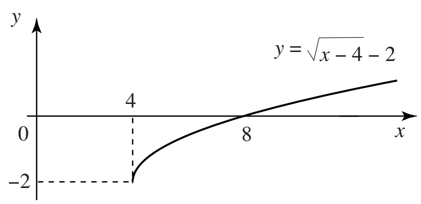
$$g(f(x)) = g\left(\frac{2x}{x+3}\right) = \frac{3\left(\frac{2x}{x+3}\right)}{2 - \frac{2x}{x+3}} = \frac{3\left(\frac{2x}{x+3}\right)}{\frac{2(x+3)}{x+3} - \frac{2x}{x+3}} = \frac{\frac{6x}{x+3}}{\frac{6}{x+3}} = \frac{6x}{x+3} \cdot \frac{x+3}{6} = x$$

The two functions are inverse of each other: we can write $g(x) = f^{-1}(x)$ or $f(x) = g^{-1}(x)$.

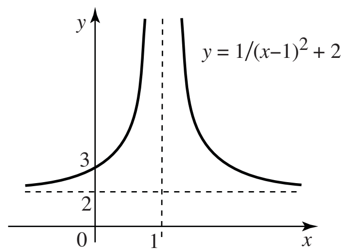
15. (a) Looking at $y = (x+2)^2 + 3$, we decide to start with the graph of $y = x^2$. First, replace x in $y = x^2$ by $x + 2$, thus getting $y = (x+2)^2$ (corresponding transformation is shift left for 2 units). Then add 3, to get $y = (x+2)^2 + 3$ (i.e., move the graph up 3 units). See below.



(b) Start with $y = \sqrt{x}$ and replace x by $x - 4$. This way, we obtain the function $y = \sqrt{x-4}$; its graph is obtained by moving the graph of $y = \sqrt{x}$ to the right for 4 units. We move this new graph 2 units down, to obtain the graph of $y = \sqrt{x-4} - 2$; see figure below. Note that the domain of this function is $x \geq 4$.



(c) To obtain $y = \dfrac{1}{(x-1)^2}$ we move the graph of $y = \dfrac{1}{x^2}$ to the right for 1 unit. Next, we move that graph up for 2 units, to obtain $y = \dfrac{1}{(x-1)^2} + 2$. The domain of this function is $x \neq 1$. See below.



(d) Move the graph of $y = |x|$ to the right by 11 units, to obtain $y = |x - 11|$; then move this new graph 10 units down, to obtain $y = |x - 11| - 10$.

(e) Recall that replacing x by $x +$ positive number creates a shift to the left. So, move the graph of $y = 1/x$ for 4 units to the left (this will create $y = \frac{1}{x+4}$) and then move it 4 units up, to obtain $y = \frac{1}{x+4} + 4$.

(f) Move the graph of $y = x^3$ to the right by 5 units (thus drawing $y = (x - 5)^3$), and then move this new graph 5 units down, to obtain $y = (x - 5)^3 - 5$.

(g) Completing the square, we obtain

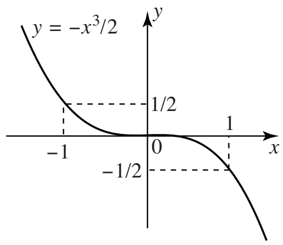
$$f(x) = x^2 + 6x - 11 = [(x + 3)^2 - 9] - 11 = (x + 3)^2 - 20$$

Move the parabola $y = x^2$ to the left by 3 units, thus obtaining the graph of $y = (x + 3)^2$; then move this new graph down by 20 units.

16. (a) To graph $y = -x^3/2$, we start from $y = x^3$, and draw the graph of $y = \frac{1}{2}x^3$ first.

Recall the general rule: to obtain the graph of $y = f(cx)$, $0 < c < 1$, we stretch the graph of $y = f(x)$ horizontally by a factor of $1/c$. Thus, we need to stretch the graph of x^3 by a factor of 2.

To obtain $y = -\frac{1}{2}x^3$ we then reflect it with respect to the x-axis. See below.



(b) Expand the graph of $y = x^3$ vertically by a factor of 2 (to obtain $y = 2x^3$), and then reflect it across the x-axis (to obtain $y = -2x^3$). Finally, move the graph down by 1 unit.

(c) Move the graph of $y = x^3$ to the right by 1 unit (to obtain $y = (x-1)^3$) and then expand vertically by a factor of 2 (to obtain $y = 2(x-1)^3$); finally move this new graph one unit up.

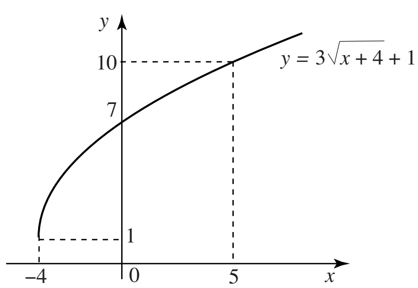
(d) Looking at the formula $y = 3\sqrt{x+4} + 1$ we see that we need to do transformations related to the numbers 3, 4 and 1. Let's break it down into steps, starting with $y = \sqrt{x}$:

Replace x by $x + 4$, thus getting $y = \sqrt{x+4}$ (transformation: shift left for 4 units).

Multiply y by 3, getting $y = 3\sqrt{x+4}$ (transformation: vertical stretch by a factor fo 3).

Finally, add 1 to y, to get $y = 3\sqrt{x+4} + 1$ (transformation: move the graph up 1 unit).

See below.



(e) Start with the graph of $y = \sqrt{x}$. Replace x by $x + 4$, thus getting $y = \sqrt{x+4}$ (transformation: shift left for 4 units). Multiply y by 3, thus getting $y = 3\sqrt{x+4}$ (transformation: vertical expansion by a factor fo 3). Finally, reflect the graph of $y = 3\sqrt{x+4}$ across the x-axis, to obtain $y = -3\sqrt{x+4}$.

(f) Move the graph of $y = \sqrt{x}$ for 4 units to the left (to get $y = \sqrt{x+4}$) then reflect it across the x-axis (to get $y = -\sqrt{x+4}$) and finally move it down by 3 units.

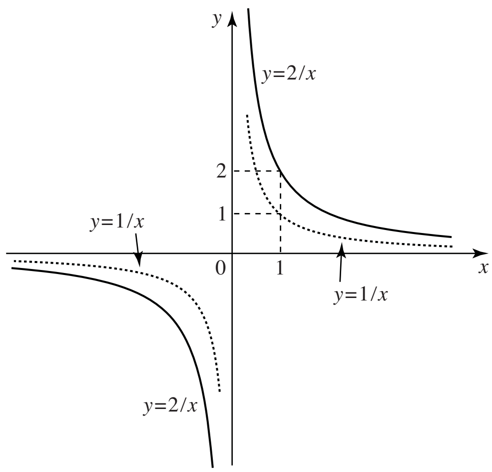
(g) Think of $y = 2/x$ as

$$y = 2 \cdot \frac{1}{x}$$

So, we expand the graph of $y = 1/x$ vertically by a factor of 2. Alternatively, write

$$y = 2 \cdot \frac{1}{x} = \frac{1}{\frac{1}{2}} \cdot \frac{1}{x} = \frac{1}{x/2}$$

So, start with the graph of $y = 1/x$ and then (because we need to replace x by $x/2$) expand it horizontally by a factor of 2. See below.

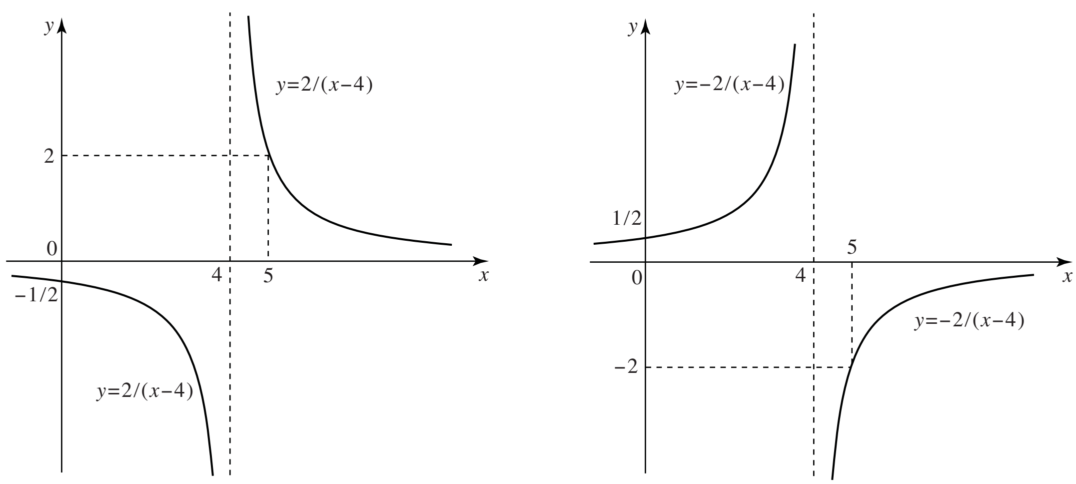


(h) Start with the graph of $y = 1/x$. Here are the steps:

replace x by $x - 4$, thus getting $y = \frac{1}{x-4}$ (transformation: shift right for 4 units)

multiply by 2, to obtain $y = \frac{2}{x-4}$ (transformation: vertical expansion by a factor of 2)

finally, multiply by -1, obtain $y = -\frac{2}{x-4}$ (transformation: reflection across the x-axis); see below.



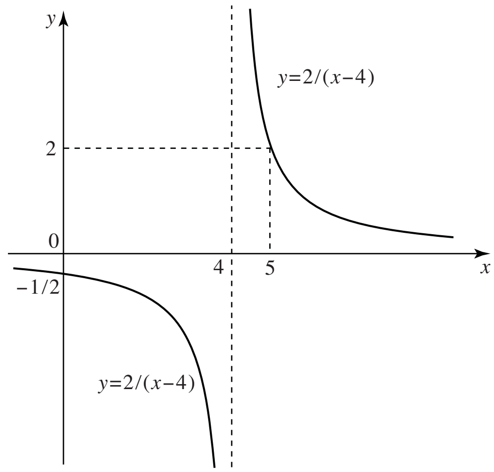
Note that the last two operations could be interchanged (i.e., reflection first, and then vertical expansion).

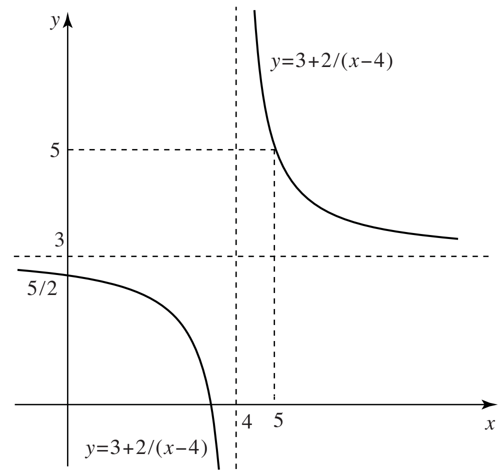
(i) Start with the graph of $y = 1/x$. Here are the steps:

replace x by $x - 4$, thus getting $y = \frac{1}{x-4}$ (transformation: shift right for 4 units)

next, multiply by 2, to obtain $y = \frac{2}{x-4}$ (transformation: vertical expansion by a factor of 2)

finally, add 3, obtain $y = 3 + \frac{2}{x-4}$ (transformation: shift up by 3 units); see below.





(j) Think of $y = 1 - |x|$ as $y = -|x| + 1$. Start with the graph of $y = |x|$, reflect it across the x-axis to obtain $y = -|x|$, and then move one unit up.

(k) Recall that $|a - b| = |b - a|$; thus, $y = |1 - x| = |x - 1|$. So, start with the graph of $y = |x|$, and (to account for replacing x by $x - 1$) shift the graph 1 unit to the right.

(l) Move the graph of $y = |x|$ for 4 units to the right (to get $y = |x - 4|$) then expand it vertically by a factor of 2 (to get $y = 2|x - 4|$) and finally move it up by 3 units.

Section 7. Trigonometric Functions

1. (a) 225° equals $225\frac{\pi}{180} = \frac{5\pi}{4}$ rad (cancel the fraction by 45).

(b) $\frac{7\pi}{6}$ rad equals $\frac{7\pi}{6}\frac{180}{\pi} = 210°$.

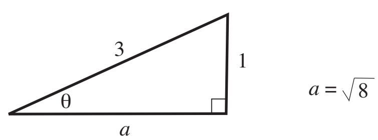
(c) 76° equals $76\frac{\pi}{180} = \frac{19\pi}{45} \approx 1.326$ rad.

(d) 3.7 rad equals $3.7\frac{180}{\pi} \approx 211.994°$.

2. Since $\csc\theta = 1/\sin\theta = 3$, it follows that $\sin\theta = 1/3$.

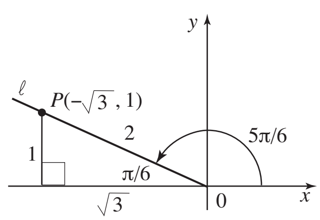
We draw the right triangle in which $\sin\theta = 1/3$ (see figure below) and calculate the remaining side using Pythagorean theorem: $3^2 = a^2 + 1^2$, so $a^2 = 8$ and $a = \sqrt{8}$.

Thus, $\cos\theta = \text{adjacent/hypotenuse} = \sqrt{8}/3$ and $\tan\theta = \text{opposite/adjacent} = 1/\sqrt{8}$.



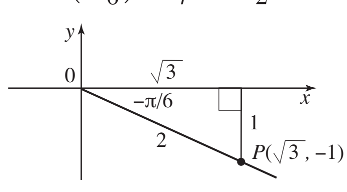
3. (a) We place the triangle that we used to compute the ratios for $\pi/6$ and $\pi/3$, as shown in figure below. We get

$$\sin\frac{5\pi}{6} = \frac{y}{r} = \frac{1}{2} \qquad \cos\frac{5\pi}{6} = \frac{x}{r} = -\frac{\sqrt{3}}{2} \qquad \tan\frac{5\pi}{6} = \frac{y}{x} = -1/\sqrt{3}$$



(b) We place the same triangle we used in (a), this time as shown in figure below. We get

$$\sin\left(-\frac{\pi}{6}\right) = \frac{y}{r} = -\frac{1}{2} \qquad \cos\left(-\frac{\pi}{6}\right) = \frac{x}{r} = \frac{\sqrt{3}}{2} \qquad \tan\left(-\frac{\pi}{6}\right) = \frac{y}{x} = -1/\sqrt{3}$$



4. The angle $\theta = -3\pi/2$ radians is equal to three right angles measured clockwise from the positive x-axis; thus, it is defined by the positive y-axis, i.e., it is equal to $\pi/2$ radians. The coordinates of the point P which is the intersection of the unit circle and the y-axis are $(0, 1)$. Thus,

$$\cos\left(-\frac{3\pi}{2}\right) = 0 \quad \text{and} \quad \sin\left(-\frac{3\pi}{2}\right) = 1$$

Since they have $\cos\left(-\frac{3\pi}{2}\right)$ in the denominator, it follows that $\tan\left(-\frac{3\pi}{2}\right)$ and $\sec\left(-\frac{3\pi}{2}\right)$ are not defined.

5. (a) Recall that the graph of $f(cx)$, $c > 1$, is obtained by compressing the graph of $f(x)$ by a factor of c. If $0 < c < 1$, then we expand the graph of $f(x)$ by a factor of $1/c$ to obtain the graph of $f(cx)$.

Thus, to obtain the graph of $\cos 3x$, we compress the graph of $\cos x$ along the x-axis by a factor of 3. In other words, we compress it so that its period is $2\pi/3$.

To obtain the graph of $\cos 0.5x$, we stretch the graph of $\cos x$ along the x-axis by a factor of 2. Therefore, the period of $\cos 0.5x$ is 4π.

(b) To obtain $-\cos x$, we reflect the graph of $\cos x$ with respect to the x-axis. Then we stretch $-\cos x$ vertically by a factor of 4 to obtain $-4\cos x$. Note that we could have stretched the graph first and then reflected it.

We reflect the graph of $\cos x$ with respect to the x-axis to obtain $-\cos x$. To obtain $-\cos x - 4$, we move the graph of $-\cos x$ vertically down for 4 units.

6. In order to figure out the horizontal shift, we need to write $f(x)$ as

$$f(x) = 2\sin(3x - \pi) = 2\sin\left(3\left(x - \frac{\pi}{3}\right)\right).$$

Thus, we see that in order to obtain the $\sin(3x - \pi)$ part of the given function, we need to replace x in $\sin(3x)$ by $x - \pi/3$.

We start with the graph of $y = \sin x$ and complete the following transformations.

Compress the graph by the factor of 3, to obtain $y = \sin(3x)$ (the period of which is $2\pi/3$).

Replace x by by $x - \pi/3$ to obtain $\sin\left(3\left(x - \frac{\pi}{3}\right)\right)$ (transformation: shift right for $\pi/3$ units).

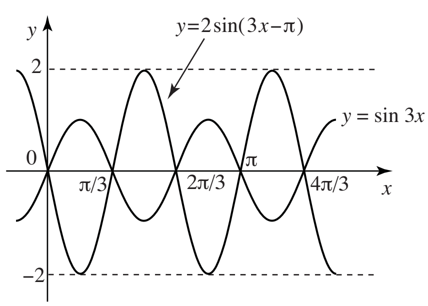
Finally, multiply by 2, to obtain $y = 2\sin(3x - \pi)$ (transformation: vertical stretch by a factor of 2).

Looking at the graph (see below) we see that the range is $[-2, 2]$.

We can calculate the range without looking at the graph. Since $\sin x$ is always between -1 and 1, we start with $-1 \leq \sin(3x - \pi) \leq 1$ and then multiply by 2 :

$$-2 \leq 2\sin(3x - \pi) \leq 2$$

Thus, the range of $f(x)$ is $[-2, 2]$.



7. We know that the period of $\tan x$ is π. As well, the graph of $\tan(ax)$ is a horizontal scaling (compression or expansion) of the graph of $\tan x$.

If $a > 1$, $\tan(ax)$ is a compression of $\tan x$ by a factor of a; so, the period will shrink by the same factor, i.e., it will be π/a.

If $0 < a < 1$, $\tan(ax)$ is an expansion of $\tan x$ by a factor of $1/a$; so, the period will expand by the same factor, i.e., it will be $(1/a)\pi = \pi/a$.

Thus, in either case, the period is π/a.

8. General strategy: we start with the one side of the identity, simplify (square the expression, or calculate common denominator, etc.) and/or use formulas (such as trig identities) to arrive at the other side.

(a) Using the addition formula for $\sin x$ (see the box *Addition and Subtraction Formulas*) we compute

$$\sin(\pi/2 + x) = \sin(\pi/2)\cos x + cos(\pi/2)\sin x = \cos x,$$

because $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$.

(b) Use the definition of $\cot x$ and then compute common denominator:

$$\cot^2 x + 1 = \frac{\cos^2 x}{\sin^2 x} + 1 = \frac{\cos^2 x}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x} = \frac{\cos^2 x + \sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} = \csc^2 x,$$

because $\cos^2 x + \sin^2 x = 1$ (basic trig identity); as well, by definition. $1/\sin x = \csc x$.

(c) We'll prove this in two different ways.

Group the first and the third terms together, and factor out $\sin^2 x$

$$\sin^2 x - \tan^2 x + \sin^2 x \tan^2 x = \sin^2 x(1 + \tan^2 x) - \tan^2 x$$

use the identity $1 + \tan^2 x = \sec^2 x$ to get

$$= \sin^2 x \sec^2 x - \tan^2 x$$

now recall that $\sin x \sec x = \sin x \frac{1}{\cos x} = \tan x$

$$= \tan^2 x - \tan^2 x = 0.$$

Alternatively, we use $\tan x = \sin x / \cos x$ and calculate the common denominator:

$$\sin^2 x - \tan^2 x + \sin^2 x \tan^2 x = \sin^2 x - \frac{\sin^2 x}{\cos^2 x} + \sin^2 x \frac{\sin^2 x}{\cos^2 x}$$

$$= \frac{\sin^2 x \cos^2 x - \sin^2 x + \sin^2 x \sin^2 x}{\cos^2 x}$$

group the first and the third terms and factor out $\sin^2 x$

$$= \frac{\sin^2 x(\cos^2 x + \sin^2 x) - \sin^2 x +}{\cos^2 x}$$

use the identity $\cos^2 x + \sin^2 x = 1$

$$= \frac{\sin^2 x - \sin^2 x +}{\cos^2 x} = 0$$

9. Write $3x = 2x + x$, and start with the addition formula for the sine function:

$$\begin{aligned} \sin 3x &= \sin(2x + x) \\ &= \sin 2x \cos x + \cos 2x \sin x \end{aligned}$$

Use the double angle formulas and simplify:

$$\begin{aligned} &= (2\sin x \cos x)\cos x + (1 - 2\sin^2 x)\sin x \\ &= 2\sin x \cos^2 x + \sin x - 2\sin^3 x \end{aligned}$$

Replace $\cos^2 x$ using the basic trigonometric identity $\cos^2 x = 1 - \sin^2 x$ and simplify:

$$\begin{aligned} &= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x \\ &= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x \\ &= 3\sin x - 4\sin^3 x \end{aligned}$$

10. We use the addition and the subtraction formulas for $\sin x$ and multiply out the terms that we get

$$\begin{aligned} \sin(x + y)\sin(x - y) &= (\sin x \cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y) \\ &= \sin^2 x \cos^2 y - \cos^2 x \sin^2 y \end{aligned}$$

(note that we used difference of squares formula!). Next, we use the basic trig identity to eliminate cos terms

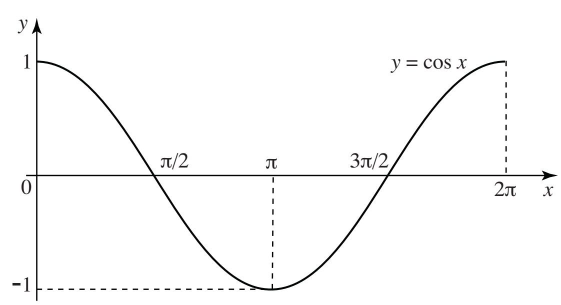
$$= \sin^2 x(1 - \sin^2 y) - (1 - \sin^2 x)\sin^2 y$$

Now multiply out and cancel

$$\begin{aligned} &= \sin^2 x - \sin^2 x \sin^2 y - \sin^2 y + \sin^2 x \sin^2 y \\ &= \sin^2 x - \sin^2 y \end{aligned}$$

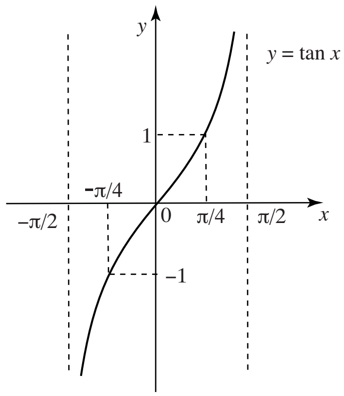
11. (a) Looking at the graph of $\cos x$, we see that there is only one solution to $\cos x = -1$ in its main period, namely $x = \pi$ (see below).

Thus, all solutions are given by $x = \pi + 2k\pi = (1 + 2k)\pi$, where k is an integer.



(b) In its main period, $\tan x$ assumes the value -1 only once, when $x = -\pi/4$ (see below). Since the period of $\tan x$ is π, we write all solutions of $\tan x = -1$ as

$$x = -\frac{\pi}{4} + k\pi$$



12. Recall the strategy: to identify the number of solutions and their location in the main period, we use the graph; to get their values, we relate them to the values in the first quadrant (which we are supposed to remember).

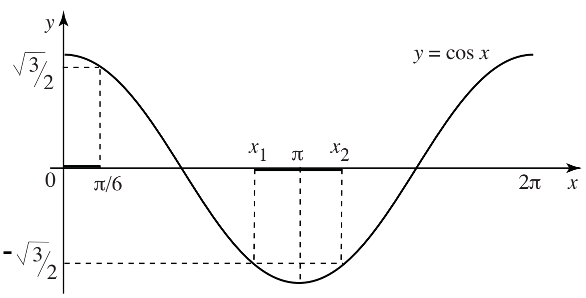
(a) The graph below shows that there are two solutions of $\cos x = -\sqrt{3}/2$ in the main period, labeled x_1 and x_2.

We know that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. Since we are looking for the values where cos is equal to $-\sqrt{3}/2$, we need to relate x_1 and x_2 to $\frac{\pi}{6}$.

The shape of the graph implies that that x_1 is $\pi/6$ units to the left of π, so $x_1 = \pi - \pi/6 = 5\pi/6$. The solution x_2 is $\pi/6$ units to the right of π; thus, $x_2 = \pi + \pi/6 = 7\pi/6$.

So, the solutions are

$$x = \frac{5\pi}{6} + 2k\pi \quad \text{and} \quad x = \frac{7\pi}{6} + 2k\pi.$$



(b) The equation $\tan x = \sqrt{3}$ has only one solution in the main period (look at the graph of $\tan x$). From memory, we know that $\tan(\pi/3) = \sqrt{3}$.

Thus, all solutions of the given equation are given by $x = \frac{\pi}{3} + k\pi$, since the period of $\tan x$ is π.

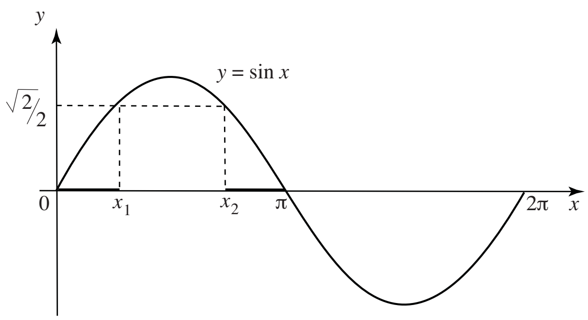
(c) The graph below shows that there are two solutions of $\sin x = \sqrt{2}/2$ in the main period, labeled x_1 and x_2. One of them is in the first quadrant, and we know (from memory) that $x_1 = \pi/4$.

We need to relate x_2 to $\frac{\pi}{4}$. We see that x_2 is $\pi/4$ units to the left of π, so $x_2 = \pi - \pi/4 = 3\pi/4$.

So, the solutions are

$$x = \frac{\pi}{4} + 2k\pi \quad \text{and} \quad x = \frac{3\pi}{4} + 2k\pi$$

See below.



13. Use the double-angle formula for $\sin x$

$$\sin 2x = \sin x$$
$$2\sin x\cos x = \sin x$$

move $\sin x$ to the left side and factor

$$2\sin x\cos x - \sin x = 0$$
$$\sin x(2\cos x - 1) = 0.$$

Thus, $\sin x = 0$ or $2\cos x - 1 = 0$, i.e., $\cos x = 1/2$.

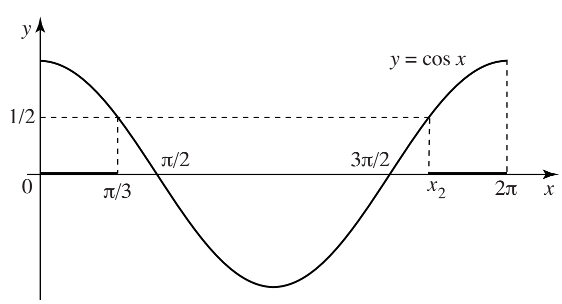
From the graph of $y = \sin x$, we see that $\sin x = 0$ when $x = k\pi$.

From the graph below, we see that there are two solutions of $\cos x = 1/2$ in the main period.

One of them is $x_1 = \frac{\pi}{3}$. Due to symmetry, the other solution is $\pi/3$ units to the left of 2π; thus, $x_2 = 2\pi - \pi/3 = 5\pi/3$. It follows that all solutions are given by $x = \frac{\pi}{3} + 2k\pi$ and $x = \frac{5\pi}{3} + 2k\pi$.

Thus, the solutions to the given equation are

$$x = k\pi \quad \text{and} \quad x = \frac{\pi}{3} + 2k\pi \quad \text{and} \quad x = \frac{5\pi}{3} + 2k\pi.$$



14. From $2\cos 2x - 1 = 0$ we get $\cos 2x = 1/2$.

In the previous exercise we solved $\cos x = 1/2$ and got $x = \frac{\pi}{3} + 2k\pi$ and $x = \frac{5\pi}{3} + 2k\pi$.

Replacing x by $2x$, we get that the solutions of $\cos 2x = 1/2$ are $2x = \frac{\pi}{3} + 2k\pi$ and $2x = \frac{5\pi}{3} + 2k\pi$ i.e., dividing by 2,

$$x = \frac{\pi}{6} + k\pi \quad \text{and} \quad x = \frac{5\pi}{6} + k\pi.$$

15. (a) Expanding $x = \frac{\pi}{2} + 2k\pi$ and $x = \frac{3\pi}{2} + 2k\pi$ we get

$$\frac{\pi}{2} + 2k\pi = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{-3\pi}{2}, \frac{-7\pi}{2}, \ldots \quad \text{and} \quad \frac{3\pi}{2} + 2k\pi = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \frac{-\pi}{2}, \frac{-5\pi}{2}, \ldots$$

i.e, the general form is

$$\frac{(\text{odd number})\ \pi}{2}.$$

We see that expanding $x = \frac{\pi}{2} + k\pi$ we get

$$\frac{\pi}{2} + k\pi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{-\pi}{2}, \frac{-3\pi}{2}, \ldots$$

which is the same thing.

(b) Expanding $(2k+1)\pi$ we get

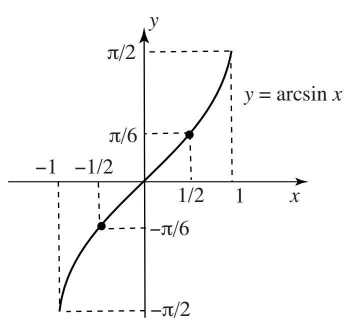
$$(2k+1)\pi = \pi, 3\pi, 5\pi, -\pi, -3\pi, \ldots$$

(i.e., odd multiples of π). Expanding $-\pi + 2k\pi$ we get $-\pi, \pi, 3\pi, -3\pi, -5\pi, \ldots$ which again contains odd multiples of π.

16. The value $\arcsin 1$ represents the unique angle in $[-\pi/2, \pi/2]$ whose sin is equal to 1. From memory, we know that $\sin(\pi/2) = 1$, and so $\arcsin 1 = \pi/2$.

Likewise, $\arcsin(-1/2)$ represents the unique angle in $[-\pi/2, \pi/2]$ whose sin is equal to $-1/2$. From Example 11 in this section in the book we see that $\arcsin(-1/2) = -\pi/6$.

Alternative: the function $\arcsin x$ is odd (its graph is symmetric with respect to the origin); see below. Thus, the values of arcsin at x and $-x$ are negatives of each other. So, if $\arcsin(1/2) = \pi/6$ (we know that is true because $\sin(\pi/6) = 1/2$) then $\arcsin(-1/2) = -\pi/6$. See the graph below.

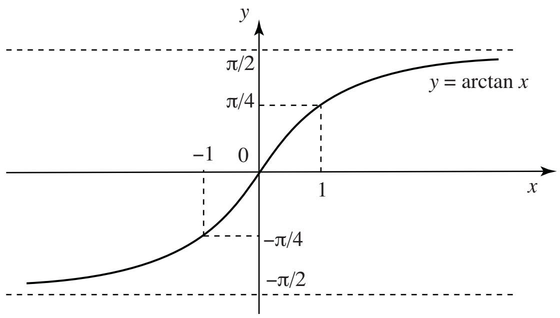


Since $\sin(\pi/4) = \sqrt{2}/2 = 1/\sqrt{2}$ and $\pi/4$ is in $[-\pi/2, \pi/2]$, we conclude that $\arcsin(1/\sqrt{2}) = \pi/4$. Using the same argument as above, we conclude that $\arcsin(-1/\sqrt{2}) = -\pi/4$.

17. To find $\arctan(-1)$ means to find the unique angle θ in $(-\pi/2, \pi/2)$ such that $\tan\theta = -1$. In the solution to Exercise 11(b) we found out that $\tan(-\pi/4) = -1$. Thus, $\arctan(-1) = -\pi/4$.

Note that, like the arcsin function (look at the previous exercise) the function arctan is odd, see the figure below. Recall that an odd function is symmetric with respect to the origin, that is, the values of arctan at x and $-x$ are negatives of each other.

Since we know (from memory) that $\tan(\pi/4) = 1$, we conclude that $\arctan 1 = \pi/4$ and thus $\arctan(-1) = -\pi/4$.



Since $\tan(\pi/6) = 1/\sqrt{3}$, we know that $\arctan(1/\sqrt{3}) = \pi/6$. As above, using the fact that tan is an odd function, we conclude that $\arctan(-1\sqrt{3}) = -\pi/6$.

From memory, we know that $\tan(\pi/3) = \sqrt{3}$, and thus $\arctan(\sqrt{3}) = \pi/3$.

18. (a) $315°$ equals $315 \cdot \frac{\pi}{180} = \frac{7\pi}{4}$ radians (we cancelled the fraction by 45).

(b) $-240°$ equals $-240 \cdot \frac{\pi}{180} = -\frac{4\pi}{3}$ radians.

(c) $-\frac{5\pi}{3}$ rad equals $-\frac{5\pi}{3}\frac{180}{\pi} = -300°$.

(d) $\frac{11\pi}{6}$ rad equals $\frac{11\pi}{6} \cdot \frac{180}{\pi} = 330°$.

(e) π rad equals $\pi\frac{180}{\pi} = 180°$.

(f) π degrees equals $\pi\frac{\pi}{180} = \frac{\pi^2}{180} \approx 0.0548$ rad.

(g) 4 rad equals $4 \cdot \frac{180}{\pi} = \frac{720}{\pi} \approx 229.183$ degrees.

(h) $4°$ equals $4° \cdot \frac{\pi}{180} = \frac{\pi}{45} \approx 0.0698$ radians.

19. (a) From Pythagorean Theorem we compute

$$c = \sqrt{a^2 + b^2} = \sqrt{21^2 + 20^2} = \sqrt{841} = 29.$$

Thus,

$$\sin\alpha = \frac{a}{c} = \frac{21}{29}, \quad \cos\alpha = \frac{b}{c} = \frac{20}{29}, \quad \sin\beta = \frac{b}{c} = \frac{20}{29}, \quad \cos\beta = \frac{a}{c} = \frac{21}{29}.$$

(b) From $\cos\beta = 12/13$ and $\cos\beta = \frac{a}{c} = \frac{a}{13}$ we get that $a = 12$. From Pythagorean Theorem we compute

$$b = \sqrt{c^2 - a^2} = \sqrt{13^2 - 12^2} = \sqrt{25} = 5.$$

Now we can calculate any trig ratio of α or β.

$$\sin\beta = \frac{b}{c} = \frac{5}{13}, \quad \tan\beta = \frac{b}{a} = \frac{5}{12}.$$

(c) From Pythagorean Theorem we compute

$$a = \sqrt{c^2 - b^2} = \sqrt{1^2 - 0.6^2} = \sqrt{0.64} = 0.8.$$

Thus,

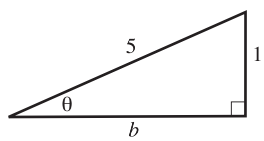
$$\sin\alpha = \frac{a}{c} = \frac{0.8}{1} = 0.8 = \frac{4}{5}, \quad \cos\alpha = \frac{b}{c} = \frac{0.6}{1} = 0.6 = \frac{3}{5}, \quad \tan\alpha = \frac{a}{b} = \frac{0.8}{0.6} = \frac{8}{6} = \frac{4}{3}.$$

The remaining three functions are the reciprocals:

$$\csc\alpha = \frac{1}{\sin\alpha} = \frac{5}{4}, \quad \sec\alpha = \frac{1}{\cos\alpha} = \frac{5}{3}, \quad \cot\alpha = \frac{1}{\tan\alpha} = \frac{3}{4}.$$

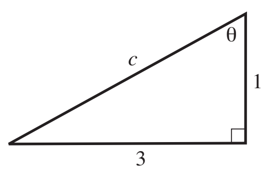
20. (a) Draw a right triangle with a side of length 1 (opposite) and the hypotenuse of length 5, so that $\sin\theta = 1/5$ (see below). The remaining side is equal to (using Pythagorean theorem) $b = \sqrt{5^2 - 1^2} = \sqrt{24} = 2\sqrt{6}$.

Using the definition of trigonometric ratios, we find $\cos\theta = \sqrt{24}/5$, $\tan\theta = 1/\sqrt{24}$, $\csc\theta = 5$, $\sec\theta = 5/\sqrt{24}$, and $\cot\theta = \sqrt{24}$.



(b) Draw a right triangle with sides 3 (opposite) and 1 (adjacent) so that $\tan\theta = 3$ (see below). The hypotenuse is equal to $c = \sqrt{3^2 + 1^2} = \sqrt{10}$.

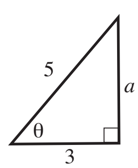
Thus, $\sin\theta = 3/\sqrt{10}$, $\cos\theta = 1/\sqrt{10}$, $\sec\theta = \sqrt{10}$, $\csc\theta = \sqrt{10}/3$, and $\cot\theta = 1/\tan\theta = 1/3$.



(c) From $\sec\theta = 5/3$ we obtain $\cos\theta = 1/\sec\theta = 3/5$.

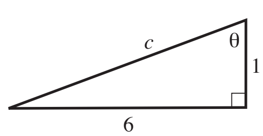
Draw a right triangle with a side of length 3 (adjacent) and the hypotenuse of length 5, so that $\cos\theta = 3/5$ (see below). The remaining side is equal to $a = \sqrt{5^2 - 3^2} = \sqrt{16} = 4$ by Pythagorean theorem.

Thus, $\sin\theta = 4/5$, $\cos\theta = 3/5$. $\tan\theta = 4/3$, $\csc\theta = 5/4$, and $\cot\theta = 3/4$.



(d) From $\cot\theta = \frac{1}{6}$ we obtain $\tan\theta = 1/\cot\theta = 6$.

Draw a right triangle with sides 6 (opposite) and 1 (adjacent) so that $\tan\theta = 6$ (see below).



The hypotenuse is equal to $c = \sqrt{6^2 + 1^2} = \sqrt{37}$.

Thus, $\sin\theta = 6/\sqrt{37}$, $\cos\theta = 1/\sqrt{37}$, $\tan\theta = 6$, $\csc\theta = \sqrt{37}/6$, and $\sec\theta = \sqrt{37}$.

21. (a) $k=-3$: $\theta = \frac{\pi}{3} - 3\pi = -\frac{8\pi}{3}$; $k=-2$: $\theta = \frac{\pi}{3} - 2\pi = -\frac{5\pi}{3}$; $k=-1$: $\theta = \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$; $k=0$: $\theta = \frac{\pi}{3} + 0 = \frac{\pi}{3}$; $k=1$: $\theta = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$; $k=2$: $\theta = \frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$; $k=3$: $\theta = \frac{\pi}{3} + 3\pi = \frac{10\pi}{3}$.

(b) $k=-3$: $\beta = \frac{\pi}{4} + 2(-3)\pi = -\frac{23\pi}{4}$; $k=-1$: $\beta = \frac{\pi}{4} + 2(-1)\pi = -\frac{7\pi}{4}$; $k=0$: $\beta = \frac{\pi}{4} + 2(0)\pi = \frac{\pi}{4}$; $k=1$: $\beta = \frac{\pi}{4} + 2(1)\pi = \frac{9\pi}{4}$; $k=4$: $\beta = \frac{\pi}{4} + 2(4)\pi = \frac{33\pi}{4}$; $k=5$: $\beta = \frac{\pi}{4} + 2(5)\pi = \frac{41\pi}{4}$.

(c) $k=-4$: $\alpha = \frac{\pi}{2} + \frac{(-4)\pi}{2} = -\frac{3\pi}{2}$; $k=-3$: $\alpha = \frac{\pi}{2} + \frac{(-3)\pi}{2} = -\frac{2\pi}{2} = -\pi$; $k=-2$: $\alpha = \frac{\pi}{2} + \frac{(-2)\pi}{2} = -\frac{\pi}{2}$; $k=-1$: $\alpha = \frac{\pi}{2} + \frac{(-1)\pi}{2} = 0$; $k=0$: $\alpha = \frac{\pi}{2} + \frac{(0)\pi}{2} = \frac{\pi}{2}$; $k=1$: $\alpha = \frac{\pi}{2} + \frac{(1)\pi}{2} = \frac{2\pi}{2} = \pi$; $k=2$: $\alpha = \frac{\pi}{2} + \frac{(2)\pi}{2} = \frac{3\pi}{2}$.

(d) Looking at the pattern, we see that each term is obtained from the previous by adding $\pi/2$. Thus, starting at any angle in the list, if we keep adding (or subtracting) $\pi/2$ we will arrive at all other terms. We write all angles as $0 + \frac{k\pi}{2} = \frac{k\pi}{2}$. "Starting at any angle in the list" means that we can also write $-\pi + \frac{k\pi}{2}$, or $\frac{3\pi}{2} + \frac{k\pi}{2}$, and so on.

Alternatively, we notice that the angles are multiples of $\frac{\pi}{2}$. So, we write $k\frac{\pi}{2} = \frac{k\pi}{2}$.

(e) Note that the angles are multiples of $\frac{2\pi}{3}$. So, we write $k\frac{2\pi}{3} = \frac{2k\pi}{3}$.

(f) Note that the angles are $2\pi/3$ units apart from each other. If we start at $\pi/3$, then we write

$$\frac{\pi}{3} + k\frac{2\pi}{3} = \frac{\pi + 2k\pi}{3} = \frac{(2k+1)\pi}{3}$$

We coud start anywhere else, and obtain the same answer. For instance, starting at π,

$$\pi + k\frac{2\pi}{3} = \frac{3\pi + 2k\pi}{3} = \frac{(2k+3)\pi}{3}$$

Now as k runs through all integer values, the values $2k+3$ give all odd integers. Likewise, as k runs through all integer values, the values $2k+1$ give all odd integers. (Note that it does not matter that the k values are not the same. For instance, when $k=3$, then $2k+3=9$; to represent 9 using $2k+1$, we use $k=4$.)

(g) The idea is to expand the two formulas to see that they indeed represent the same angles. By taking k values between, say, -4 and 4 we obtain

$$k\pi = -4\pi, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi$$

Likewise,

$$2\pi + k\pi = -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$$

In both cases, we obtain multiples of π.

(Note that the same k value does not produce the same term in both sequences.)

We could do this in a more formal way: consider $k_1\pi$ and $2\pi + k_2\pi$ where k_1 and k_2 are integers. From $k_1\pi = 2\pi + k_2\pi$ we obtain, after cancelling by π, $k_1 = 2 + k_2$. What does this mean? It tells us exactly how the terms in the two sequences match: for instance, by taking $k_1 = 4$, we obtain $k_1\pi = 4\pi$. From $k_1 = 2 + k_2$ we get $4 = 2 + k_2$ and so $k_2 = 2$. With that value, $2\pi + k_2\pi = 2\pi + 2\pi = 4\pi$.

(h) Expand the two formulas to see that they indeed represent the same angles. By taking k values between, say, -3 and 3 we obtain

$$\begin{aligned} -\frac{\pi}{2} + 2k\pi &= -\frac{\pi}{2} - 6\pi, -\frac{\pi}{2} - 4\pi, -\frac{\pi}{2} - 2\pi, -\frac{\pi}{2} + 0\pi, -\frac{\pi}{2} + 2\pi, -\frac{\pi}{2} + 4\pi, -\frac{\pi}{2} + 6\pi, \\ &= -\frac{13\pi}{2}, -\frac{9\pi}{2}, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2} \end{aligned}$$

Likewise,

$$\begin{aligned}\frac{3\pi}{2}+2k\pi &= \frac{3\pi}{2}-6\pi, \frac{3\pi}{2}-4\pi, \frac{3\pi}{2}-2\pi, \frac{3\pi}{2}+0\pi, \frac{3\pi}{2}+2\pi, \frac{3\pi}{2}+4\pi, \frac{3\pi}{2}+6\pi,\\ &= -\frac{9\pi}{2}, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \frac{15\pi}{2}\end{aligned}$$

(i) We expand both formulas (use k values between, say, -3 and 3):

$$\begin{aligned}\frac{\pi}{2}+2k\pi &= \frac{\pi}{2}-6\pi, \frac{\pi}{2}-4\pi, \frac{\pi}{2}-2\pi, \frac{\pi}{2}+0\pi, \frac{\pi}{2}+2\pi, \frac{\pi}{2}+4\pi, \frac{\pi}{2}+6\pi,\\ &= -\frac{11\pi}{2}, -\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}\end{aligned}$$

and

$$\begin{aligned}\frac{3\pi}{2}+2k\pi &= \frac{3\pi}{2}-6\pi, \frac{3\pi}{2}-4\pi, \frac{3\pi}{2}-2\pi, \frac{3\pi}{2}+0\pi, \frac{3\pi}{2}+2\pi, \frac{3\pi}{2}+4\pi, \frac{3\pi}{2}+6\pi,\\ &= -\frac{9\pi}{2}, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \frac{15\pi}{2}\end{aligned}$$

The combined list contains terms of the form (odd number) $\pi/2$. Writing an odd number as $2k+1$ (where k is an integer), we obtain

$$\frac{(2k+1)\pi}{2} = \frac{2k\pi+\pi}{2} = \frac{2k\pi}{2}+\frac{\pi}{2} = k\pi+\frac{\pi}{2}$$

22. Keep in mind that one full revolution is 360 degrees or 2π radians.

(a) We break down the given angle into multiples of 2π and a remainder part:

$$\frac{36\pi}{7} = \frac{28\pi+8\pi}{7} = \frac{28\pi}{7}+\frac{8\pi}{7} = 4\pi+\frac{8\pi}{7}.$$

The angle of 4π brings us back to the start (positive x-axis); so, the question is – in what quadrant does $\frac{8\pi}{7}$ lie? Clearly, $\frac{8\pi}{7} > \pi$. Since

$$\frac{8\pi}{7} = \frac{16\pi}{14} < \frac{21\pi}{14} = \frac{3\pi}{2},$$

the angle $\frac{8\pi}{7}$ lies in the third quadrant.

Alternatively, $\frac{8\pi}{7}$ radians is $\frac{8\pi}{7}\frac{180}{\pi} = 205.7$ degrees, so it must be in the third quadrant.

(b) Integer multiples of 360 are 360, 720, 1080, etc.

Since $999 = 720 + 279$, we see that the angle of 999 degrees is the same as the angle of 279 degrees, which is in the fourth quadrant.

(c) We can proceed as in (b), or argue like this: dividing 989 by 360, we get 2.747. So, it's two full revolutions, plus 0.747 of a revolution. Since 0.747 is between the half and the three-quarters of a revolution, the angle must be in the third quadrant.

Alternatively, we multiply 0.747 by 360 to get 268.92 degrees, which is in the third quadrant.

(d) We argue as in (a), or like this: since $44/5 = 8.8$, we write

$$\frac{44\pi}{8} = 8.8\pi = 8\pi + 0.8\pi.$$

Now 8π is four full revolutions, so we are back at the start. Since

$$\frac{\pi}{2} = 0.5\pi < 0.8\pi < \pi$$

we conclude that 0.8π – and thus the given angle – are in the second quadrant.

(e) Integer multiples of 360 are 360, 720, 1080, etc.

Start with $-550°$ and keep adding $360°$ until we find an angle which is easier to identify.

So $-550° + 360° = -190°$. Counting -190 degrees clockwise, end up in the second quadrant.

Alternatively: $-550° + 360° + 360° = 170°$. This angle is larger than $90°$ and smaller than $180°$, so it lies in the second quadrant.

(f) Integer multiples of 360 are 360, 720, 1080, etc.

Start with $-530°$ and keep adding $360°$ until we find an angle which is easier to identify.

So $-530° + 360° = -170°$. Counting -170 degrees clockwise, end up in the third quadrant.

Alternatively: $-530° + 360° + 360° = 190°$. This angle is larger than $180°$ and smaller than $270°$, so it lies in the third quadrant.

(g) 10 means 10 radians. Convert to degrees: $10 = 10 \cdot \frac{180}{\pi} \approx 573°$; Now argue as in (e) and (f): $573° - 360° = 212°$. Thus, the angle of $573°$ lies in the same quadrant as the angle of $212°$, i.e., in the third quadrant.

Alternatively, reason in radians (one way or another decimal numbers are involved). We need the quadrants in radians:

first quadrant: angles from 0 rad to $\pi/2 = 1.57080$ rad;

second quadrant: angles from $\pi/2 = 1.57080$ rad to $\pi = 3.14159$ rad;

third quadrant: angles from $\pi = 3.14159$ rad to $3\pi/2 = 4.71239$ rad;

fourth quadrant: angles from $3\pi/2 = 4.71239$ rad to $2\pi = 6.28319$ rad.

Now 10 radians minus one full revolution is $10 - 2\pi \approx 3.71681$; so it is in the third quadrant.

(h) 11 means 11 radians. Convert to degrees: $11 = 11 \cdot \frac{180}{\pi} \approx 630.3°$; Now argue as in (e) and (f): $630.3° - 360° = 270.3°$. Thus, the angle of $630.3°$ lies in the same quadrant as the angle of $270.3°$, i.e., in the fourth quadrant.

Alternatively, reason in radians (see the solution to (g) for the quadrants in radians): 11 radians minus one full revolution is $11 - 2\pi \approx 4.71681$; so the angle is in the fourth quadrant.

(i) Convert to degrees: $-4 = -4 \cdot \frac{180}{\pi} \approx -229°$; going clockwise, we end up in the second quadrant. (Or, $-229° + 360° = 131°$, now going counterclockwise; second quadrant.)

In radians: $-4 + 2\pi \approx 2.28319$; referring to part (g), we see that this angle lies in the second quadrant.

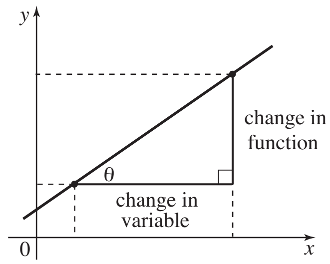
(j) Convert to degrees: $-13 \cdot \frac{180}{\pi} \approx -745°$. Repeatedly adding 360 degrees, $-745° + 360° + 360° = -25°$; fourth quadrant.

In radians: $-13 + 2\pi + 2\pi + 2\pi \approx 5.84956$; referring to part (g), we see that this angle lies in the fourth quadrant.

23. Recall that the slope is defined as the ratio

$$\frac{\text{change in the function}}{\text{change in the variable}}$$

(in our case, the function is a line). From the figure below, we see that this is exactly the ratio of the opposite to the adjacent sides for the angle θ. Thus, the slope is equal to $\tan\theta$.



24. Since the question is about the signs (and not the values) of trig functions, all we need to do is to figure out where (in which quadrant) the angles lie.

(a) Note that $13\pi/3 = 12\pi/3 + \pi/3 = 4\pi + \pi/3$, i.e., the angle $13\pi/3$ is the same as $\pi/3$. In other words, $13\pi/3$ lies in the first quadrant, and hence $\tan(13\pi/3) = \tan(\pi/3)$ is positive.

(b) Since $500 = 360 + 140$, it follows that $\sin(500°) = \sin(140°)$. Since the angle of $140°$ is in the second quadrant, we conclude that $\sin(500°)$ is positive.

(c) From $37\pi = 36\pi + \pi = 18(2\pi) + \pi$ we conclude that the angle of 37π equals 18 full revolutions plus π. Thus, as angles, 37π and π are equal. It follows that $\cos(37\pi) = \cos(\pi) = -1$, i.e., $\cos(37\pi)$ is negative.

(d) Since both angles $\pi/12$ and $\pi/7$ lie in the first quadrant, and sin and cos are positive there, we conclude that $\sin(\pi/12) + \cos(\pi/7)$ is positive.

(e) We need to identify the quadrant which contains the angle of 8 radians. Convert to degrees: $8 = 8 \cdot \frac{180}{\pi} \approx 458.4°$ Because $458.4° - 360° = 98.4°$, the angle lies in the second quadrant. Because $\cos x$ is negative in the second quadrant, then so is $\sec x = 1/\cos x$. Thus, $\sec 8$ is negative.

(f) We need to identify the quadrant which contains the angle of -3 radians. Convert to degrees: $-3 = -3 \cdot \frac{180}{\pi} \approx -171.9°$. This angle lies in the third quadrant. Because $\sin x$ is negative in the third quadrant, we conclude that $\sin(-3) < 0$.

(g) We need to identify the quadrant which contains the angle of 14 radians. Convert to degrees: $14 = 14 \cdot \frac{180}{\pi} \approx 802.1°$ Because $802.1° - 360° - 360° = 82.1°$, the angle lies in the first quadrant. We know that $\cos x$ is positive in the first quadrant, and thus $\cos 14 > 0$.

(h) We need to identify the quadrant which contains the angle of -6 radians. Since $-6 + 2\pi \approx 0.283$, the angle lies in the first quadrant. Thus, $\tan(-6)$ is positive.

(Recall that in Exercise 22(g) we identified the quadrants in radians: first quadrant: angles from 0 rad to $\pi/2 = 1.57080$ rad; second quadrant: angles from $\pi/2 = 1.57080$ rad to $\pi = 3.14159$ rad; third quadrant: angles from $\pi = 3.14159$ rad to $3\pi/2 = 4.71239$ rad; fourth quadrant: angles from $3\pi/2 = 4.71239$ rad to $2\pi = 6.28319$ rad.)

25. Recall the convention for measuring angles – if no unit is mentioned, then it is radians.

To compare, we convert to the same unit. One radian is equal to $180/\pi = 57.3$ degrees (so the angle of 1 radian lies in the first quadrant).

The angles between $\pi/2 = 1.57$ and $\pi = 3.14$ radians constitute the second quadrant. Thus, the angle of 2 radians lies in the second quadrant (angle measure of 2 radians is equal to $2(180/\pi) = 114.6$ degrees).

(a) Both 1° and 1 rad $= 57.3°$ lie in the first quadrant. Since sin is increasing there (i.e., values of $\sin x$ increase as angle x increases) we conclude that $\sin 1$ (radian) is larger than $\sin 1°$.

(b) The angle of 2° is in the first quadrant, and so $\cos 2°$ is positive. Since the angle of 2 radians is in the second quadrant, we know that $\cos 2$ is negative. Thus $\cos 2°$ is larger than $\cos 2$.

(c) Since $\tan x$ is increasing in the first quadrant, using exactly the same argument as in (a) we conclude that $\tan 1$ (radian) is larger than $\tan 1°$.

26. (a) Recall that the period of $\cos(ax)$ is $2\pi/a$. Thus, the period of $\cos 3x$ is (here we use $a = 3$) $2\pi/3$.

(b) The period of $\sin(ax)$ is $2\pi/a$. Thus, the period of $\sin 2.5x$ (here we use $a = 2.5$) is $2\pi/2.5$. One way to simplify is this:

$$\frac{2\pi}{2.5} = \frac{2\pi}{\frac{5}{2}} = 2\pi \cdot \frac{2}{5} = \frac{4\pi}{5}$$

(c) Recall that the period of $\tan(ax)$ is π/a (look at Exercise 7). Thus, the period of $\tan(x/6)$ is (note that $a = 1/6$):

$$\frac{\pi}{a} = \frac{\pi}{\frac{1}{6}} = 6\pi$$

(d) Recall that the period of $\cos(ax)$ is $2\pi/a$. Thus, the period of $\cos(2x/5)$ is (here $a = 2/5$)

$$\frac{2\pi}{a} = \frac{2\pi}{\frac{2}{5}} = 5\pi$$

(e) Since multiplying a function by 2 results in a vertical transformation (expansion), it does not affect the period of $2\cos(4x)$, so we ignore it. Recall that the period of $\cos(ax)$ is $2\pi/a$. Here $a = 4$; thus, the period of $2\cos(4x)$ is

$$\frac{2\pi}{a} = \frac{2\pi}{4} = \frac{\pi}{2}$$

(f) Since multiplying a function by 4 results in a vertical transformation (expansion), it does not affect the period of $4\sin(2x)$, so we ignore it. Recall that the period of $\sin(ax)$ is $2\pi/a$. Here $a = 2$; thus, the period of $4\sin(2x)$ is

$$\frac{2\pi}{a} = \frac{2\pi}{2} = \pi$$

(g) Recall that the period of $\tan(ax)$ is π/a (look at Exercise 7). We can ignore the $+6$ term, since it represents vertical shift, and thus does not affect the period.

We conclude that the period of $\tan 0.4x + 6$ is (here $a = 0.4$)

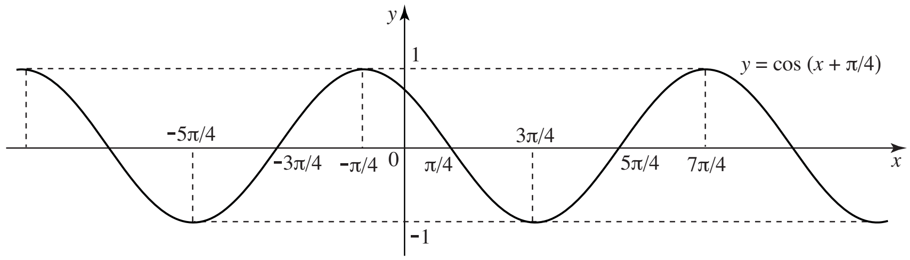
$$\frac{\pi}{a} = \frac{\pi}{0.4} = 2.5\pi$$

(h) Recall that the period of $\cos(ax)$ is $2\pi/a$. Thus, the period of $\cos 15x$ (here $a = 15$) is

$$\frac{2\pi}{a} = \frac{2\pi}{15}$$

27. (a) Recall that replacing x by x plus a positive number results in shift towards the left. Thus, the graph of $\cos(x + \pi/4)$ is obtained by moving the graph of $y = \cos x$ to the left for $\pi/4$ units; see below.

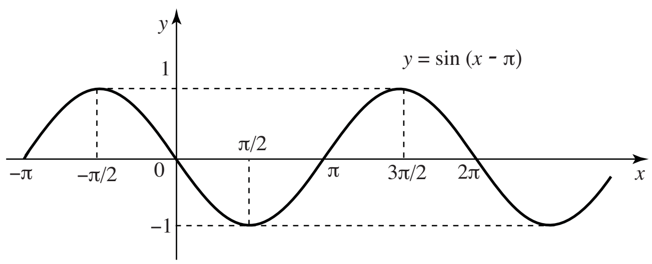
Looking at the graph, we see that the range is the set $[-1, 1]$. (Actually we do not need to look at the graph - horizontal shifts do not change the range of values of a function, and so the range of $\cos(x + \pi/4)$ is the same as the range of $\cos x$.)



(b) Recall that replacing x by $4x$ results in a compression by a factor of 4. To obtain the graph of $\cos 4x$ we compress the graph of $\cos x$ horizontally by a factor of 4 (so that the length of its period is $2\pi/4 = \pi/2$).

(c) We shift the graph of $\sin x$ for π units to the right to obtain the graph of $\sin(x - \pi)$; see below.

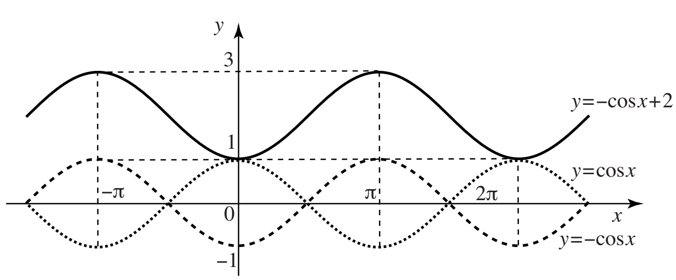
Looking at the graph, we see that the range is the set $[-1, 1]$. (Actually we do not need to look at the graph - horizontal shifts do not change the range of values of a function, and so the range of $\sin(x - \pi)$ is the same as the range of $\sin x$.)



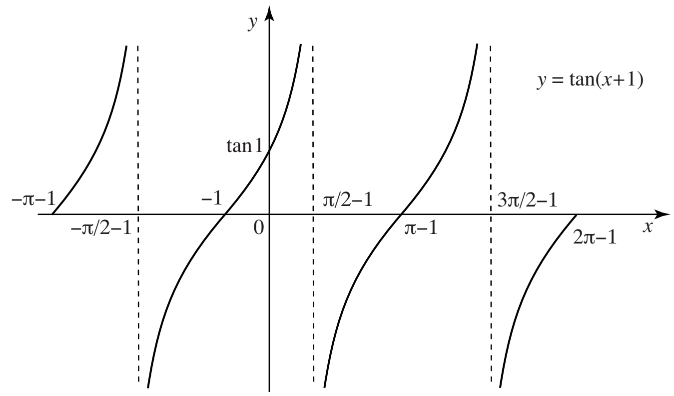
(d) Shift the graph of $\sin x$ down for $\pi/2$ units. Note that if it read $\sin(x - \pi/2)$ then we would shift the graph of $\sin x$ right for $\pi/2$ units.

(e) Write $2 - \cos x$ as $-\cos x + 2$.

Start with the graph of $\cos x$, reflect it across the x-axis (thus drawing $-\cos x$) and then move up for 2 units. See below.



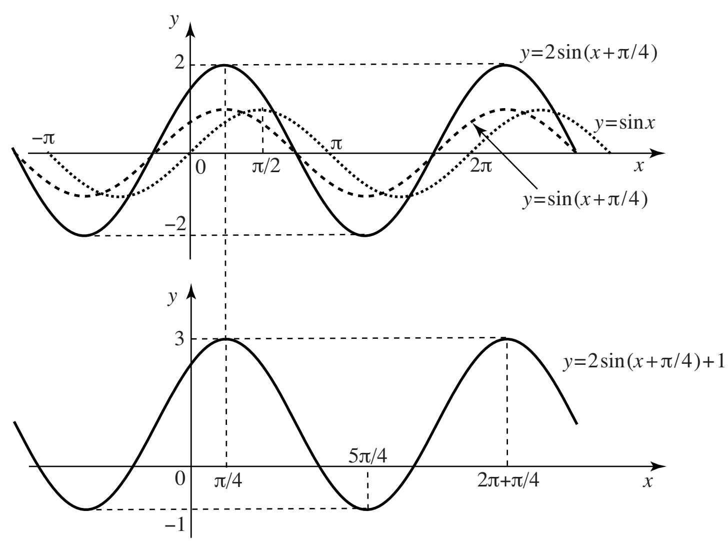
(f) Start with $\tan x$ and move it one unit to the left; see below. Note that the y-intercept is $\tan 1 = 1.557$. The range is the set of all real numbers.



(g) Compress the graph of $\cos x$ horizontally by a factor of 3 (thus shrinking its period to $2\pi/3$) and then expand it vertically by a factor of 2 (so that its range becomes $-2 \le 2\cos 3x \le 2$).

(h) Expand the graph of $\cos x$ horizontally by a factor of 4 (to obtain $\cos x/4$); this transformation expands the period to 8π. Then reflect across the x-axis (to obtain $-\cos x/4$), and finally expand this graph vertically by a factor of 3. Thus, the range of $-3\cos x/4$ is $[-3, 3]$. (Note that we can switch the order of expanding and reflecting.)

(i) Move the graph of $\sin x$ to the left by $\pi/4$ units (to draw $\sin(x + \pi/4)$) then expand it vertically by a factor of 2 (to draw $2\sin(x + \pi/4)$) and finally move up one unit. See below.



(j) Factor 2 out:

$$\sin(2x + \pi/4) = \sin 2\left(x + \frac{\pi}{8}\right)$$

Thus, we start with the graph of $\sin x$, compress it horizontally by a factor of 2 (to obtain $\sin 2x$); to account for the change x to $x + \frac{\pi}{8}$, we shift it $\pi/8$ units to the left.

(k) Factor 4 out:

$$\cos(4x + \pi) = \cos 4\left(x + \frac{\pi}{4}\right)$$

Thus, we start with the graph of $\cos x$, compress it horizontally by a factor of 4 to obtain $\cos 4x$. Then we shift it $\pi/4$ units to the left.

(l) Factor 4 out:

$$-2\cos(4x+\pi) = -2\cos 4\left(x+\frac{\pi}{4}\right)$$

Start with the graph of $\cos x$, compress it horizontally by a factor of 4 to obtain $\cos 4x$. Then shift it $\pi/4$ units to the left, thus drawing $\cos 4\left(x+\frac{\pi}{4}\right)$. Next, expand it vertically by a factor of 2 and then reflect across the x-axis.

(m) Factor 1/2 out:

$$\sin(x/2-\pi/3) = \sin\frac{1}{2}\left(x-\frac{2\pi}{3}\right)$$

We start with the graph of $\sin x$, expand it horizontally it by a factor of 2 to obtain $\sin(x/2)$. Then we shift it $2\pi/3$ units to the right.

(n) To graph $\tan 3x+10$, start with the grapf of $\tan x$ and compress it by a factor of 3 (so that its main period is between $-\pi/6$ and $\pi/6$). Then move the graph 10 units up.

If, instead, we had to graph $\tan(3x+10)$, then we would start by factoring, to obtain $\tan 3(x+10/3)$. Now, starting with $\tan x$, we would construct $\tan 3x$ as described above. Then, we would shift that graph 10/3 units to the left.

(o) Factor 3 out, and write

$$1+2\cos(3x+4) = 1+2\cos\left(3\left(x+\frac{4}{3}\right)\right)$$

Start with the graph of $\cos x$ and follow these steps:

replace x by $3x$ to obtain $\cos 3x$ (transformation: compress by a factor of 3)

replace x in $\cos 3x$ by $x+4/3$ to obtain $\cos 3(x+4/3) = \cos(3x+4)$ (transformation: move left for 4/3 units)

multiply $\cos(3x+4)$ by 2 to obtain $2\cos(3x+4)$ (transformation: expand vertically by a factor of 2)

add 1 to $2\cos(3x+4)$ to obtain $1+2\cos(3x+4)$ (transformation: shift up by one unit).

(p) Rewrite the given function as

$$5-4\sin(3x-2) = -4\sin\left(3\left(x-\frac{2}{3}\right)\right)+5$$

Start with the graph of $\sin x$ and follow these steps:

replace x by $3x$ to obtain $\sin 3x$ (transformation: compress by a factor of 3)

replace x in $\sin 3x$ by $x-2/3$ to obtain $\sin 3(x-2/3) = \sin(3x-2)$ (transformation: move right for 2/3 units)

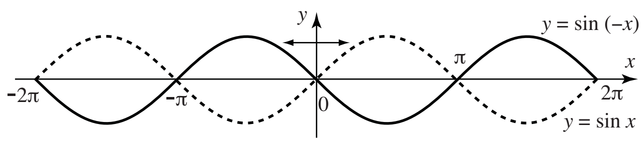
multiply $\sin(3x-2)$ by 4 to obtain $4\sin(3x-2)$ (transformation: expand vertically by a factor of 4)

insert the minus sign in front of $4\sin(3x-2)$ to obtain $-4\sin(3x-2)$ (transformation: reflect acoross the x-axis)

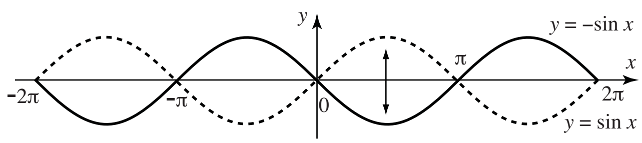
add 5 to $-4\sin(3x-2)$ to obtain $-4\sin(3x-2)+5$ (transformation: shift up by 5 units).

28. (a) Start with the graph of $\sin x$. Reflect across the y-axis to obtain $\sin(-x)$. Then take $\sin x$ again and reflect across the x-axis to obtain $-\sin x$. The two resulting graphs are identical, thus $\sin(-x) = -\sin x$; see below.

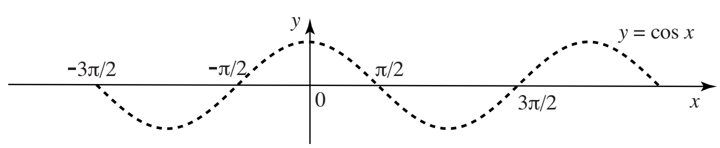
reflection across the y-axis



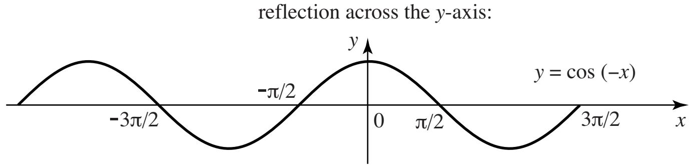
reflection across the x-axis



(b) Start with the graph of $\cos x$. Reflect across the y-axis to obtain $\cos(-x)$. The two graphs are identical, thus $\cos(-x) = \cos x$; see below.

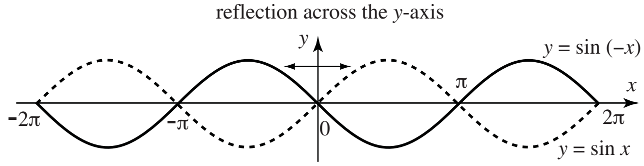


reflection across the y-axis:

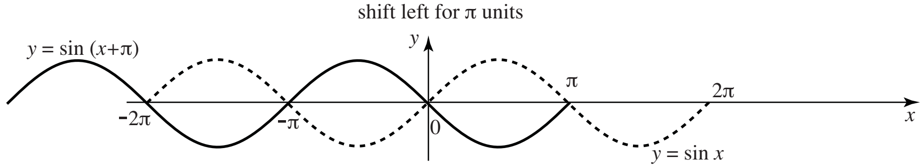


(c) Start with the graph of $\sin x$. Reflect across the y-axis to obtain $\sin(-x)$. Then take $\sin x$ again and move it left for π units. The two resulting graphs are identical, thus $\sin(x + \pi) = -\sin x$; see below.

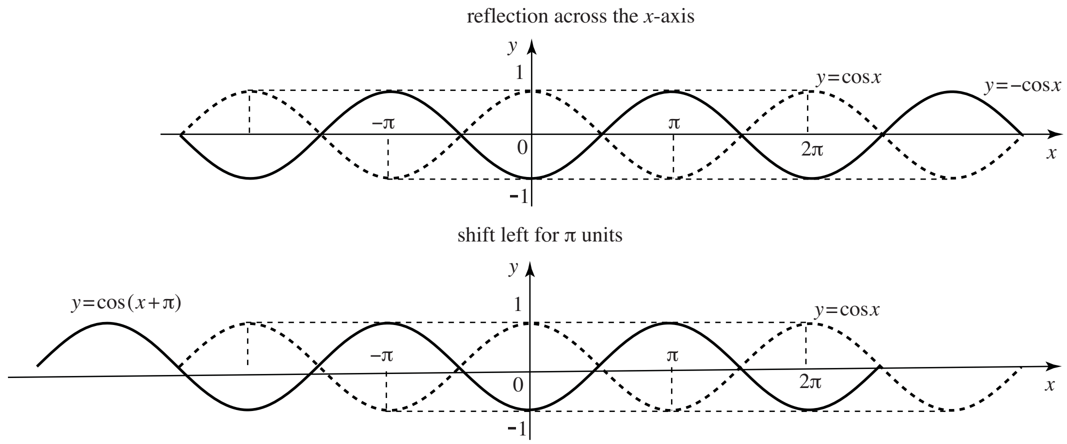
reflection across the y-axis



shift left for π units



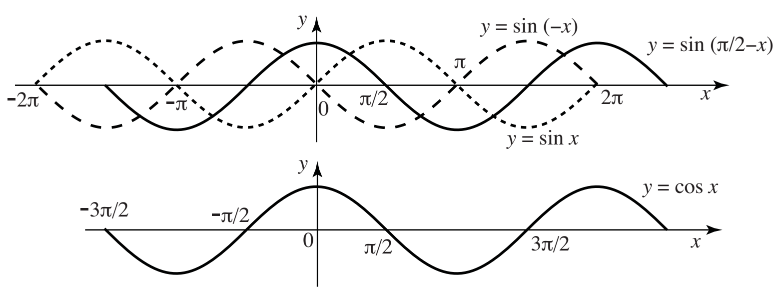
(d) Start with the graph of $\cos x$. Reflect across the x-axis to obtain $-\cos x$. Then take $\cos x$ again and move it left for π units. The two resulting graphs are identical, thus $\cos(x + \pi) = -\cos x$; see below.



(e) Write

$$\sin\left(\frac{\pi}{2} - x\right) = \sin\left(-x + \frac{\pi}{2}\right) = \sin\left(-\left(x - \frac{\pi}{2}\right)\right)$$

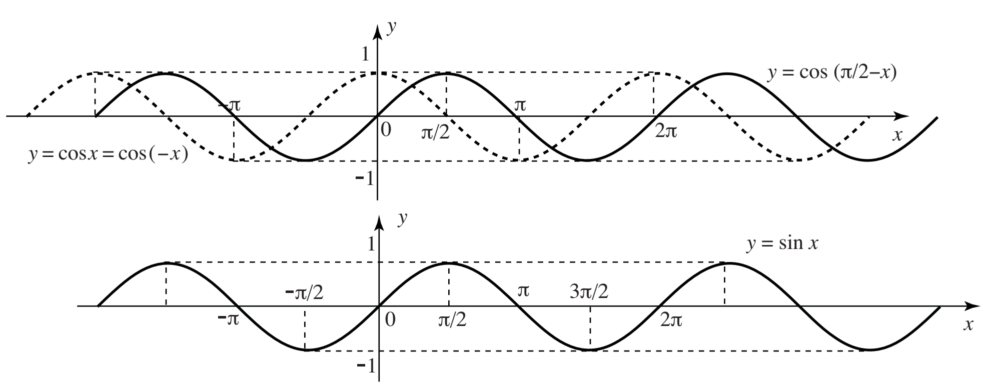
Start with the graph of $\sin x$, reflect across the y-axis to obtain $\sin(-x)$ and then (since we need to replace x by $x - \pi/2$) move the graph $\pi/2$ units to the right. The resulting graph is identical to the graph of $\cos x$; see below.



(f) Write

$$\cos\left(\frac{\pi}{2} - x\right) = \cos\left(-x + \frac{\pi}{2}\right) = \cos\left(-\left(x - \frac{\pi}{2}\right)\right)$$

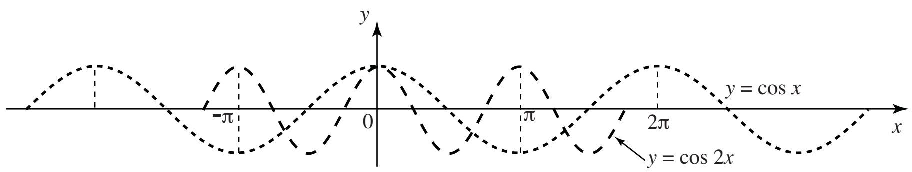
Start with the graph of $\cos x$, reflect across the y-axis to obtain $\cos(-x)$ (no change, the two graphs are identical; see part (b)); and then (since we need to replace x by $x - \pi/2$) move the graph $\pi/2$ units to the right. The resulting graph is identical to the graph of $\sin x$; see below.

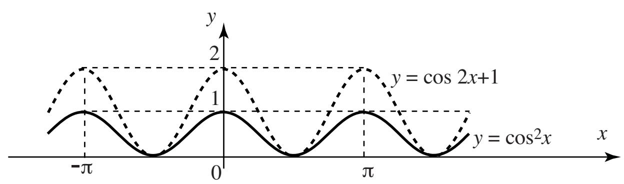


29. For graphing purpose, we rewrite:

$$y = \cos^2 x = \frac{1 + \cos 2x}{2} = \frac{1}{2}(\cos 2x + 1)$$

Start with the graph of $\cos x$, compress it horizontally by a factor of 2 (to get $\cos 2x$) and move it up by 1 unit (to get $\cos 2x + 1$); finally, compress it vertically by a factor of 2.

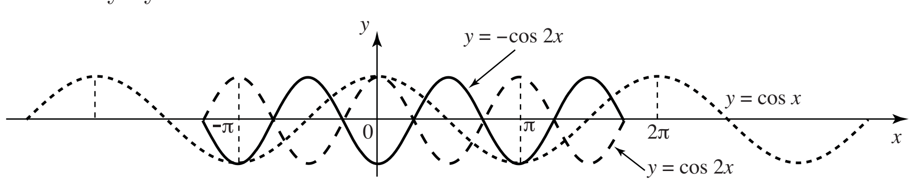


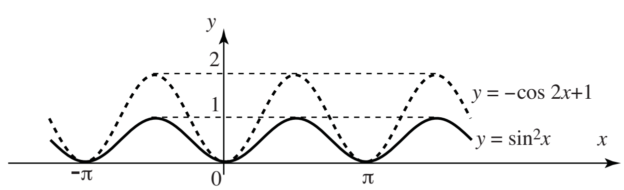


30. For graphing purpose, we rewrite:

$$y = \sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2}(-\cos 2x + 1)$$

Start with the graph of $\cos x$, compress it horizontally by a factor of 2 (to get $\cos 2x$) and then reflect across the x-axis (to get $-\cos 2x$). Next, move it up by 1 unit (to get $-\cos 2x + 1$), and finally compress vertically by a factor of 2.





31. (a) Using the fundamental identity $\sin^2 x + \cos^2 x = 1$ we get

$$\begin{aligned}\sec^2 x - \sin^2 x - \cos^2 x &= \sec^2 x - (\sin^2 x + \cos^2 x)\\ &= \sec^2 x - 1\\ &= \frac{1}{\cos^2 x} - 1\\ &= \frac{1 - \cos^2 x}{\cos^2 x} = \frac{\sin^2 x}{\cos^2 x} = \tan^2 x.\end{aligned}$$

Note that in this calculation we also proved the fact that $\sec^2 x - 1 = \tan^2 x$.

(b) Write $\tan x = \sin x / \cos x$ and compute the common denominator

$$\begin{aligned}\frac{\cos x}{1+\sin x} + \tan x &= \frac{\cos x}{1+\sin x} + \frac{\sin x}{\cos x} \\ &= \frac{\cos x \cos x}{(1+\sin x)\cos x} + \frac{\sin x(1+\sin x)}{(1+\sin x)} \\ &= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos x(1+\sin x)} \\ &= \frac{1+\sin x}{\cos x(1+\sin x)} = \frac{1}{\cos x} = \sec x.\end{aligned}$$

Near the end, we used $\sin^2 x + \cos^2 x = 1$.

(c) Compute the common denominator (note the difference of squares!)

$$\begin{aligned}\frac{\sin x}{1+\cos x} + \frac{\sin x}{1-\cos x} &= \frac{\sin x(1-\cos x)}{(1+\cos x)(1-\cos x)} + \frac{\sin x(1+\cos x)}{(1-\cos x)(1+\cos x)} \\ &= \frac{\sin x - \sin x\cos x + \sin x + \sin x \cos x}{1-\cos^2 x} \\ &= \frac{2\sin x}{\sin^2 x} \\ &= \frac{2}{\sin x} = 2\csc x\end{aligned}$$

(d) Square both binomials

$$\begin{aligned}&(\sin x + \cos x)^2 + (\sin x - \cos x)^2 \\ &\qquad = \sin^2 x + 2\sin x\cos x + \cos^2 x + \sin^2 x - 2\sin x \cos x + \cos^2 x \\ &\qquad = 2(\sin^2 x + \cos^2 x) = 2.\end{aligned}$$

32. (a) From the graph of $\cos x$ we see that $\cos x = -1$ in the main period only when $x = \pi$. Accounting for periodicity, we write all solutions as

$$x = \pi + 2k\pi = (2k+1)\pi$$

Note that, instead, we could have written $x = (2k-1)\pi$, since in both cases we include all odd multiples of π.

(b) From (a), we learned that $\cos A = -1$ when $A = (2k+1)\pi$ (or, equivalently, when $A = (2k-1)\pi$). In this case $A = 4x$, and thus $4x = (2k+1)\pi$ and $x = (2k+1)\pi/4$. Alternatively, $4x = (2k-1)\pi$ and $x = (2k-1)\pi/4$.

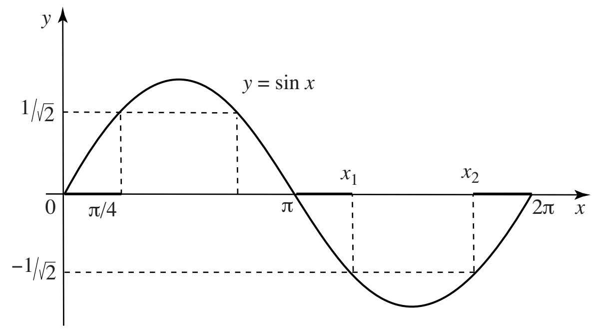
(c) From the graph of $y = \cos A$ we see that $\cos A = -1$ when $A = -\pi + 2k\pi = (2k-1)\pi$, where k is an integer [i.e., $A =$ odd multiple of π; and could also be written as $A = (2k+1)\pi$].

Replace A by $4x - \pi$, to obtain $4x - \pi = (2k-1)\pi$, $4x = 2k\pi$ and $x = k\pi/2$.

(d) The graph below shows that there are two solutions to $\sin x = -1/\sqrt{2} = -\sqrt{2}/2$ in the main period of $\sin x$ (labelled x_1 and x_2).

We know (from memory) that $\sin(\pi/4) = 1/\sqrt{2}$. Due to the symmetry of the graph, we see that x_1 is $\pi/4$ units to the right of π, so $x_1 = \pi + \pi/4 = 5\pi/4$. As well, x_2 is $\pi/4$ units to the left of 2π, so $x_2 = 2\pi - \pi/4 = 7\pi/4$.

Thus, the solutions are $x = \frac{5\pi}{4} + 2k\pi$ and $x = \frac{7\pi}{4} + 2k\pi$.



(e) In (d) we showed that the solutions to the equation $\sin A = -1/\sqrt{2}$ are given by $A = \frac{5\pi}{4} + 2k\pi$ and $A = \frac{7\pi}{4} + 2k\pi$. To solve $\sin(x/3) = -1/\sqrt{2}$ we replace A by $x/3$. Thus, the solutions are $\frac{x}{3} = \frac{5\pi}{4} + 2k\pi$ and $\frac{x}{3} = \frac{7\pi}{4} + 2k\pi$, i.e., $x = \frac{15\pi}{4} + 6k\pi$ and $x = \frac{21\pi}{4} + 6k\pi$.

(f) In (d) we showed that the solutions to the equation $\sin A = -1/\sqrt{2}$ are given by $A = \frac{5\pi}{4} + 2k\pi$ and $A = \frac{7\pi}{4} + 2k\pi$. To solve $\sin(x/3 - 2) = -1/\sqrt{2}$ we replace A by $x/3 - 2$. Thus, the solutions are $\frac{x}{3} - 2 = \frac{5\pi}{4} + 2k\pi$ and $\frac{x}{3} - 2 = \frac{7\pi}{4} + 2k\pi$, i.e., $x = \frac{15\pi}{4} + 6k\pi + 6$ and $x = \frac{21\pi}{4} + 6k\pi + 6$.

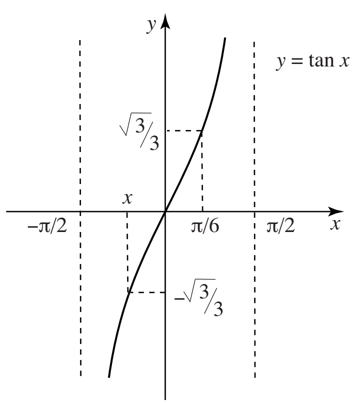
(g) We employ our usual strategy of relating values of trig functions to their values in the first quadrant.

From memory, we know that

$$\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}.$$

Due to the symmetry, we see that $x = -\pi/6$; see figure below. As well, $x = -\pi/6$ is the only solution of the given equation in the main period of tan.

Thus, the solutions of the given equation are $x = -\frac{\pi}{6} + k\pi$.



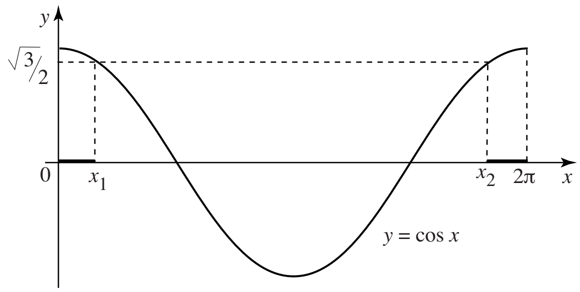
(h) Computing the reciprocal of $\cot x = -1$ we get $\tan x = -1$. We solved this equation in Exercise 11(b), the solution is $x = -\frac{\pi}{4} + k\pi$. We can also write it as $x = \frac{3\pi}{4} + k\pi$.

(i) The graph below shows that there are two solutions of $\cos x = \sqrt{3}/2$ in the main period, labeled x_1 and x_2.

We know that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$; so $x_1 = \frac{\pi}{6}$.

Due to the symmetry of the graph of $\cos x$, we see that x_2 is $\pi/6$ units to the left of 2π, i.e. $x_2 = 2\pi - \pi/6 = 11\pi/6$.

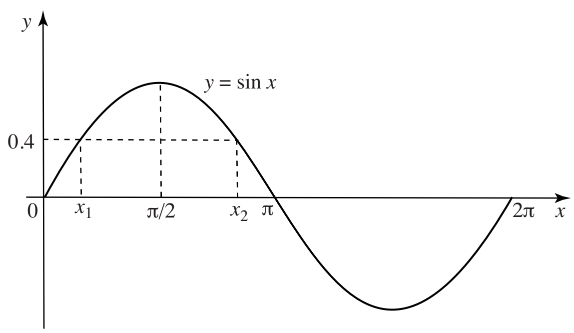
So, the solutions are $x = \frac{\pi}{6} + 2k\pi$ and $x = \frac{11\pi}{6} + 2k\pi$.



(j) From the graph (below) we see that there are two solutions of $\sin x = 0.4$ in the main period (labeled x_1 and x_2); x_1 is in the first quadrant, x_2 is in the second quadrant, and they ared related by $x_2 = \pi - x_1$.

To solve $\sin x = 0.4$ we write $x_1 = \arcsin(0.4)$ and use the calculator (working in radians!) to find that $x_1 \approx 0.41152$ rad. Then $x_2 = \pi - x_1 \approx 3.14159 - 0.41152 = 2.73007$ rad.

Accounting for periodicity, we obtain all solutions: $x \approx 0.41152 + 2k\pi$ and $x \approx 2.73007 + 2k\pi$.

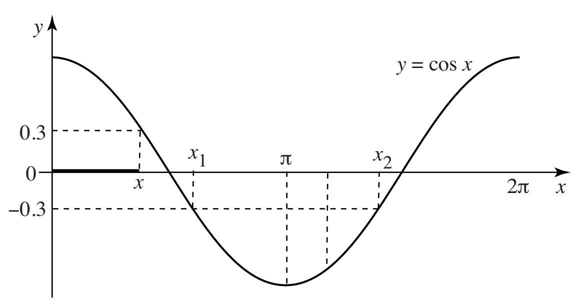


(k) From the graph (below) we see that there are two solutions of $\cos x = -0.3$ in the main period (labeled x_1 and x_2); x_1 is in the second quadrant, x_2 is in the third quadrant.

Using the inverse cosine function, we will find the solution to $\cos x = 0.3$ in the first quadrant (we call it x). Looking at the graph, we realize that $x_1 = \pi - x$ and $x_2 = \pi + x$.

To solve $\cos x = 0.3$ wed write $x = \arccos(0.3)$ and use the calculator (working in radians!) to find that $x \approx 1.26610$ rad. Then $x_1 = \pi - x \approx 3.14159 - 1.26610 = 1.87549$ rad and $x_2 = \pi + x \approx 3.14159 + 1.26610 = 4.40769$ rad.

Accounting for periodicity, we obtain all solutions: $x \approx 1.87549 + 2k\pi$ and $x \approx 4.40769 + 2k\pi$.



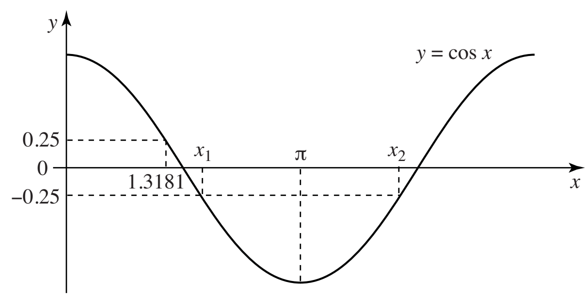
(l) In its main period $(-\pi/2, \pi/2)$, $y = \tan x$ is an increasing function, so it crosses the horizontal line at $y = 12$ only once. We conclude that the equation $\tan x = 12$ has one solution in the main period, given by $x = \arctan 12 \approx 1.488$ (radians).

The solution to the given equation is $x = \arctan 12 + k\pi \approx 1.488 + k\pi$, where k is an integer.

(m) Writing $\sec x = 1/\cos x$, we get $\cos x = 1/(-4) = -0.25$.

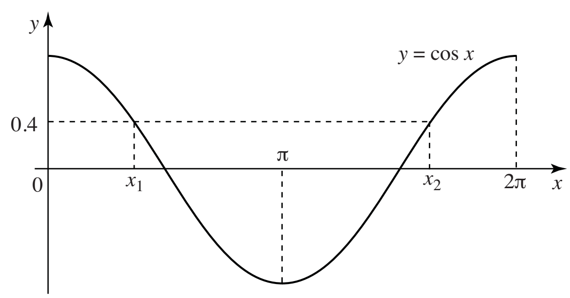
Using a calculator, we find that (in the first quadrant) $\cos 1.3181 = 0.25$ (press $\cos^{-1}$ key, enter 0.25 and press the equals key). From the graph below, we see that there are two solutions to $\cos x = -0.25$ in the main period; due to symmetry of the graph of $\cos x$, these solutions lie at the distance of 1.318 from π. Thus, $x_1 = \pi - 1.3181 \approx 1.8234$ and $x_2 = \pi + 1.3181 \approx 4.4596$

Accounting for periodicity, we obtain all solutions: $x \approx 1.8234 + 2k\pi$ and $x \approx 4.4596 + 2k\pi$.



(n) From $\sec x = 2.5$ we find $\cos x = 1/2.5 = 0.4$. Using a calculator, we find that (in the first quadrant) $\cos 1.15928 \approx 0.4$ (press $\cos^{-1}$ key, enter 0.4 and press the equals key). Thus, $x_1 \approx 1.15928$ and then $x_2 = 2\pi - x_1 \approx 2(3.14159) - 1.15928 = 5.12390$.

Accounting for periodicity, we obtain all solutions: $x \approx 1.15928 + 2k\pi$ and $x \approx 5.12390 + 2k\pi$.



(o) In its main period $(-\pi/2, \pi/2)$, $y = \tan x$ is an increasing function, so it crosses the horizontal line at $y = -1.5$ only once. We conclude that the equation $\tan x = -1.5$ has one solution in the main period, given by $x = \arctan(-1.5) \approx -0.98279$ (radians).

The solution to the given equation is $x = \arctan 12 + k\pi \approx -0.98279 + k\pi$, where k is an integer.

33. (a) Since the largest value $\sin x$ and $\cos x$ can attain is 1, their sum can never exceed 2. So there are no solutions.

(b) Use definitions of $\tan x$ and $\cot x$ and calculate common denominator

$$\tan x + \cot x = 0.5$$
$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = 0.5$$

$$\frac{\sin x \sin x}{\cos x \sin x} + \frac{\cos x \cos x}{\sin x \cos x} = 0.5$$
$$\frac{\sin^2 x + \cos^2 x}{\cos x \sin x} = 0.5$$
$$\frac{1}{\cos x \sin x} = 0.5$$

now take reciprocals of both sides to get

$$\cos x \sin x = 2$$

Since $\sin x \leq 1$ and $\cos x \leq 1$, their product satisfies $\sin x \cos x \leq 1$ as well. Thus, the above equation has no solutions.

(c) Factor the equation

$$\cos^2 x - \cos x - 2 = 0$$
$$(\cos x - 2)(\cos x + 1) = 0$$

If $\cos x - 2 = 0$, then $\cos x = 2$, and there are no solutions (keep in mind that $\cos x \leq 1$ for all x).

If $\cos x + 1 = 0$, then $\cos x = -1$, and $x = \pi + 2k\pi$; see Exercise 11(a).

Thus, all solutions are given by $x = \pi + 2k\pi$.

34. (a) Use the addition formula for cos:

$$\cos(x + \pi) = \cos x \cos \pi - \sin x \sin \pi = \cos x(-1) - \sin x(0) = -\cos x$$

(b) Use the subtraction formula for cos:

$$\cos\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2}\right)\cos x + \sin\left(\frac{\pi}{2}\right)\sin x = (0)\cos x + (1)\sin x = \sin x$$

(c) Use the subtraction formula for sin:

$$\sin\left(\frac{\pi}{2} - x\right) = \sin\left(\frac{\pi}{2}\right)\cos x - \cos\left(\frac{\pi}{2}\right)\sin x = (1)\cos x + (0)\sin x = \cos x$$

(d) Reduce sec to cos and use the subtraction formula for cos:

$$\sec\left(\frac{\pi}{2} - x\right) = \frac{1}{\cos(\frac{\pi}{2} - x)} = \frac{1}{\cos(\frac{\pi}{2})\cos x + \sin(\frac{\pi}{2})\sin x} = \frac{1}{(0)\cos x + (1)\sin x} = \frac{1}{\sin x} = \csc x$$

(e) Write $\cot x = \cos x / \sin x$ and then use the subtraction formulas for sin and cos:

$$\cot(\pi/2 - x) = \frac{\cos(\pi/2 - x)}{\sin(\pi/2 - x)}$$
$$= \frac{\cos(\pi/2)\cos x + \sin(\pi/2)\sin x}{\sin(\pi/2)\cos x - \cos(\pi/2)\sin x} = \frac{\sin x}{\cos x} = \tan x$$

Recall that $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$.

(f) Reduce sec to cos and use the addition formula for cos:

$$\sec(x + \pi) = \frac{1}{\cos(x + \pi)}$$
$$= \frac{1}{\cos x \cos \pi - \sin x \sin \pi} = \frac{1}{\cos x(-1) + \sin x(0)} = \frac{1}{-\cos x} = -\sec x$$

35. Recall that, to prove an identity, we start with one side and keep transforming until we arrive at the other side. Sometimes, we transform both sides simultaneously until we arrive at an identity. We illustrate both approaches in the solution to part (a) below.

(a) Start with the left side, and multiply and divide by $1 + \sin x$

$$\frac{1-\sin x}{\cos x} = \frac{(1-\sin x)(1+\sin x)}{\cos x(1+\sin x)}$$

Multiply the terms in the numerator using the difference of squares, and then simplify with the help of the basic trig identity

$$= \frac{1-\sin^2 x}{\cos x(1+\sin x)}$$
$$= \frac{\cos^2 x}{\cos x(1+\sin x)} = \frac{\cos x}{1+\sin x}$$

Alternatively, we start with the identity we have to prove and cross-multiply

$$\frac{1-\sin x}{\cos x} = \frac{\cos x}{1+\sin x}$$
$$(1-\sin x)(1+\sin x) = \cos^2 x$$
$$1-\sin^2 x = \cos^2 x$$
$$\cos^2 x = \cos^2 x$$

Done!

(b) Start with the left side – use definitions of $\tan x$ and $\cot x$

$$\frac{1}{1+\tan^2 x} + \frac{1}{1+\cot^2 x} = \frac{1}{1+\frac{\sin^2 x}{\cos^2 x}} + \frac{1}{1+\frac{\cos^2 x}{\sin^2 x}}$$

Multiply the first fraction by $\cos^2 x$ and the second by $\sin^2 x$

$$= \frac{\cos^2 x}{\cos^2 x + \sin^2 x} + \frac{\sin^2 x}{\sin^2 x + \cos^2}$$
$$= \frac{\cos^2 x}{1} + \frac{\sin^2 x}{1} = 1$$

after repeated use of the identity $\cos^2 x + \sin^2 x = 1$.

36. (a) To find $\arcsin(\sqrt{3}/2)$ means to find the angle x in $[-\pi/2, \pi/2]$ for which $\sin x = \sqrt{3}/2$. From memory (it's good to remember the values of trigonometric functions for special angles in the first quadrant) we recall that $\sin(\pi/3) = \sqrt{3}/2$. Thus, $\arcsin(\sqrt{3}/2) = \pi/3$.

(b) To find $\arcsin(-\sqrt{3}/2)$ means to find the angle x in $[-\pi/2, \pi/2]$ for which $\sin x = -\sqrt{3}/2$. From memory we recall that $\sin(\pi/3) = \sqrt{3}/2$; using the graph of $\sin x$, we see that $\sin(-\pi/3) = -\sqrt{3}/2$. Thus, $\arcsin(-\sqrt{3}/2) = -\pi/3$.

(c) To find $\arcsin(-1)$ means to find the angle x in $[-\pi/2, \pi/2]$ for which $\sin x = -1$. Looking at the graph of $y = \sin x$ (or otherwise), we see that $\sin(-\pi/2) = -1$. Thus, $\arcsin(-1) = -\pi/2$.

(d) To find $\arcsin \pi$ means to find the angle x in $[-\pi/2, \pi/2]$ for which $\sin x = \pi$. Since $\sin x$ is never larger than 1, it could not be equal to π. Thus, $\arcsin \pi$ is not defined.

(e) To find $\arctan(-\sqrt{3})$ means to find the angle x in $(-\pi/2, \pi/2)$ for which $\tan x = -\sqrt{3}$. Looking at the graph of $y = \tan x$ (recalling that $\tan(\pi/3) = \sqrt{3}$ and arguing by symmetry), we see that $\tan(-\pi/3) = -\sqrt{3}$. Thus, $\arctan(-\sqrt{3}) = -\pi/3$.

(f) To find $\arctan(-1)$ means to find the angle x in $(-\pi/2, \pi/2)$ for which $\tan x = -1$. Looking at the graph of $y = \tan x$ (or otherwise, for instance recalling that $\tan(\pi/4) = 1$ and arguing by symmetry), we see that $\tan(-\pi/4) = -1$. Thus, $\arctan(-1) = -\pi/4$.

(g) To find $\arctan(-2)$ means to find the angle x in $(-\pi/2, \pi/2)$ for which $\tan x = -2$. Using a calculator (inverse tangent key), we find that $\tan(-1.10715) = -2$. Thus, $\arctan(-2) = -1.10715$.

(h) To find $\arctan(24)$ means to find the angle x in $(-\pi/2, \pi/2)$ for which $\tan x = 24$. Using a calculator, we find that $\tan(1.52915) = 24$. Thus, $\arctan(24) = 1.52915$.

37. (a) To find $\arccos(0)$ means to find the angle x in $[0, \pi]$ for which $\cos x = 0$. Since $\cos(\pi/2) = 0$, and $\pi/2$ lies in the interval $[0, \pi]$, we conclude that $\arccos(0) = \pi/2$.

(b) To find $\arccos(-1)$ means to find the angle x in $[0, \pi]$ for which $\cos x = -1$. Looking at the graph of $y = \cos x$ (or otherwise), we see that $\cos \pi = -1$. Thus, $\arccos(-1) = \pi$.

(c) To find $\arccos \sqrt{2}$ means to find the angle x in $[0, \pi]$ for which $\cos x = \sqrt{2}$. Since $\cos x$ cannot be larger than 1, it could not be equal to $\sqrt{2}$. Thus, $\arccos \sqrt{2}$ is not defined.

(d) To find $\arccos(1/\sqrt{2})$ means to find the (unique) angle x in $[0, \pi]$ for which $\cos x = 1/\sqrt{2}$. Since $\cos(\pi/4) = 1/\sqrt{2}$, and $\pi/4$ is in $[0, \pi]$, we conclude that $\arccos(1/\sqrt{2}) = \pi/4$.

Section 8. Exponential and Logarithmic Functions

1. (a) To get the graph of $y = 2^x + 4$, move the graph of $y = 2^x$ up 4 units.

(b) We see that x in $y = 2^x$ has been replaced by $x - 4$. So, we move the graph of $y = 2^x$ to the right for 4 units.

Alternatively, we write $y = 2^{x-4} = 2^{-4}2^x = \frac{1}{16}2^x$; so we need to compress the graph of $y = 2^x$ vertically by a factor of 16.

(c) To draw $y = -2^x$, we reflect the graph of $y = 2^x$ with respect to the x-axis.

(d) Reflect the graph of $y = 2^x$ with respect to the x-axis (thus getting $y = -2^x$) and then with respect to the y-axis (to get $y = -2^{-x}$).

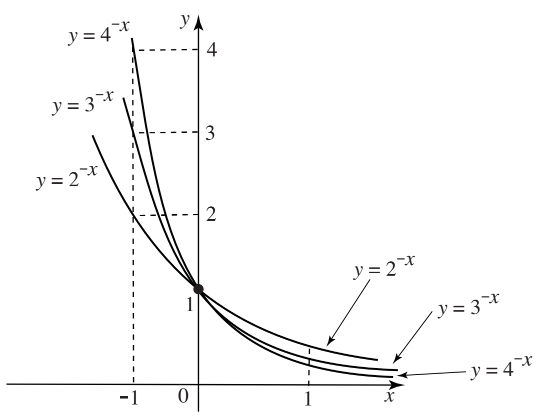
Reversing the order of the two reflections would result in the same graph.

Alternatively, write $-2^{-x} = -(2^{-1})^x = -(1/2)^x$; so we need to reflect the graph of $y = (1/2)^x$ with respect to the x-axis.

2. We plot points to draw the graphs (see below).

We see that as x approaches $-\infty$, all graphs $y = 2^{-x}$, $y = 3^{-x}$ and $y = 4^{-x}$ approach ∞. Note that a^{-x} approaches ∞ more quickly as a increases.

As x approaches ∞, all three graphs approach zero – but at different rates: a^{-x} approaches zero more quickly as a increases.



x	2^{-x}	3^{-x}	4^{-x}
0	1	1	1
1	1/2	1/3	1/4
2	1/4	1/9	1/16
−1	2	3	4
−2	4	9	16
...	...	...	...

3. (a) Write $25 = 5^2$; we get

$$5^{x-2} \cdot 25^{3-x} = 5^{x-2} \cdot (5^2)^{3-x} = 5^{x-2} \cdot 5^{2(3-x)} = 5^{x-2} \cdot 5^{6-2x} = 5^{x-2+6-2x} = 5^{-x+4}$$

(b) Use the fact that $9 = 3^2$ and $27 = 3^3$; thus

$$\begin{aligned} 3^{x-1} \cdot 9^{x-2} \cdot 27^{x-3} &= 3^{x-1} \cdot (3^2)^{x-2} \cdot (3^3)^{x-3} \\ &= 3^{x-1} \cdot 3^{2(x-2)} \cdot 3^{3(x-3)} \\ &= 3^{x-1} \cdot 3^{2x-4} \cdot 3^{3x-9} \\ &= 3^{x-1+2x-4+3x-9} \\ &= 3^{6x-14} \end{aligned}$$

(c) We recognize the numbers involved as powers of 2; i.e., $8 = 2^3$ and $16 = 2^4$. Thus

$$\frac{8^{x+4}}{16^{x-2}} = \frac{(2^3)^{x+4}}{(2^4)^{x-2}} = \frac{2^{3(x+4)}}{2^{4(x-2)}} = \frac{2^{3x+12}}{2^{4x-8}} = 2^{3x+12-(4x-8)} = 2^{-x+20}$$

4. (a) Simplify so that both sides have the same base:

$$0.5^{x^2} = 0.125$$
$$0.5^{x^2} = 0.5^3$$

Thus $x^2 = 3$ and $x = \pm\sqrt{3}$.

(b) From $3^x(3^x - 3) = 0$ it follows that either $3^x = 0$ or $3^x - 3 = 0$.

Since a^x is positive for all $a > 0$ and for any x, we conclude that the equation $3^x = 0$ has no solutions.

From $3^x - 3 = 0$ we get $3^x = 3 = 3^1$, and so $x = 1$.

Thus, $x = 1$ is the only solution.

(c) In this solution we will use the fact that $(2^x)^2 = 2^{2x}$ (recall that $(a^m)^n = a^{mn}$).

Write $2^{2x} - 5 \cdot 2^x + 4 = 0$ as $(2^x)^2 - 5 \cdot 2^x + 4 = 0$ and factor

$$(2^x - 4)(2^x - 1) = 0.$$

If $2^x - 4 = 0$, then $2^x = 4 = 2^2$, and so $x = 2$. If $2^x - 1 = 0$, then $2^x = 1 = 2^0$, and so $x = 0$.

We conclude that there are two solutions, $x = 0$ and $x = 2$.

5. We keep in mind that $\log_a n = m$ is equivalent to $a^m = n$

So, if $\log_{10} 10 = 1$, then $10^1 = 10$.

The remaining entries are done analogously: Reading the above statement with $a = 3$, $m = -2$ and $n = 1/9$, we see that $3^{-2} = 1/9$ is equivalent to $\log_3(1/9) = -2$

In the same way, we fill out the remaining entries:

If $\log_5 0.04 = -2$, then $5^{-2} = 0.04$.

From $10^{-3} = 0.001$ we conclude that $\log_{10} 0.001 = -3$, etc. See table below.

Exponential	Logarithm
$10^1 = 10$	$\log_{10} 10 = 1$
$3^{-2} = 1/9$	$\log_3(1/9) = -2$
$5^{-2} = 0.04$	$\log_5 0.04 = -2$
$10^{-3} = 0.001$	$\log_{10} 0.001 = -3$
$10^0 = 1$	$\log_{10} 1 = 0$
$10^6 = 1,000,000$	$\log_{10} 1,000,000 = 6$

6. To provide counterexamples, we need convenient values for the logarithms involved, such as the following: if $a > 0$, then $\log_a 1 = 0$ (because $a^0 = 1$) and $\log_a a = 1$ (because $a^1 = a$); as well, $\log_a a^n = n$ for any real number n.

To show that $\log_a(x+y) = \log_a x + \log_a y$ does not work, we try simplest possible values - take $x = y = 1$; then $\log_a x = 0$ and $\log_a y = 0$, and thus $\log_a x + \log_a y = 0$.

However, $\log_a(x+y) = \log_a 2 \neq 0$. (Why is $\log_a 2 \neq 0$? Think of the definition: $\log_a 2$ has the property that a raised to must give 2. If $\log_a 2$ were zero, then a raised to it would be 1, and not 2).

To prove that $\log_{10}(x-y) = \log_{10} x - \log_{10} y$ is not correct we could take $x = y = 1$ again. The left side is $\log_{10} 0$, which is not defined; however, the right side is equal to zero.

Alternatively, take $x = 2$, $y = 1$; then $\log_{10}(x-y) = \log_{10} 1 = 0$, whereas $\log_{10} x - \log_{10} y = \log_{10} 2 - \log_{10} 1 = \log_{10} 2 \neq 0$.

7. (a) One answer is: since it is log of a sum, it cannot be simplified.

However, the sum $y^3 + 8$ can be written as a product! Since

$$y^3 + 8 = y^3 + 2^3 = (y+2)(y^2 - 2y + 4)$$

(to review this, look at the sum of cubes formula in Section 2). Thus

$$\log_{12}(y^3+8) = \log_{12}(y+2)(y^2-2y+4) = \log_{12}(y+2) + \log_{12}(y^2-2y+4)$$

using the product law of logarithms, $\log_{12}(AB) = \log_{12} A + \log_{12} B$.

(b) Using laws of logarithms (for products and quotients), we get

$$\begin{aligned}\log_{10}\frac{100a^4c^{-1}}{a+b} &= \log_{10}(100a^4c^{-1}) - \log_{10}(a+b)\\ &= \log_{10} 100 + \log_{10} a^4 + \log_{10} c^{-1} - \log_{10}(a+b)\\ &= 2 + 4\log_{10} a - \log_{10} c - \log_{10}(a+b)\end{aligned}$$

because $\log_{10} 100 = \log_{10} 10^2 = 2\log_{10} 10 = 2(1) = 2$. Note the we used the formula $\log_{10} a^n = n\log_{10} a$.

(c) Simplify

$$\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{1}{2^{1/2}} = 2^{-1/2}.$$

Thus

$$\log_2\sqrt{\frac{1}{2}} = \log_2 2^{-1/2} = -\frac{1}{2}\log_2 2 = -\frac{1}{2},$$

since $\log_2 2 = 1$.

8. We make use of the conversion formula

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Thus,

$$\log_4 x = \frac{\log_{16} x}{\log_{16} 4} = \frac{\log_{16} x}{\log_{16} 16^{1/2}} = \frac{\log_{16} x}{\frac{1}{2}\log_{16} 16} = 2\log_{16} x,$$

since $\log_{16} 16 = 1$.

Likewise,

$$\log_{0.5} x = \frac{\log_2 x}{\log_2 0.5} = \frac{\log_2 x}{\log_2 2^{-1}} = \frac{\log_2 x}{-\log_2 2} = -\log_2 x.$$

(in the denominator, we used $\log_2 2^{-1} = (-1)\log_2 2 = -1$, since $\log_2 2 = 1$).

To find the numeric value of $\log_{11} 1.45$ we convert to base 10 and then use a calculator:

$$\log_{11} 1.45 = \frac{\log_{10} 1.45}{\log_{10} 11} \approx \frac{0.161368}{1.041393} \approx 0.154954.$$

Instead of converting to base 10, we could convert to base e, i.e., use ln; in this case,

$$\log_{0.4} 44 = \frac{\ln 44}{\ln 0.4} \approx \frac{3.784190}{-0.916291} \approx -4.129900.$$

9. (a) We rewrite $\log_x 4 = 1/2$ in exponential form: $x^{1/2} = 4$, i.e., $\sqrt{x} = 4$. Thus, $x = 16$.

(b) As in (a), we rewrite $\log_3 x = 5$ as $3^5 = x$; thus, $x = 243$.

(c) From $\log_2 x^3 = \log_2(4x)$ we conclude that $x^3 = 4x$. Seems that we canceled by $\log_2$; is that allowed? Yes! To remove $\log_2$ from $\log_2 x^3 = \log_2(4x)$ we apply the exponential function 2^x:

$$2^{\log_2 x^3} = 2^{\log_2(4x)}$$

(now recall that $2^{\log_2 A} = A$ for all $A > 0$). Therefore, $x^3 = 4x$.

To solve $x^3 = 4x$, we write $x^3 - 4x = 0$ and then factor:

$$x(x^2 - 4) = x(x + 2)(x - 2) = 0.$$

It follows that $x = 0$, $x = -2$ and $x = 2$.

Note that $x = 0$ and $x = -2$ are not in the domain of the two functions $\log_2 x^3$ and $\log_2(4x)$ and thus cannot be solutions. It follows that $x = 2$ is the only solution.

(d) From $16^{\log_4 x} = 4 = \sqrt{16} = 16^{1/2}$ we get $\log_4 x = 1/2$. Thus, $x = 4^{1/2} = 2$.

10. (a) Simplify using the product law of logarithms $\ln A + \ln B = \ln(AB)$ to get

$$e^{\ln 4 + \ln 5} = e^{\ln(4 \cdot 5)} = e^{\ln 20} = 20,$$

since $e^{\ln A} = A$ for all positive numbers A.

(b) Using $\ln(e^A) = A$ we write $2 = \ln e^2$; thus

$$4 \ln 2 + \ln 3 + 2 = \ln 4^2 + \ln 3 + \ln e^2 = \ln(4^2 \cdot 3 \cdot e^2) = \ln(48e^2).$$

Note that we also used $\ln A^n = n \ln A$ and the product law of logarithms $\ln A + \ln B = \ln(AB)$.

(c) Applying ln to both sides of $e^{3x-2} = 4$ we get $3x - 2 = \ln 4$ and thus $x = (\ln 4 + 2)/3$.

(d) From $\ln(x^2 + x - 1) = 0$ we get

$$e^{\ln(x^2+x-1)} = e^0,$$

and

$$x^2 + x - 1 = 1.$$

Thus, $x^2 + x - 2 = 0$, and, after factoring, $(x + 2)(x - 1) = 0$. It follows that $x = -2$ and $x = 1$. Both $x = -2$ and $x = 1$ belong to the domain of $\ln(x^2 + x - 1)$ and hence are solutions of the given equation.

11. (a) $\log_x 243 = 5$

(b) $A = e^{13}$

(c) $\log_A B = n$

(d) $\log_{10} 0.001 = x$

(e) $y = e^x$

(f) $\log_6 D = C$

12. (a) Using base 10, we get

$$10 \cdot 100^2 \cdot 1000^4 = 10 \cdot (10^2)^2 \cdot (10^3)^4 = 10 \cdot 10^{2 \cdot 2} \cdot 10^{3 \cdot 4} = 10 \cdot 10^4 \cdot 10^{12} = 10^{1+4+12} = 10^{17}.$$

(b) Write $6 = 2 \cdot 3$ so that

$$3^7 + 6 \cdot 3^6 = 3^7 + 2 \cdot 3 \cdot 3^6 = 3^7 + 2 \cdot 3^{1+6} = 3^7 + 2 \cdot 3^7 = 3 \cdot 3^7 = 3^{1+7} = 3^8.$$

Alternatively, start by factoring out 3^6

$$3^7 + 6 \cdot 3^6 = 3^6(3 + 6) = 3^6 \cdot 9 = 3^6 \cdot 3^2 = 3^{6+2} = 3^8.$$

(c) Write $9 = 3^2$ and $27 = 3^3$ to get

$$9 \cdot 27^3 + 2 \cdot 3^{11} = 3^2 \cdot (3^3)^3 + 2 \cdot 3^{11} = 3^2 \cdot 3^9 + 2 \cdot 3^{11} = 3^{11} + 2 \cdot 3^{11} = 3 \cdot 3^{11} = 3^{12}.$$

(d) Since $36 = 6^2$, we get

$$\frac{36^{n+3}}{6^{2n+5}} = \frac{(6^2)^{n+3}}{6^{2n+5}} = \frac{6^{2(n+3)}}{6^{2n+5}} = \frac{6^{2n+6}}{6^{2n+5}} = 6^{2n+6-(2n+5)} = 6^1 = 6.$$

Note that we used the formula $A^m/A^n = A^{m-n}$.

(e) Factor 6 and 10 and then use $(ab)^n = a^n b^n$:

$$\frac{5^{x+3}}{3^{2x}} \cdot \frac{6^{3x}}{10^x} = \frac{5^{x+3}}{3^{2x}} \cdot \frac{(2 \cdot 3)^{3x}}{(2 \cdot 5)^x} = \frac{5^x \cdot 5^3 \cdot 2^{3x} \cdot 3^{3x}}{3^{2x} \cdot 2^x \cdot 5^x} = 5^3 \cdot 2^{2x} \cdot 3^x$$

Note that we cancelled the fraction by 5^x. As well, $2^{3x}/2^x = 2^{3x-x} = 2^{2x}$ and $3^{3x}/3^{2x} = 3^{3x-2x} = 3^x$.

(f) Looking at all powers of 5, we see that $x + 2$ is the smallest, so we factor out 5^{x+2}. As well, 7^x is the smallest power of 7. Thus:

$$\begin{aligned} 5^{x+4}7^x - 2 \cdot 5^{x+2}7^{x+1} + 5^{x+3}7^{x+2} &= 5^{x+2} \cdot 7^x \left(5^2 - 2 \cdot 7^1 + 5^1 \cdot 7^2\right) \\ &= 5^{x+2} \cdot 7^x (25 - 14 + 245) \\ &= 256 \cdot 5^{x+2} \cdot 7^x \end{aligned}$$

(g) Note that 4 and 9 are squares and use $(a^m)^n = a^{mn}$:

$$4^x - 9^x = \left(2^2\right)^x - \left(3^2\right)^x = 2^{2x} - 3^{2x} = (2^x)^2 - (3^x)^2 = (2^x - 3^x)(2^x + 3^x)$$

(h) Use $(e^x)^2 = e^{x \cdot 2} = e^{2x}$, $(e^{-x})^2 = e^{-x \cdot 2} = e^{-2x}$ and $e^x e^{-x} = e^{x-x} = e^0 = 1$:

$$\begin{aligned} \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 &= \frac{(e^x)^2 + 2e^x e^{-x} + (e^{-x})^2}{4} - \frac{(e^x)^2 - 2e^x e^{-x} + (e^{-x})^2}{4} \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4}{4} = 1 \end{aligned}$$

(i) Keep in mind that $e^a/e^b = e^{a-b}$:

$$\frac{e^x + e^{-2x}}{2e^x - e^{-3x}} = \frac{\frac{e^x}{e^x} + \frac{e^{-2x}}{e^x}}{\frac{2e^x}{e^x} - \frac{e^{-3x}}{e^x}} = \frac{1 + e^{-3x}}{2 - e^{-4x}}$$

(j) Keep in mind that $e^a/e^b = e^{a-b}$:

$$\frac{e^{4x}-3e^{-x}}{e^x+e^{-7x}} = \frac{\frac{e^{4x}}{e^{-x}} - \frac{3e^{-x}}{e^{-x}}}{\frac{e^x}{e^{-x}} + \frac{e^{-7x}}{e^{-x}}} = \frac{e^{5x}-3}{e^{2x}-e^{-6x}}$$

To simplify the second term in the denominator, we wrote $e^{-7x}/e^{-x} = e^{-7x-(-x)} = e^{-6x}$. Other terms were simplified in the same way.

13. (a) Using laws of exponents,

$$\frac{11^{-2}}{11^2} = 11^{-2-2} = 11^{-4} = \frac{1}{11^4} = \frac{1}{14641}$$

(b) Write $49 = 7^2$ and cancel:

$$\frac{2\cdot 7^4}{49} = \frac{2\cdot 7^4}{7^2} = 2\cdot 7^2 = 98$$

(c) Split into three fractions:

$$\frac{2^5+2^6+2^7}{2^5} = \frac{2^5}{2^5} + \frac{2^6}{2^5} + \frac{2^7}{2^5} = 1+2+2^2 = 1+2+4 = 7$$

Alternatively, factor 2^5 out and cancel:

$$\frac{2^5+2^6+2^7}{2^5} = \frac{2^5(1+2^1+2^2)}{2^5} = 1+2+2^2 = 7$$

(d) Rewrite all terms as powers of 2 and use laws of exponents:

$$4^{-1}\cdot 16^{-2}\cdot 2^8 = (2^2)^{-1}\cdot(2^4)^{-2}\cdot 2^8 = 2^{-2}\cdot 2^{-8}\cdot 2^8 = 2^{-2} = \frac{1}{4}$$

(e) Write $0.5 = 2^{-1}$ and $0.1 = 10^{-1}$ and use laws of exponents

$$\frac{(0.5\cdot 10)^{-3}}{16\cdot 0.1^4} = \frac{(2^{-1}\cdot 10)^{-3}}{16\cdot(10^{-1})^4} = \frac{(2^{-1})^{-3}\cdot(10)^{-3}}{2^4\cdot 10^{-4}} = \frac{2^3\cdot 10^{-3}}{2^4\cdot 10^{-4}}$$
$$= 2^{3-4}\cdot 10^{-3-(-4)} = 2^{-1}\cdot 10^1 = \frac{10}{2} = 5$$

Alternatively,

$$\frac{(0.5\cdot 10)^{-3}}{16\cdot 0.1^4} = \frac{5^{-3}}{2^4\cdot 0.1^4} = \frac{5^{-3}}{0.2^4} = \frac{1}{5^3\ 0.2^4}$$
$$= \frac{1}{5^3\ 0.2^3\ 0.2^1} = \frac{1}{(5\cdot 0.2)^3 0.2^1}$$
$$= \frac{1}{(1)^3 0.2} = \frac{1}{0.2} = \frac{10}{2} = 5$$

(f) Write $0.2 = 2\cdot 0.1$ and $0.1 = 10^{-1}$ to get

$$0.2^{-4}\cdot 16 = (2\cdot 0.1)^{-4}\cdot 2^4 = 2^{-4}\cdot(10^{-1})^{-4}\cdot 2^4 = 2^{-4}\cdot 2^4\cdot 10^{(-1)(-4)}$$
$$= 2^{-4+4}\cdot 10^4 = 2^0\cdot 10^4 = 10^4$$

since $2^0 = 1$.

(g) Note that $32 = 2^5$; thus

$$-32\cdot\left(\frac{1}{2}\right)^4 = -2^5\frac{1^4}{2^4} = -2^5\cdot 2^{-4} = -2^{5-4} = -2$$

There are other ways of simplifying (here, and in other questions in this exercise); for instance,

$$-32 \cdot \left(\frac{1}{2}\right)^4 = -32\frac{1^4}{2^4} = -32\frac{1}{16} = -\frac{32}{16} = -2$$

(h) Using laws of logarithms,

$$\log_{10} 0.0001 = \log_{10} 10^{-4} = -4\log_{10} 10 = -4,$$

since $\log_{10} 10 = 1$. Alternatively, after the first equals sign, we use $\log_a a^n = n$ with $a = 10$ and $n = -4$.

(i) Reduce $1/100$ to a power of 10:

$$\log_{10} \frac{1}{100} = \log_{10} \frac{1}{10^2} = \log_{10} 10^{-2} = -2\log_{10} 10 = -2.$$

(j) Reduce 100^{-3} to a power of 10:

$$\log_{10} 100^{-3} = \log_{10}(10^2)^{-3} = \log_{10} 10^{-6} = -6$$

(k) Since $25 = 5^2$, we find

$$\log_5 25 = \log_5 5^2 = 2$$

(l) Since $25 = 5^2$, we find

$$\log_5 25^2 = \log_5(5^2)^2 = \log_5 5^4 = 4$$

(m) As in previous exercises, reduce to a power of 5:

$$\log_5 \frac{1}{5} = \log_5 5^{-1} = -1$$

(n) By definition, $\log_4 2 = x$ is equivalent to $4^x = 2$. Rewrite it as $(2^2)^x = 2^{2x} = 2^1$; thus $2x = 1$ and $x = 1/2$.

Alternatively,

$$\log_4 2 = \log_4 4^{1/2} = \frac{1}{2}\log_4 4 = \frac{1}{2}$$

since $\log_4 4 = 1$.

(o) Recall that $\log_4 2^5 = x$ is equivalent to $4^x = 2^5$. Now we solve this exponential equation.

$$\begin{aligned} 4^x &= 2^5 \\ (2^2)^x &= 2^5 \\ 2^{2x} &= 2^5 \end{aligned}$$

Thus, $2x = 5$ and $x = 5/2$.

Alternatively:

$$\log_4 2^5 = \log_4(4^{1/2})^5 = \log_4 4^{5/2} = \frac{5}{2}\log_4 4 = \frac{5}{2}$$

because $\log_4 4 = 1$.

(p) As in (n), reduce $\log_4 2$ to $\log_4$ of 4 to some power:

$$(\log_4 2)^5 = (\log_4 4^{1/2})^5 = \left(\frac{1}{2}\log_4 4\right)^5 = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

14. (a) Using $n \ln A = \ln A^n$, we get
$$e^{(1/2)\ln 16} = e^{\ln 16^{1/2}} = 16^{1/2} = \sqrt{16} = 4.$$
We used the cancellation law $e^{\ln A} = A$.

(b) Recall the general cancellation law: $a^{\log_a x} = x$, i.e., if we take a number (call in x), apply $\log_a$ to it and then exponentiate it (with the base a) we get our number back.

When $a = 10$, the formula states that $10^{\log_{10} x} = x$. Thus, $10^{\log_{10} 5} = 5$.

(c) $\log_{10} 100000 = \log_{10} 10^5 = 5 \log_{10} 10 = 5$.

(d) Using the cancellation formula that we reviewed in (b),
$$10^{3\log_{10} 4} = 10^{\log_{10} 4^3} = 4^3 = 64.$$

(e) $\log_{1000} 10 = x$ is equivalent to $1000^x = 10$. Write is as $1000^x = (10^3)^x = 10^{3x} = 10^1$, from where we conclude that $3x = 1$. Thus $x = 1/3$.

Alternatively,
$$\log_{1000} 10 = \log_{1000} 1000^{1/3} = \frac{1}{3}\log_{1000} 1000 = \frac{1}{3}$$
since $\log_{1000} 1000 = 1$.

(f) Recalling that $a^{\log_a x} = x$, we write $6^{\log_6 7} = 7$.

(g) Since $\ln e^x = x$ for all x, we conclude that $\ln(e^4) = 4$.

(h) Using laws of logarithms,
$$\ln(5e^4) - \ln 5 = (\ln 5 + \ln e^4) - \ln 5 = \ln e^4 = 4$$
Alternatively,
$$\ln(5e^4) - \ln 5 = \ln \frac{5e^4}{5} = \ln e^4 = 4$$

(i) Start by using $\ln(a/b) = \ln a - \ln b$:
$$\begin{aligned}\ln \frac{1}{2e^6} + \ln 2 &= \ln 1 - \ln(2e^6) + \ln 2\\ &= 0 - (\ln 2 + \ln e^6) + \ln 2\\ &= -\ln 2 - 6 + \ln 2 = -6\end{aligned}$$
Note that we used the fact that $\ln 1 = 0$ and the property $\ln e^x = x$ with $x = 6$.

(j) Using laws of exponents and logarithms,
$$e^{-4\ln 3} = e^{\ln 3^{-4}} = 3^{-4} = \frac{1}{3^4} = \frac{1}{81}$$

(k) Recalling that $a^{\log_a x} = x$, we write
$$12^{2\log_{12} 11} = 12^{\log_{12} 11^2} = 11^2 = 121$$

(l) Recalling that $\log_a 1 = 0$ for all $a > 0$, we find $\log_{10} 1 - \log_6 1 + \ln 1 = 0$.

(m) Rewrite $1/9$ as a power of 3:
$$\log_3 \frac{1}{9} = \log_3 \left(\frac{1}{3}\right)^2 = \log_3 \left(3^{-1}\right)^2 = \log_3 3^{-2} = -2.$$
In the last step we used the cancellation law $\log_a a^x = x$.

(n) Applying $e^{\ln A} = A$ to $A = e^2$ we get $e^{\ln(e^2)} = e^2$. Thus,

$$\ln\left(e^{\ln(e^2)}\right) = \ln(e^2) = 2.$$

In the last step we used the cancellation law $\ln e^x = x$ which is the special case (let $a = e$) of the law we reviewed in (b).

(o) $e^{-\ln 23} = e^{\ln 23^{-1}} = 23^{-1} = 1/23$.

(p) Recall that $\log_{1/9} 3 = x$ is equivalent to $(1/9)^x = 3$. Thus

$$\left(\frac{1}{9}\right)^x = \left(\frac{1}{3^2}\right)^x = (3^{-2})^x = 3^{-2x} = 3$$

implies that $-2x = 1$ and $x = -1/2$.

15. (a) By the cancellation law $a^{\log_a x} = x$, we get $6^{\log_6 x^3} = x^3$.

(b) By the same cancellation law that we used in (a), $2^{\log_2 e^x} = e^x$.

(c) By the same cancellation law, $10^{\log_{10}(x^2+1)} = x^2 + 1$.

(d) As in (a)-(c), $e^{\ln x^2} = x^2$.

(e) Likewise, $e^{\ln(\ln x)} = \ln x$.

(f) By the cancellation law $\log_a a^x = x$, we get $\ln(e^{x^2}) = x^2$.

(g) $\ln(5e^{x^2}) = \ln 5 + \ln e^{x^2} = \ln 5 + x^2$, by the same cancellation law that we used in (f).

(h) Simplifying the logarithm term, we obtain

$$e^{2\ln x^2} = e^{\ln(x^2)^2} = e^{\ln x^4} = x^4$$

16. (a) Recall the conversion formula

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$\log_5 x = \dfrac{\log_{10} x}{\log_{10} 5} \approx \dfrac{\log_{10} x}{0.698970} \approx 1.430677 \log_{10} x$.

Likewise, $\log_5 x = \dfrac{\log_{12} x}{\log_{12} 5} \approx \dfrac{\log_{12} x}{0.647685} \approx 1.543960 \log_{12} x$.

We calculated $\log_{12} 5$ using conversion $\log_{12} 5 = \log_{10} 5/\log_{10} 12 \approx 0.698970/1.079181 \approx 0.647685$.

$\log_5 x = \dfrac{\ln x}{\ln 5} \approx \dfrac{\ln x}{1.609438} \approx 0.621335 \ln x$.

(b) $\ln x = \dfrac{\log_{10} x}{\log_{10} e} \approx \dfrac{\log_{10} x}{.434294} \approx 2.302585 \log_{10} x$.

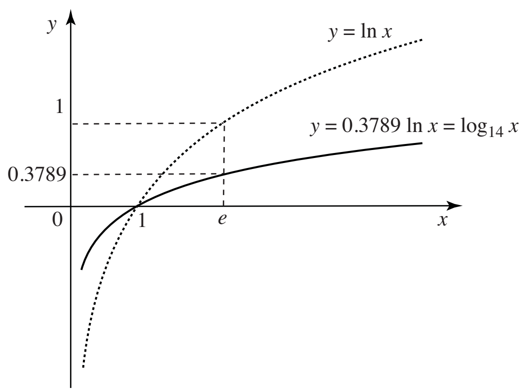
Finally, $\ln x = \dfrac{\log_{12} x}{\log_{12} e} \approx \dfrac{\log_{12} x}{} \approx 2.484910 \log_{12} x$.

We calculated $\log_{12} e$ using conversion $\log_{12} e = \log_{10} e/\log_{10} 12 \approx 0.434294/1.079181 \approx 0.402429$.

(c) Using the conversion formula $\log_a x = \frac{\log_b x}{\log_b a}$ we find

$$\log_{14} x = \frac{\ln x}{\ln 14} \approx \frac{\ln x}{2.6391} = 0.3789 \ln x$$

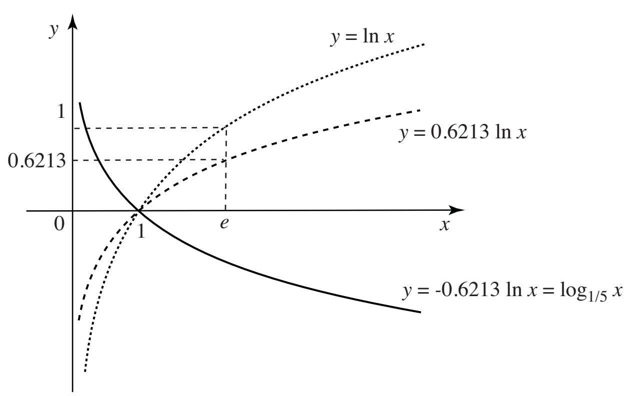
To obtain the graph of $\log_{14} x$, we scale the graph of $\ln x$ vertically by a factor of 0.3789 (alternatively, we say that we compress it vertically by a factor if $1/0.3789 = 2.6391$). See below.



(d) Using the conversion formula $\log_a x = \frac{\log_b x}{\log_b a}$ we find

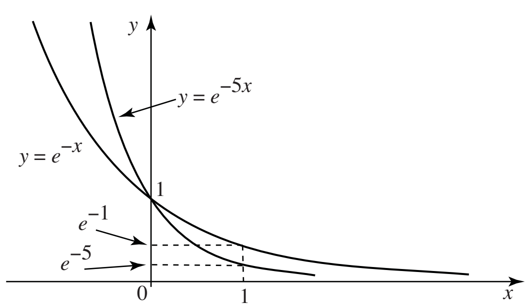
$$\log_{1/5} x = \frac{\ln x}{\ln(1/5)} \approx \frac{\ln x}{-1.6094} = -0.6213 \ln x$$

To obtain the graph of $\log_{1/5} x$, we scale the graph of $\ln x$ vertically by a factor of 0.6213 (alternatively, we say that we compress it vertically by a factor of $1/0.6213 = 1.6094$) and then reflect across the x-axis. See below.



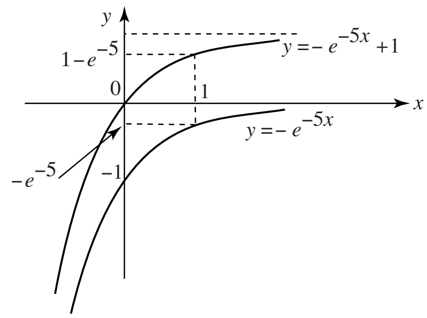
17. (a) Stretch the graph of e^x vertically by a factor of 5.

(b) Start with the graph of $y = e^x$, compress it horizontally by a factor of 5 (to obtain e^{5x}) and then reflect across the y-axis. Or, start with $y = e^{-x}$, and compress it horizontally by a factor of 5. See below (not drawn to scale).



(c) Think of $g(x) = 1 - e^{-5x}$ as $g(x) = -e^{-5x} + 1$.

Start with the finished graph in (b), reflect across the x-axis (to obtain $-e^{-5x}$) and then shift up by 1 unit. See below (not drawn to scale).



(d) Think of $f(x) = 1 - 4e^{-5x}$ as $f(x) = -4e^{-5x} + 1$.

Start with the graph of $y = e^x$, compress it horizontally by a factor of 5 (to obtain e^{5x}) and then reflect across the y-axis (to obtain e^{-5x}).

Next, stretch the graph vertically by a factor of 4 (to obtain $4e^{-5x}$) and reflect across the x-axis (to obtain $-4e^{-5x}$).

Finally, move the graph up for 1 unit.

(e) We obtain $f(x) = 6^{-x}$ by replacing x in $y = 6^x$ by $-x$. So, start with the graph of $y = 6^x$ (exponentially increasing graph) and reflect it across the y-axis.

(f) Write $y = 6^{-x-1} - 1$ as $y = 6^{-(x+1)} - 1$. Starting with $y = 6^x$, we do the following:

replace x in $y = 6^x$ by $-x$, to obtain $y = 6^{-x}$ (transformation: reflect across the y-axis)

replace x in $y = 6^{-x}$ by $x + 1$, to obtain $y = 6^{-(x+1)}$ (transformation: shift left by 1 unit)

subtract 1 from $y = 6^{-(x+1)}$ to obtain $y = 6^{-(x+1)} - 1$ (transformation: shift down by 1 unit).

(g) Starting with $y = 6^x$, we do the following:

replace x in $y = 6^x$ by $-x$, to obtain $y = 6^{-x}$ (transformation: reflect across the y-axis)

replace x in $y = 6^{-x}$ by $4x$, to obtain $y = 6^{-4x}$ (transformation: horizontal compression by a factor of 4).

Note: in this case, the order of the two operations can be switched.

(h) Because of the term +2, we need to identify what horizontal shift is involved. Factor −4 in the exponent, to obtain

$$f(x) = 6^{-4x+2} = 6^{-4(x-1/2)}$$

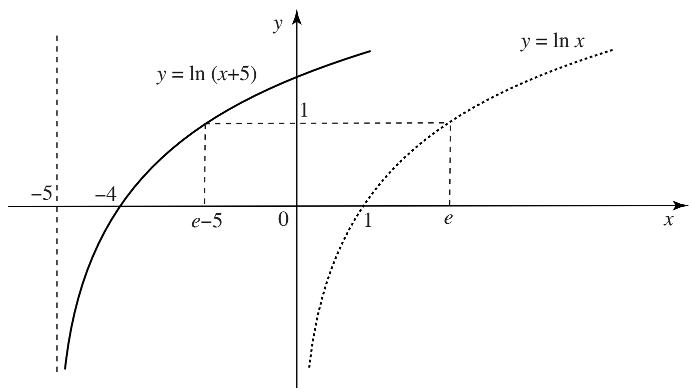
Starting with $y = 6^x$, we do the following:

replace x in $y = 6^x$ by $-x$, to obtain $y = 6^{-x}$ (transformation: reflect across the y-axis)

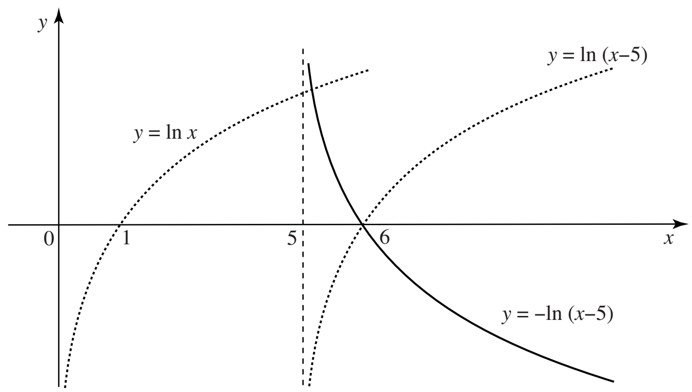
replace x in $y = 6^{-x}$ by $4x$, to obtain $y = 6^{-4x}$ (transformation: horizontal compression by a factor of 4)

replace x in $y = 6^{-4x}$ by $x - 1/2$, to obtain $y = 6^{-4(x-1/2)}$ (transformation: shift right by 1/2 units).

(i) Shift the graph of $y = \ln x$ to the left for 5 units. See below.



(j) Start with the graph of $y = \ln x$, shift it right by 5 units (to obtain $\ln(x-5)$), and then reflect across the x-axis. See below.



(k) Write $y = 1 - \ln x = -\ln x + 1$. Start with the graph of $\ln x$, reflect it across the x-axis (to obtain $-\ln x$) and move up for 1 unit.

(l) Start with the graph of $\ln x$, shift it right by 3 units (to obtain $\ln(x-3)$) and then expand vertically by a factor of 2 (to obtain $2\ln(x-3)$). Finally, move this graph one unit up.

(m) Start with the graph of $y = \log_{10} x$ and compress it horizontally by a factor of 2.

Alternatively, write

$$y = \log_{10}(2x) = \log_{10} 2 + \log_{10} x$$

so start with the graph of $y = \log_{10} x$ and shift it up by $\log_{10} 2 \approx 0.301$ units.

(n) Start with the graph of $y = \log_{10} x$ and shift it up for 2 units.

(o) Start with the graph of $y = \log_{10} x$ and shift it left for 1 unit, to obtain $y = \log_{10}(x+1)$. Then expand it vertically by a factor of 3.

(p) We need to identify what horizontal shift is involved, so we factor out 2:

$$g(x) = \log_{10}(2x+1) = \log_{10} 2\left(x + \frac{1}{2}\right)$$

Start with the graph of $y = \log_{10} x$ and do the following:

replace x in $y = \log_{10} x$ by $2x$, to obtain $y = \log_{10} 2x$ (transformation: horizontal compression by a factor of 2) replace x in $y = \log_{10} 2x$ by $x + 1/2$, to obtain $y = \log_{10} 2(x + 1/2)$ (transformation: shift left by 1/2 units).

18. (a) Incorrect. Take $x = y = 1$; then $\ln(x+y) = \ln 2$, whereas $\ln x + \ln y = \ln 1 + \ln 1 = 0 + 0 = 0$.

(b) Incorrect. Take $x = y = 1$; then $\ln(x + y) = \ln 2$, whereas $\ln x \cdot \ln y = \ln 1 \cdot \ln 1 = 0$.

(c) Correct (law of logarithms).

(d) The formula $\ln(3x) = 3 \ln x$ is incorrect. Take, for instance, $x = 1$; then $\ln(3x) = \ln 3$, whereas $3 \ln x = 3 \ln 1 = 3(0) = 0$.

Note: of course, other choices for x work as well. For instance, if $x = e$, then $\ln(3e) = \ln 3 + \ln e = \ln 3 + 1 \approx 2.098$ whereas $3 \ln x = 3 \ln e = 3$.

(e) Correct (law of exponents).

(f) Correct (law of exponents).

(g) Take $x = 0$. Then $e^{-4x} = e^0 = 1$, whereas $-4e^x = -4e^0 = -4$. Thus, $e^{-4x} = -4e^x$ is incorrect.

(h) Incorrect. Take $x = 2$; then $\ln(x/2) = \ln 1 = 0$, whereas $\ln \sqrt{x} = \ln \sqrt{2} = \frac{1}{2} \ln 2 \neq 0$.

(i) Correct; by laws of exponents and logarithms,

$$\frac{1}{2} \ln x = \ln x^{1/2} = \ln \sqrt{x}$$

19. (a) Recalling that $\ln(ab) = \ln a + \ln b$, we write

$$\ln(3e^x \sin x) = \ln 3 + \ln e^x + \ln \sin x = \ln 3 + x + \ln \sin x$$

(b) Cannot be done. Remember that $\ln(a + b)$ and $\ln(a - b)$ cannot be expanded.

(c) We start with $\ln(a/b) = \ln a - \ln b$ and then expand log of the product $3x^2$:

$$\log_{10} \frac{3x^2}{100} = \log_{10}(3x^2) - \log_{10} 100 = \log_{10} 3 + \log_{10} x^2 - \log_{10} 10^2 = \log_{10} 3 + 2 \log_{10} x - 2$$

Alternatively, write $1/100 = 1/10^2 = 10^{-2}$:

$$\log_{10} \frac{3x^2}{100} = \log_{10} \left(10^{-2}\, 3x^2\right) = \log_{10} 10^{-2} + \log_{10} 3 + \log_{10} x^2 = -2 + \log_{10} 3 + 2 \log_{10} x$$

(d) Starting by replacing the square root by the power of 1/2 and applying laws of logarithms we get

$$\begin{aligned}
\log_{10} x^{-3}\sqrt{yz^4} &= \log_{10} x^{-3}(yz^4)^{1/2} \\
&= \log_{10} x^{-3} + \log_{10}(yz^4)^{1/2} \\
&= -3 \log_{10} x + \frac{1}{2} \log_{10}(yz^4) \\
&= -3 \log_{10} x + \frac{1}{2} \left(\log_{10} y + \log_{10} z^4\right) \\
&= -3 \log_{10} x + \frac{1}{2} \left(\log_{10} y + 4 \log_{10} z\right) \\
&= -3 \log_{10} x + \frac{1}{2} \log_{10} y + 2 \log_{10} z
\end{aligned}$$

(e) Start by replacing the square root by the power of 1/2 and applying laws of logarithms:

$$\begin{aligned}\ln\sqrt{\frac{e^{2x}x^3}{x+1}} &= \ln\left(\frac{e^{2x}x^3}{x+1}\right)^{1/2}\\ &= \frac{1}{2}\ln\frac{e^{2x}x^3}{x+1}\\ &= \frac{1}{2}\left(\ln e^{2x}+\ln x^3-\ln(x+1)\right)\\ &= \frac{1}{2}\left(2x+3\ln x-\ln(x+1)\right)\end{aligned}$$

(f) Using laws of logarithms,

$$\begin{aligned}\ln\frac{\sqrt[3]{x}}{x^2(e^x+1)} &= \ln\sqrt[3]{x}-\ln\left(x^2(e^x+1)\right)\\ &= \ln x^{1/3}-\left(\ln x^2+\ln(e^x+1)\right)\\ &= \tfrac{1}{3}\ln x-2\ln x-\ln(e^x+1) = -\tfrac{5}{3}\ln x-\ln(e^x+1)\end{aligned}$$

20. (a) Think of laws of logarithms "in reverse" and use $a=\ln e^a$:

$$\ln x-\ln y+4=\ln x-\ln y+\ln e^4=\ln x+\ln e^4-\ln y=\ln(xe^4)-\ln y=\ln\left(\frac{xe^4}{y}\right)$$

(b) Once in the form of a fraction, we use the difference of squares formula to factor the numerator.

$$\ln(x^2-y^2)-\ln(x-y)=\ln\frac{x^2-y^2}{x-y}=\ln\frac{(x-y)(x+y)}{x-y}=\ln(x+y)$$

(c) Remove coefficients in front of ln first:

$$\begin{aligned}\ln x-3\ln(x-4)+\frac{1}{2}\ln x &= \ln x-\ln(x-4)^3+\ln x^{\frac{1}{2}}\\ &= \ln x+\ln x^{\frac{1}{2}}-\ln(x-4)^3\\ &= \ln(x\cdot x^{\frac{1}{2}})-\ln(x-4)^3\\ &= \ln(x^{\frac{3}{2}})-\ln(x-4)^3=\ln\frac{x^{3/2}}{(x-4)^3}\end{aligned}$$

(d) We find

$$\begin{aligned}\log_{10}x-3\log_{10}y^2+4\log_{10}(xy^3) &= \log_{10}x+\log_{10}\left(xy^3\right)^4-\log_{10}\left(y^2\right)^3\\ &= \log_{10}x+\log_{10}x^4y^{12}-\log_{10}y^6\\ &= \log_{10}x\cdot\frac{x^4y^{12}}{y^6}\\ &= \log_{10}\frac{x^5y^{12}}{y^6}=\log_{10}x^5y^6\end{aligned}$$

(e) Remove coefficients in front of ln first:

$$\begin{aligned}\frac{1}{3}\log_{10}x+\frac{1}{5}\log_{10}x^2-\frac{2}{3}\log_{10}8 &= \log_{10}x^{1/3}+\log_{10}(x^2)^{1/5}-\log_{10}8^{2/3}\\ &= \log_{10}x^{1/3}+\log_{10}x^{2/5}-\log_{10}(2^3)^{2/3}\\ &= \log_{10}(x^{1/3}\cdot x^{2/5})-\log_{10}2^2\\ &= \log_{10}(x^{11/15})-\log_{10}4\end{aligned}$$

$$= \log_{10} \frac{x^{11/15}}{4}$$

In this calculation, we used $x^{1/3} \cdot x^{2/5} = x^{1/3+2/5} = x^{5/15+6/15} = x^{11/15}$ and the fact that $(2^3)^{2/3} = 2^{3\cdot(2/3)} = 2^2$.

21. (a) Use the fact that $0.1 = 10^{-1}$ and $100 = 10^2$; we get

$$\begin{aligned} 0.1^x &= 100 \\ (10^{-1})^x &= 10^2 \\ 10^{-x} &= 10^2 \end{aligned}$$

Thus, $-x = 2$ and $x = -2$.

Alternatively, we can use logarithm to any base – we take ln

$$\begin{aligned} 0.1^x &= 100 \\ \ln 0.1^x &= \ln 100 \\ x \ln 0.1 &= \ln 10^2 \\ x &= \frac{\ln 10^2}{\ln 0.1} = \frac{2 \ln 10}{\ln 10^{-1}} = \frac{2 \ln 10}{(-1) \ln 10} = -2. \end{aligned}$$

(b) We use the same strategy as in (a) – reduce to the same base:

$$\begin{aligned} \left(\frac{1}{4}\right)^x &= 2 \\ \left(\frac{1}{2^2}\right)^x &= 2 \\ (2^{-2})^x &= 2 \\ 2^{-2x} &= 2 = 2^1 \end{aligned}$$

Thus, $-2x = 1$ and $x = -1/2$.

(c) Note that $0.25 = 1/4$; thus

$$\begin{aligned} 0.25^x &= 16 \\ \left(\frac{1}{4}\right)^x &= 16 \\ (4^{-1})^x &= 16 \\ 4^{-x} &= 4^2 \end{aligned}$$

So the solution is $-x = 2$, i.e., $x = -2$.

(d) Divide both sides of the equation by 7 and apply ln:

$$\begin{aligned} 7e^{5x} &= 21 \\ e^{5x} &= 3 \\ \ln e^{5x} &= \ln 3 \\ 5x &= \ln 3 \\ x &= \frac{\ln 3}{5} \end{aligned}$$

(e) We find

$$\begin{aligned}4e^{2x-1} &= 27\\ e^{2x-1} &= \frac{27}{4}\\ \ln e^{2x-1} &= \ln\frac{27}{4}\\ 2x-1 &= \ln\frac{27}{4}\\ x &= \frac{\ln\frac{27}{4}+1}{2}\end{aligned}$$

(f) Simplify so that there is a single exponential term; to do this, we multiply both sides by e^{-7x} (which is the same as dividing by e^{7x})

$$\begin{aligned}4e^{-3x+1} &= 6e^{7x}\\ 4e^{-3x+1}e^{-7x} &= 6e^{7x}e^{-7x}\\ 4e^{-10x+1} &= 6\\ e^{-10x+1} &= \frac{3}{2}\\ -10x+1 &= \ln(3/2)\\ 10x &= 1-\ln(3/2)\\ x &= \frac{1-\ln(3/2)}{10}\end{aligned}$$

(g) Isolate 6^x and then apply ln:

$$\begin{aligned}5\cdot 6^x &= 8\\ 6^x &= \frac{8}{5}\\ \ln 6^x &= \ln\frac{8}{5}\\ x\ln 6 &= \ln\frac{8}{5}\\ x &= \frac{\ln(8/5)}{\ln 6}\end{aligned}$$

Alternatively, we can use the definition of logarithm with base 6 to write $6^x = 8/5$ as $x = \log_6(8/5)$. When we apply the conversion formula expressing $\log_6 x$ in terms of $\ln x$ we obtain the above answer.

(h) We apply ln to both sides, then expand and gather x terms together:

$$\begin{aligned}\ln(5\cdot 6^x) &= \ln(8^{3x-1})\\ \ln 5+\ln 6^x &= (3x-1)\ln 8\\ \ln 5 + x\ln 6 &= 3x\ln 8-\ln 8\\ x\ln 6-3x\ln 8 &= -\ln 8-\ln 5\\ x(\ln 6-3\ln 8) &= -\ln 8-\ln 5\\ x &= \frac{-\ln 8-\ln 5}{\ln 6-3\ln 8}\end{aligned}$$

(i) We apply ln to both sides, then expand and gather x terms together:

$$\begin{aligned}\ln(3e^{-3x+1}) &= \ln 4^{-x}\\ \ln 3 + \ln e^{-3x+1} &= -x\ln 4\\ \ln 3 + (-3x+1) &= -x\ln 4\\ -3x + x\ln 4 &= -\ln 3 - 1\\ x(-3+\ln 4) &= -\ln 3 - 1\\ x &= \frac{-\ln 3 - 1}{-3+\ln 4}\end{aligned}$$

(j) We use the fact that $0.1 = 10^{-1}$ to get

$$\begin{aligned}0.1^{x+2} &= 100^{1/3}\\ (10^{-1})^{x+2} &= (10^2)^{1/3}\\ 10^{-1(x+2)} &= 10^{2/3}\end{aligned}$$

We conclude that $-x-2 = 2/3$, and thus $x = -2 - 2/3 = -8/3$.

(k) We use the fact that $(e^x)^2 = e^{2x}$ (recall that $(a^m)^n = a^{mn}$).

Write $e^{2x} + 2e^x - 8 = 0$ as $(e^x)^2 + 2e^x - 8 = 0$ and factor

$$(e^x+4)(e^x-2) = 0.$$

If $e^x + 4 = 0$, then $e^x = -4$, and so there are no solutions (since $e^x > 0$ for all x). If $e^x - 2 = 0$, then $e^x = 2$, and so $x = \ln 2$.

We conclude that there is one solution, $x = \ln 2$.

(l) Rewrite $100^{\cos x} = 10$ as $(10^2)^{\cos x} = 10$ i.e., $10^{2\cos x} = 10^1$. Thus, $2\cos x = 1$ and $\cos x = 1/2$.

In Example 11 in Section 7 we solved this equation and got $x = \frac{\pi}{3} + 2k\pi$ and $x = \frac{5\pi}{3} + 2k\pi$.

(m) Apply the exponential function to both sides:

$$\begin{aligned}\ln(4x-2) &= 17\\ e^{\ln(4x-2)} &= e^{17}\\ 4x-2 &= e^{17}\\ 4x &= e^{17}+2\\ x &= \frac{e^{17}+2}{4}\end{aligned}$$

(n) Rewrite $\log_{10}$ using exponential function:

$$\begin{aligned}\log_{10}(x^2-1) &= 5\\ x^2 - 1 &= 10^5\\ x^2 &= 10^5 + 1\\ x &= \pm\sqrt{10^5+1}\end{aligned}$$

(o) Combine the ln terms on the left side:

$$\begin{aligned}\ln x + \ln(x+1) &= \ln 2\\ \ln(x(x+1)) &= \ln 2\end{aligned}$$

$$\begin{aligned} x(x+1) &= 2 \\ x^2 + x - 2 &= 0 \\ (x+2)(x-1) &= 0 \end{aligned}$$

We obtain $x = -2$ and $x = 1$. When $x = -2$, the two ln terms on the left side are not defined, and so $x = -2$ is not a solution. On the other hand, $x = 1$ is a solution, as the two ln terms on the left side are defined.

(p) Rewrite $\log_{10}(x - 98) = 4$ as $x - 98 = 10^4$. Thus $x = 10^4 + 98 = 10098$.

(q) Apply the exponential function to both sides: $e^{\ln(\ln x)} = e^1$ and so $\ln x = e$. Apply the exponential function once more: $e^{\ln x} = e^e$, i.e., $x = e^e$.

Alternatively, recall that $\ln A = B$ implies $A = e^B$. Apply this to $A = \ln x$ and $B = 1$ to obtain $A = \ln x = e^1 = e$, i.e., $\ln x = e$. Apply the same to $A = x$ and $B = e$ to obtain $A = x = e^e$.

(r) We are solving $\log_{10}(\log_{10} x) = 0$.

Recall that $\log_{10} A = B$ means that $A = 10^B$. Apply this with $A = \log_{10} x$ and $B = 0$, to obtain $A = \log_{10} x = 10^0 = 1$. From $\log_{10} x = 1$ (use the above again, with $A = x$ and $B = 1$) we obtain $x = 10^1 = 10$.

Alternatively, apply the exponential function to both sides and use the fact that $10^{\log_{10} x} = x$:

$$\begin{aligned} \log_{10}(\log_{10} x) &= 0 \\ 10^{\log_{10}(\log_{10} x)} &= 10^0 \\ \log_{10} x &= 1 \\ 10^{\log_{10} x} &= 10^1 \\ x &= 10 \end{aligned}$$

22. Using the conversion formula, we write

$$\log_4 7 = \frac{\ln 7}{\ln 4} \approx \frac{1.945910}{1.386294} \approx 1.403677.$$

Section 9. Mathematical Language; Mathematical Thinking and Logic

1. Recall that, first, we need to identify two parts: what is taken to be true (i.e., assumption(s)), and what is claimed to be true (i.e., conclusion(s)); then we write it in the form

If <assumption(s)> then <conclusion(s)> .

(a) The statement is about a square of an even number (that's what's assumed); the claim is that such number is even. Thus, we write

If a number is even, then its square is even.

(b) The statement is about increasing functions, and it claims that those functions have inverses. Thus

If a function is increasing, then it has an inverse function.

(c) Likewise,

If x is a real number, then $|x| \geq 0$.

Note that the meaning of the indefinite article "a" is not "one" or "some" but "any" or "all" (see the very last note (just before *Additional Exercises*) in this section in the book).

(d) The statement is about how the equality between two numbers a and b (assumption) affects the given equation (conclusion). Thus,

If $a = b$, then the equation $x^2/a^2 + y^2/b^2 = 1$ represents a circle.

2. A is what is assumed, and B is what is claimed to follow from A. Recall that the converse of "if A, then B" is the statement "if B, then A".

(a) From

If x is positive, then e^x is positive.

we identify A="x is positive", B="e^x is positive".

This statement is true, because exponential function a^x for any $a > 0$ and any real number x is positive (or, look at the graph of e^x!).

The converse reads

If e^x is positive, then x is positive.

This is a false statement; to prove it, we need to find a counterexample. Take $x = -1$. Then $e^x = e^{-1}$ is positive, but $x = -1$ is not positive.

(b) We see that A="$a = b$"; B="$a^2 = b^2$". This statement is true: if two numbers are equal, then their squares must be equal as well.

The converse statement is

If $a^2 = b^2$, then $a = b$.

This is not true: numbers $a = 5$ and $b = -5$ squared give the same number (25), but are not equal.

(c) A="$m_1m_2 = -1$"; B="the lines $y = m_1x$ and $y = m_2x$ are perpendicular".

This is a true statement, see Section 5 (note that we did not prove it, just used it).

The converse reads

If the lines $y = m_1x$ and $y = m_2x$ are perpendicular, then $m_1m_2 = -1$

Again, this is a true statement, as explained (but not proved) in Section 5.

3. Recall that the contrapositive of "if A, then B" is "if *not* B, then *not* A."

(a) From

$$\text{If } \sin x \neq 0, \text{ then } \csc x \text{ is defined}$$

we see that A="$\sin x \neq 0$"; B="$\csc x$ is defined".

Thus, *not* A="$\sin x = 0$" and *not* B="$\csc x$ is not defined".

The contrapositive is

$$\text{If } \csc x \text{ is not defined, then } \sin x = 0.$$

(b) Reading the statement

$$\text{If } x_1 \neq x_2, \text{ then } f(x_1) \neq f(x_2)$$

we see that A="$x_1 \neq x_2$"; B="$f(x_1) \neq f(x_2)$".

Thus, *not* A="$x_1 = x_2$" and *not* B="$f(x_1) = f(x_2)$".

The contrapositive is

$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

(c) We see that A="curve is a straight line"; B="same slope at all points".

Thus, *not* A="curve is not a straight line" and *not* B="not same slope at all points".

The contrapositive is

If a curve does not have the same slope at all points, then it is not a line.

4. (a) We start by experimenting: we take a prime number and check whether that number plus two is also a prime.

If $p = 3$, Then $p + 2 = 5$, which is a prime. If $p = 5$, Then $p + 2 = 7$, which is a prime. If $p = 7$, Then $p + 2 = 9$, which is not a prime!

Thus, we found a counterexample: a prime number which does not yield a prime number when we add 2 to it. Consequently, the given statement is false.

Note that by examining cases we could never prove that some statement implicating all prime numbers is true (since we would have to check that it holds for all infinitely many prime numbers). However, to prove that a statement involving universal quantifier is not true, all we need is one example where it does not work (that's called a counterexample).

(b) If we can find one example, i.e., one prime number for which the statement holds, we will prove that it is true.

We have already found two examples in part (a). As well, $p = 11$ is prime, and so is $p + 2 = 13$.

(c) The statement is true, by the definition of the square root. (Note that examining numbers one by one would not work.)

(d) Recall that if we wish to prove that this statement is true, we have to show that it holds for at least one real number. To disprove it, we must show that it does not hold for *any one* real number.

We see that the discriminant $D = b^2 - 4ac = 3^2 - 4(7) = -19$ is negative, and so the equation $x^2 + 3x + 7 = 0$ has no real number solutions.

This shows that *no* real number satisfies $x^2 + 3x + 7 = 0$; so this statement is false:

$$\text{There is a real number } x \text{ such that } x^2 + 3x + 7 = 0.$$

5. (a) The statement is about a property of the square root, and the numbers involved (a and b) are assumed to be positive. Thus, we write

$$\text{If } a \text{ and } b \text{ are positive numbers, then } \sqrt{ab} = \sqrt{a}\sqrt{b}.$$

(b) The statement is about the exponential function $y = a^x$ in the case when $a > 1$ (so, that's the assumption). Thus,

If $a > 1$, then the exponential function $y = a^x$ is increasing.

(c) The statement is about the sequence which is bounded and convergent (that's assumed). Such sequence is claimed to be convergent (conclusion). We write

If a sequence is bounded and increasing, then it is convergent.

(d) "As long as ..." suggests that we need to take an even number; in that case its square is divisible by 4. Thus,

If a number is even, then its square is divisible by 4.

(e) The sentence is clear about what is assumed and what is claimed. Thus, we write

If a function $f(x)$ is differentiable and has a relative maximum at $x = a$, then $f'(a) = 0$.

(f) The sentence states that two positive numbers m and n such that $m > n$ (that is what we assume) satisfy certain inequality. Thus, we write

$$\text{If } m \text{ and } n \text{ are positive numbers such that } m > n, \text{ then } \frac{1}{m} < \frac{1}{n}.$$

(g) We assume that we are given parallel lines, and claim a property of their slopes. Thus,

If lines are parallel, then their slopes are equal.

(h) The formula $|x| > 0$ is a consequence (conclusion) of a certain property of x (assumption). We write

If x is any non-zero real number, then $|x| > 0$.

(i) "As long as ..." suggests that what follows is an assumption. Thus,

If $f(x)$ is continuous, then the composition $\sin f(x)$ is continuous.

(j) The sentence clearly suggests that the assumption is $m_1 \neq m_2$. Thus,

If $m_1 \neq m_2$, then the lines $y = m_1 x + b_1$ and $y = m_2 x + b_2$ intersect at a point.

6. Recall that given an implication "if A, then B" its converse is "if B, then A", and the contrapositive is "if *not* B, then *not* A."

(a) We identify A="$x \neq 0$" and B="$1/x$ is a real number". Thus, *not* A = "$x = 0$" and *not* B = "$1/x$ is not a real number".

Converse: If $1/x$ is a real number, then $x \neq 0$.

Contrapositive: If $1/x$ is not a real number, then $x = 0$.

(b) We identify A="$f(x)$ is an increasing function" and B="$f(x)$ satisfies the horizontal line test". Thus, *not* A = "$f(x)$ is not an increasing function" and *not* B = "$f(x)$ does not satisfy the horizontal line test".

Converse: If $f(x)$ satisfies the horizontal line test, then $f(x)$ is an increasing function.

Contrapositive: If $f(x)$ does not satisfy the horizontal line test, then $f(x)$ is not an increasing function.

(c) We see that A="it rains" and B="the roads are wet". Thus, *not* A = "it does not rain" and *not* B = "the roads are not wet".

Converse: If the roads are wet, then it rains.

Contrapositive: If the roads are not wet, then it does not rain.

(d) We see that A="$\ln x = \ln y + \ln z$" and B="$x = yz$". Thus, *not* A = "$\ln x \neq \ln y + \ln z$" and *not* B = "$x \neq yz$".

Converse: If $x = yz$, then $\ln x = \ln y + \ln z$.

Contrapositive: If $x \neq yz$, then $\ln x \neq \ln y + \ln z$.

(e) We identify A="$f(g(x)) = x$" and B="$f(x)$ is an inverse function of $g(x)$". Thus, *not* A = "$f(g(x)) \neq x$" and *not* B = "$f(x)$ is not an inverse function of $g(x)$".

Converse: If $f(x)$ is an inverse function of $g(x)$, then $f(g(x)) = x$.

Contrapositive: If $f(x)$ is not an inverse function of $g(x)$, then $f(g(x)) \neq x$.

(f) We identify A="$f(x)$ has a relative extreme value at $x = a$" and B="$x = a$ is a critical point". Thus, *not* A = "$f(x)$ does not have a relative extreme value at $x = a$" and *not* B = "$x = a$ is not a critical point".

Converse: If $x = a$ is a critical point, then $f(x)$ has a relative extreme value at $x = a$.

Contrapositive: If $x = a$ is not a critical point, then $f(x)$ does not have a relative extreme value at $x = a$.

(g) We identify A="a sequence a_n is convergent" and B="it is bounded"; in this case (for language reasons) it might be more appropriate to use: B="a sequence a_n is bounded". Thus, *not* A = "a sequence a_n is not convergent" and *not* B = "a sequence a_n is not bounded".

Converse: If a sequence a_n is bounded, then a sequence a_n is convergent. Using better language: If a sequence a_n is bounded, then it is convergent.

Contrapositive: If a sequence a_n is not bounded, then a sequence a_n is not convergent. Using better language: If a sequence a_n is not bounded, then it is not convergent.

(h) We identify A="$m_1 = m_2$" and B="the lines $y = m_1x + b_1$ and $y = m_2x + b_2$ are parallel". Thus, *not* A = "$m_1 \neq m_2$" and *not* B = "the lines $y = m_1x + b_1$ and $y = m_2x + b_2$ are not parallel".

Converse: If the lines $y = m_1x + b_1$ and $y = m_2x + b_2$ are parallel, then $m_1 = m_2$.

Contrapositive: If the lines $y = m_1x + b_1$ and $y = m_2x + b_2$ are not parallel, then $m_1 \neq m_2$.

(i) We see that A="$m_1 \neq m_2$" and B="the lines $y = m_1x + b_1$ and $y = m_2x + b_2$ are not parallel". Thus, *not* A = "$m_1 = m_2$" and *not* B = "the lines $y = m_1x + b_1$ and $y = m_2x + b_2$ are parallel".

Converse: If the lines $y = m_1x + b_1$ and $y = m_2x + b_2$ are not parallel, then $m_1 \neq m_2$.

Contrapositive: If the lines $y = m_1x + b_1$ and $y = m_2x + b_2$ are parallel, then $m_1 = m_2$.

(j) We see that A="$x_1 \neq x_2$" and B="$f(x_1) = f(x_2)$". Thus, *not* A = "$x_1 = x_2$" and *not* B = "$f(x_1) \neq f(x_2)$".

Converse: If $f(x_1) = f(x_2)$, then $x_1 \neq x_2$.

Contrapositive: If $f(x_1) \neq f(x_2)$, then $x_1 = x_2$.

7. (a) We see that A="$\ln x = \ln y$" and B="$x = y$".

To prove the implication "If $\ln x = \ln y$ then $x = y$" we assume that $\ln x = \ln y$. Then, by composing with the exponential function, we obtain $e^{\ln x} = e^{\ln y}$ and $x = y$. Thus, this implication is true.

Converse: "If $x = y$ then $\ln x = \ln y$." This statement is true: since $x = y$, x and y are actually the same number, and so their logarithms are equal.

(b) We see that A="$x > 4$" and B="$|x| > 4$".

To prove the implication "If $x > 4$, then $|x| > 4$" we assume that $x > 4$. Since x is positive, the definition of the absolute value states that $|x| = x$. Thus, $|x|$ must be greater than 4 as well. Thus, this implication is true.

Converse: "If $|x| > 4$ then $x > 4$." This statement is not true (we need a counterexample); for instance, $x = -5$ satisfies $|x| = |-5| = 5 > 4$, but $x = -5 > 4$ is not true.

(c) We identify A="$\cos x = 0$" and B="$x = 0$".

To prove that the implication "If $\cos x = 0$, then $x = 0$" is not true, we recall that all solutions of $\cos x = 0$ are given by $x = \frac{\pi}{2} + k\pi$; no matter what integer value for k we take, we cannot make x equal to zero.

Converse: "If $x = 0$ then $\cos x = 0$." This statement is not true; when $x = 0$, we find $\cos x = \cos 0 = 1$, and not zero.

(d) We identify A="$x = 7$" and B="$x^2 = 49$".

To prove that the implication "If $x = 7$ then $x^2 = 49$" is true, we start with (the assumption) $x = 7$, and square it to obtain (the conclusion) $x^2 = 49$.

Converse: "If $x^2 = 49$ then $x = 7$." This statement is not true. If $x^2 = 49$, Then x could be 7, or it could be -7.

(e) We identify A="$f(g(x)) = g(f(x))$" and B="$f(x)$ and $g(x)$ are inverse of each other".

Just because two functions commute, it does not mean that they are inverse to each other. If, on top, both compositions are equal to x, ie, if $f(g(x)) = g(f(x)) = x$, then $f(x)$ and $g(x)$ are inverse of each other. Thus the implication "If $f(g(x)) = g(f(x))$, then $f(x)$ and $g(x)$ are inverse of each other" is not true.

Alternative proof: let $f(x) = x^2$ and $g(x) = x^3$. Then $f(g(x)) = f(x^3) = (x^3)^2 = x^6$ is equal to $g(f(x)) = g(x^2) = (x^2)^3 = x^6$, but $f(x) = x^2$ and $g(x) = x^3$ are not inverse to each other.

Converse: "If $f(x)$ and $g(x)$ are inverse of each other then $f(g(x)) = g(f(x))$." This statement is true, as it is part of the property of being inverse.

In other words: if $f(x)$ and $g(x)$ are inverse of each other then $f(g(x)) = g(f(x)) = x$, holds; so, a weaker condition $f(g(x)) = g(f(x))$ holds as well.

(f) We identify A="x is a real number" and B="$1/x$ is a real number".

To show that the implication "If x is a real number, then $1/x$ is a real number" is not true, we recall the fact that when $x = 0$ the fraction $1/x$ is not defined (i.e., it is not a real number). The implication would have been true if the assumption read "x is a *non-zero* real number".

Converse: "If $1/x$ is a real number then x is a real number." This statement is true. Actually, if $1/x$ is a real number then we know that x is a real number such that $x \neq 0$.

(g) We identify A="a real number satisfies $x > 1$" and B="its square satisfies $x^2 > 1$".

Multiplying a number greater than 1 by itself produces a number greater than 1. (Geometrically, the graph of $y = x^2$ lies above the line $y = 1$ for all $x > 1$.) Thus, the implication "If a real number satisfies $x > 1$, then its square satisfies $x^2 > 1$" is true.

Converse: "If the square of a real number satisfies $x^2 > 1$, then $x > 1$." This statement is not true. For instance, the number $x = -10$ satisfies $x^2 = (-10)^2 = 100 > 1$, but $x = -10$ is not larger than 1.

(h) We identify A="a line is horizontal" and B="its slope is zero"; or, B="a slope of a line is zero".

Think of the slope as "rise over run."For a horizontal line, the rise is zero, and thus the slope is zero. We conclude that the implication "If a line is horizontal, then its slope is zero" is true.

Converse: "If the slope of a line is zero, then the line is horizontal." This statement is true. From slope = rise/run = 0 we conclude that rise = 0, and so the line is horizontal.

8. (a) We would need to check that every single country in South America is not landlocked.

(b) We would need to identify one country which is landlocked. [By the way, this statement is true: for instance, Bolivia is landlocked.]

(c) We would have to find one car in Canada which does not have four wheels.

(d) We would have to check that every single car in Canada has four wheels.

(e) We would have to check that there is no living dinosaur anywhere on Earth.

(f) We would have to find one living dinosaur somewhere on Earth.

(g) We would have to check that every single dog in Ontario barks.

(h) We would need to find one dog in Ontario which does not bark.

(i) We would have to find one non-zero real number x for which $\ln x^2 = 2 \ln x$ does not hold. [For instance, when $x = -5$, the left side is $\ln 25$, whereas the right side is not defined.]

(j) We would have to find one line which does not satisfy the horizontal line test. [Horizontal line!]

9. (a) We know that the formula $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ is not true in general (i.e., does not work for *all* choices of a and b). But we are asked whether or not we can find one pair of values for a and b when it does work.

There are no formulas nor procedures to find this out, so we just try different numbers (of course, we start with small numbers, and in particular 0 and 1).

If $a = 1$ and $b = 1$, then $\sqrt{a+b} = \sqrt{2}$ and $\sqrt{a} + \sqrt{b} = \sqrt{1} + \sqrt{1}$. Not good, the two sides are not equal.

However, if $a = 0$ and $b = 1$, then $\sqrt{a+b} = \sqrt{1} = 1$ and $\sqrt{a} + \sqrt{b} = \sqrt{0} + \sqrt{1} = 1$. The two sides are equal, so we proved that "There exist real numbers a and b such that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$" is true.

(By the way, the formula holds if we take any $a \geq 0$ and $b = 0$. But for the purpose of this question, one example suffices.)

(b) This formula is false. To prove it, we need to find one pair of values for a and b so that the two sides of the formula give different values.

For instance, if $a = 4$ and $b = 1$, then $\sqrt{a+b} = \sqrt{4+1} = \sqrt{5}$ and $\sqrt{a} + \sqrt{b} = \sqrt{4} + \sqrt{1} = 2 + 1 = 3$. Since $\sqrt{5} \neq 3$, we are done.

(c) We claim that the statement is true. "There exists ..." means that we need to find one real number $x > 0$ such that $1 - \frac{1}{x} > 0$.

To make $1 - \frac{1}{x}$ positive, what we subtract from 1 should be smaller than 1; thus, we need to make $1/x$ smaller than 1. That can be done, take for example $x = 13$.

To summarize: if we take $x = 13 > 0$, then $1 - \frac{1}{x} = 1 - \frac{1}{13} = \frac{12}{13} > 0$.

(d) In this case, we cannot check real numbers one by one, but need a general argument, i.e., the reasoning which which works for all $x > 0$.

If a number is positive ($x > 0$) then its reciprocal $1/x > 0$ is positive (as the quotient of two positive numbers). Adding 1 to the inequality $1/x > 0$ we obtain $1 + \frac{1}{x} > 1 + 0 = 1 > 0$, and our proof is complete.

(e) True. Recall that the geometric transformation resulting from replacing x by ax is a *horizontal* transformation (contraction in this case; but it does not matter). Because it is horizontal, such transformation does not affect the range of a function.

(f) Let's compute the range of $y = b \sin x$. We know that $-1 \leq \sin x \leq 1$; by multiplying by $b > 0$ we obtain $-b \leq b \sin x \leq b$. Using interval notation, we write the range of $y = b \sin x$ as $[-b, b]$.

We claim that there is no b for which $[-b, b] = [0, 2]$. This is true, since b would have to satisfy $-b = 0$ and $b = 2$ at the same time. Consequently, the given statement is false.

(g) Let's find the range of $y = c + \sin x$. From $-1 \leq \sin x \leq 1$, by adding c to all three sides, we obtain $c - 1 \leq c + \sin x \leq c + 1$. Using interval notation, we write the range of $y = c + \sin x$ as $[c - 1, c + 1]$.

We see that if we take $c = 1$, then $[c - 1, c + 1] = [1 - 1, 1 + 1] = [0, 2]$. Thus, the given statement is true.

(h) True. To prove this, we need to identify one value of a for which $y = a^x$ is decreasing function. Recalling the graphs of exponential functions, we realize that any number a such $0 < a < 1$ will generate a decreasing exponential function $y = a^x$.

(i) To prove that this statement is false, we need a counterexample, i.e., we need to find one $a > 0$ for which $y = a^x$ is not decreasing.

As a matter of fact, for any $a > 1$ the function $y = a^x$ is increasing. So, for instance, $a = 6$ and $y = 6^x$ constitute a counterexample.

(j) To prove that this statement is false, we need a counterexample, i.e., we need to find one real number x for which $f(x) = xe^{-x+4} < 0$.

For instance, if $x = -1$, then $f(-x) = -e^5 < 0$, since the exponential function is always positive (i.e., $e^5 > 0$). Done.

(k) False. We need to show that no matter what real number x we take, the value of the function $f(x) = e^{-x+4}$ cannot be negative.

Recall that the exponential function is always positive, i.e., $e^A > 0$ no matter what A is. Thus, for any real number x, $e^{-x+4} > 0$. Done.