

# CALCULUS \* FEAR NO MORE

## SOLUTIONS TO ALL EXERCISES

© 2010, 2011 Miroslav Lovric. This is an open-access document. You are allowed to download, print and share this document, or use it otherwise for noncommercial purposes. Making copies and distributing this document (in any format) for profit is prohibited. Version 1 August 2011.

## FOREWORD

This file contains solutions to all exercises from the book *Calculus Fear No More* [Review and Reference for First Year College and University Courses] by Miroslav Lovrić, published by Nelson.

In most cases all details of a solution are provided. Sometimes, related theoretical concepts, methods or formulas are recalled. Keep in mind that – in many cases – there are multiple correct ways of solving a question (and/or simplifying an answer); just because your answer looks different from the one given here does not mean that it's incorrect. Some exercises require that you use a calculator.

Do not read this manual! It is far more beneficial to try to solve an exercise on your own. Start and see how far you can go. If you get stuck, identify the problem first – try to understand why you are having difficulties, and then look up the solution. This way, you will learn not only what the problem is (or what you have problems with), but also how that particular problem has been resolved. If you just read a solution, you might not recognize the hard part(s) - or, even worse, you might miss the whole point of the exercise.

Big Thank you! to Andrijana Burazin (University of Waterloo) for writing solutions to all exercises. I accept full responsibility for errors and will be grateful to anybody who brings them to my attention. Your comments and suggestions will be greatly appreciated.

Miroslav Lovrić  
Department of Mathematics and Statistics  
McMaster University  
Hamilton, Ontario, Canada L8S 4K1  
e-mail: lovric@mcmaster.ca

January 2010, August 2011

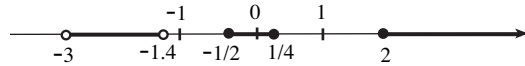
## CONTENTS

1. Numbers and Operations .....	1
2. Basic Algebra .....	4
3. Basic Facts and Formulas from Geometry .....	[no exercises]
4. Equations and Inequalities .....	11
5. Elements of Analytic Geometry .....	24
6. Functions .....	31
7. Trigonometric Functions .....	38
8. Exponential and Logarithmic Functions .....	52
9. Mathematical Language; Mathematical Thinking and Logic .....	61

---

**Section 1. Numbers and Operations**

1. Remember: we use round bracket or empty dot to say that a point (real number) is not included, and square bracket or filled dot when we wish to include a point (real number).



2. The leftmost interval includes all negative numbers up to and including  $-2$ , so we denote it by  $(-\infty, -2]$ . The middle interval contains all numbers between  $-1$  and  $2$ , not including  $-1$  but including  $2$ ; so, we write  $(-1, 2]$ . The rightmost interval is  $(3, 4)$ .

3. (a) Since  $\pi$  is positive,  $|\pi| = \pi$ .

By definition,  $|-1/2| = -(-1/2) = 1/2$ , and thus  $-|-1/2| = -1/2$ .

Since  $-0.33$  is negative,  $|-0.33| = -(-0.33) = 0.33$ .

Note: of course, we could have written right away that  $|-1/2| = 1/2$  or  $|-0.33| = 0.33$ , since absolute value returns positive numbers. We included intermediate steps to practice the definition.

Why is the definition of absolute value so messy if all it does is to convert a number into a positive number? The reason is that we want to do more than just do calculations with numbers. The definition, as we will see, will be applied to algebraic expressions.

(b) Recall that the distance between two numbers on a number line is computed as the absolute value of their difference; so, the distance between  $-3$  and  $-4$  is  $|-3 - (-4)| = |-3 + 4| = |1| = 1$ .

$$(c) \frac{|4 - 4.5|}{4.5 - 5} = \frac{|-0.5|}{-0.5} = \frac{0.5}{-0.5} = -1.$$

$$(d) \frac{2 + |3 - (-4)|}{|-3 - 4|} = \frac{2 + |7|}{|-7|} = \frac{2 + 7}{7} = \frac{9}{7}.$$

$$4. (a) (-3)^4 = (-3)(-3)(-3)(-3) = 81.$$

$$(b) 4^{-3} = \frac{1}{4^3} = \frac{1}{64}.$$

$$(c) 0^{-3} = \frac{1}{0^3} = \frac{1}{0} \text{ is undefined, since division by zero does not yield a real number.}$$

$$(d) (1/2)^{-3} = \frac{1}{(1/2)^3} = \frac{1}{1/8} = 1 \frac{8}{1} = 8. \text{ (Remember: it's top times the reciprocal of the bottom.)}$$

$$5. (a) \sqrt[3]{\frac{-125}{64}} = \left(\frac{-125}{64}\right)^{1/3} = \frac{(-125)^{1/3}}{64^{1/3}} = \frac{((-5)^3)^{1/3}}{(4^3)^{1/3}} = \frac{(-5)^{3 \cdot (1/3)}}{4^{3 \cdot (1/3)}} = \frac{(-5)^1}{4^1} = \frac{-5}{4}.$$

For the record: the formulas that we used in the calculation above, from first to last equals sign:  $\sqrt[n]{A} = A^{1/n}$  (with  $n = 3$ ),  $(A/B)^n = A^n/B^n$  (with  $n = 1/3$ ); next, we recall that  $64 = 4^3$  and  $125 = 5^3$ , so that  $-125 = (-5)^3$ ; then  $(A^m)^n = A^{mn}$  (with  $m = 3$  and  $n = 1/3$ ), and finally we simplify exponents and use the fact that  $A^1 = A$ .

Note: there are other ways to simplify this expression; for instance

$$\sqrt[3]{\frac{-125}{64}} = \sqrt[3]{\frac{(-5)^3}{4^3}} = \sqrt[3]{\left(\frac{-5}{4}\right)^3} = \frac{-5}{4};$$

in the last step we used the formula  $\sqrt[n]{A^n} = A$  which holds for all odd exponents  $n$  (in our case  $n = 3$ ).

Try to find other ways of correctly simplifying this!

(b) One way to simplify is:  $\frac{\sqrt{100}}{\sqrt{200}} = \frac{10}{\sqrt{100 \cdot 2}} = \frac{10}{\sqrt{100}\sqrt{2}} = \frac{10}{10\sqrt{2}} = \frac{1}{\sqrt{2}}$ .

If needed, we rationalize the denominator and write

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Alternatively, we could simplify given expression as

$$\frac{\sqrt{100}}{\sqrt{200}} = \sqrt{\frac{100}{200}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

In the first and the last steps we used the formula  $\sqrt{A/B} = \sqrt{A}/\sqrt{B}$ .

(c)  $\left(\frac{4}{9}\right)^{-3/2} = \frac{4^{-3/2}}{9^{-3/2}} = \frac{(2^2)^{-3/2}}{(3^2)^{-3/2}} = \frac{2^{2 \cdot (-3/2)}}{3^{2 \cdot (-3/2)}} = \frac{2^{-3}}{3^{-3}} = \frac{\frac{1}{2^3}}{\frac{1}{3^3}} = \frac{1}{2^3} \frac{3^3}{1} = \frac{3^3}{2^3} = \frac{27}{8}$ .

As usual, there are other ways of simplifying; for instance:

$$\left(\frac{4}{9}\right)^{-3/2} = \left(\frac{2^2}{3^2}\right)^{-3/2} = \left(\left(\frac{2}{3}\right)^2\right)^{-3/2} = \left(\frac{2}{3}\right)^{2 \cdot (-3/2)} = \left(\frac{2}{3}\right)^{-3} = \frac{1}{\left(\frac{2}{3}\right)^3} = \frac{1}{\left(\frac{2^3}{3^3}\right)} = \frac{1}{\left(\frac{8}{27}\right)} = \frac{27}{8}.$$

(d) Likewise,  $16^{-3/2}9^{3/2} = (4^2)^{-3/2}(3^2)^{3/2} = 4^{2 \cdot (-3/2)}3^{2 \cdot (3/2)} = 4^{-3}3^3 = \frac{3^3}{4^3} = \frac{27}{64}$ .

**6.** (a) Using laws of exponents,

$$\left(\frac{x^{0.4}x^{4.5}}{x^{-1.3}}\right)^{0.9} = \left(\frac{x^{0.4+4.5}}{x^{-1.3}}\right)^{0.9} = \left(\frac{x^{4.9}}{x^{-1.3}}\right)^{0.9} = \left(x^{4.9-(-1.3)}\right)^{0.9} = (x^{6.2})^{0.9} = x^{6.2 \cdot 0.9} = x^{5.58}.$$

First equals sign is due to the formula  $x^m \cdot x^n = x^{m+n}$ , and the third one due to  $x^m/x^n = x^{m-n}$ . We computed  $(x^{6.2})^{0.9}$  using  $(x^m)^n = x^{mn}$ .

(b)  $\left(\frac{x^3y^{-2}}{x^{-1}y^4}\right)^{-1/2} \left(x^{-1/2}y^{3/4}\right)^{-1} = (x^4y^{-6})^{-1/2} \left(x^{1/2}y^{-3/4}\right) = (x^{-2}y^3) \left(x^{1/2}y^{-3/4}\right) = x^{-3/2}y^{9/4}$ .

In the first step we simplified the fraction

$$\frac{x^3y^{-2}}{x^{-1}y^4} = \frac{x^3}{x^{-1}} \frac{y^{-2}}{y^4} = x^{3-(-1)}y^{-2-4} = x^2y^{-6}.$$

The second equals sign is due to the formulas  $(AB)^n = A^nB^n$ , here applied with  $A = x^4$ ,  $B = y^{-6}$  and  $n = -1/2$ , and  $(x^m)^n = x^{mn}$ :

$$(x^4y^{-6})^{-1/2} = (x^4)^{-1/2} (y^{-6})^{-1/2} = x^{4 \cdot (-1/2)}y^{-6 \cdot (-1/2)} = x^{-2}y^3.$$

Finally, in the last step we use the fact that multiplying two powers with the same base results in the addition of their exponents.

(c) We get

$$\frac{(x-a)^{-2}}{(x-a)^{-3}}(x-a)^0 = (x-a)^{-2-(-3)}(x-a)^0 = (x-a)^1(x-a)^0 = x-a,$$

since  $(x-a)^1 = x-a$  and  $(x-a)^0 = 1$ .

## Section 2. Basic Algebra

1. (a) Think of  $-3$  as  $-3 = 0 \cdot x + (-3)$ . Comparing  $mx + b = -3 = 0x + (-3)$  we see that  $m = 0$  and  $b = -3$ .

(b) From  $mx + b = x = 1 \cdot x + 0$  we get  $m = 1$  and  $b = 0$ .

(c) From  $mx + b = 0 = 0 \cdot x + 0$  we get  $m = 0$  and  $b = 0$ .

(d) From  $ax^2 + bx + c = -x^2 = (-1)x^2 + 0 \cdot x + 0$  we see that  $a = -1$ ,  $b = 0$ ,  $c = 0$

(e) Comparing  $ax^2 + bx + c = 1 - x = 0 \cdot x^2 + (-1) \cdot x + 1$  we get  $a = 0$ ,  $b = -1$ ,  $c = 1$ .

(f) Comparing  $ax^2 + bx + c = 0 = 0 \cdot x^2 + 0 \cdot x + 0$  we get  $a = 0$ ,  $b = 0$  and  $c = 0$ .

2. (a) Multiply each with each, to get

$$\begin{aligned}(x^2 - x + 1)(3x - 4) &= (x^2)(3x) - x(3x) + 1(3x) + (x^2)(-4) - x(-4) + 1(-4) \\ &= 3x^3 - 3x^2 + 3x - 4x^2 + 4x - 4 = 3x^3 - 7x^2 + 7x - 4\end{aligned}$$

(b) Using the formula for the square of the difference, we compute

$$\begin{aligned}(x^2 - 0.2)^2 &= (x^2)^2 - 2(x^2)(0.2) + (0.2)^2 \\ &= x^4 - 0.4x^2 + 0.04\end{aligned}$$

(c) We multiply the first two terms first

$$\begin{aligned}x(x^3 - 1)(x^3 + 1) &= (x^4 - x)(x^3 + 1) \\ &= (x^4)(x^3) - x(x^3) + x^4 - x \\ &= x^7 - x^4 + x^4 - x = x^7 - x\end{aligned}$$

(d) Using the cube of the difference formula, we get

$$\begin{aligned}(2x - 1)^3 &= (2x)^3 - 3(2x)^2(1) + 3(2x)(1)^2 - (1)^3 \\ &= 8x^3 - 3(4x^2) + 6x - 1 = 8x^3 - 12x^2 + 6x - 1\end{aligned}$$

3. (a) We add and subtract  $(b/2)^2 = (16/2)^2 = 64$ , to get

$$x^2 + 8x = x^2 + 8x + (64 - 64) = (x^2 + 8x + 64) - 64 = (x + 8)^2 - 64$$

Keep in mind that the three terms in the bracket are equal to  $(x + \frac{b}{2})^2$ ; i.e.,  $x^2 + 8x + 64 = (x + 8)^2$ .

(b) We add and subtract  $(b/2)^2 = (7/2)^2 = 49/4$ , to get

$$x^2 + 7x + 4 = \left(x^2 + 7x + \frac{49}{4}\right) - \frac{49}{4} + 4 = \left(x + \frac{7}{2}\right)^2 - \frac{33}{4}$$

(c) Factor 3 out first

$$\begin{aligned}3x^2 - 12x &= 3[x^2 - 4x] = 3[(x^2 - 4x + 4) - 4] \\ &= 3[(x - 2)^2 - 4] = 3(x - 2)^2 - 12\end{aligned}$$

(d) As in (c), we factor out the coefficient of  $x$  first

$$\begin{aligned} 3x^2 - 9x + 2 &= 3[x^2 - 3x] + 2 = 3\left[\left(x^2 - 3x + \frac{9}{4}\right) - \frac{9}{4}\right] + 2 \\ &= 3\left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4}\right] + 2 = 3\left(x - \frac{3}{2}\right)^2 - \frac{27}{4} + 2 = 3\left(x - \frac{3}{2}\right)^2 - \frac{23}{4} \end{aligned}$$

4. (a) Write  $x^2 - 0.04 = (x)^2 - (0.2)^2$  and use the difference of squares formula  $a^2 - b^2 = (a+b)(a-b)$  with  $a = x$  and  $b = 0.2$  to get

$$x^2 - 0.04 = (x)^2 - (0.2)^2 = (x - 0.2)(x + 0.2).$$

(b)  $5x^2 - 125 = 5(x^2 - 25) = 5((x)^2 - (5)^2) = 5(x - 5)(x + 5)$

(c)  $5x^2 - 55 = 5(x^2 - 11) = 5((x)^2 - (\sqrt{11})^2) = 5(x - \sqrt{11})(x + \sqrt{11})$

(d) Start with  $x^4 - 2 = (x^2)^2 - (\sqrt{2})^2$  to get

$$x^4 - 2 = (x^2 - \sqrt{2})(x^2 + \sqrt{2}).$$

Now  $x^2 + \sqrt{2}$  cannot be factored any further, but  $x^2 - \sqrt{2}$  can:

$$x^2 - \sqrt{2} = (x)^2 - (\sqrt[4]{2})^2 = (x - \sqrt[4]{2})(x + \sqrt[4]{2}).$$

Thus

$$x^4 - 2 = (x^2 - \sqrt{2})(x^2 + \sqrt{2}) = (x - \sqrt[4]{2})(x + \sqrt[4]{2})(x^2 + \sqrt{2}).$$

5. (a) We hope to factor the given expression in the form  $x^2 + 3x - 10 = (x + a)(x + b)$ ; thus, we need to find two numbers whose sum is 3 and product is  $-10$ . We find that 5 and  $-2$  satisfy these requirements, and so

$$x^2 + 3x - 10 = (x - 2)(x + 5).$$

(b) Start with  $2x^2 + 7x - 4 = (2x + a)(x + b)$  and try combinations of  $a$  and  $b$  whose product is  $-4$ . Try  $a = 1$ ,  $b = -4$ :  $(2x + 1)(x - 4) = 2x^2 - 8x + x - 4 = 2x^2 - 7x - 4$ , which does not work. Choosing  $a = -1$  and  $b = 4$ , we get  $(2x - 1)(x + 4) = 2x^2 + 8x - x - 4 = 2x^2 + 7x - 4$  - works!

Had it not worked, we would have tried  $a = 2$ ,  $b = -2$  or  $a = -2$ ,  $b = 2$ , etc.

(c) Start by factoring 2 out:  $2x^2 + 2x - 24 = 2(x^2 + x - 12)$ . Now, we are looking for two numbers whose sum is 1 and whose product is  $-12$ . The numbers 4 and  $-3$  work, and so

$$2x^2 + 2x - 24 = 2(x^2 + x - 12) = 2(x - 3)(x + 4).$$

(d) Write  $x^6 + 1 = (x^2)^3 + (1)^3$  and use the sum of the cubes formula  $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$  with  $a = x^2$  and  $b = 1$ :

$$x^6 + 1 = (x^2)^3 + (1)^3 = (x^2 + 1)((x^2)^2 - (x^2)1 + (1)^2) = (x^2 + 1)(x^4 - x^2 + 1).$$

(e) Factor  $x$  out and then use the same strategy as in (a) or (c):

$$x^3 - 2x^2 - 15x = x(x^2 - 2x - 15) = x(x - 3)(x + 5).$$

(f) Because  $x^4 - 16 = (x^2)^2 - (4)^2$ , we use the difference of squares

$$x^4 - 16 = (x^2 - 4)(x^2 + 4).$$

The first factor is the difference of squares again, and thus

$$x^4 - 16 = (x - 2)(x + 2)(x^2 + 4).$$

**6.** (a) We group the terms and then factor:

$$2x^2y + y^2 - 4x^2 - 2y = (2x^2y + y^2) - (4x^2 + 2y) = y(2x^2 + y) - 2(2x^2 + y) = (2x^2 + y)(y - 2).$$

Alternatively, we group the first and the third terms and the second and the fourth terms:

$$2x^2y + y^2 - 4x^2 - 2y = (2x^2y - 4x^2) + (y^2 - 2y) = 2x^2(y - 2) + y(y - 2) = (y - 2)(2x^2 + y).$$

(b) Factor out  $x^2$  and then use the difference of squares:

$$x^4 - x^2 = x^2(x^2 - 1) = x^2(x - 1)(x + 1).$$

(c) Factor 3 out:  $27y^2 - 75x^2 = 3(9y^2 - 25x^2)$  and use the difference of squares:

$$3(9y^2 - 25x^2) = 3((3y)^2 - (5x)^2) = 3(3y - 5x)(3y + 5x).$$

(d) Using the difference of squares twice, we get

$$x^5 - x = x(x^4 - 1) = x(x^2 - 1)(x^2 + 1) = x(x - 1)(x + 1)(x^2 + 1).$$

(e)  $8x^2 - 50 = 2(4x^2 - 25) = 2((2x)^2 - (5)^2) = 2(2x - 5)(2x + 5)$

(f) Again, it's the difference of squares:

$$\frac{x^2}{16} - \frac{y^2}{49} = \left(\frac{x}{4}\right)^2 - \left(\frac{y}{7}\right)^2 = \left(\frac{x}{4} - \frac{y}{7}\right)\left(\frac{x}{4} + \frac{y}{7}\right)$$

(g)  $4x^2 - 12 = 4(x^2 - 3) = 4(x - \sqrt{3})(x + \sqrt{3})$ .

(h) We factor the given expression in the form  $4x^2 - 11x + 6 = (4x + a)(x + b)$  and try combinations of  $a$  and  $b$  for which  $ab = 6$ . We get

$$4x^2 - 11x + 6 = (4x - 3)(x - 2).$$

(i) We factor  $3x^2 - 8x - 3 = (3x + a)(x + b)$  and check integer values of  $a$  and  $b$  for which  $ab = -3$ . We get

$$3x^2 - 8x - 3 = (3x + 1)(x - 3)$$

**7.** (a) The common denominator of  $x - 2$  and  $x - 4$  is their product; therefore

$$\begin{aligned} \frac{x}{x-2} - \frac{2x-1}{x-4} &= \frac{x(x-4)}{(x-2)(x-4)} - \frac{(2x-1)(x-2)}{(x-2)(x-4)} \\ &= \frac{x^2 - 4x - (2x^2 - x - 4x + 2)}{(x-2)(x-4)} = \frac{-x^2 + x - 2}{(x-2)(x-4)} \end{aligned}$$



(b) The least common multiple of  $x$ ,  $x - 1$  and  $(x - 1)^2$  is  $x(x - 1)^2$ . Thus

$$\begin{aligned} \frac{3}{x} - \frac{2x}{x-1} + \frac{4}{(x-1)^2} &= \frac{3(x-1)^2}{x(x-1)^2} - \frac{2x(x)(x-1)}{x(x-1)^2} + \frac{4x}{x(x-1)^2} \\ &= \frac{3(x^2 - 2x + 1) - 2x^2(x-1) + 4x}{x(x-1)^2} \\ &= \frac{3x^2 - 6x + 3 - 2x^3 + 2x^2 + 4x}{x(x-1)^2} = \frac{-2x^3 + 5x^2 - 2x + 3}{(x-2)(x-4)} \end{aligned}$$

(c) Since  $x^2 - 5x + 6 = (x - 2)(x - 3)$  we see that the common denominator is  $(x - 2)(x - 3)$ :

$$\begin{aligned} \frac{2x+1}{x^2-5x+6} - \frac{1}{x-2} + \frac{x}{x-3} &= \frac{2x+1}{(x-2)(x-3)} - \frac{x-3}{(x-2)(x-3)} + \frac{x(x-2)}{(x-2)(x-3)} \\ &= \frac{2x+1-x+3+x^2-2x}{(x-2)(x-3)} = \frac{x^2-x+4}{(x-2)(x-3)} \end{aligned}$$

**8.** (a) Factoring using the difference of cubes and the difference of squares formulas and then canceling the fraction, we get

$$\frac{x^3 - 27}{x^2 - 9} = \frac{(x)^3 - (3)^3}{(x)^2 - (3)^2} = \frac{(x-3)(x^2 + 3x + 9)}{(x-3)(x+3)} = \frac{x^2 + 3x + 9}{x+3}$$

(b) Factor, and then cancel:

$$\frac{x^4 - x^2}{2x^2 + 3x + 1} = \frac{x^2(x^2 - 1)}{(2x+1)(x+1)} = \frac{x^2(x-1)(x+1)}{(2x+1)(x+1)} = \frac{x^2(x-1)}{x+1}$$

In the denominator, we factored by writing  $2x^2 + 3x + 1 = (2x + a)(x + b)$  and then checking integer values of  $a$  and  $b$  such that  $ab = 1$ .

(c) Start by calculating common denominators:

$$1 - \frac{3}{x} + \frac{9}{x^2} = \frac{\frac{x}{x} - \frac{3}{x}}{\frac{x^2}{x^2} - \frac{6x}{x^2} + \frac{9}{x^2}} = \frac{\frac{x-3}{x}}{\frac{x^2-6x+9}{x^2}} = \frac{x-3}{x} \cdot \frac{x^2}{x^2-6x+9}$$

In the last step, we simplified using the “top times the reciprocal of the bottom” rule. Next, we cancel the fraction

$$\frac{x-3}{x} \cdot \frac{x^2}{x^2-6x+9} = \frac{x-3}{x} \cdot \frac{x^2}{(x-3)^2} = \frac{x}{x-3}$$

(d) Start by calculating common denominators, then simplify the double fraction and cancel:

$$\frac{1 + \frac{1}{x-1}}{1 - \frac{1}{x-1}} = \frac{\frac{x-1}{x-1} + \frac{1}{x-1}}{\frac{x-1}{x-1} - \frac{1}{x-1}} = \frac{\frac{x}{x-1}}{\frac{x-2}{x-1}} = \frac{x}{x-1} \cdot \frac{x-1}{x-2} = \frac{x}{x-2}$$

9. (a)

$$\begin{array}{r}
 x^2 + 4x + 13 \\
 x - 4 \overline{) \begin{array}{r} x^3 - 3x + 5 \\ - (x^3 - 4x^2) \\ \hline 4x^2 - 3x \\ - (4x^2 - 16x) \\ \hline 13x + 5 \\ - (13x - 52) \\ \hline 57 \end{array}
 \end{array}$$

Thus,  $\frac{x^3 - 3x + 5}{x - 4} = x^2 + 4x + 13 + \frac{57}{x - 4}$ .

(b)

$$\begin{array}{r}
 x + 6 \\
 x + 1 \overline{) \begin{array}{r} x^2 + 7x - 2 \\ - (x^2 + x) \\ \hline 6x - 2 \\ - (6x + 6) \\ \hline -8 \end{array}
 \end{array}$$

Thus,  $\frac{x^2 + 7x - 2}{x + 1} = x + 6 - \frac{8}{x + 1}$ .

10. (a) Subtract and add 15, to get

$$\frac{6x}{2x - 5} = \frac{6x - 15 + 15}{2x - 5} = \frac{6x - 15}{2x - 5} + \frac{15}{2x - 5} = \frac{3(2x - 5)}{2x - 5} + \frac{15}{2x - 5} = 3 + \frac{15}{2x - 5}$$

Check:

$$\begin{array}{r}
 3 \\
 2x - 5 \overline{) \begin{array}{r} 6x \\ - (6x - 15) \\ \hline 15 \end{array}
 \end{array}$$

(b) Add and subtract 2, thus getting

$$\frac{2x^2}{x^2 + 1} = \frac{2x^2 + 2 - 2}{x^2 + 1} = \frac{2x^2 + 2}{x^2 + 1} + \frac{-2}{x^2 + 1} = \frac{2(x^2 + 1)}{x^2 + 1} - \frac{2}{x^2 + 1} = 2 - \frac{2}{x^2 + 1}$$

Check:

$$\begin{array}{r} x^2 + 1 \overline{) 2x^2} \\ \underline{-(2x^2 + 2)} \\ -2 \end{array}$$

11. (a) Multiplying and dividing by  $\sqrt{7} + 3$ , we get

$$\frac{4}{\sqrt{7} - 3} = \frac{4}{\sqrt{7} - 3} \frac{\sqrt{7} + 3}{\sqrt{7} + 3} = \frac{4(\sqrt{7} + 3)}{(\sqrt{7})^2 - (3)^2} = \frac{4\sqrt{7} + 12}{-2} = -2\sqrt{7} - 6$$

(b) We multiply and divide by  $\sqrt{x} + \sqrt{x-1}$

$$\begin{aligned} \frac{3}{\sqrt{x} - \sqrt{x-1}} &= \frac{3}{\sqrt{x} - \sqrt{x-1}} \frac{\sqrt{x} + \sqrt{x-1}}{\sqrt{x} + \sqrt{x-1}} \\ &= \frac{3(\sqrt{x} + \sqrt{x-1})}{(\sqrt{x})^2 - (\sqrt{x-1})^2} \\ &= \frac{3(\sqrt{x} + \sqrt{x-1})}{x - (x-1)} = 3(\sqrt{x} + \sqrt{x-1}) \end{aligned}$$

(c) As above,

$$\frac{4 - \sqrt{x}}{2 + \sqrt{x}} = \frac{4 - \sqrt{x}}{2 + \sqrt{x}} \frac{2 - \sqrt{x}}{2 - \sqrt{x}} = \frac{(4 - \sqrt{x})(2 - \sqrt{x})}{(2)^2 - (\sqrt{x})^2} = \frac{8 - 6\sqrt{x} - x}{4 - x}$$

12. We try to identify a root of  $x^3 + 2x^2 - 11x - 12$  by guessing. Recall that, the root must divide the free coefficient  $-12$ . Substituting  $x = 1$ , we get  $(1)^3 + 2(1)^2 - 11(1) - 12 = -20$ , so 1 is not a root. Substituting  $x = -1$ , we get  $(-1)^3 + 2(-1)^2 - 11(-1) - 12 = 0$ ! So,  $x = -1$  is a root of the given polynomial, i.e.,  $x + 1$  is its factor.

Thus

$$x^3 + 2x^2 - 11x - 12 = (x + 1)q(x)$$

where the unknown polynomial  $q(x)$  is identified using long division:

$$\begin{array}{r} x^2 + x - 12 \\ x + 1 \overline{) x^3 + 2x^2 - 11x - 12} \\ \underline{-(x^3 + x^2)} \\ x^2 - 11x \\ \underline{-(x^2 + x)} \\ -12x - 12 \\ \underline{-(-12x - 12)} \\ 0 \end{array}$$

We write

$$\frac{x^3 + 2x^2 - 11x - 12}{x + 1} = x^2 + x - 12$$

i.e.,

$$x^3 + 2x^2 - 11x - 12 = (x + 1)(x^2 + x - 12).$$

Now we factor  $x^2 + x - 12 = (x + 4)(x - 3)$  and so

$$x^3 + x^2 - 10x + 8 = (x + 1)(x + 4)(x - 3)$$

Of course, we could have initially guessed some other root ( $x = -4$  or  $x = 3$ ). There would be no change, however – we follow the same procedure and arrive at the same factorization.

### Section 4. Equations and Inequalities

1. (a) Substituting  $x = 1$  into the given equation, we compute

$$\frac{1}{2x+1} - \frac{1}{x+1} = \frac{1}{2(1)+1} - \frac{1}{1+1} = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6} \neq 0.$$

So,  $x = 1$  is not a solution.

(b)  $x = -2$  is not a solution because

$$\frac{1}{2x+1} - \frac{1}{x+1} = \frac{1}{2(-2)+1} - \frac{1}{-2+1} = -\frac{1}{3} + 1 = \frac{2}{3}$$

is not zero.

(c) Substitute  $x = 0$ :

$$\frac{1}{2x+1} - \frac{1}{x+1} = \frac{1}{2(0)+1} - \frac{1}{0+1} = 1 - 1 = 0$$

So,  $x = 0$  is a solution of the given equation.

2. (a) Simplify

$$\begin{aligned} 4(3x - 0.5) + 11 &= -3(x + 2) \\ 12x - 2 + 11 &= -3x - 6 \end{aligned}$$

gather like terms together

$$\begin{aligned} 12x + 3x &= -6 + 2 - 11 \\ 15x &= -15 \end{aligned}$$

Dividing by 15, we get  $x = -1$ .

(b) Multiplying the equation by 52 ( $52 = 4 \cdot 13$  is the lowest common denominator), we get

$$\begin{aligned} \frac{3}{4}x - \frac{7}{13} &= 1 \\ \frac{3 \cdot 4 \cdot 13}{4}x - \frac{7 \cdot 4 \cdot 13}{13} &= 52 \\ 39x - 28 &= 52 \\ 39x &= 80 \\ x &= \frac{80}{39} \end{aligned}$$

(c) Simplify and gather like terms together

$$\begin{aligned} 3.4x - 6.2 &= -3.3(2 - 3x) \\ 3.4x - 6.2 &= -6.6 + 9.9x \\ 3.4x - 9.9x &= -6.6 + 6.2 \\ -6.5x &= -0.4 \\ x &= \frac{-0.4}{-6.5} = \frac{4}{65} = 0.061538 \end{aligned}$$

(d) Gather like terms together, divide and simplify

$$\begin{aligned}\sqrt{3}x - 3 &= 2\sqrt{3} \\ \sqrt{3}x &= 2\sqrt{3} + 3 \\ x &= \frac{2\sqrt{3} + 3}{\sqrt{3}} \\ x &= \frac{2\sqrt{3}}{\sqrt{3}} + \frac{3}{\sqrt{3}} = 2 + \sqrt{3}\end{aligned}$$

In the last step we rationalized the denominator (which might, or might not be required):

$$\frac{3}{\sqrt{3}} = \frac{3}{\sqrt{3}} \frac{\sqrt{3}}{\sqrt{3}} = \sqrt{3}$$

Note that we could have canceled the fraction by  $\sqrt{3}$  instead.

**3.** (a) Multiplying the equation by 20 (20 is the lowest common denominator), we get

$$\begin{aligned}4(4x - 2) - 5(3x - 11) &= 0 \\ 16x - 8 - 15x + 55 &= 0 \\ x &= -47\end{aligned}$$

(b) The lowest common denominator of  $2x - 1$  and  $x + 3$  is their product  $(2x - 1)(x + 3)$ . Multiplying the given equation by  $(2x - 1)(x + 3)$ , we get

$$\begin{aligned}\frac{-2}{2x - 1}(2x - 1)(x + 3) + \frac{4}{x + 3}(2x - 1)(x + 3) &= 0 \\ -2(x + 3) + 4(2x - 1) &= 0 \\ 6x &= -10 \\ x &= \frac{-10}{-6} = \frac{5}{3}\end{aligned}$$

(c) We multiply the given equation by  $x(x + 1)$  (or, cross-multiply!)

$$\begin{aligned}\frac{\sqrt{5}}{x} &= \frac{2\sqrt{5}}{x + 1} \\ \frac{\sqrt{5}}{x}x(x + 1) &= \frac{2\sqrt{5}}{x + 1}x(x + 1) \\ \sqrt{5}(x + 1) &= 2\sqrt{5}x \\ \sqrt{5}x + \sqrt{5} &= 2\sqrt{5}x \\ -\sqrt{5}x &= -\sqrt{5} \\ x &= 1\end{aligned}$$

**4.** (a) The lowest common denominator is  $6x$ . Multiplying the given equation by  $6x$  yields

$$\begin{aligned}\frac{2}{x}(6x) - \frac{5}{2x}(6x) + \frac{4}{3x}(6x) &= \frac{1}{2}(6x) \\ 12 - 15 + 8 &= 3x \\ 3x &= 5\end{aligned}$$

Dividing by 3, we get  $x = 5/3$ .

(b) We cross multiply to get rid of fractions:

$$\begin{aligned}\frac{x}{x-3} &= \frac{x+3}{x+5} \\ x(x+5) &= (x-3)(x+3) \\ x^2 + 5x &= x^2 - 9 \\ 5x &= -9 \\ x &= -\frac{9}{5}\end{aligned}$$

(c) The common denominator is  $2x - 4 = 2(x - 2)$ ; thus

$$\begin{aligned}\frac{2}{x-2} + \frac{1}{2} &= \frac{3}{2x-4} \\ \frac{2}{x-2}(2x-4) + \frac{1}{2}(2x-4) &= \frac{3}{2x-4}(2x-4) \\ 4 + (x-2) &= 3 \\ x &= 1\end{aligned}$$

5. (a) Since the square of a real number cannot be negative, the given equation has no solutions.

(b) Computing the square root of both sides, we get  $x - 6 = \pm\sqrt{12}$ ; i.e.,  $x = 6 \pm \sqrt{12}$ . Recall that  $\sqrt{12}$  can be simplified as follows

$$\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \sqrt{3} = 2\sqrt{3}$$

and so  $x = 6 \pm 2\sqrt{3}$ .

(c) We are asked to solve  $x^2 + 10x + 24 = 0$ . Factoring, we get

$$x^2 + 10x + 24 = (x+4)(x+6) = 0$$

and so  $x = -4$  and  $x = -6$ .

(d) We have to solve  $3x^2 - x - 4 = 0$ . Using quadratic formula, we get

$$x = \frac{1 \pm \sqrt{1+48}}{6} = \frac{1 \pm \sqrt{49}}{6} = \frac{1 \pm 7}{6}$$

Thus, the solutions are  $x = 8/6 = 4/3$  and  $x = -6/6 = -1$ .

6. (a) We add and subtract the square of one half of the coefficient of  $x$ , which is  $(-1/2)^2 = 1/4$ , and write  $x^2 - x + 1/4 - 1/4 + 5 = 0$ , i.e.,

$$\left(x^2 - x + \frac{1}{4}\right) - \frac{1}{4} + 5 = 0$$

Next, we recall that the square term involves one half of the coefficient of  $x$  (i.e., one half of  $-1$ )

$$\left(x - \frac{1}{2}\right)^2 = -\frac{21}{4}$$

Since the square of every number is zero or positive, the above equation has no solutions.

(b) First, divide the equation by 2 so that the coefficient of  $x^2$  is 1

$$\begin{aligned} 2x^2 - 2x - 7 &= 0 \\ x^2 - x - \frac{7}{2} &= 0 \end{aligned}$$

Now complete the square: one half of the coefficient of  $x$  is  $-1/2$ ; square it (so, it's  $1/4$ ) and add and subtract on the left side

$$\begin{aligned} x^2 - x + \frac{1}{4} - \frac{1}{4} - \frac{7}{2} &= 0 \\ \left(x^2 - x + \frac{1}{4}\right) - \frac{1}{4} - \frac{7}{2} &= 0 \\ \left(x - \frac{1}{2}\right)^2 &= \frac{15}{4} \end{aligned}$$

Compute the square root of both sides

$$x - \frac{1}{2} = \pm \frac{\sqrt{15}}{2}$$

Thus,

$$x = \frac{1}{2} \pm \frac{\sqrt{15}}{2}.$$

(c) Divide the equation by 3 so that the coefficient of  $x^2$  is 1

$$\begin{aligned} 3x^2 - x - 5 &= 0 \\ x^2 - \frac{x}{3} - \frac{5}{3} &= 0 \end{aligned}$$

Complete the square: one half of the coefficient of  $x$  is  $-1/6$ ; square it (so, it's  $1/36$ ) and add to and subtract from the expression on the left side

$$\begin{aligned} x^2 - \frac{x}{3} + \frac{1}{36} - \frac{1}{36} - \frac{5}{3} &= 0 \\ \left(x^2 - \frac{x}{3} + \frac{1}{36}\right) - \frac{1}{36} - \frac{5}{3} &= 0 \\ \left(x - \frac{1}{6}\right)^2 &= \frac{61}{36} \end{aligned}$$

Compute the square root of both sides

$$x - \frac{1}{6} = \pm \frac{\sqrt{61}}{6}$$

Thus,

$$x = \frac{1}{6} \pm \frac{\sqrt{61}}{6}.$$

**7.** (a) Since  $x^2 - 18x + 81 = (x - 9)^2 = 0$ , we conclude that  $x - 9 = 0$ , or  $x = 9$ . Alternatively, we could have used the quadratic equation.



(b) Multiplying the equation by the common denominator  $(x-3)(x+3) = x^2 - 9$ , we get

$$\begin{aligned}\frac{2}{x^2-9} &= \frac{1}{x+3} + \frac{3}{x-3} \\ \frac{2}{x^2-9}(x^2-9) &= \frac{1}{x+3}(x-3)(x+3) + \frac{3}{x-3}(x-3)(x+3) \\ 2 &= x-3 + 3(x+3) \\ 4x &= -4\end{aligned}$$

and  $x = -1$ . Note that all expressions involved are defined at  $-1$ , so  $x = -1$  is indeed a solution.

(c) Squaring both sides, we get

$$\begin{aligned}(\sqrt{2x+4})^2 &= (\sqrt{6x+1} - 1)^2 \\ 2x+4 &= 6x+1 - 2\sqrt{6x+1} + 1 \\ 2\sqrt{6x+1} &= 4x-2\end{aligned}$$

Divide by 2 and square again

$$\begin{aligned}\sqrt{6x+1} &= 2x-1 \\ 6x+1 &= 4x^2-4x+1 \\ 4x^2-10x &= 0\end{aligned}$$

Factoring, we get

$$4x^2 - 10x = 2x(2x - 5) = 0,$$

and thus  $x = 0$  or  $2x - 5 = 0$ , i.e.,  $x = 5/2$ .

Note that  $x = 0$  is not a solution: the left side of the given equation is  $\sqrt{2x+4} = \sqrt{4} = 2$ , whereas the right side is  $\sqrt{6x+1} - 1 = \sqrt{1} - 1 = 0$ .

Substituting  $x = 5/2$  into the left side, we get

$$\sqrt{2x+4} = \sqrt{2 \cdot \frac{5}{2} + 4} = \sqrt{9} = 3.$$

The right side is

$$\sqrt{6x+1} - 1 = \sqrt{6 \cdot \frac{5}{2} + 1} - 1 = \sqrt{16} - 1 = 3.$$

So,  $x = 5/2$  is the only solution of the given equation.

(d) Multiplying the equation by the common denominator  $(x-1)(x+1) = x^2 - 1$ , we get

$$\begin{aligned}\frac{3}{x+1} - \frac{2}{x-1} &= -1 \\ \frac{3}{x+1}(x-1)(x+1) - \frac{2}{x-1}(x-1)(x+1) &= -1(x^2-1) \\ 3(x-1) - 2(x+1) &= -x^2+1 \\ x^2+x-6 &= 0\end{aligned}$$

Factoring

$$x^2 + x - 6 = (x+3)(x-2) = 0$$

we get that  $x = -3$  and  $x = 2$ . Since the denominators  $x - 1$  and  $x + 2$  are not zero when  $x = -3$  and  $x = 2$ , the two values of  $x$  are indeed solutions.

8. (a) Using the quadratic formula, we solve  $x^2 + x - 1 = 0$  for  $x$ :

$$x = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

Thus,

$$x^2 + x - 1 = \left(x - \frac{-1 + \sqrt{5}}{2}\right) \left(x - \frac{-1 - \sqrt{5}}{2}\right) = \left(x + \frac{1 - \sqrt{5}}{2}\right) \left(x + \frac{1 + \sqrt{5}}{2}\right)$$

(b) Instead of writing  $4x^2 - 4x - 15 = (4x + a)(x + b)$  and  $4x^2 - 4x - 15 = (2x + a)(2x + b)$  and checking which values for  $a$  and  $b$  work, we solve the quadratic equation  $4x^2 - 4x - 15 = 0$ :

$$x = \frac{4 \pm \sqrt{256}}{8} = \frac{4 \pm 16}{8}.$$

So,  $x = 20/8 = 5/2$  and  $x = -12/8 = -3/2$  are the solutions, and we factor

$$4x^2 - 4x - 15 = 4 \left(x - \frac{5}{2}\right) \left(x - \frac{-3}{2}\right) = 2 \left(x - \frac{5}{2}\right) 2 \left(x + \frac{3}{2}\right) = (2x - 5)(2x + 3).$$

(c) We solve  $2x^2 + 2x - 3 = 0$  for  $x$ :

$$x = \frac{-2 \pm \sqrt{28}}{4} = \frac{-2 \pm 2\sqrt{7}}{4} = \frac{-1 \pm \sqrt{7}}{2}.$$

Note that we simplified  $\sqrt{28} = \sqrt{4 \cdot 7} = \sqrt{4}\sqrt{7} = 2\sqrt{7}$ . Thus,

$$2x^2 + 2x - 3 = 2 \left(x - \frac{-1 + \sqrt{7}}{2}\right) \left(x - \frac{-1 - \sqrt{7}}{2}\right) = 2 \left(x + \frac{1 - \sqrt{7}}{2}\right) \left(x + \frac{1 + \sqrt{7}}{2}\right)$$

9. (a) We rearrange the terms

$$\begin{aligned} 13x - 17 &> 4x + 1 \\ 9x &> 18 \\ x &> 2 \end{aligned}$$

In interval notation, the solution is  $(2, \infty)$ .

(b) We break the inequality into two inequalities.

Solving  $1 - x \geq 3 - 2x$ , we get  $x \geq 2$ .

Solving  $3 - 2x \geq x - 6$ , we get  $-3x \geq -9$  and (after dividing by  $-3$ )  $x \leq 3$ .

The solution  $x$  must satisfy both  $x \geq 2$  and  $x \leq 3$ . Thus,  $2 \leq x \leq 3$ ; using interval notation, we write the solution as  $[2, 3]$ .

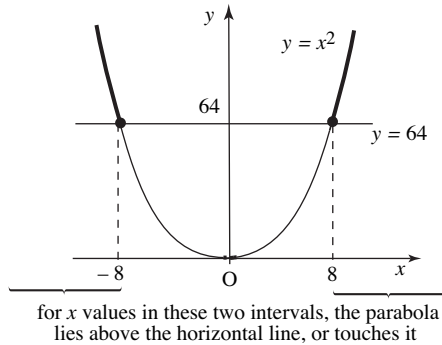
(c) Because only one term involves  $x$ , we can work with both inequalities at the same time. Adding 2 to all sides and dividing by 3, we get

$$\begin{aligned} 3 &\leq 3x - 2 \leq 4 \\ 5 &\leq 3x \leq 6 \end{aligned}$$

$$\frac{5}{3} < x \leq 2$$

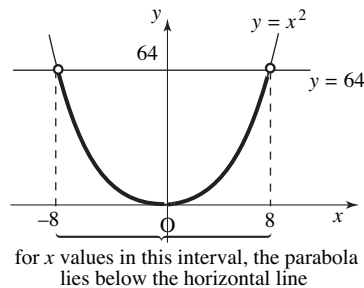
Thus, the solution is the interval  $[5/3, 2]$ .

**10.** (a) We solve  $x^2 \geq 64$  geometrically: we identify those parts of the parabola  $y = x^2$  that lie above the horizontal line  $y = 64$  or touch it; see figure below. The corresponding values of  $x$  lie in the two intervals  $(-\infty, -8]$  and  $[8, \infty)$ .



Of course, we could have solved the inequality algebraically: starting from  $x^2 \geq 64$  we write  $x^2 - 64 \geq 0$ , and then analyze the factors in the inequality  $(x - 8)(x + 8) \geq 0$ .

(b) This time, we are looking for  $x$  values for which the points on the parabola  $y = x^2$  lie below the horizontal line  $y = 64$ ; see figure below. The solution is  $(-8, 8)$ .



(c) The linear functions involved change the sign at 2 and  $-2$ . Thus, we need to analyze intervals  $(-\infty, -2)$ ,  $(-2, 2)$  and  $(2, \infty)$ .

We organize the calculations in the table below.

	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
$x - 2$	-	-	+
$x + 2$	-	+	+
$\frac{x+2}{x-2}$	+	-	+

The fraction  $\frac{x+2}{x-2}$  is negative on the interval  $(-2, 2)$ . Note that we also need to include the case

$$\frac{x+2}{x-2} = 0$$

which happens if  $x + 2 = 0$ , i.e., if  $x = -2$ . So, the solution of the given inequality is  $[-2, 2)$ .

(d) We simplify using a common denominator and then factor:

$$\begin{aligned}\frac{x+2}{x-2} &\leq 1 \\ \frac{x+2}{x-2} - 1 &\leq 0 \\ \frac{x+2}{x-2} - \frac{x-2}{x-2} &\leq 0 \\ \frac{4}{x-2} &\leq 0.\end{aligned}$$

Since the numerator is positive, the denominator has to be negative, so  $x - 2 < 0$  and  $x < 2$ . Note that the fraction is never equal to zero (since its numerator is not zero). Thus, the solution is  $(-\infty, 2)$

(e) Move all terms to the left side and factor:

$$\begin{aligned}x^2 &> 21 - 4x \\ x^2 + 4x - 21 &> 0 \\ (x+7)(x-3) &> 0\end{aligned}$$

The solutions of the equation  $(x+7)(x-3) = 0$  are  $x = -7$  and  $x = 3$ . The numbers  $-7$  and  $3$  divide the number line into three intervals:

$$(-\infty, -7), (-7, 3) \text{ and } (3, \infty)$$

We check the sign of each factor on each interval and record it in the table below.

	$(-\infty, -7)$	$(-7, 3)$	$(3, \infty)$
$x - 3$	-	-	+
$x + 7$	-	+	+
$(x - 3)(x + 7)$	+	-	+

The solution (the product has to be positive!) consists of two intervals,  $(-\infty, -7)$  and  $(3, \infty)$ .

(f) Move 2 to the left side and compute the common denominator:

$$\begin{aligned}\frac{6}{x-5} &\leq 2 \\ \frac{6}{x-5} - 2 &\leq 0 \\ \frac{6}{x-5} - \frac{2(x-5)}{x-5} &\leq 0 \\ \frac{-2x+16}{x-5} &\leq 0 \\ \frac{-2(x-8)}{x-5} &\leq 0\end{aligned}$$

The linear terms involved change the sign at 5 and 8, so we check the intervals  $(-\infty, 5)$ ,  $(5, 8)$  and  $(8, \infty)$ .

	$(-\infty, 5)$	$(5, 8)$	$(8, \infty)$
$x - 8$	-	-	+
$x - 5$	-	+	+
$\frac{-2(x-8)}{x-5}$	-	+	-

Note that the signs in the last row of the table include the factor of  $-2$ . From the table we conclude that the intervals  $(-\infty, 5)$ , and  $(8, \infty)$  are part of the solution.

Since equality is allowed, the numerator  $-2(x - 8)$  can be zero, i.e.,  $x$  can be equal to 8. Thus, the solution is  $(-\infty, 5)$  and  $[8, \infty)$ .

(g) Since  $x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1)$ , we need to solve

$$x(x - 1)(x + 1) \leq 0.$$

The factors change their sign at  $-1$ ,  $0$  and  $1$ .

	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
$x - 1$	-	-	-	+
$x$	-	-	+	+
$x + 1$	-	+	+	+
$x(x - 1)(x + 1)$	-	+	-	+

So, the intervals  $(-\infty, -1)$  and  $(0, 1)$  are part of the solution. Since the inequality allows  $x^3 - x$  to be equal to zero, we must include  $x = 0$ ,  $x = -1$  and  $x = 1$ . Thus, the solution to the given inequality is  $(-\infty, -1]$  and  $[0, 1]$ .

(h) Rewrite the given inequality as  $x^2 - x - 2 \leq 0$  and factor

$$(x - 2)(x + 1) \leq 0$$

	$(-\infty, -1)$	$(-1, 2)$	$(2, \infty)$
$x - 2$	-	-	+
$x + 1$	-	+	+
$(x - 2)(x + 1)$	+	-	+

So, the solution is the interval  $[-1, 2]$ .

Alternatively – let us use test values and checked the whole product at once. To test the interval  $(-\infty, -1)$  pick  $x = -10$ ; then  $(x - 2)(x + 1) = (-12)(-9) = 108 > 0$ , and so  $(-\infty, -1)$  is not part of the solution.

To test the interval  $(-1, 2)$ , we use  $x = 0$ :  $(x - 2)(x + 1) = (-2)(1) = -2 < 0$ , so the interval  $(-1, 2)$  is part of the solution.

To test the interval  $(2, \infty)$  we use  $x = 5$ :  $(x - 2)(x + 1) = (3)(6) = 18 > 0$ , so the interval  $(2, \infty)$  does not belong to the solution.

So, we have the interval  $(-1, 2)$ . At the endpoints, the product  $(x - 2)(x + 1)$  is zero, and so the solution of  $(x - 2)(x + 1) \leq 0$  is  $[-1, 2]$ .

**11.** Keep in mind that  $|A - B|$  is the distance between two numbers  $A$  and  $B$  on the number line. Thus,  $|A| = |A - 0|$  is the distance between the number  $A$  and the origin.

(a) The interval  $[-8, 8]$  contains all numbers whose distance from the origin is 8 or less; thus, the desired inequality is  $|x| \leq 8$ .

(b) The midpoint of the interval  $(3, 4)$  is 3.5; we see that  $(3, 4)$  contains all numbers whose distance from 3.5 is strictly less than 0.5. So, the desired equation is  $|x - 3.5| < 0.5$ .

(c) The two intervals contain numbers whose distance from the origin is 1 or more than 1; thus, the equation is  $|x| \geq 1$ .

(d) The half-way between 0 and 4 is 2. We see that the points in the given interval have the property that their distance from 2 is equal to 2 or larger than 2. Thus, the desired inequality is  $|x - 2| \geq 2$ .

**12.** (a) Using the definition of absolute value, we write

$$|2x - 3| = \begin{cases} 2x - 3 & \text{if } 2x - 3 \geq 0 \\ -(2x - 3) & \text{if } 2x - 3 < 0 \end{cases} = \begin{cases} 2x - 3 & \text{if } x \geq 3/2 \\ -2x + 3 & \text{if } x < 3/2 \end{cases}$$

Thus, the equation  $|2x - 3| = 7$  breaks up into two equations:

If  $x \geq 3/2$ , it reads  $2x - 3 = 7$  (and the solution is  $x = 5$ ).

If  $x < 3/2$ , it reads  $-2x + 3 = 7$  (and the solution is  $x = -2$ ).

Thus, there are two solutions,  $x = 5$  and  $x = -2$ .

Geometric approach: rewrite  $|2x - 3| = 7$  as  $2|x - \frac{3}{2}| = 7$  i.e.,

$$\left| x - \frac{3}{2} \right| = \frac{7}{2}.$$

Thus, we are looking for  $x$  whose distance from  $3/2$  is  $7/2$ . There are two solutions,  $x = 3/2 + 7/2 = 5$  and  $x = 3/2 - 7/2 = -2$ .

(b) As in (a),

$$\left| \frac{x}{2} - 4 \right| = \begin{cases} x/2 - 4 & \text{if } x/2 - 4 \geq 0 \\ -(x/2 - 4) & \text{if } x/2 - 4 < 0 \end{cases} = \begin{cases} x/2 - 4 & \text{if } x \geq 8 \\ -x/2 + 4 & \text{if } x < 8 \end{cases}$$

Thus, the equation  $\left| \frac{x}{2} - 4 \right| = 3$  breaks up into two equations:

If  $x \geq 8$ , it reads  $x/2 - 4 = 3$  (and the solution is  $x = 14$ ).

If  $x < 8$ , it reads  $-x/2 + 4 = 3$  (and the solution is  $x = 2$ ).

Thus, there are two solutions,  $x = 2$  and  $x = 14$ .

To interpret  $\left| \frac{x}{2} - 4 \right| = 3$  geometrically, we need to have the left side in the form  $|x - \text{some number}|$ . So, multiply by 2 to get

$$|x - 8| = 6.$$

The solutions are those numbers on the number line whose distance from 8 is 6; thus  $x = 8 - 6 = 2$  and  $x = 8 + 6 = 14$ .

**13.** All exercises (a)-(f) can be solved either algebraically or geometrically.

(a) Recall that  $|A| < a$  if and only if  $-a < A < a$ ; thus  $|2x - 3| < 2$  is equivalent to the pair of inequalities  $-2 < 2x - 3 < 2$  that we solve simultaneously

$$\begin{aligned} -2 &< 2x - 3 < 2 \\ 1 &< 2x < 5 \\ 1/2 &< x < 5/2 \end{aligned}$$

Now the geometric approach: to interpret the absolute value in  $|2x - 3| < 2$  as distance, we need to get rid of 2 in front of  $x$ . So, we factor it out:  $2|x - \frac{3}{2}| < 2$ , then divide the equation by 2 to get  $|x - \frac{3}{2}| < 1$ . We are looking for all  $x$  whose distance from  $\frac{3}{2}$  is strictly less than 1; all such  $x$  belong to the interval  $(\frac{1}{2}, \frac{5}{2})$ .

(b) Divide  $|3x + 4| \geq 4$  by 3, to get  $|x + \frac{4}{3}| \geq \frac{4}{3}$ , i.e.,

$$\left| x - \left( -\frac{4}{3} \right) \right| \geq \frac{4}{3}$$

The solution consists of all  $x$  whose distance from  $-4/3$  is greater than or equal to  $4/3$ ; thus  $x \geq -4/3 + 4/3 = 0$  and  $x \leq -4/3 - 4/3 = -8/3$ . In other words, the solutions belong to one of the intervals  $(-\infty, -8/3]$  and  $[0, \infty)$ .

(c) The inequality  $|3x + 4| < 4$  is equivalent to

$$\begin{aligned} -4 &< 3x + 4 < 4 \\ -8 &< 3x < 0 \\ -8/3 &< x < 0 \end{aligned}$$

Using interval notation, the solution is  $(-8/3, 0)$ .

(d) Simplify  $3|x - 5| + 2 < 7$  first, by moving 2 over and dividing by 3; we get  $|x - 5| < 5/3$ . We are looking for all  $x$  whose distance from 5 is smaller than  $5/3$ . Thus,  $x < 5 + 5/3 = 20/3$  and  $x > 5 - 5/3 = 10/3$ . The solution is  $(10/3, 20/3)$ .

(e) Multiply  $\frac{2}{3}|x - 5| < 1$  by  $\frac{3}{2}$  to get  $|x - 5| < \frac{3}{2}$ . Thus

$$\begin{aligned} -\frac{3}{2} &< x - 5 < \frac{3}{2} \\ 5 - \frac{3}{2} &< x < 5 + \frac{3}{2} \\ \frac{7}{2} &< x < \frac{13}{2} \end{aligned}$$

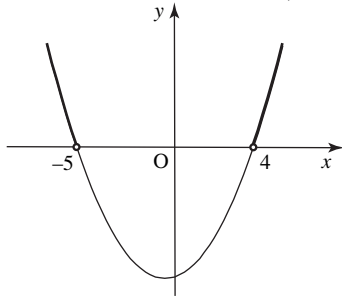
In interval notation, the solution is  $(7/2, 13/2)$ .

(f) Move 4 over and then multiply by 4, to get  $|x + 3| < 32$ . Writing it as

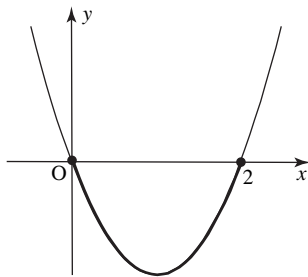
$$|x - (-3)| < 32$$

we realize that we are looking for those  $x$  whose distance from  $-3$  is less than  $32$ . Thus,  $x < -3 + 32 = 29$  and  $x > -3 - 32 = -35$ . The solution is  $(-35, 29)$ .

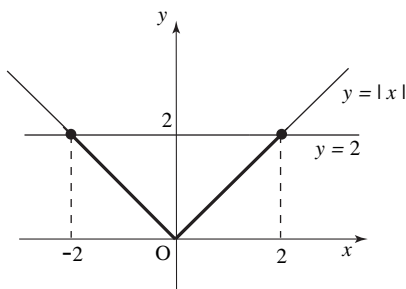
**14.** (a) The  $x$ -intercepts of the parabola  $y = x^2 + x - 20 = (x + 5)(x - 4)$  are  $-5$  and  $4$ . We are looking for those  $x$  for which the points on the parabola lie above the  $x$ -axis (see figure below). The answer consists of two intervals,  $(-\infty, -5)$  and  $(4, \infty)$ .



(b) The  $x$ -intercepts of the parabola  $y = x^2 - 2x$  are  $x = 0$  and  $x = 2$ . We are looking for the values of  $x$  for which the parabola lies below the  $x$ -axis, or touches it. From the figure below we see that the solution is the interval  $[0, 2]$ .



(c) We are looking for all  $x$  such that the graph of the absolute value  $y = |x|$  lies below, or touches, the horizontal line  $y = 2$ . See figure below. The solution is  $[-2, 2]$ .



**15.** (a) From  $x + 2y = -11$  we get that  $x = -11 - 2y$ . Substituting into the remaining equation, we get

$$\begin{aligned} 2x - y &= 13 \\ 2(-11 - 2y) - y &= 13 \\ -22 - 4y - y &= 13 \end{aligned}$$



$$-5y = 35$$

$$y = -7$$

Thus,  $x = -11 - 2y = -11 - 2(-7) = 3$ . The solution of the system is  $x = 3, y = -7$ .

(b) We compute  $y$  from  $x - y + 4 = 0$  and substitute into the remaining equation. Thus,  $y = x + 4$  and so

$$y = x^2 + 2x + 2$$

$$x + 4 = x^2 + 2x + 2$$

$$0 = x^2 + x - 2$$

$$0 = (x + 2)(x - 1)$$

So, there are two solutions,  $x = -2$  and  $x = 1$ . If  $x = -2$ , then  $y = x + 4 = -2 + 4 = 2$ . When  $x = 1$ , then  $y = x + 4 = 1 + 4 = 5$ .

Thus, the solutions of the given system are  $x = 1, y = 5$  and  $x = -2, y = 2$ .

### Section 5. Elements of Analytic Geometry

1. (a) The distance between the points  $(-3, 4)$  and  $(4, -2)$  is

$$\sqrt{(-3 - 4)^2 + (4 - (-2))^2} = \sqrt{49 + 36} = \sqrt{85}$$

- (b) Distance between  $(-4, 2)$  and  $(0, 5)$  is  $\sqrt{(-4 - 0)^2 + (2 - 5)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$ .

Distance between  $(0, 1)$  and  $(0, 5)$  is  $\sqrt{(0 - 0)^2 + (1 - 5)^2} = \sqrt{16} = 4$ .

Distance between  $(3, 1)$  and  $(0, 5)$  is  $\sqrt{(3 - 0)^2 + (1 - 5)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$ .

Thus, of the three points, the point  $(0, 1)$  is closest to  $(0, 5)$ .

2. The slope of the line through the points  $(1, 3)$  and  $(2, -4)$  is

$$m = \frac{-4 - 3}{2 - 1} = 7.$$

Using the point-slope form of the equation of the line, we get

$$y - 3 = -7(x - 1)$$

i.e.,  $y = -7x + 10$ . We used the point  $(1, 3)$  to calculate the equation. If we use  $(2, -4)$ , we get

$$y - (-4) = -7(x - 2)$$

which, of course, gives the same equation,  $y = -7x + 10$ .

3. Whenever possible, we transform the given equation to the slope-intercept form  $y = mx + b$ , from which we read the slope ( $m$ ) and the  $y$ -intercept ( $b$ ).

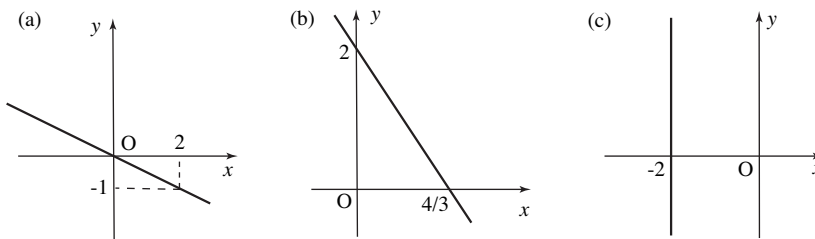
(a) From  $-x - 2y = 0$  we get  $y = -x/2$ ; i.e.,  $m = -1/2$  and  $b = 0$ . Thus, the graph is a line through the origin of slope  $-1/2$ ; see figure below, (a).

(b) From  $3x + 2y - 4 = 0$  we get  $2y = -3x + 4$  and  $y = -\frac{3}{2}x + 2$ . So, it's a line of slope  $-3/2$  and  $y$ -intercept 2; see figure below, (b). Note that the slope determined by  $(0, 2)$  and  $(4/3, 0)$  is equal to

$$m = \frac{0 - 2}{\frac{4}{3} - 0} = -\frac{2}{\frac{4}{3}} = -2\frac{3}{4} = -\frac{3}{2},$$

as it should.

(c) This time, we cannot convert to the point-slope form – instead, we write the given equation as  $x = -2$ . It represents the vertical line through  $(-2, 0)$ ; figure below, (c).



4. (a) Recall that parallel lines have equal slopes.

From  $x + 2y + 4 = 0$  we get  $2y = -x - 4$  and  $y = -\frac{x}{2} - 2$ . Thus, the slope is  $-1/2$ . Since the slope of the line  $y = -3x + 4$  is  $-3$ , the two lines are not parallel.

(b) Recall that the lines with slopes  $m_1$  and  $m_2$  are perpendicular if and only if  $m_2 = -1/m_1$ .

From  $2x + y - 4 = 0$  we get that  $y = -2x + 4$ ; i.e., the slope of the given line is  $-2$ . We conclude that the slope of a line perpendicular to it is  $-\frac{1}{-2} = \frac{1}{2}$ . We now use the point-slope form to get the desired equation:

$$y - (-2) = \frac{1}{2}(x - 1); \text{ i.e., } y = \frac{1}{2}x - \frac{5}{2}$$

(c) Recall that the condition  $m_2 = -1/m_1$  (perpendicular lines) can be written as  $m_1m_2 = -1$ . (Done for convenience only.)

From  $3x - 2y = 6$  we get  $2y = 3x - 6$  and  $y = \frac{3}{2}x - 3$ ; so,  $m_1 = \frac{3}{2}$ .

From  $2x + 3y - 12 = 0$  we compute  $3y = -2x + 12$  and  $y = -\frac{2}{3}x + 4$ ; so  $m_2 = -\frac{2}{3}$ .

Since  $m_1m_2 = \frac{3}{2}(-\frac{2}{3}) = -1$ , the two lines are perpendicular.

**5.** (a) In general, a line parallel to the  $x$ -axis has the equation  $y = b$ , where  $b$  represents the  $y$ -intercept. So, the desired equation is  $y = -4$ .

(b) A line parallel to the  $y$ -axis has the equation  $x = a$ , where  $a$  represents the  $x$ -intercept. So, the desired equation is  $x = 3$ .

(c) Solving  $x - 3y - 7 = 0$  for  $y$ , we get  $y = \frac{1}{3}x - \frac{7}{3}$ . Thus the slope of the given line (and, consequently, the slope of the line we are looking for) is  $1/3$ . Using the point-slope equation, we get  $y - (-4) = \frac{1}{3}(x - 3)$ ; i.e.,  $y = \frac{1}{3}x - 5$ .

(d) Because the slope of the given line is  $1/3$  (from (c)), the slope of a line perpendicular to it is

$$-\frac{1}{1/3} = -3.$$

We now use the point-slope form to get the desired equation:

$$y - (-4) = (-3)(x - 3); \text{ i.e., } y = -3x + 5$$

**6.** (a) Rewrite  $x + y = 0$  as  $y = -x$  and compare with  $y = mx + b$ . Thus, the slope is  $-1$  and the intercept is  $0$ .

(b) Solve  $2x + 3 = 0$  for  $x$ , to get  $x = -3/2$ . It's a vertical line, and its slope is not defined (i.e., it is not a real number). There is no  $y$ -intercept.

(c)  $y + 4 = 0$  represents the horizontal line ( $y = -4$ ) whose  $y$ -intercept is  $-4$ . As any horizontal line, its slope is zero.

(d) Solve  $3x - 5y - 1 = 0$  for  $y$ :  $5y = 3x - 1$  and  $y = \frac{3}{5}x - \frac{1}{5}$ . Thus, the slope is  $3/5$  and the  $y$ -intercept is  $-1/5$ .

**7.** In this exercise we need to use the technique of completing the square. If you need to review, read about it in Section 2.

(a) Start with  $x^2 + y^2 - 5x + 1 = 0$  and group the  $x$  terms and the  $y$  terms together

$$x^2 - 5x + y^2 = -1$$

and then complete the square:

$$\begin{aligned} x^2 - 5x + \frac{25}{4} - \frac{25}{4} + y^2 &= -1 \\ \left(x^2 - 5x + \frac{25}{4}\right) + y^2 &= -1 + \frac{25}{4} \\ \left(x - \frac{5}{2}\right)^2 + y^2 &= \frac{21}{4} \end{aligned}$$

Thus, the given equation represents the circle of radius  $\sqrt{21}/2$  centered at  $(5/2, 0)$ .

(b) Again, start with the given equation  $x^2 + y^2 - 2x + 8y = 4$ , group the  $x$  terms and the  $y$  terms together

$$x^2 - 2x + y^2 + 8y = 4$$

and complete the square:

$$\begin{aligned} (x^2 - 2x + 1 - 1) + (y^2 + 8y + 16 - 16) &= 4 \\ (x^2 - 2x + 1) + (y^2 + 8y + 16) &= 4 + 1 + 16 \\ (x - 1)^2 + (y + 4)^2 &= 21 \end{aligned}$$

We conclude that the given equation represents the circle of radius  $\sqrt{21}$  centered at  $(1, -4)$ .

(c) Divide the equation by 2, getting  $x^2 + y^2 - 3 = 0$  and  $x^2 + y^2 = 3$ . Thus, it represents the circle centred at the origin of radius  $\sqrt{3}$ .

8. (a) Divide the equation by 9

$$\frac{x^2}{9} + \frac{y^2}{3} = 1$$

Comparing with  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  we see that  $a^2 = 9$  and  $b^2 = 3$ .

The given equation represents an ellipse with semi-axes  $a = \sqrt{9} = 3$  and  $b = \sqrt{3}$ , centred at the origin. Its  $x$ -intercepts are  $\pm a = \pm 3$ , and its  $y$ -intercepts are  $\pm b = \pm\sqrt{3}$ .

(b) Complete the square

$$\begin{aligned} x^2 + y^2 - 2x - 6y &= 0 \\ x^2 - 2x + y^2 - 6y &= 0 \\ (x^2 - 2x + 1 - 1) + (y^2 - 6y + 9 - 9) &= 0 \\ (x^2 - 2x + 1) + (y^2 - 6y + 9) &= 1 + 9 \\ (x - 1)^2 + (y - 3)^2 &= 10 \end{aligned}$$

The equation represents the circle centered at  $(1, 3)$ , of radius  $\sqrt{10}$ .

(c) Divide  $2x^2 + 3y^2 = 4$  by 4

$$\frac{x^2}{2} + \frac{3y^2}{4} = 1 \quad \text{i.e.,} \quad \frac{x^2}{2} + \frac{y^2}{4/3} = 1$$

(thus  $a^2 = 2$  and  $b^2 = 4/3$ ).

The equation represents the ellipse with semi-axes  $\sqrt{2}$  and  $2/\sqrt{3}$ , centred at the origin. Its  $x$ -intercepts are  $\pm\sqrt{2}$ , and its  $y$ -intercepts are  $\pm 2/\sqrt{3}$ .

(d) Because the coefficients of  $x^2$  and  $y^2$  are equal, we know that the equation represents a circle.

Complete the square

$$\begin{aligned}x^2 + y^2 - y &= 0 \\x^2 + \left(y^2 - y + \frac{1}{4} - \frac{1}{4}\right) &= 0 \\x^2 + \left(y^2 - y + \frac{1}{4}\right) &= \frac{1}{4} \\x^2 + \left(y - \frac{1}{2}\right)^2 &= \frac{1}{4}\end{aligned}$$

The equation represents the circle centered at  $(0, 1/2)$  of radius  $1/2$ .

9. (a) Since  $a = 1$ , the parabola  $y = x^2 + 6x + 3$  opens upward. We solve for  $x$ -intercepts:

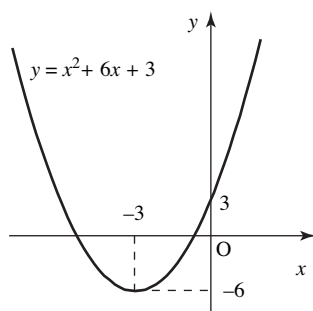
$$x = \frac{-6 \pm \sqrt{24}}{2} = \frac{-6 \pm 2\sqrt{6}}{2} = -3 \pm \sqrt{6}.$$

(Note that  $\sqrt{24} = \sqrt{4 \cdot 6} = \sqrt{4}\sqrt{6} = 2\sqrt{6}$ .) The  $x$  coordinate of the vertex is half-way between the  $x$ -intercepts  $-3 - \sqrt{6}$  and  $-3 + \sqrt{6}$ , i.e., it is equal to  $-3$ . (Alternatively, we can use  $x = -6/2(1) = -3$ .)

The  $y$ -coordinate of the vertex is obtained by substituting  $x = -3$  into the equation  $y = x^2 + 6x + 3$  of the parabola; thus,  $y = (-3)^2 + 6(-3) + 3 = -6$ .

The  $y$ -intercept (substitute  $x = 0$  into  $y = x^2 + 6x + 3$ ) is  $y = 3$ .

So we need to draw a parabola that opens upward, whose vertex is at  $(-3, -6)$ , that crosses the  $x$ -axis at  $-3 - \sqrt{6}$  and  $-3 + \sqrt{6}$ , and that crosses the  $y$ -axis at  $-6$ ; see the figure below.

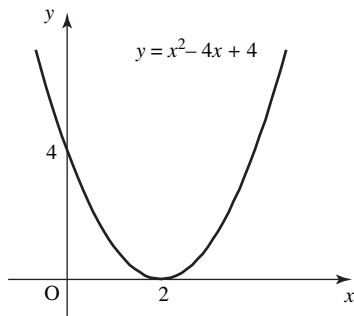


(b) The parabola  $y = x^2 - 4x + 4$  opens upward since its  $x^2$  coefficient is positive. Note that  $y = (x - 2)^2$ ; thus, it has one  $x$ -intercept, namely  $x = 2$ .

The  $x$ -coordinate of the vertex is  $x = -b/2a = 2$ ; the corresponding  $y$ -coordinate is 0. Thus, the vertex is at  $(2, 0)$ .

The  $y$ -intercept is  $y = 4$ .

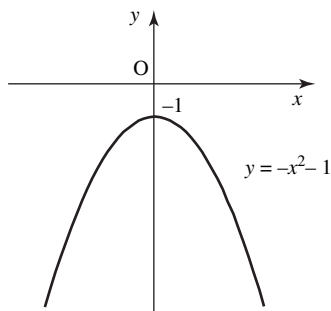
The graph is a parabola that opens upward, has vertex at  $(2, 0)$ ,  $y$ -intercept at 4 and  $x = 2$  is its only  $x$ -intercept. See figure below.



(c) The coefficient of  $x^2$  is  $-1$ , and thus the parabola  $y = -x^2 - 1$  opens downward. From  $-x^2 - 1 = 0$  we get the equation  $x^2 = -1$  which has no solutions. Thus, there are no  $x$ -intercepts.

The  $x$ -coordinate of the vertex is  $x = -b/2a = 0$ ; the corresponding  $y$ -coordinate is  $-1$ ; so the vertex is at  $(0, -1)$ .

The graph is a parabola that opens downward and lies entirely below the  $x$ -axis; its vertex is at  $(0, -1)$ . See figure below.



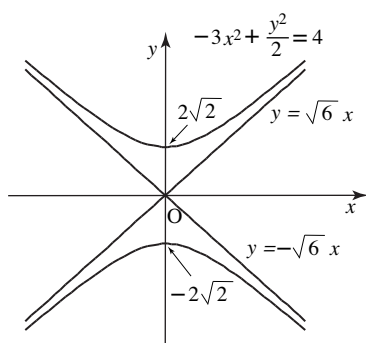
10. (a) Dividing both sides of the equation  $-3x^2 + y^2/2 = 4$  by 4, we get

$$-\frac{3x^2}{4} + \frac{y^2}{8} = 1, \quad \text{i.e.,} \quad -\frac{x^2}{4/3} + \frac{y^2}{8} = 1$$

Because one of the signs is a minus, the equation represents a hyperbola. From  $a^2 = 4/3$  we get  $a = 2/\sqrt{3}$ ; from  $b^2 = 8$  we get  $b = \sqrt{8} = 2\sqrt{2}$ . It follows that the asymptotes are given by

$$y = \pm \frac{b}{a}x = \pm \frac{2\sqrt{2}}{\frac{2}{\sqrt{3}}} = \pm\sqrt{6}x.$$

Substituting  $y = 0$  into the given equation (we are looking for  $x$ -intercepts) we get  $-\frac{3x^2}{4} = 1$ , i.e.,  $x^2 = -\frac{4}{3}$ . Since this equation has no solutions, there are no  $x$ -intercepts. Substituting  $x = 0$ , we get  $\frac{y^2}{8} = 1$ , i.e.,  $y^2 = 8$ , and  $y = \pm 2\sqrt{2}$ . So, there are two  $y$ -intercepts,  $2\sqrt{2}$  and  $-2\sqrt{2}$ . The hyperbola lies in the regions above and below its asymptotes; see the figure below.



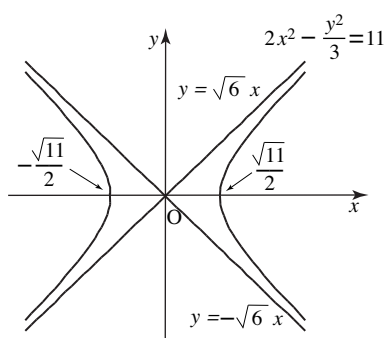
(b) We divide both sides of the equation  $2x^2 - y^2/3 = 11$  by 11, to get

$$\frac{2x^2}{11} - \frac{y^2}{33} = 1, \quad \text{i.e.,} \quad \frac{x^2}{11/2} - \frac{y^2}{33} = 1$$

As in (a), one of the signs is a minus, and thus the equation represents a hyperbola. From  $a^2 = 11/2$  we get  $a = \sqrt{11/2}$ ; from  $b^2 = 33$  we get  $b = \sqrt{33}$ . It follows that the asymptotes are given by

$$y = \pm \frac{b}{a}x = \pm \frac{\sqrt{33}}{\frac{\sqrt{11}}{\sqrt{2}}} = \pm \frac{\sqrt{3}\sqrt{11}}{\frac{\sqrt{11}}{\sqrt{2}}} = \pm \sqrt{6}x.$$

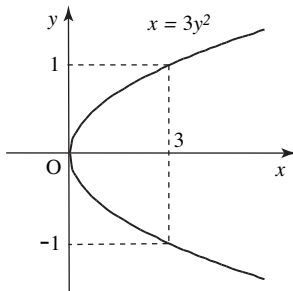
Substituting  $y = 0$  into the given equation (we are looking for  $x$ -intercepts) we get  $2x^2 = 11$ , i.e.,  $x^2 = 11/2$ . So, there are two  $x$ -intercepts,  $x = \pm\sqrt{11/2}$ . Substituting  $x = 0$ , we get  $-y^2/3 = 11$ , i.e.,  $y^2 = -33$ . Thus, there are no  $y$ -intercepts. The hyperbola lies in the regions to the right and to the left of its asymptotes; see the figure below.



(c) Switching  $x$  and  $y$  we get  $y = 3x^2$ , which is a parabola that opens upward, with its vertex at the origin. Drawing the mirror image of it with respect to  $y = x$ , we obtain the graph of  $x = 3y^2$ ; see figure below.

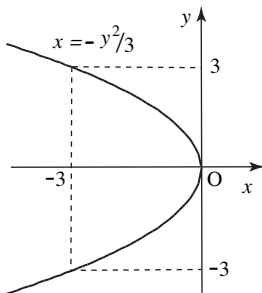
Alternatively, solving  $x = 3y^2$  for  $y$ , we get  $y^2 = x/3$  and  $y = \pm\sqrt{x/3}$ . Thus, to draw the graph of  $x = 3y^2$ , we sketch the graph of  $y = \sqrt{x/3}$ , and then add to it its symmetric image (with respect to  $x$ -axis)

It is useful to remember that the equation  $x = ay^2$  for  $a > 0$  represents a parabola that opens towards the  $x$ -axis.



(d) We can use general fact – the equation  $x = ay^2$  for  $a < 0$  represents a parabola that opens towards the negative  $x$ -axis, and its vertex is at the origin.

Alternatively, we proceed as in (c); Switching  $x$  and  $y$  we get  $y = -x^2/3$ , which is a parabola that opens downward, with its vertex at the origin. Drawing the mirror image of it with respect to  $y = x$ , we obtain the graph of  $x = -y^2/3$ ; see figure below.





---

**Section 6. Functions**

1. (a) Since the fourth root is defined for positive numbers and zero, we need  $x - 2 \geq 0$ ; moreover,  $x - 2 \neq 0$  guarantees that the denominator is not zero. So,  $x$  is in the domain of the given function if  $x - 2 > 0$ , i.e.,  $x > 2$ . Using set notation,  $D = \{x \mid x > 2\}$ .

(b) The domain consists of all real numbers except those for which  $x^3 - x = 0$ . Factoring

$$x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1) = 0$$

we see that  $x = 0$ ,  $x = 1$  or  $x = -1$ . Therefore,  $D = \{x \mid x \neq 0, 1, -1\}$ .

(c) Because of  $1/x$ , we require that  $x \neq 0$ . On top of that, to make sure that we can calculate the sixth root (recall that  $A^{1/6} = \sqrt[6]{A}$ ) we need

$$1 - \frac{1}{x} \geq 0.$$

To solve this inequality, we calculate common denominator

$$1 - \frac{1}{x} = \frac{x - 1}{x} \geq 0$$

and then analyze the signs of the numerator and the denominator:

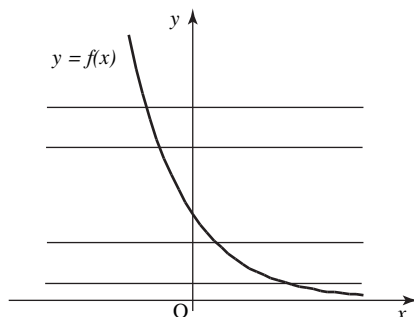
	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$x - 1$	-	-	+
$x$	-	+	+
$\frac{x-1}{x}$	+	-	+

From the table, we conclude that  $1 - \frac{1}{x} \geq 0$  when  $x$  is in  $(-\infty, 0]$  or  $[1, \infty)$ .

Remembering that  $x \neq 0$ , we see that the domain is  $D = \{x \mid x < 0 \text{ or } x \geq 1\} = (-\infty, 0) \cup [1, \infty)$ .

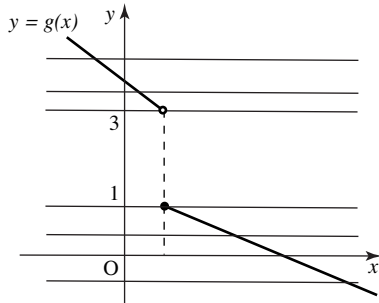
2. Recall that a number  $c$  (on the  $y$ -axis) is in the range of a function if the horizontal line through  $c$  intersects the graph of the function.

(a) Any horizontal line that crosses the  $y$ -axis at some positive number intersects the graph. On the other hand, the  $x$ -axis and the horizontal lines that lie below it do not cross the graph. Thus, the range consists of all positive numbers; i.e.,  $R = (0, \infty)$ . See below.

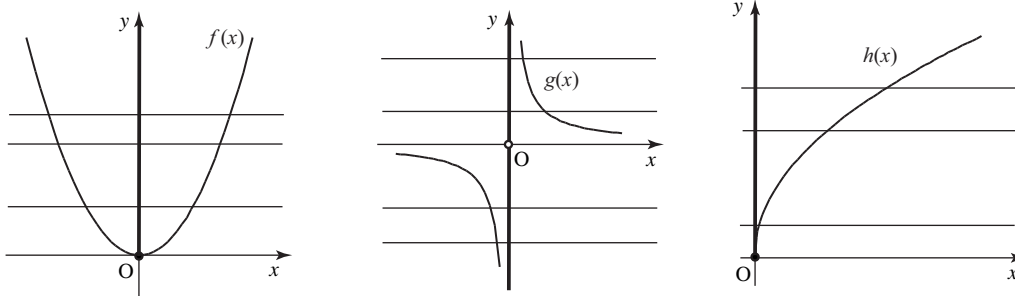


(b) Horizontal lines that cross the  $y$ -axis between 1 and 3 do not intersect the graph. The line  $y = 1$  intersects the graph, but  $y = 3$  does not. All horizontal lines below  $y = 1$  and above  $y = 3$  intersect

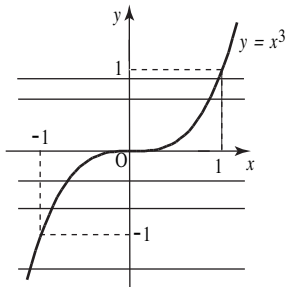
the graph. Thus, the range consists of all real numbers smaller than or equal to 1, or greater than 3. In symbols,  $R = (-\infty, 1] \cup (3, \infty)$ . See figure below.



**3.** The graphs are shown in figure below; we use horizontal lines (as in previous exercise) to test for the range. The range is shown in thicker line on the  $y$ -axis.



**4.** (a) Since we can calculate the cube of any number, the domain is the set of all real numbers. From the graph of  $f(x) = x^3$  (see figure below) we see that the range consists of all real numbers. Alternatively, we pick a real number  $c$  and try to find  $x$  such that  $f(x) = x^3 = c$ . Clearly,  $x = \sqrt[3]{c}$ ; check:  $f(\sqrt[3]{c}) = (\sqrt[3]{c})^3 = c$ . In symbols,  $D = \mathbb{R}$ ,  $R = \mathbb{R}$ .



(b) To avoid division by zero, we need to remove  $x = 0$  from the domain. Thus,  $D = \{x \mid x \neq 0\}$ . Again, looking at the graph (below) we see that the range consists of all positive numbers.

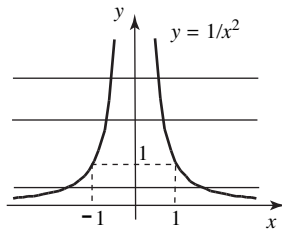
To prove this claim algebraically, we show that no matter what  $c > 0$  we take, we can find a value  $x$  such that  $g(x) = c$ . From

$$g(x) = \frac{1}{x^2} = c$$

we get

$$x^2 = \frac{1}{c} \quad \text{and thus} \quad x = \pm\sqrt{\frac{1}{c}}.$$

In symbols,  $D = \{x \mid x \neq 0\}$ ,  $R = \{y \mid y > 0\}$ .



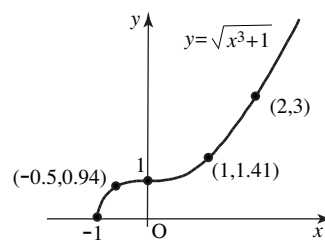
(c) By definition, the domain of  $h(x) = |x|$  is the set of all real numbers (i.e., we can calculate the absolute value of any real number). As well, the definition states that the absolute value is a positive number, or zero. Thus,  $D = \mathbb{R}$ ,  $R = \{y \mid y \geq 0\}$ .

Alternatively, we sketch the graph of  $h(x) = |x|$  and determine the range from it.

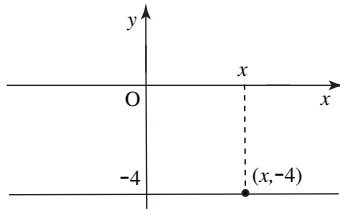
**5.** The domain of  $f$  is determined from the requirement  $x^3 + 1 \geq 0$ , i.e.,  $x^3 \geq -1$ . Looking at the graph of  $x^3$  (see (a) in previous exercise) we see that  $x \geq -1$ .

To sketch the graph, we calculate the table of values (below).

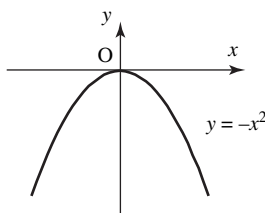
$x$	$\sqrt{x^3 + 1}$
-1	0
-0.5	0.94
0	1
1	1.41
2	3
2.5	4.08
...	...



**6.** (a) The formula  $f(x) = -4$  states that, no matter what the  $x$  value is, the value of the function  $f(x)$  is  $-4$ . Thus, the graph of  $f(x)$  contains all points of the form  $(x, -4)$ , where  $x$  is a real number. So, the graph is a horizontal line that crosses the  $y$ -axis at  $y = -4$ . See below.



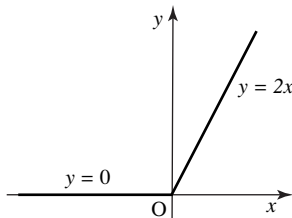
(b) By plotting points, or by reflecting the graph of  $y = x^2$  with respect to the  $x$ -axis, we obtain the graph below.



(c) Using the definition of absolute value, we get

$$f(x) = |x| + x = \begin{cases} x + x & \text{if } x \geq 0 \\ -x + x & \text{if } x < 0 \end{cases} = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Thus, the graph of this piecewise defined function consists of the line of slope 2 through the origin (for  $x \geq 0$ ) and the  $x$ -axis (for  $x < 0$ ). See below.



7. (a) Using the definition of composition,

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x}{2}\right) = 3\left(\frac{x}{2}\right) + 4 = \frac{3x}{2} + 4$$

(b)  $(f \circ f)(x) = f(f(x)) = f(3x + 4) = 3(3x + 4) + 4 = 9x + 16.$

(c)  $(g \circ f)(x) = g(f(x)) = g(3x + 4) = \frac{3x + 4}{2}.$

8. (a)  $(f \circ g)(x) = f(g(x)) = f(x^2 + x + 1) = (x^2 + x + 1)^5.$

(b)  $(f \circ f)(x) = f(f(x)) = f(x^5) = (x^5)^5 = x^{25}.$

(c)  $(g \circ f)(x) = g(f(x)) = g(x^5) = (x^5)^2 + x^5 + 1 = x^{10} + x^5 + 1.$

9. (a) Replace  $f(x)$  in  $f(x) = \frac{2}{x-4}$  by  $y$  and solve for  $x$

$$\begin{aligned} y &= \frac{2}{x-4} \\ \frac{1}{y} &= \frac{x-4}{2} \\ \frac{2}{y} &= x-4 \\ x &= \frac{2}{y} + 4. \end{aligned}$$

Thus,

$$f^{-1}(y) = \frac{2}{y} + 4$$

In many situations we prefer to use  $x$  as independent variable; so, we write

$$f^{-1}(x) = \frac{2}{x} + 4.$$

(b) We start with  $f(x) = 4\sqrt[3]{2-x} - 1$ , switch  $x$  and  $y$  to get  $x = 4\sqrt[3]{2-y} - 1$  and solve for  $y$  :

$$\begin{aligned} x+1 &= 4\sqrt[3]{2-y} \\ \frac{x+1}{4} &= \sqrt[3]{2-y} \\ \left(\frac{x+1}{4}\right)^3 &= 2-y \\ y &= 2 - \left(\frac{x+1}{4}\right)^3 \end{aligned}$$

Thus,

$$f^{-1}(x) = 2 - \left(\frac{x+1}{4}\right)^3.$$

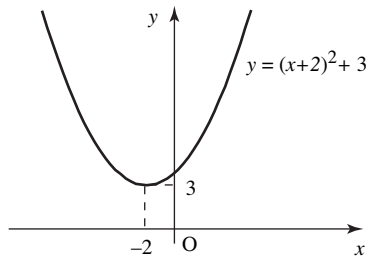
(c) We solve  $y = 1 - x^3$  for  $x$  :

$$\begin{aligned} y &= 1 - x^3 \\ x^3 &= 1 - y \\ x &= \sqrt[3]{1-y} \end{aligned}$$

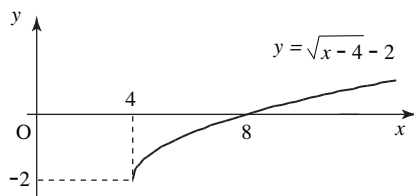
Thus,

$$f^{-1}(y) = \sqrt[3]{1-y}, \quad \text{or} \quad f^{-1}(x) = \sqrt[3]{1-x}.$$

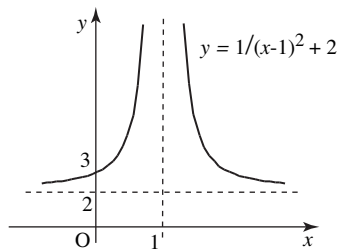
10. (a) Looking at  $y = (x+2)^2 + 3$ , we decide to start with the graph of  $y = x^2$ . First, replace  $x$  in  $y = x^2$  by  $x+2$ , thus getting  $y = (x+2)^2$  (corresponding transformation is shift left for 2 units). Then add 3, to get  $y = (x+2)^2 + 3$  (i.e., move the graph up 3 units). See below.



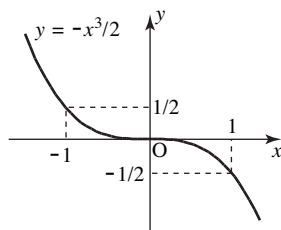
(b) Start with  $y = \sqrt{x}$  and replace  $x$  by  $x - 4$ . This way, we obtain the function  $y = \sqrt{x - 4}$ ; its graph is obtained by moving the graph of  $y = \sqrt{x}$  to the right for 4 units. We move this new graph 2 units down, to obtain the graph of  $y = \sqrt{x - 4} - 2$ ; see figure below. Note that the domain of this function is  $x \geq 4$ .



(c) To obtain  $y = \frac{1}{(x-1)^2}$  we move the graph of  $y = \frac{1}{x^2}$  to the right for 1 unit. Next, we move that graph up for 2 units, to obtain  $y = \frac{1}{(x-1)^2} + 2$ . The domain of this function is  $x \neq 1$ . See below.



**11.** (a) To obtain the graph of  $y = -x^3/2$ , we start from  $y = x^3$  and first draw the graph of  $y = \frac{1}{2}x^3$ . Recall the general rule: to obtain the graph of  $y = f(cx)$ ,  $0 < c < 1$ , we stretch the graph of  $y = f(x)$  horizontally by a factor of  $1/c$ . Thus, we need to stretch the graph of  $x^3$  by a factor of 2. To obtain  $y = -\frac{1}{2}x^3$  we then reflect it with respect to the  $x$ -axis. See below.



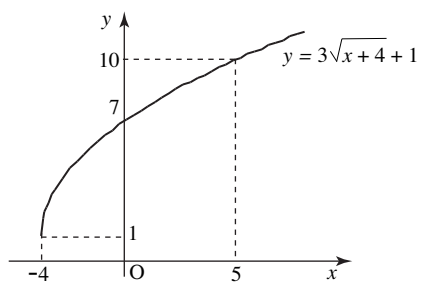
(b) Looking at the formula  $y = 3\sqrt{x+4} + 1$  we see that we need to do transformations related to the numbers 3, 4 and 1. Let's break it down into steps, starting with  $y = \sqrt{x}$  :

Replace  $x$  by  $x + 4$ , thus getting  $y = \sqrt{x+4}$  (transformation: shift left for 4 units).

Multiply  $y$  by 3, getting  $y = 3\sqrt{x+4}$  (transformation: vertical stretch by a factor fo 3).

Finally, add 1 to  $y$ , to get  $y = 3\sqrt{x+4} + 1$  (transformation: move the graph up 1 unit).

See figure below.



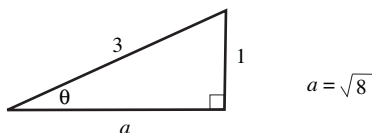
## Section 7. Trigonometric Functions

1. (a)  $225^\circ$  equals  $225 \frac{\pi}{180} = \frac{5\pi}{4}$  rad (cancel the fraction by 45).  
 (b)  $\frac{7\pi}{6}$  rad equals  $\frac{7\pi}{6} \frac{180}{\pi} = 210^\circ$ .  
 (c)  $76^\circ$  equals  $76 \frac{\pi}{180} = \frac{19\pi}{45} \approx 1.326$  rad.  
 (d)  $3.7$  rad equals  $3.7 \frac{180}{\pi} \approx 211.994^\circ$ .

2. Since  $\csc \theta = 1/\sin \theta = 3$ , it follows that  $\sin \theta = 1/3$ .

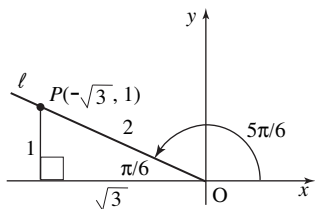
We draw the right triangle in which  $\sin \theta = 1/3$  (see figure below) and calculate the remaining side using Pythagorean theorem:  $3^2 = a^2 + 1^2$ , so  $a^2 = 8$  and  $a = \sqrt{8}$ .

Thus,  $\cos \theta = \text{adjacent/hypotenuse} = \sqrt{8}/3$  and  $\tan \theta = \text{opposite/adjacent} = 1/\sqrt{8}$ .



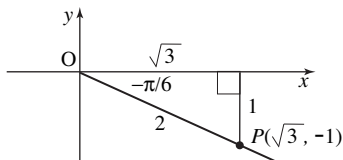
3. (a) We place the triangle that we used to compute the ratios for  $\pi/6$  and  $\pi/3$ , as shown in figure below. We get

$$\sin \frac{5\pi}{6} = \frac{y}{r} = \frac{1}{2} \quad \cos \frac{5\pi}{6} = \frac{x}{r} = -\frac{\sqrt{3}}{2} \quad \tan \frac{5\pi}{6} = \frac{y}{x} = -1/\sqrt{3}$$



- (b) We place the same triangle we used in (a), this time as shown in figure below. We get

$$\sin \left(-\frac{\pi}{6}\right) = \frac{y}{r} = -\frac{1}{2} \quad \cos \left(-\frac{\pi}{6}\right) = \frac{x}{r} = \frac{\sqrt{3}}{2} \quad \tan \left(-\frac{\pi}{6}\right) = \frac{y}{x} = -1/\sqrt{3}$$



4. The angle  $\theta = -3\pi/2$  radians is equal to three right angles measured clockwise from the positive  $x$ -axis; thus, it is defined by the positive  $y$ -axis, i.e., it is equal to  $\pi/2$  radians. The coordinates of



the point  $P$  which is the intersection of the unit circle and the  $y$ -axis are  $(0, 1)$ . Thus,

$$\cos\left(-\frac{3\pi}{2}\right) = 0 \quad \text{and} \quad \sin\left(-\frac{3\pi}{2}\right) = 1$$

Since they have  $\cos\left(-\frac{3\pi}{2}\right)$  in the denominator, it follows that  $\tan\left(-\frac{3\pi}{2}\right)$  and  $\sec\left(-\frac{3\pi}{2}\right)$  are not defined.

**5.** (a) Recall that the graph of  $f(cx)$ ,  $c > 1$ , is obtained by compressing the graph of  $f(x)$  by a factor of  $c$ . If  $0 < c < 1$ , then we expand the graph of  $f(x)$  by a factor of  $1/c$  to obtain the graph of  $f(cx)$ .

Thus, to obtain the graph of  $\cos 3x$ , we compress the graph of  $\cos x$  along the  $x$ -axis by a factor of 3. In other words, we compress it so that its period is  $2\pi/3$ .

To obtain the graph of  $\cos 0.5x$ , we stretch the graph of  $\cos x$  along the  $x$ -axis by a factor of 2. Therefore, the period of  $\cos 0.5x$  is  $4\pi$ .

(b) To obtain  $-\cos x$ , we reflect the graph of  $\cos x$  with respect to the  $x$ -axis. Then we stretch  $-\cos x$  vertically by a factor of 4 to obtain  $-4\cos x$ . Note that we could have stretched the graph first and then reflected it.

(b) As in (a), we reflect the graph of  $\cos x$  with respect to the  $x$ -axis to obtain  $-\cos x$ . To obtain  $-\cos x - 4$ , we move the graph of  $-\cos x$  vertically down for 4 units.

**6.** In order to figure out the horizontal shift, we need to write  $f(x)$  as

$$f(x) = 2 \sin(3x - \pi) = 2 \sin\left(3\left(x - \frac{\pi}{3}\right)\right).$$

Thus, we see that in order to obtain the  $\sin(3x - \pi)$  part of the given function, we need to replace  $x$  in  $\sin(3x)$  by  $x - \pi/3$ .

We start with the graph of  $y = \sin x$  and complete the following transformations.

Compress the graph by the factor of 3, to obtain  $y = \sin(3x)$  (the period of which is  $2\pi/3$ ).

Replace  $x$  by  $x - \pi/3$  to obtain  $\sin\left(3\left(x - \frac{\pi}{3}\right)\right)$  (transformation: shift right for  $\pi/3$  units).

Finally, multiply by 2, to obtain  $y = 2 \sin(3x - \pi)$  (transformation: vertical stretch by a factor of 2).

Looking at the graph (see below) we see that the range is  $[-2, 2]$ .

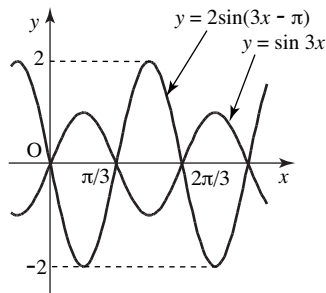
We can calculate the range without looking at the graph. Since  $\sin x$  is always between  $-1$  and  $1$ , we start with

$$-1 \leq \sin(3x - \pi) \leq 1$$

and then multiply by 2 :

$$-2 \leq 2 \sin(3x - \pi) \leq 2$$

Thus, the range of  $f(x)$  is  $[-2, 2]$ .



**7.** We know that the period of  $\tan x$  is  $\pi$ . As well, the graph of  $\tan(ax)$  is a horizontal scaling (compression or expansion) of the graph of  $\tan x$ .

If  $a > 1$ ,  $\tan(ax)$  is a compression of  $\tan x$  by a factor of  $a$ ; so, the period will shrink by the same factor, i.e., it will be  $\pi/a$ .

If  $0 < a < 1$ ,  $\tan(ax)$  is an expansion of  $\tan x$  by a factor of  $1/a$ ; so, the period will expand by the same factor, i.e., it will be  $(1/a)\pi = \pi/a$ .

Thus, in either case, the period is  $\pi/a$ .

**8.** General strategy: we start with the one side of the identity, simplify (square the expression, or calculate common denominator, etc.) and/or use formulas (such as trig identities) to arrive at the other side.

(a) Using the addition formula for  $\sin x$  (see the box *Addition and Subtraction Formulas*) we compute

$$\sin(\pi/2 + x) = \sin(\pi/2)\cos x + \cos(\pi/2)\sin x = \cos x,$$

because  $\sin(\pi/2) = 1$  and  $\cos(\pi/2) = 0$ .

(b) Use the definition of  $\cot x$  and then compute common denominator:

$$\cot^2 x + 1 = \frac{\cos^2 x}{\sin^2 x} + 1 = \frac{\cos^2 x}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x} = \frac{\cos^2 x + \sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} = \csc^2 x,$$

because  $\cos^2 x + \sin^2 x = 1$  (basic trig identity); as well, by definition.  $1/\sin x = \csc x$ .

(c) We'll prove this in two different ways.

Group the first and the third terms together, and factor out  $\sin^2 x$

$$\sin^2 x - \tan^2 x + \sin^2 x \tan^2 x = \sin^2 x(1 + \tan^2 x) - \tan^2 x$$

use the identity  $1 + \tan^2 x = \sec^2 x$  to get

$$= \sin^2 x \sec^2 x - \tan^2 x$$

now recall that  $\sin x \sec x = \sin x \frac{1}{\cos x} = \tan x$

$$= \tan^2 x - \tan^2 x = 0.$$

Alternatively, we use  $\tan x = \sin x / \cos x$  and calculate the common denominator:

$$\begin{aligned}\sin^2 x - \tan^2 x + \sin^2 x \tan^2 x &= \sin^2 x - \frac{\sin^2 x}{\cos^2 x} + \sin^2 x \frac{\sin^2 x}{\cos^2 x} \\ &= \frac{\sin^2 x \cos^2 x - \sin^2 x + \sin^2 x \sin^2 x}{\cos^2 x}\end{aligned}$$

group the first and the third terms and factor out  $\sin^2 x$

$$= \frac{\sin^2 x (\cos^2 x + \sin^2 x) - \sin^2 x}{\cos^2 x}$$

use the identity  $\cos^2 x + \sin^2 x = 1$

$$= \frac{\sin^2 x - \sin^2 x}{\cos^2 x} = 0.$$

**9.** Write  $3x = 2x + x$ , and start with the addition formula for the cosine function:

$$\begin{aligned}\cos 3x &= \sin(2x + x) \\ &= \sin 2x \cos x + \cos 2x \sin x\end{aligned}$$

Use the double angle formulas and simplify:

$$\begin{aligned}&= (2 \sin x \cos x) \cos x + (1 - 2 \sin^2 x) \sin x \\ &= 2 \sin x \cos^2 x + \sin x - 2 \sin^3 x\end{aligned}$$

Replace  $\cos^2 x$  using the basic trigonometric identity  $\cos^2 x = 1 - \sin^2 x$  and simplify:

$$\begin{aligned}&= 2 \sin x (1 - \sin^2 x) + \sin x - 2 \sin^3 x \\ &= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x \\ &= 3 \sin x - 4 \sin^3 x\end{aligned}$$

**10.** We use the addition and the subtraction formulas for  $\sin x$  and multiply out the terms that we get

$$\begin{aligned}\sin(x + y) \sin(x - y) &= (\sin x \cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y) \\ &= \sin^2 x \cos^2 y - \cos^2 x \sin^2 y\end{aligned}$$

(note that we used difference of squares formula!). Next, we use the basic trig identity to eliminate cos terms

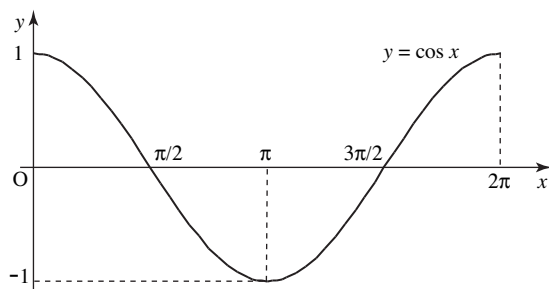
$$= \sin^2 x (1 - \sin^2 y) - (1 - \sin^2 x) \sin^2 y$$

Now multiply out and cancel

$$\begin{aligned}&= \sin^2 x - \sin^2 x \sin^2 y - \sin^2 y + \sin^2 x \sin^2 y \\ &= \sin^2 x - \sin^2 y\end{aligned}$$

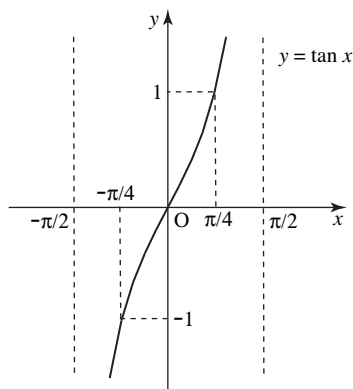
**11.** (a) Looking at the graph of  $\cos x$ , we see that there is only one solution to  $\cos x = -1$  in its main period, namely  $x = \pi$  (see below).

Thus, all solutions are given by  $x = \pi + 2k\pi = (1 + 2k)\pi$ , where  $k$  is an integer.



(b) In its main period,  $\tan x$  assumes the value  $-1$  only once, when  $x = -\pi/4$  (see below). Since the period of  $\tan x$  is  $\pi$ , we write all solutions of  $\tan x = -1$  as

$$x = -\frac{\pi}{4} + k\pi$$



**12.** Recall the strategy: to identify the number of solutions and their location in the main period, we use the graph; to get their values, we relate them to the values in the first quadrant (which we are supposed to remember).

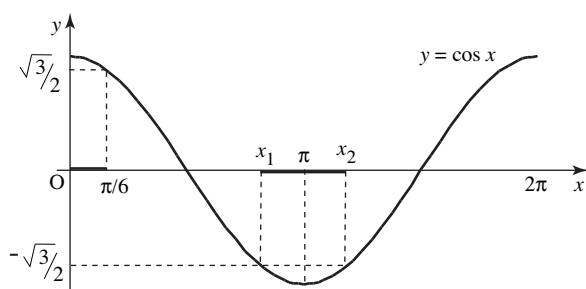
(a) The graph below shows that there are two solutions of  $\cos x = -\sqrt{3}/2$  in the main period, labeled  $x_1$  and  $x_2$ .

We know that  $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ . Since we are looking for the values where  $\cos$  is equal to  $-\sqrt{3}/2$ , we need to relate  $x_1$  and  $x_2$  to  $\frac{\pi}{6}$ .

We see that  $x_1$  is  $\pi/6$  units to the left of  $\pi$ , so  $x_1 = \pi - \pi/6 = 5\pi/6$ . The solution  $x_2$  is  $\pi/6$  units to the right of  $\pi$ ; thus,  $x_2 = \pi + \pi/6 = 7\pi/6$ .

So, the solutions are

$$x = \frac{5\pi}{6} + 2k\pi \quad \text{and} \quad x = \frac{7\pi}{6} + 2k\pi.$$



(b) The equation  $\tan x = \sqrt{3}$  has only one solution in the main period (look at the graph of  $\tan x$ ). From memory, we know that  $\tan(\pi/3) = \sqrt{3}$ .

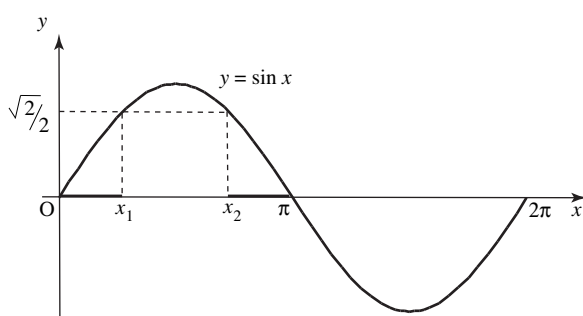
Thus, all solutions of the given equation are given by  $x = \frac{\pi}{3} + k\pi$ , since the period of  $\tan x$  is  $\pi$ .

(c) The graph below shows that there are two solutions of  $\sin x = \sqrt{2}/2$  in the main period, labeled  $x_1$  and  $x_2$ . One of them is in the first quadrant, and we know (from memory) that  $x_1 = \pi/4$ .

We need to relate  $x_2$  to  $\frac{\pi}{4}$ . We see that  $x_2$  is  $\pi/4$  units to the left of  $\pi$ , so  $x_2 = \pi - \pi/4 = 3\pi/4$ .

So, the solutions are

$$x = \frac{\pi}{4} + 2k\pi \quad \text{and} \quad x = \frac{3\pi}{4} + 2k\pi.$$



**13.** Use the double-angle formula for  $\sin x$

$$\sin 2x = 2 \sin x \cos x$$

$$2 \sin x \cos x = \sin x$$

move  $\sin x$  to the left side and factor

$$2 \sin x \cos x - \sin x = 0$$

$$\sin x(2 \cos x - 1) = 0.$$

Thus,  $\sin x = 0$  or  $2 \cos x - 1 = 0$ , i.e.,  $\cos x = 1/2$ .

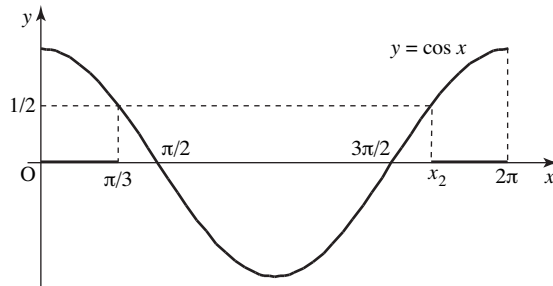
From the graph of  $y = \sin x$ , we see that  $\sin x = 0$  when  $x = k\pi$ .

From the graph below, we see that there are two solutions of  $\cos x = 1/2$  in the main period.

One of them is  $x_1 = \frac{\pi}{3}$ . Due to symmetry, the other solution is  $\pi/3$  units to the left of  $2\pi$ ; thus,  $x_2 = 2\pi - \pi/3 = 5\pi/3$ . It follows that all solutions are given by  $x = \frac{\pi}{3} + 2k\pi$  and  $x = \frac{5\pi}{3} + 2k\pi$ .

Thus, the solutions to the given equation are

$$x = k\pi \quad \text{and} \quad x = \frac{\pi}{3} + 2k\pi \quad \text{and} \quad x = \frac{5\pi}{3} + 2k\pi.$$



**14.** From  $2 \cos 2x - 1 = 0$  we get  $\cos 2x = 1/2$ .

In the previous exercise we solved  $\cos x = 1/2$  and got  $x = \frac{\pi}{3} + 2k\pi$  and  $x = \frac{5\pi}{3} + 2k\pi$ .

Replacing  $x$  by  $2x$ , we get that the solutions of  $\cos 2x = 1/2$  are  $2x = \frac{\pi}{3} + 2k\pi$  and  $2x = \frac{5\pi}{3} + 2k\pi$  i.e., dividing by 2,

$$x = \frac{\pi}{6} + k\pi \quad \text{and} \quad x = \frac{5\pi}{6} + k\pi.$$

**15.** (a) Expanding  $x = \frac{\pi}{2} + 2k\pi$  and  $x = \frac{3\pi}{2} + 2k\pi$  we get

$$\frac{\pi}{2} + 2k\pi = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{-3\pi}{2}, \frac{-7\pi}{2}, \dots \quad \text{and} \quad \frac{3\pi}{2} + 2k\pi = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \frac{-\pi}{2}, \frac{-5\pi}{2}, \dots$$

i.e., the general form is

$$\frac{(\text{odd number}) \pi}{2}.$$

We see that expanding  $x = \frac{\pi}{2} + k\pi$  we get

$$\frac{\pi}{2} + k\pi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{-\pi}{2}, \frac{-3\pi}{2}, \dots$$

which is the same thing.

(b) Expanding  $(2k+1)\pi$  we get

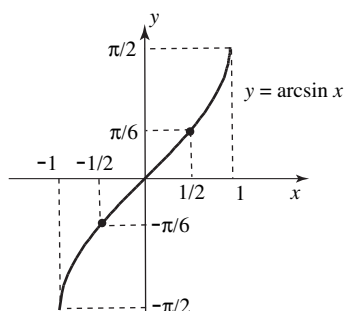
$$(2k+1)\pi = \pi, 3\pi, 5\pi, -\pi, -3\pi, \dots$$

(i.e., odd multiples of  $\pi$ ). Expanding  $-\pi + 2k\pi$  we get  $(2k+1)\pi = -\pi, \pi, 3\pi, -3\pi, -5\pi, \dots$  which again contains odd multiples of  $\pi$ .

**16.** The value  $\arcsin 1$  represents the unique angle in  $[-\pi/2, \pi/2]$  whose sin is equal to 1. From memory, we know that  $\sin(\pi/2) = 1$ , and so  $\arcsin 1 = \pi/2$ .

Likewise,  $\arcsin(-1/2)$  represents the unique angle in  $[-\pi/2, \pi/2]$  whose sin is equal to  $-1/2$ . From Example 11 in this section in the book we see that  $\arcsin(-1/2) = -\pi/6$ .

Alternative: the function  $\arcsin x$  is odd (its graph is symmetric with respect to the origin); see below. Thus, the values of  $\arcsin$  at  $x$  and  $-x$  are negatives of each other. So, if  $\arcsin(1/2) = \pi/6$  (we know that is true because  $\sin(\pi/6) = 1/2$ ) then  $\arcsin(-1/2) = -\pi/6$ . See the graph below.

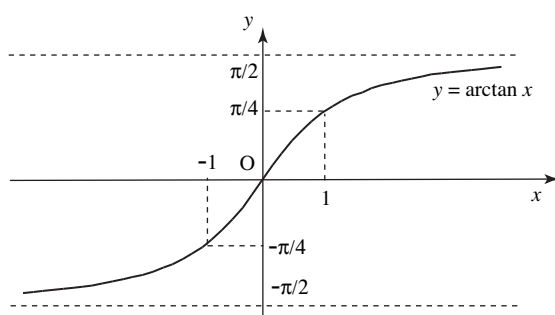


Since  $\sin(\pi/4) = \sqrt{2}/2 = 1/\sqrt{2}$  and  $\pi/4$  is in  $[-\pi/2, \pi/2]$ , we conclude that  $\arcsin(1/\sqrt{2}) = \pi/4$ .

Using the same argument as above, we conclude that  $\arcsin(-1/\sqrt{2}) = -\pi/4$ .

**17.** To find  $\arctan(-1)$  means to find the unique angle  $\theta$  in  $(-\pi/2, \pi/2)$  such that  $\tan \theta = -1$ . In the solution to Exercise 11(b) we found out that  $\tan(-\pi/4) = -1$ . Thus,  $\arctan(-1) = -\pi/4$ .

Note that, like the arcsin function (look at the previous exercise) the function  $\arctan$  is odd, see the figure below. Recall that an odd function is symmetric with respect to the origin, that is, the values of  $\arctan$  at  $x$  and  $-x$  are negatives of each other. Since we know (from memory) that  $\tan(\pi/4) = 1$ , we conclude that  $\arctan(1) = \pi/4$  and thus  $\arctan(-1) = -\pi/4$ .



Since  $\tan(\pi/6) = 1/\sqrt{3}$ , we know that  $\arctan(1/\sqrt{3}) = \pi/6$ . As above, using the fact that  $\tan$  is an odd function, we conclude that  $\arctan(-1/\sqrt{3}) = -\pi/6$ .

From memory, we know that  $\tan(\pi/3) = \sqrt{3}$ , and thus  $\arctan(\sqrt{3}) = \pi/3$ .

**18.** (a) From Pythagorean Theorem we compute

$$c = \sqrt{a^2 + b^2} = \sqrt{20^2 + 21^2} = \sqrt{841} = 29.$$

Thus,

$$\sin \alpha = \frac{a}{c} = \frac{21}{29}, \quad \cos \alpha = \frac{b}{c} = \frac{20}{29}, \quad \sin \beta = \frac{b}{c} = \frac{20}{29}, \quad \cos \beta = \frac{a}{c} = \frac{21}{29}.$$

(b) From  $\cos \beta = 12/13$  and  $\cos \beta = \frac{a}{c} = \frac{a}{13}$  we get that  $a = 12$ . From Pythagorean Theorem we compute

$$b = \sqrt{c^2 - a^2} = \sqrt{13^2 - 12^2} = \sqrt{25} = 5.$$

Now we can calculate any trig ratio of  $\alpha$  or  $\beta$ .

$$\sin \beta = \frac{b}{c} = \frac{5}{13}, \quad \tan \beta = \frac{b}{a} = \frac{5}{12}.$$

(c) From Pythagorean Theorem we compute

$$a = \sqrt{c^2 - b^2} = \sqrt{1^2 - 0.6^2} = \sqrt{0.64} = 0.8.$$

Thus,

$$\sin \alpha = \frac{a}{c} = \frac{0.8}{1} = 0.8 = \frac{4}{5}, \quad \cos \alpha = \frac{b}{c} = \frac{0.6}{1} = 0.6 = \frac{3}{5}, \quad \tan \alpha = \frac{a}{b} = \frac{0.8}{0.6} = \frac{8}{6} = \frac{4}{3}.$$

The remaining three functions are the reciprocals:

$$\csc \alpha = \frac{1}{\sin \alpha} = \frac{5}{4}, \quad \sec \alpha = \frac{1}{\cos \alpha} = \frac{5}{3}, \quad \cot \alpha = \frac{1}{\tan \alpha} = \frac{3}{4}.$$

**19.** Keep in mind that one full revolution is 360 degrees or  $2\pi$  radians.

(a) We break down the given angle into multiples of  $2\pi$  and a remainder part:

$$\frac{36\pi}{7} = \frac{28 + 8\pi}{7} = \frac{28\pi}{7} + \frac{8\pi}{7} = 4\pi + \frac{8\pi}{7}.$$

The angle of  $4\pi$  brings us back to the start (positive  $x$ -axis); so, the question is – in what quadrant does  $\frac{8\pi}{7}$  lie? Clearly,  $\frac{8\pi}{7} > \pi$ . Since

$$\frac{8\pi}{7} = \frac{16\pi}{14} < \frac{21\pi}{14} = \frac{3\pi}{2},$$

the angle  $\frac{8\pi}{7}$  lies in the third quadrant.

Alternatively,  $\frac{8\pi}{7}$  radians is  $\frac{8\pi}{7} \frac{180}{\pi} = 205.7$  degrees, so it must be in the third quadrant.

(b) Integer multiples of 360 are 360, 720, 1080, etc.

Since  $999 = 720 + 279$ , we see that the angle of 999 degrees is the same as the angle of 279 degrees, which is in the fourth quadrant.

(c) We can proceed as in (b), or argue like this: dividing 989 by 360, we get 2.747. So, it's two full revolutions, plus 0.747 of a revolution. Since 0.747 is between the half and the three-quarters of a revolution, the angle must be in the third quadrant.

Alternatively, we multiply 0.747 by 360 to get 268.92 degrees, which is in the third quadrant.

(d) We argue as in (a), or like this: since  $44/5 = 8.8$ , we write

$$\frac{44\pi}{5} = 8.8\pi = 8\pi + 0.8\pi.$$

Now  $8\pi$  is four full revolutions, so we are back at the start. Since

$$\frac{\pi}{2} = 0.5\pi < 0.8\pi < \pi$$

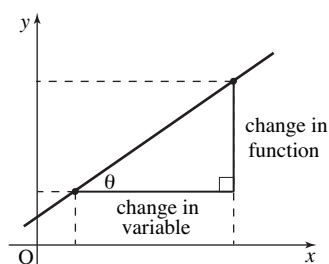


we conclude that  $0.8\pi$  – and thus the given angle – are in the second quadrant.

**20.** Recall that the slope is defined as the ratio

$$\frac{\text{change in the function}}{\text{change in the variable}}$$

(in our case, the function is a line). From the figure below, we see that this is exactly the ratio of the opposite to the adjacent sides for the angle  $\theta$ . Thus, the slope is equal to  $\tan \theta$ .



**21.** Since the question is about the signs (and not the values) of trig functions, all we need to do is to figure out where (in which quadrant) the angles lie.

(a) Note that  $13\pi/3 = 12\pi/3 + \pi/3 = 4\pi + \pi/3$ , i.e., the angle  $13\pi/3$  is the same as  $\pi/3$ . In other words,  $13\pi/3$  lies in the first quadrant, and hence  $\tan(13\pi/3) = \tan(\pi/3)$  is positive.

(b) Since  $500 = 360 + 140$ , it follows that  $\sin(500^\circ) = \sin(140^\circ)$ . Since the angle of  $140^\circ$  is in the second quadrant, we conclude that  $\sin(500^\circ)$  is positive.

(c) From  $37\pi = 36\pi + \pi = 18(2\pi) + \pi$  we conclude that the angle of  $37\pi$  equals 18 full revolutions plus  $\pi$ . Thus, as angles,  $37\pi$  and  $\pi$  are equal. It follows that  $\cos(37\pi) = \cos(\pi) = -1$ , i.e.,  $\cos(37\pi)$  is negative.

(d) Since both angles  $\pi/12$  and  $\pi/7$  lie in the first quadrant, and  $\sin$  and  $\cos$  are positive there, we conclude that  $\sin(\pi/12) + \cos(\pi/7)$  is positive.

**22.** Recall the convention for measuring angles – if no unit is mentioned, then it is radians.

To compare, we convert to the same unit. One radian is equal to  $180/\pi = 57.3$  degrees (so the angle of 1 radian lies in the first quadrant).

The angles between  $\pi/2 = 1.57$  and  $\pi = 3.14$  radians constitute the second quadrant. Thus, the angle of 2 radians lies in the second quadrant (angle measure of 2 radians is equal to  $2(180/\pi) = 114.6$  degrees).

(a) Both  $1^\circ$  and  $1 \text{ rad} = 57.3^\circ$  lie in the first quadrant. Since  $\sin$  is increasing there (i.e., values of  $\sin x$  increase as angle  $x$  increases) we conclude that  $\sin 1$  (radian) is larger than  $\sin 1^\circ$ .

(b) The angle of  $2^\circ$  is in the first quadrant, and so  $\cos 2^\circ$  is positive. Since the angle of 2 radians is in the second quadrant, we know that  $\cos 2$  is negative. Thus  $\cos 2^\circ$  is larger than  $\cos 2$ .

(c) Since  $\tan x$  is increasing in the first quadrant, using exactly the same argument as in (a) we conclude that  $\tan 1$  (radian) is larger than  $\tan 1^\circ$ .

**23.** (a) Using the fundamental identity  $\sin^2 x + \cos^2 x = 1$  we get

$$\begin{aligned} \sec^2 x - \sin^2 x - \cos^2 x &= \sec^2 x - (\sin^2 x + \cos^2 x) \\ &= \sec^2 x - 1 \\ &= \frac{1}{\cos^2 x} - 1 \\ &= \frac{1 - \cos^2 x}{\cos^2 x} = \frac{\sin^2 x}{\cos^2 x} = \tan^2 x. \end{aligned}$$

Note that in this calculation we also proved the fact that  $\sec^2 x - 1 = \tan^2 x$ .

(b) Write  $\tan x = \sin x / \cos x$  and compute the common denominator

$$\begin{aligned} \frac{\cos x}{1 + \sin x} + \tan x &= \frac{\cos x}{1 + \sin x} + \frac{\sin x}{\cos x} \\ &= \frac{\cos x \cos x}{(1 + \sin x) \cos x} + \frac{\sin x(1 + \sin x)}{(1 + \sin x)} \\ &= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos x(1 + \sin x)} \\ &= \frac{1 + \sin x}{\cos x(1 + \sin x)} = \frac{1}{\cos x} = \sec x. \end{aligned}$$

Near the end, we used  $\sin^2 x + \cos^2 x = 1$ .

(c) Compute the common denominator (note the difference of squares!)

$$\begin{aligned} \frac{\sin x}{1 + \cos x} + \frac{\sin x}{1 - \cos x} &= \frac{\sin x(1 - \cos x)}{(1 + \cos x)(1 - \cos x)} + \frac{\sin x(1 + \cos x)}{(1 - \cos x)(1 + \cos x)} \\ &= \frac{\sin x - \sin x \cos x + \sin x + \sin x \cos x}{1 - \cos^2 x} \\ &= \frac{2 \sin x}{\sin^2 x} \\ &= \frac{2}{\sin x} = 2 \csc x \end{aligned}$$

(d) Square both binomials

$$\begin{aligned} (\sin x + \cos x)^2 + (\sin x - \cos x)^2 &= \sin^2 x + 2 \sin x \cos x + \cos^2 x + \sin^2 x - 2 \sin x \cos x + \cos^2 x \\ &= 2(\sin^2 x + \cos^2 x) = 2. \end{aligned}$$

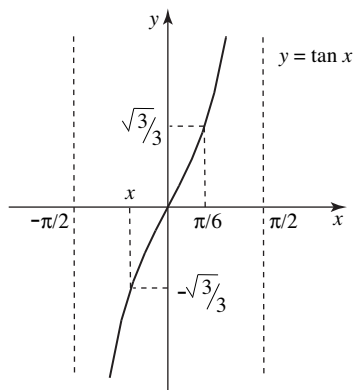
**24.** (a) We employ our usual strategy of relating values of trig functions to their values in the first quadrant.

From memory, we know that

$$\tan \frac{\pi}{6} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}.$$

Due to the symmetry, we see that  $x = -\pi/6$ ; see figure below. As well,  $x = -\pi/6$  is the only solution of the given equation in the main period of  $\tan$ .

Thus, the solutions of the given equation are  $x = -\frac{\pi}{6} + k\pi$ .



(b) Computing the reciprocal of  $\cot x = -1$  we get  $\tan x = -1$ . We solved this equation in Exercise 11(b), the solution is  $x = \frac{-\pi}{4} + k\pi$ . We can also write it as  $x = \frac{3\pi}{4} + k\pi$ .

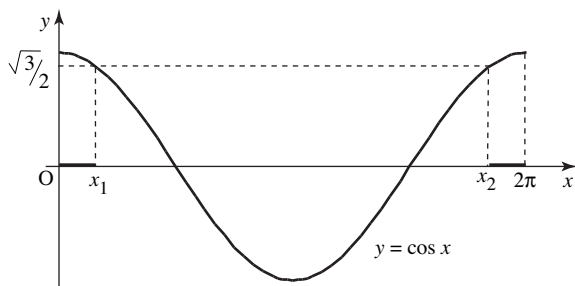
(c) The graph below shows that there are two solutions of  $\cos x = \sqrt{3}/2$  in the main period, labeled  $x_1$  and  $x_2$ .

We know that  $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ ; so  $x_1 = \frac{\pi}{6}$ .

Due to the symmetry of the graph of  $\cos x$ , we see that  $x_2$  is  $\pi/6$  units to the left of  $2\pi$ , i.e.  $x_2 = 2\pi - \pi/6 = 11\pi/6$ .

So, the solutions are

$$x = \frac{\pi}{6} + 2k\pi \quad \text{and} \quad x = \frac{11\pi}{6} + 2k\pi.$$



**25.** (a) This question has already been solved in Exercise 12(c), sorry! The answer is  $x = \frac{\pi}{4} + 2k\pi$  and  $x = \frac{3\pi}{4} + 2k\pi$

(b) Use definitions of  $\tan x$  and  $\cot x$  and calculate common denominator

$$\begin{aligned} \tan x + \cot x &= 0.5 \\ \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} &= 0.5 \\ \frac{\sin x \sin x}{\cos x \sin x} + \frac{\cos x \cos x}{\sin x \cos x} &= 0.5 \end{aligned}$$

$$\frac{\sin^2 x + \cos^2 x}{\cos x \sin x} = 0.5$$

$$\frac{1}{\cos x \sin x} = 0.5$$

now take reciprocals of both sides to get

$$\cos x \sin x = 2$$

Since  $\sin x \leq 1$  and  $\cos x \leq 1$ , their product satisfies  $\sin x \cos x \leq 1$  as well. Thus, the above equation has no solutions.

(c) Factor the equation

$$\cos^2 x - \cos x - 2 = 0$$

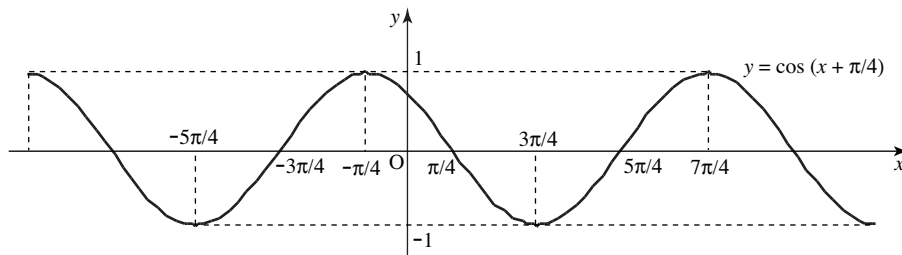
$$(\cos x - 2)(\cos x + 1) = 0$$

If  $\cos x - 2 = 0$ , then  $\cos x = 2$ , and there are no solutions (keep in mind that  $\cos x \leq 1$  for all  $x$ ).

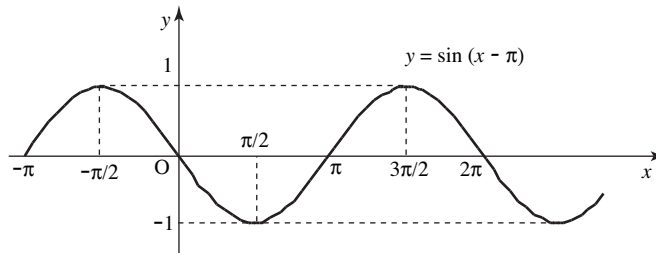
If  $\cos x + 1 = 0$ , then  $\cos x = -1$ , and  $x = \pi + 2k\pi$ ; see Exercise 11(a).

Thus, all solutions are given by  $x = \pi + 2k\pi$ .

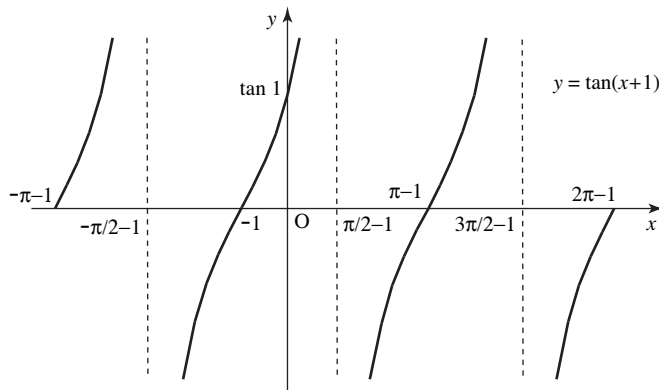
**26.** (a) Recall that replacing  $x$  by  $x$  plus a positive number results in shift towards the left. Thus, the graph of  $\cos(x + \pi/4)$  is obtained by moving the graph of  $y = \cos x$  to the left for  $\pi/4$  units; see below.



(b) We shift the graph of  $\sin x$  for  $\pi$  units to the right to obtain the graph of  $\sin(x - \pi)$ ; see below.



(c) Start with  $\tan x$  and move it one unit to the left; see below. Note that the  $y$ -intercept is  $\tan 1 = 1.557$ .



**27.** Recall that, to prove an identity, we start with one side and keep transforming until we arrive at the other side. Sometimes, we transform both sides simultaneously until we arrive at an identity. We illustrate both approaches in the solution to part (a) below.

(a) Start with the left side, and multiply and divide by  $1 + \sin x$

$$\frac{1 - \sin x}{\cos x} = \frac{(1 - \sin x)(1 + \sin x)}{\cos x(1 + \sin x)}$$

Multiply the terms in the numerator using the difference of squares, and then simplify with the help of the basic trig identity

$$\begin{aligned} &= \frac{1 - \sin^2 x}{\cos x(1 + \sin x)} \\ &= \frac{\cos^2 x}{\cos x(1 + \sin x)} = \frac{\cos x}{1 + \sin x} \end{aligned}$$

Alternatively, we start with the identity we have to prove and cross-multiply

$$\begin{aligned} \frac{1 - \sin x}{\cos x} &= \frac{\cos x}{1 + \sin x} \\ (1 - \sin x)(1 + \sin x) &= \cos^2 x \\ 1 - \sin^2 x &= \cos^2 x \\ \cos^2 x &= \cos^2 x \end{aligned}$$

Done!

(b) Start with the left side – use definitions of  $\tan x$  and  $\cot x$

$$\frac{1}{1 + \tan^2 x} + \frac{1}{1 + \cot^2 x} = \frac{1}{1 + \frac{\sin^2 x}{\cos^2 x}} + \frac{1}{1 + \frac{\cos^2 x}{\sin^2 x}}$$

Multiply the first fraction by  $\cos^2 x$  and the second by  $\sin^2 x$

$$\begin{aligned} &= \frac{\cos^2 x}{\cos^2 x + \sin^2 x} + \frac{\sin^2 x}{\sin^2 x + \cos^2 x} \\ &= \frac{\cos^2 x}{1} + \frac{\sin^2 x}{1} = 1 \end{aligned}$$

after repeated use of the identity  $\cos^2 x + \sin^2 x = 1$ .

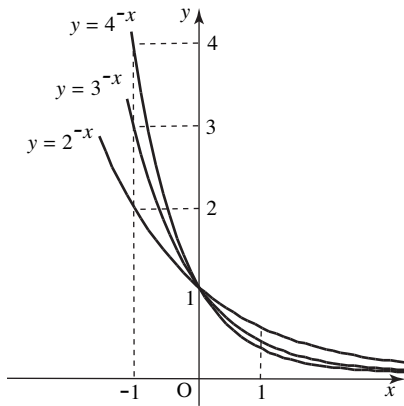
### Section 8. Exponential and Logarithmic Functions

1. (a) To get the graph of  $y = 2^x + 4$ , move the graph of  $y = 2^x$  up 4 units.  
 (b) We see that  $x$  in  $y = 2^x$  has been replaced by  $x - 4$ . So, we move the graph of  $y = 2^x$  to the right for 4 units.  
 Alternatively, we write  $y = 2^{x-4} = 2^{-4}2^x = \frac{1}{16}2^x$ ; so we need to compress the graph of  $y = 2^x$  vertically by a factor of 16.  
 (c) To draw  $y = -2^x$ , we reflect the graph of  $y = 2^x$  with respect to the  $x$ -axis.  
 (d) Reflect the graph of  $y = 2^x$  with respect to the  $x$ -axis (thus getting  $y = -2^x$ ) and then with respect to the  $y$ -axis (to get  $y = -2^{-x}$ ).  
 Reversing the order of the two reflections would result in the same graph.  
 Alternatively, write  $-2^{-x} = -(2^{-1})^x = -(1/2)^x$ ; so we need to reflect the graph of  $y = (1/2)^x$  with respect to the  $x$ -axis.

2. We plot points to draw the graphs (see below).

We see that as  $x$  approaches  $-\infty$ , all graphs  $y = 2^{-x}$ ,  $y = 3^{-x}$  and  $y = 4^{-x}$  approach  $\infty$ . Note that  $a^{-x}$  approaches  $\infty$  faster as  $a$  increases.

As  $x$  approaches  $\infty$ , all three graphs approach zero – but at different rates:  $a^{-x}$  approaches zero faster as  $a$  increases.



$x$	$2^{-x}$	$3^{-x}$	$4^{-x}$
0	1	1	1
1	1/2	1/3	1/4
2	1/4	1/9	1/16
-1	2	3	4
-2	4	9	16
...	...	...	...

3. (a) Write  $25 = 5^2$ ; we get

$$5^{x-2} \cdot 25^{3-x} = 5^{x-2} \cdot (5^2)^{3-x} = 5^{x-2} \cdot 5^{2(3-x)} = 5^{x-2} \cdot 5^{6-2x} = 5^{x-2+6-2x} = 5^{-x+4}$$

- (b) Use the fact that  $9 = 3^2$  and  $27 = 3^3$ ; thus

$$\begin{aligned} 3^{x-1} \cdot 9^{x-2} \cdot 27^{x-3} &= 3^{x-1} \cdot (3^2)^{x-2} \cdot (3^3)^{x-3} \\ &= 3^{x-1} \cdot 3^{2(x-2)} \cdot 3^{3(x-3)} \\ &= 3^{x-1} \cdot 3^{2x-4} \cdot 3^{3x-9} \\ &= 3^{x-1+2x-4+3x-9} = 3^{6x-14} \end{aligned}$$

(c) We recognize the numbers involved as powers of 2; i.e.,  $8 = 2^3$  and  $16 = 2^4$ . Thus

$$\frac{8^{x+4}}{16^{x-2}} = \frac{(2^3)^{x+4}}{(2^4)^{x-2}} = \frac{2^{3(x+4)}}{2^{4(x-2)}} = \frac{2^{3x+12}}{2^{4x-8}} = 2^{3x+12-(4x-8)} = 2^{-x+20}$$

4. (a) Simplify so that both sides have the same base:

$$0.5^{x^2} = 0.125$$

$$0.5^{x^2} = 0.5^3$$

Thus  $x^2 = 3$  and  $x = \pm\sqrt{3}$ .

(b) From  $3^x(3^x - 3) = 0$  it follows that either  $3^x = 0$  or  $3^x - 3 = 0$ .

Since  $a^x$  is positive for all  $a > 0$  and for any  $x$ , we conclude that the equation  $3^x = 0$  has no solutions.

From  $3^x - 3 = 0$  we get  $3^x = 3 = 3^1$ , and so  $x = 1$ .

Thus,  $x = 1$  is the only solution.

(c) In this solution we will use the fact that  $(2^x)^2 = 2^{2x}$  (recall that  $(a^m)^n = a^{mn}$ ).

Write  $2^{2x} - 5 \cdot 2^x + 4 = 0$  as  $(2^x)^2 - 5 \cdot 2^x + 4 = 0$  and factor

$$(2^x - 4)(2^x - 1) = 0.$$

If  $2^x - 4 = 0$ , then  $2^x = 4 = 2^2$ , and so  $x = 2$ . If  $2^x - 1 = 0$ , then  $2^x = 1 = 2^0$ , and so  $x = 0$ .

We conclude that there are two solutions,  $x = 0$  and  $x = 2$ .

5. We keep in mind that  $\log_a n = m$  is equivalent to  $a^m = n$

So, if  $\log_{10} 10 = 1$ , then  $10^1 = 10$ .

The remaining entries are done analogously: Reading the above statement with  $a = 3$ ,  $m = -2$  and  $n = 1/9$ , we see that  $3^{-2} = 1/9$  is equivalent to  $\log_3(1/9) = -2$

In the same way, we fill out the remaining entries:

If  $\log_5 0.04 = -2$ , then  $5^{-2} = 0.04$ .

From  $10^{-3} = 0.001$  we conclude that  $\log_{10} 0.001 = -3$ , etc. See table below.

Exponential	Logarithm
$10^1 = 10$	$\log_{10} 10 = 1$
$3^{-2} = 1/9$	$\log_3(1/9) = -2$
$5^{-2} = 0.04$	$\log_5 0.04 = -2$
$10^{-3} = 0.001$	$\log_{10} 0.001 = -3$
$10^0 = 1$	$\log_{10} 1 = 0$
$10^6 = 1,000,000$	$\log_{10} 1,000,000 = 6$

**6.** To provide counterexamples, we need convenient values for the logarithms involved, such as the following: if  $a > 0$ , then  $\log_a 1 = 0$  (because  $a^0 = 1$ ) and  $\log_a a = 1$  (because  $a^1 = a$ ); as well,  $\log_a a^n = n$  for any real number  $n$ .

To show that  $\log_a(x + y) = \log_a x + \log_a y$  does not work, we try simplest possible values - take  $x = y = 1$ ; then  $\log_a x = 0$  and  $\log_a y = 0$ , and thus  $\log_a x + \log_a y = 0$ .

However,  $\log_a(x + y) = \log_a 2 \neq 0$ . (Why is  $\log_a 2 \neq 0$ ? Think of the definition:  $\log_a 2$  has the property that  $a$  raised to must give 2. If  $\log_a 2$  were zero, then  $a$  raised to it would be 1, and not 2).

To prove that  $\log_{10}(x - y) = \log_{10} x - \log_{10} y$  is not correct we could take  $x = y = 1$  again. The left side is  $\log_{10} 0$ , which is not defined; however, the right side is equal to zero.

Alternatively, take  $x = 2$ ,  $y = 1$ ; then  $\log_{10}(x - y) = \log_{10} 1 = 0$ , whereas  $\log_{10} x - \log_{10} y = \log_{10} 2 - \log_{10} 1 = \log_{10} 2 \neq 0$ .

**7.** (a) Since

$$y^3 + 8 = y^3 + 2^3 = (y + 2)(y^2 - 2y + 4)$$

(to review this, look at the sum of cubes formula in Section 2). Thus

$$\log_{12}(y^3 + 8) = \log_{12}(y + 2)(y^2 - 2y + 4) = \log_{12}(y + 2) + \log_{12}(y^2 - 2y + 4)$$

using the product law of logarithms,  $\log_{12}(AB) = \log_{12} A + \log_{12} B$ .

(b) Using laws of logarithms (for products and quotients), we get

$$\begin{aligned} \log_{10} \frac{100a^4c^{-1}}{a+b} &= \log_{10}(100a^4c^{-1}) - \log_{10}(a+b) \\ &= \log_{10} 100 + \log_{10} a^4 + \log_{10} c^{-1} - \log_{10}(a+b) \\ &= 2 + 4\log_{10} a - \log_{10} c - \log_{10}(a+b) \end{aligned}$$

because  $\log_{10} 100 = \log_{10} 10^2 = 2\log_{10} 10 = 2(1) = 2$ . Note that we used the formula  $\log_{10} a^n = n\log_{10} a$ .

(c) Simplify

$$\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{1}{2^{1/2}} = 2^{-1/2}.$$

Thus

$$\log_2 \sqrt{\frac{1}{2}} = \log_2 2^{-1/2} = -\frac{1}{2} \log_2 2 = -\frac{1}{2},$$

since  $\log_2 2 = 1$ .

**8.** We make use of the conversion formula

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Thus,

$$\log_4 x = \frac{\log_{16} x}{\log_{16} 4} = \frac{\log_{16} x}{\log_{16} 16^{1/2}} = \frac{\log_{16} x}{\frac{1}{2} \log_{16} 16} = 2 \log_{16} x,$$

since  $\log_{16} 16 = 1$ .



Likewise,

$$\log_{0.5} x = \frac{\log_2 x}{\log_2 0.5} = \frac{\log_2 x}{\log_2 2^{-1}} = \frac{\log_2 x}{-\log_2 2} = -\log_2 x.$$

(in the denominator, we used  $\log_2 2^{-1} = (-1)\log_2 2 = -1$ , since  $\log_2 2 = 1$ .)

To find the numeric value of  $\log_{11} 1.45$  we convert to base 10 and then use a calculator:

$$\log_{11} 1.45 = \frac{\log_{10} 1.45}{\log_{10} 11} \approx \frac{0.161368}{1.041393} \approx 0.154954.$$

Instead of converting to base 10, we could convert to base  $e$ , i.e., use  $\ln$ ; in this case,

$$\log_{0.4} 44 = \frac{\ln 44}{\ln 0.4} \approx \frac{3.784190}{-0.916291} \approx -4.129900.$$

**9.** (a) We rewrite  $\log_x 4 = 1/2$  in exponential form:  $x^{1/2} = 4$ , i.e.,  $\sqrt{x} = 4$ . Thus,  $x = 16$ .

(b) As in (a), we rewrite  $\log_3 x = 5$  as  $3^5 = x$ ; thus,  $x = 243$ .

(c) From  $\log_2 x^3 = \log_2(4x)$  we conclude that  $x^3 = 4x$ . Seems that we canceled by  $\log_2$ ; is that allowed? Yes! To remove  $\log_2$  from  $\log_2 x^3 = \log_2(4x)$  we apply the exponential function  $2^x$ :

$$2^{\log_2 x^3} = 2^{\log_2(4x)}$$

(now recall that  $2^{\log_2 A} = A$  for all  $A > 0$ ). Therefore,  $x^3 = 4x$ .

To solve  $x^3 = 4x$ , we write  $x^3 - 4x = 0$  and then factor:

$$x(x^2 - 4) = x(x + 2)(x - 2) = 0.$$

It follows that  $x = 0$ ,  $x = -2$  and  $x = 2$ .

Note that  $x = 0$  and  $x = -2$  are not in the domain of the two functions  $\log_2 x^3$  and  $\log_2(4x)$  and thus cannot be solutions. It follows that  $x = 2$  is the only solution.

(d) From  $16^{\log_4 x} = 4 = \sqrt{16} = 16^{1/2}$  we get  $\log_4 x = 1/2$ . Thus,  $x = 4^{1/2} = 2$ .

**10.** (a) Simplify using the product law of logarithms  $\ln A + \ln B = \ln(AB)$  to get

$$e^{\ln 4 + \ln 5} = e^{\ln(4 \cdot 5)} = e^{\ln 20} = 20,$$

since  $e^{\ln A} = A$  for all positive numbers  $A$ .

(b) Using  $\ln(e^A) = A$  we write  $2 = \ln e^2$ ; thus

$$4 \ln 2 + \ln 3 + 2 = \ln 4^2 + \ln 3 + \ln e^2 = \ln(4^2 \cdot 3 \cdot e^2) = \ln(48e^2).$$

Note that we also used  $\ln A^n = n \ln A$  and the product law of logarithms  $\ln A + \ln B = \ln(AB)$ .

(c) Applying  $\ln$  to both sides of  $e^{3x-2} = 4$  we get  $3x - 2 = \ln 4$  and thus  $x = (\ln 4 + 2)/3$ .

(d) From  $\ln(x^2 + x - 1) = 0$  we get

$$e^{\ln(x^2 + x - 1)} = e^0,$$

and

$$x^2 + x - 1 = 1.$$

Thus,  $x^2 + x - 2 = 0$ , and, after factoring,  $(x + 2)(x - 1) = 0$ . It follows that  $x = -2$  and  $x = 1$ . Both  $x = -2$  and  $x = 1$  belong to the domain of  $\ln(x^2 + x - 1)$  and hence are solutions of the given equation.

11. (a) Using base 10, we get

$$10 \cdot 100^2 \cdot 1000^4 = 10 \cdot (10^2)^2 \cdot (10^3)^4 = 10 \cdot 10^{2 \cdot 2} \cdot 10^{3 \cdot 4} = 10 \cdot 10^4 \cdot 10^{12} = 10^{1+4+12} = 10^{17}.$$

(b) Write  $6 = 2 \cdot 3$  so that

$$3^7 + 6 \cdot 3^6 = 3^7 + 2 \cdot 3 \cdot 3^6 = 3^7 + 2 \cdot 3^{1+6} = 3^7 + 2 \cdot 3^7 = 3 \cdot 3^7 = 3^{1+7} = 3^8.$$

Alternatively, start by factoring out  $3^6$

$$3^7 + 6 \cdot 3^6 = 3^6(3 + 6) = 3^6 \cdot 9 = 3^6 \cdot 3^2 = 3^{6+2} = 3^8.$$

(c) Write  $9 = 3^2$  and  $27 = 3^3$  to get

$$9 \cdot 27^3 + 2 \cdot 3^{11} = 3^2 \cdot (3^3)^3 + 2 \cdot 3^{11} = 3^2 \cdot 3^9 + 2 \cdot 3^{11} = 3^{11} + 2 \cdot 3^{11} = 3 \cdot 3^{11} = 3^{12}.$$

(d) Since  $36 = 6^2$ , we get

$$\frac{36^{n+3}}{6^{2n+5}} = \frac{(6^2)^{n+3}}{6^{2n+5}} = \frac{6^{2(n+3)}}{6^{2n+5}} = \frac{6^{2n+6}}{6^{2n+5}} = 6^{2n+6-(2n+5)} = 6^1 = 6.$$

Note that we used the formula  $A^m/A^n = A^{m-n}$ .

12. (a) Write  $0.5 = 2^{-1}$  and  $0. = 10^{-1}$  and use laws of exponents

$$\begin{aligned} \frac{(0.5 \cdot 10)^{-3}}{16 \cdot 0.1^4} &= \frac{(2^{-1} \cdot 10)^{-3}}{16 \cdot (10^{-1})^4} = \frac{(2^{-1})^{-3} \cdot (10)^{-3}}{2^4 \cdot 10^{-4}} = \frac{2^3 \cdot 10^{-3}}{2^4 \cdot 10^{-4}} \\ &= 2^{3-4} \cdot 10^{-3-(-4)} = 2^{-1} \cdot 10^1 = \frac{10}{2} = 5. \end{aligned}$$

Alternatively,

$$\begin{aligned} \frac{(0.5 \cdot 10)^{-3}}{16 \cdot 0.1^4} &= \frac{5^{-3}}{2^4 \cdot 0.1^4} = \frac{5^{-3}}{0.2^4} = \frac{1}{5^3 \cdot 0.2^4} \\ &= \frac{1}{5^3 \cdot 0.2^3 \cdot 0.2^1} = \frac{1}{(5 \cdot 0.2)^3 \cdot 0.2^1} \\ &= \frac{1}{(1)^3 \cdot 0.2} = \frac{1}{0.2} = \frac{10}{2} = 5 \end{aligned}$$

(b) Write  $0.2 = 2 \cdot 0.1$  and  $0.1 = 10^{-1}$  to get

$$\begin{aligned} 0.2^{-4} \cdot 16 &= (2 \cdot 0.1)^{-4} \cdot 2^4 = 2^{-4} \cdot (10^{-1})^{-4} \cdot 2^4 = 2^{-4} \cdot 2^4 \cdot 10^{(-1)(-4)} \\ &= 2^{-4+4} \cdot 10^4 = 2^0 \cdot 10^4 = 10^4 \end{aligned}$$

since  $2^0 = 1$ .

(c) Note that  $32 = 2^5$ ; thus

$$-32 \cdot \left(\frac{1}{2}\right)^4 = -2^5 \frac{1^4}{2^4} = -2^5 \cdot 2^{-4} = -2^{5-4} = -2.$$

There are other ways of simplifying (here, and in other questions in this exercise); for instance,

$$-32 \cdot \left(\frac{1}{2}\right)^4 = -32 \frac{1^4}{2^4} = -32 \frac{1}{16} = -\frac{32}{16} = -2.$$

(d) Using laws of logarithms,

$$\log_{10} 0.0001 = \log_{10} 10^{-4} = -4 \log_{10} 10 = -4,$$

since  $\log_{10} 10 = 1$ .

(e)  $\log_{10} \frac{1}{100} = \log_{10} \frac{1}{10^2} = \log_{10} 10^{-2} = -2 \log_{10} 10 = -2.$

(f) Recall that  $\log_4 2^5 = x$  is equivalent to  $4^x = 2^5$ . Now we solve this exponential equation.

$$\begin{aligned} 4^x &= 2^5 \\ (2^2)^x &= 2^5 \\ 2^{2x} &= 2^5. \end{aligned}$$

Thus,  $2x = 5$  and  $x = 5/2$ .

Alternatively:

$$\log_4 2^5 = \log_4 (4^{1/2})^5 = \log_4 4^{5/2} = \frac{5}{2} \log_4 4 = \frac{5}{2}$$

because  $\log_4 4 = 1$ .

(g) As in (f), reduce  $\log_4 2$  to  $\log_4$  of 4 to some power:

$$(\log_4 2)^5 = (\log_4 4^{1/2})^5 = \left(\frac{1}{2} \log_4 4\right)^5 = \left(\frac{1}{2}\right)^5 = \frac{1}{32}.$$

**13.** (a) Using  $n \ln A = \ln A^n$ , we get

$$e^{(1/2) \ln 16} = e^{\ln 16^{1/2}} = 16^{1/2} = \sqrt{16} = 4.$$

We used the cancellation law  $e^{\ln A} = A$ .

(b) Recall the general cancellation law:  $a^{\log_a x} = x$ , i.e., if we take a number (call it  $x$ ), apply  $\log_a$  to it and then exponentiate it (with the base  $a$ ) we get our number back.

When  $a = 10$ , the formula states that  $10^{\log_{10} x} = x$ . Thus,  $10^{\log_{10} 5} = 5$ .

(c)  $\log_{10} 100000 = \log_{10} 10^5 = 5 \log_{10} 10 = 5.$

(d) Using the cancellation formula that we reviewed in (b),

$$10^{3 \log_{10} 4} = 10^{\log_{10} 4^3} = 4^3 = 64.$$

**14.** Recall the conversion formula

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$\log_5 x = \frac{\log_{10} x}{\log_{10} 5} \approx \frac{\log_{10} x}{0.698970} \approx 1.430677 \log_{10} x.$$

Likewise,  $\log_5 x = \frac{\log_{12} x}{\log_{12} 5} \approx \frac{\log_{12} x}{0.647685} \approx 1.543960 \log_{12} x$ . We calculated  $\log_{12} 5$  using conversion  $\log_{12} 5 = \log_{10} 5 / \log_{10} 12 \approx 0.698970 / 1.079181 \approx 0.647685$ .

$$\log_5 x = \frac{\ln x}{\ln 5} \approx \frac{\ln x}{1.609438} \approx 0.621335 \ln x.$$

Next, we convert  $\ln$ .

$$\ln x = \frac{\log_{10} x}{\log_{10} e} \approx \frac{\log_{10} x}{.434294} \approx 2.302585 \log_{10} x.$$

$$\text{Finally, } \ln x = \frac{\log_{12} x}{\log_{12} e} \approx \frac{\log_{12} x}{.207918} \approx 2.484910 \log_{12} x.$$

We calculated  $\log_{12} e$  using conversion  $\log_{12} e = \log_{10} e / \log_{10} 12 \approx 0.434294 / 1.079181 \approx 0.402429$ .

**15.** Starting by replacing the square root by the power of  $1/2$  and applying laws of logarithms we get

$$\begin{aligned} \log_{10} x^{-3} \sqrt{yz^4} &= \log_{10} x^{-3} (yz^4)^{1/2} \\ &= \log_{10} x^{-3} + \log_{10} (yz^4)^{1/2} \\ &= -3 \log_{10} x + \frac{1}{2} \log_{10} (yz^4) \\ &= -3 \log_{10} x + \frac{1}{2} (\log_{10} y + \log_{10} z^4) \\ &= -3 \log_{10} x + \frac{1}{2} (\log_{10} y + 4 \log_{10} z) \\ &= -3 \log_{10} x + \frac{1}{2} \log_{10} y + 2 \log_{10} z \end{aligned}$$

**16.** (a)  $\log_3(1/9) = \log_3(1/3^2) = \log_3 3^{-2} = -2$ .

In the last step we used the cancellation law  $\log_a a^x = x$ .

(b) Applying  $e^{\ln A} = A$  to  $A = e^2$  we get  $e^{\ln(e^2)} = e^2$ . Thus,

$$\ln(e^{\ln(e^2)}) = \ln(e^2) = 2.$$

In the last step we used the cancellation law  $\ln e^x = x$  which is the special case (let  $a = e$ ) of the law we reviewed in (a).

(c)  $e^{-\ln 23} = e^{\ln 23^{-1}} = 23^{-1} = 1/23$ .

**17.** (a) By the cancellation law  $a^{\log_a x} = x$ , we get  $6^{\log_6 14} = 14$ .

(b) By the same cancellation law that we used in (a),  $2^{\log_2 e} = e$ .

(c) By the same cancellation law,  $10^{\log_{10}(x^2+1)} = x^2 + 1$ .

(d) As in (a)-(c),  $e^{\ln x^2} = x^2$ .

(e) Likewise,  $e^{\ln(\ln x)} = \ln x$ .

(f) By the cancellation law  $\log_a a^x = x$ , we get  $\ln(e^{x^2}) = x^2$ .

(g)  $\ln(5e^{x^2}) = \ln 5 + \ln e^{x^2} = \ln 5 + x^2$ , by the same cancellation law that we used in (f).

$$(h) e^{2 \ln x^2} = e^{\ln(x^2)^2} = e^{\ln x^4} = x^4.$$

**18.** (a) Use the fact that  $0.1 = 10^{-1}$  and  $100 = 10^2$ ; we get

$$\begin{aligned} 0.1^x &= 100 \\ (10^{-1})^x &= 10^2 \\ 10^{-x} &= 10^2 \end{aligned}$$

Thus,  $-x = 2$  and  $x = -2$ .

Alternatively, we can use logarithm to any base – we take  $\ln$

$$\begin{aligned} 0.1^x &= 100 \\ \ln 0.1^x &= \ln 100 \\ x \ln 0.1 &= \ln 10^2 \\ x &= \frac{\ln 10^2}{\ln 0.1} = \frac{2 \ln 10}{\ln 10^{-1}} = \frac{2 \ln 10}{(-1) \ln 10} = -2. \end{aligned}$$

(b) We use the same strategy as in (a) – reduce to the same base:

$$\begin{aligned} \left(\frac{1}{4}\right)^x &= 2 \\ \left(\frac{1}{2^2}\right)^x &= 2 \\ (2^{-2})^x &= 2 \\ 2^{-2x} &= 2 = 2^1 \end{aligned}$$

Thus,  $-2x = 1$  and  $x = -1/2$ .

(c) Note that  $0.25 = 1/4$ ; thus

$$\begin{aligned} 0.25^x &= 16 \\ \left(\frac{1}{4}\right)^x &= 16 \\ (4^{-1})^x &= 16 \\ 4^{-x} &= 4^2 \end{aligned}$$

So the solution is  $-x = 2$ , i.e.,  $x = -2$ .

**19.** (a) We use the fact that  $0.1 = 10^{-1}$  to get

$$\begin{aligned} 0.1^{x+2} &= 100^{1/3} \\ (10^{-1})^{x+2} &= (10^2)^{1/3} \\ 10^{-1(x+2)} &= 10^{2/3} \end{aligned}$$

We conclude that  $-x - 2 = 2/3$ , and thus  $x = -2 - 2/3 = -8/3$ .

(b) We use the fact that  $(e^x)^2 = e^{2x}$  (recall that  $(a^m)^n = a^{mn}$ ).

Write  $e^{2x} + 2e^x - 8 = 0$  as  $(e^x)^2 + 2e^x - 8 = 0$  and factor

$$(e^x + 4)(e^x - 2) = 0.$$

If  $e^x + 4 = 0$ , then  $e^x = -4$ , and so there are no solutions (since  $e^x > 0$  for all  $x!$ ). If  $e^x - 2 = 0$ , then  $e^x = 2$ , and so  $x = \ln 2$ .

We conclude that there is one solution,  $x = \ln 2$ .

(c) Rewrite  $100^{\cos x} = 10$  as  $(10^2)^{\cos x} = 10$  i.e.,  $10^{2\cos x} = 10^1$ . Thus,  $2\cos x = 1$  and  $\cos x = 1/2$ .

In Example 11 in Section 7 we solved this equation and got  $x = \frac{\pi}{3} + 2k\pi$  and  $x = \frac{5\pi}{3} + 2k\pi$ .

**20.** Using the conversion formula, we write

$$\log_4 7 = \frac{\ln 7}{\ln 4} \approx \frac{1.945910}{1.386294} \approx 1.403677.$$

### Section 9. Mathematical Language; Mathematical Thinking and Logic

1. Recall that, first, we need to identify two parts: what is taken to be true (i.e., assumption(s)), and what is claimed to be true (i.e., conclusion(s)); then we write it in the form

If  $\langle \text{assumption(s)} \rangle$  then  $\langle \text{conclusion(s)} \rangle$ .

(a) The statement is about a square of an even number (that's what's assumed); the claim is that such number is even. Thus, we write

If a number is even, then its square is even.

(b) The statement is about increasing functions, and it claims that those functions have inverses. Thus

If a function is increasing, then it has an inverse function.

(c) Likewise,

If  $x$  is a real number, then  $|x| \geq 0$ .

Note that the meaning of the indefinite article “a” is not “one” or “some” but “any” or “all” (see the very last note in this section in the book).

(d) The statement is about how the equality between two numbers  $a$  and  $b$  (assumption) affects the given equation (conclusion). Thus,

If  $a = b$ , then the equation  $x^2/a^2 + y^2/b^2 = 1$  represents a circle.

2. A is what is assumed, and B is what is claimed to follow from A. Recall that the converse of “if A, then B” is the statement “if B, then A”.

(a) From

If  $x$  is positive, then  $e^x$  is positive.

we identify A=“ $x$  is positive”, B=“ $e^x$  is positive”.

This statement is true, because exponential function  $a^x$  for any  $a > 0$  and any real number  $x$  is positive (or, look at the graph of  $e^x$ !).

The converse reads

If  $e^x$  is positive, then  $x$  is positive.

This is a false statement; to prove it, we need to find a counterexample. Take  $x = -1$ . Then  $e^x = e^{-1}$  is positive, but  $x = -1$  is not positive.

(b) We see that A=“ $a = b$ ”; B=“ $a^2 = b^2$ ”. This statement is true: if two numbers are equal, then their squares must be equal as well.

The converse statement is

If  $a^2 = b^2$ , then  $a = b$ .

This is not true: numbers  $a = 5$  and  $b = -5$  squared give the same number (25), but are not equal.

(c) A=“ $m_1 m_2 = -1$ ”; B=“the lines  $y = m_1 x$  and  $y = m_2 x$  are perpendicular”.

This is a true statement, see Section 5 (note that we did not prove it, just used it).

The converse reads

If the lines  $y = m_1 x$  and  $y = m_2 x$  are perpendicular, then  $m_1 m_2 = -1$

Again, this is a true statement, as explained (but not proved) in Section 5.

**3.** Recall that the contrapositive of “if A, then B” is “if *not* B, then *not* A.”

(a) From

If  $\sin x \neq 0$ , then  $\csc x$  is defined

we see that  $A = “\sin x \neq 0”$ ;  $B = “\csc x$  is defined”.

Thus, *not* A = “ $\sin x = 0$ ” and *not* B = “ $\csc x$  is not defined”.

The contrapositive is

If  $\csc x$  is not defined, then  $\sin x = 0$ .

(b) Reading the statement

If  $x_1 \neq x_2$ , then  $f(x_1) \neq f(x_2)$

we see that  $A = “x_1 \neq x_2”$ ;  $B = “f(x_1) \neq f(x_2)”$ .

Thus, *not* A = “ $x_1 = x_2$ ” and *not* B = “ $f(x_1) = f(x_2)”$ .

The contrapositive is

If  $f(x_1) = f(x_2)$ , then  $x_1 = x_2$ .

(c) We see that  $A = “\text{curve is a straight line}”$ ;  $B = “\text{same slope at all points}”$ .

Thus, *not* A = “curve is not a straight line” and *not* B = “not same slope at all points”.

The contrapositive is

If a curve does not have the same slope at all points, then it is not a line.

**4.** (a) We start by experimenting: we take a prime number and check whether that number plus two is also a prime.

If  $p = 3$ , Then  $p + 2 = 5$ , which is a prime. If  $p = 5$ , Then  $p + 2 = 7$ , which is a prime. If  $p = 7$ , Then  $p + 2 = 9$ , which is not a prime!

Thus, we found a counterexample: a prime number which does not yield a prime number when we add 2 to it. Consequently, the given statement is false.

Note that by examining cases we could never prove that some statement implicating all prime numbers is true (since we would have to check that it holds for all infinitely many prime numbers). However, to prove that a statement involving universal quantifier is not true, all we need is one example where it does not work (that’s called a counterexample).

(b) If we can find one example, i.e., one prime number for which the statement holds, we will prove that it is true.

We have already found two examples in part (a). As well,  $p = 11$  is prime, and so is  $p + 2 = 13$ .

(c) The statement is true, by the definition of the square root. (Note that examining numbers one by one would not work.)

(d) Recall that if we wish to prove that this statement is true, we have to show that it holds for at least one real number. To disprove it, we must show that it does not hold for *any one* real number.

We see that the discriminant  $D = b^2 - 4ac = 3^2 - 4(7) = -19$  is negative, and so the equation  $x^2 + 3x + 7 = 0$  has no real number solutions.



This shows that *no* real number satisfies  $x^2 + 3x + 7 = 0$ . In other words, we proved that the statement

There is a real number  $x$  such that  $x^2 + 3x + 7 = 0$ .

is false.