Math@Mac Online Mathematics Competition

Wednesday, November 27, 2013

SOLUTIONS

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1. The number of pairs (m, n) of positive integers such that $m^4 + n = 100000001$ is

   (A) 150

   (B) 125

   (C) 100

   (D) 75

Answer: 100 (71% correct)

Solution:

Write $n = 100000001 - m^4$, and note that $100000001 = 100^4 + 1$, and $n = 100^4 + 1 - m^4$. Now m can be any number for which the right side is positive, i.e., $m = 1, 2, 3 \ldots, 100$. We conclude that are 100 pairs (m, n) with $m, n > 0$ which solve the given equation.

2. If $x + y + z = 0$, then

$$\left(\frac{y-z}{x} + \frac{z-x}{y} + \frac{x-y}{z}\right)\left(\frac{x}{y-z} + \frac{y}{z-x} + \frac{z}{x-y}\right)$$

is equal to

(A) 9

(B) 0

(C) 4

(D) 12

Answer: 9 (19% **correct)**

Solution:

Short way: All answers are constants, which means that no matter what x, y and z are used (as long as they satisfy $x + y + z = 0$, and the denominators are not zero) the given expression has to have the same value. Substituting $x = 1$, $y = 2$ and $z = -3$ into

$$\left(\frac{y-z}{x} + \frac{z-x}{y} + \frac{x-y}{z}\right)\left(\frac{x}{y-z} + \frac{y}{z-x} + \frac{z}{x-y}\right)$$

we obtain 9. A long way to prove this consists of multiplying out the terms in the parentheses and simplifying.

Note: Our apologies - there was a typo in the question, which made the expression non-symmetric. The first term in the second factor was incorrectly typed as $\frac{x}{y-x}$. It should have been $\frac{x}{y-z}$ (as it is now typed above). With the question as stated, none of the answers offered work. We are glad to learn that many groups caught the error and fixed it.

3. Serena and Venus have been playing table tennis. To win the game, a player must have 11 or more points, AND must also have two more points than the other player. Right now, Serena has 13 points and Venus has 12. At any time, each player has a 50% chance of scoring the next point. What is the probability that Serena will win the game?

(A) 1/2

(B) 2/3

(C) 3/4

(D) 4/5

Answer: 3/4 (34% correct)

Solution:

With a probability of 1/2, Serena will score the next point and win. The other 1/2 of the time, Venus scores the next point and the two players are tied. Once tied, the situation is symmetric, and each player wins the game with a probability of 1/2. Thus, Serena's probability of winning is $1/2 + (1/2)(1/2) = 3/4$.

4. How many non-negative integers n are there such that $n+2$ divides $(n+18)^2$?

(A) 1

(B) 3

(C) 8

(D) more than 20

Answer: 8 (29% correct)

Solution:

Using long division, we show that $(n+18)^2 = (n+2)(n+34)+256$. In order for $n+2$ to divide $(n+18)^2$, it must divide 256. The divisors of 256 are $1, 2, 2^2, 2^3, \ldots,$ and $2^8 = 256$. So $n+2 = 1, 2, 2^2, 2^3, \ldots 2^8$ and $n = -1, 0, 2, 6, \ldots 254$. Since n has to be non-negative, we count 8 solutions.

5. If 2^{2013} has m digits and 5^{2013} has n digits, then $m+n$ is

(A) 2012

(B) 2013

(C) 2014

(D) 2015

Answer: 2014 (37% correct)

Solution:

If a number n has a digits, then $10^{a-1} \leq n < 10^a$. Thus, $10^{m-1} \leq 2^{2013} < 10^m$ and $10^{n-1} \leq 5^{2013} < 10^n$. Moreover, all inequalities are sharp, since neither of the two numbers 2^{2013} or 5^{2013} is a power of 10. Multiplying the two inequalities, we obtain

$$10^{m+n-2} < 10^{2013} < 10^{m+n}.$$

Thus, $2013 = m+n-1$, i.e., $m+n = 2014$.

6. The pages of a book are 20cm tall and 10cm wide. If a page is folded appropriately, a corner of the page can stick out above the top of the book. What is the maximum amount that a page can protrude above the top of the book without tearing the page or separating it from the binding?

(A) $5\sqrt{2}$

(B) $\sqrt{10} - \sqrt{5}$

(C) $\sqrt{20} - \sqrt{10}$

(D) $10(\sqrt{5} - 2)$

Answer: $10(\sqrt{5} - 2)$ **(34% correct)**

Solution:

If the page is bound at the left edge, even the highest point after folding must be connected to the bound bottom left corner. The point farthest away from the the bottom left corner is the opposite upper right corner, at a distance of

$$\sqrt{20^2 + 10^2} = \sqrt{500} = 10\sqrt{5}.$$

Draw a diagonal line (from the bottom left corner to the upper right corner) and consider that a radius. Fold the paper repeatedly so that this radius points straight up, protruding a distance of $10\sqrt{5} - 20$ above the book.

7. A student rolled a fair die until the sum of her rolls was a prime number. She rolled three times until this occurred. What is the probability that her last roll was a 6?

(A) 0

(B) 1/24

(C) 2/24

(D) 12/73

Answer: 0 (18% **correct)**

Solution:

The possible prime numbers one can get after exactly three rolls, and with the third roll being a 6 are 11, 13 and 17 (since no larger prime can be obtained in three rolls, and the smaller primes cannot be written as a sum of 6 along with two positive integers). We see that if the sum were 11, then the sum of the first two rolls must have been 5, which is a prime. Similarly, if the totals were 13 or 17, the sum of the first two rolls would have been 7 or 11 respectively, both prime numbers. Thus, the event described in this question cannot occur, and the probability is zero.

8. The last three digits of the number $625^{376} + 376^{625}$ are

(A) 001

(B) 011

(C) 021

(D) 111

Answer: 001 (65% **correct)**

Solution:

We note that $625^2 = 390625$, so it ends with 625. In that case,

$$625^3 = 625^2 \cdot 625 = (390000 + 625) \cdot 625 = m + 625^2,$$

where the last three digits of m are 000. Thus, the last three digits of 625^3 are 625. We conclude that any positive integer power of 625 has 625 as the last three digits. Likewise, $376^2 = 141376$; by the same argument, we realize that any power of 376 has 376 as the last three digits. Thus, the last three digits of $625^{376} + 376^{625}$ are the last three digits of $625 + 376$, i.e., 001.

9. Take a unit square and draw a circle of radius 1 centred at each of its four vertices. What is the total area of the shaded regions?

(A) $4 - 2\sqrt{3} - \frac{\pi}{3}$

(B) $2 - 2\sqrt{3} - \frac{2\pi}{3}$

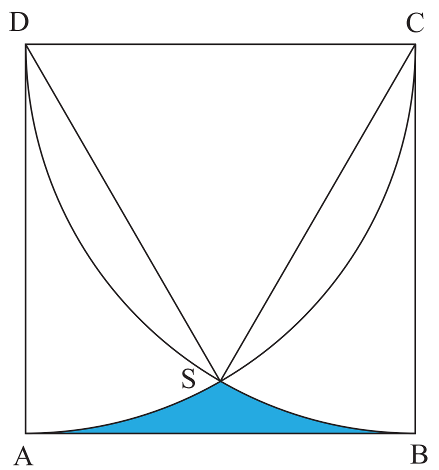
(C) $8 - \sqrt{3} - \frac{\pi}{3}$

(D) $4 - \sqrt{3} - \frac{2\pi}{3}$

Answer: $4 - \sqrt{3} - \frac{2\pi}{3}$ **(**44% **correct)**

Solution:

By S we label the intersection of the circles centred at C and D. Since all of DS, SC and CD are the radii (and thus equal to 1), the triangle ΔDSC is equilateral, and its area is $\sqrt{3}/4$. Because the angle $\angle$SDC=60^{O} it follows that $\angle$ADS=30^{O} and the area of the circle sector ADS is $\pi/12$. Thus, the area of the shaded region ABS is $1 - \sqrt{3}/4 - 2(\pi/12)$. We obtain the final answer by multiplying by 4.



10. Assume that a polynomial $P(x)$ satisfies $(x+1)P(x) = (x-2)P(x+1)$ and $P(3) = 12$. Then the coefficient of x^3 in $P(x)$ is

(A) 0

(B) 1

(C) 2

(D) 3

Answer: 2 (33% correct)

Solution:

Substitute $x = 2$ into the given equation to obtain $3P(2) = 0$, i.e., $P(2) = 0$. When $x = -1$, the same formula implies that $P(0) = 0$, and when $x = 1$, we obtain $2P(1) = -P(2)$ and $P(1) = 0$. We conclude that $x = 0$, $x = 1$ and $x = 2$ are the roots of $P(x)$, and write $P(x) = x(x-1)(x-2)Q(x)$, for some polynomial $Q(x)$. Substituting $P(x)$ into the given equation, we obtain

$$(x+1)x(x-1)(x-2)Q(x) = (x-2)(x+1)x(x-1)Q(x+1)$$

and thus $Q(x) = Q(x+1)$. This means that $Q(x)$ is a constant polynomial, i.e., $Q(x) = c$, and in that case, $P(x) = cx(x-1)(x-2)$. From $P(3) = 12$ we compute that $c = 2$. The coefficient of x^3 is 2.

The End