Math@Mac Online Mathematics Competition

Wednesday, November 26, 2014

SOLUTIONS

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1. Two towns, A and B, are located on the Croatian coast. At sunrise, Jakov begins walking south from A to B along the coast, while simultaneously Diana begins walking north from B to A. Each person walks at a constant speed, and they cross paths at noon. Jakov arrives in B at 5 pm while Diana reaches A at 11:15 pm. When was the sunrise?

(A) 4:30 am

(B) 5:30 am

(C) 5:45 am

(D) 6:15 am

Answer: 4:30 am

Solution:

Let t represent the elapsed time between sunrise and the moment that Jakov and Diana crossed paths at noon. It took Diana 11.25 hours to traverse the distance Jakov covered in t hours, and it took Jakov 5 hours to traverse the distance Diana covered in t hours. Since each person walks at a constant speed, the ratios of their times are the same, i.e.,

$$\frac{11.25}{t} = \frac{t}{5}$$

Cross-multiplying, we obtain $t^2 = 56.25$ and since $t > 0$, we have $t = 7.5$. Therefore, Jakov and Diana had been walking for 7.5 hours when they met at noon, which means that they left at 4:30 am (sunrise).

2. There are three positive integers, A, B, and C. The following operations are performed:

- Begin with the value of A
- Square it
- Multiply the value you obtained by 35
- Add to that the value of 5 times B
- Add to that the value of C squared

Which one of the following numbers could be the result of such a calculation?

(A) 664502

(B) 664503

(C) 664508

(D) 664509

Answer: 664509

Solution:

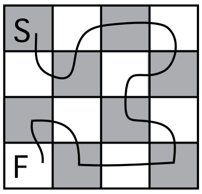
We will figure out what the last digit of the resulting number could be.

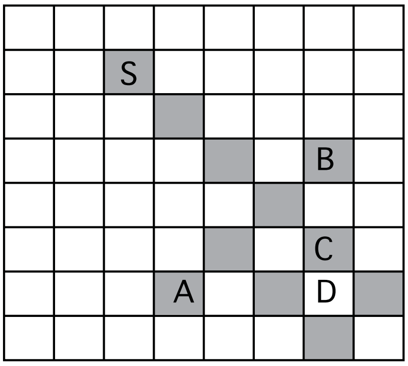
The square of any integer must end with one of 0, 1, 4, 9, 6, or 5. Thus, the last digit of A^2 is 0, 1, 4, 9, 6, or 5. The last digit of $35A^2$ is 0 or 5 (since multiplying each of 0, 1, 4, 9, 6, or 5 by 5 produces 0 or 5 as the last digit).

The last digit of 5B is 0 or 5, and so the last digit of $5B + 35A^2$ is 0 or 5. Since the last digit of C^2 is 0, 1, 4, 9, 6, or 5, the last digit of $5B + 35A^2 + C^2$ is (add 0 or 5 to each of 0, 1, 4, 9, 6, or 5) 0, 1, 4, 9, 6, or 5.

Therefore, only 664509 is a possible result (created with A=137, B=5, C=87).

3. In the 4x4 grid below, a path starts at S and proceeds through adjacent squares (horizontally or vertically, not diagonally) visiting each square exactly once and finishing at F. Consider the 100x100 grid whose top left corner is pictured below. If one were to create such a path starting at S, only one of the lettered squares, A, B, C, or D, could be the end of such a path through adjacent squares visiting each of the 10,000 squares exactly once. Which lettered square could it be?





(A) square A

(B) square B

(C) square C

(D) square D

Answer: square D

Solution:

Consider colouring the squares black and white like a checkerboard. The path must pass through squares of alternating colours: black, white, black, white, and so on. Because there is an even number of squares (10,000), half must be white and half must be black. If S is a black square, then the final square must be white. The only square which is white is D. The other 3 squares are black, and thus cannot be the end of a path that passes through each square exactly once.

4. Let $M = 124567891011121415\ldots282940414244\ldots99929994999599969997999899999$ (i.e., M is obtained by writing all numbers from 1 to 9999 that do not contain digit 3). Which statement is true for M?

(A) M has 25,422 digits and is divisible by 3

(B) M has 25,422 digits and is not divisible by 3

(C) M has 25,424 digits and is not divisible by 3

(D) M has 25,424 digits and is divisible by 3

Answer: M has 25,424 digits and is divisible by 3

Solution:

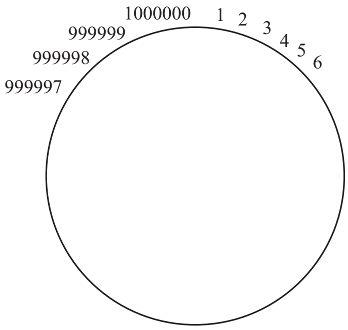
We count the digits first: M starts with eight one-digit numbers, so that's 8 digits. Think of a two-digit number as AB: there are 8 possibilities for A (cannot be 0 or 3) and 9 possibilities for B (cannot be 3). Thus, there are $8 \cdot 9 = 72$ two-digit numbers in M, which contribute $72 \cdot 2 = 144$ digits.
Continue in the same way: in a three digit number ABC there are 8 possibilities for A (cannot be 0 or 3) and 9 possibilities for each of B and C (cannot be 3). Thus, there are $8 \cdot 9 \cdot 9 = 648$ three-digit numbers in M, which contribute $648 \cdot 3 = 1944$ digits. Likewise, there are $8 \cdot 9 \cdot 9 \cdot 9 = 5832$ four-digit numbers in M, which contribute $5832 \cdot 4 = 23328$ digits. So, M has $8 + 144 + 1944 + 23328 = 25424$ digits.

Next, we compute the sum of all digits in M. Consider the sequence of all numbers from 0 to 9999, written as 0000, 0001, 0002, 0003, ..., 0010, 0011, 0012, ... ,9997, 9998, 9999. In this sequence of 10,000 numbers there are 40,000 digits, and so each digit appears 4,000 times. So, the sum of the digits in M is

$$4000 \cdot 1 + 4000 \cdot 2 + 4000 \cdot 3 + \cdots + 4000 \cdot 9 = 4000(42) = 168,000$$

Since $1 + 6 + 8 = 15$ is divisible by 3 but not by 9, M is divisible by 3 and not by 9.

5. Positive integers from 1 to 1000000 are arranged around a circle, as shown below. Start at number 1, and as you move clockwise, cross every fifteenth number (1, 16, 31, 46, and so on). Keep moving around the circle, crossing every fifteenth number until you realize that no new numbers are crossed. The numbers that you crossed are counted again - so if you cross number 111, then the next number you cross is 126, even though some numbers between 111 and 126 might already be crossed. How many numbers are left uncrossed?



(A) 520,000

(B) 660,000

(C) 800,000

(D) 824,000

Answer: 800,000

Solution:

Experiment with a smaller number, say 100. We cross 1, 16, 31, 46, 61, 76, 91, and then (91+51=106) 6, then 21, 36, 51, 66, 81, 96, then (96+15=111) 11, 26, 41, 56, 71, 86, then (86+15=101) 1. So we crossed 20 numbers. Looking at the pattern, we see that we crossed all numbers whose remainder, when divided by 5, is equal to 1. Why 5? Because it is the greatest common divisor (GCD) of 100 and 15.

Since the GCD of 1,000,000 and 15 is 5, we will cross all numbers whose remainder, when divided by 5, is 1. Since there are 5 possible remainders when a number is divided by 5, there are 1,000,000/5=200,000 numbers whose remainder is 1 (and these numbers are crossed). So, there are 800,000 numbers which are not crossed.

6. What is the probability that in drawing three cards, without replacement, from a deck of 52 cards that one will draw a spade, a queen, and a diamond in that order?

(A) 1/510

(B) 13/2550

(C) 1/208

(D) 1/204

Answer: 1/204

Solution:

The following four disjoint events cover all possibilities (disjoint means that the events have nothing in common):

1) the first card is the queen of spades and the second is the queen of diamonds
2) the first card is the queen of spades and the second is the queen of hearts or clubs
3) the first card is a spade, but not the queen and the second card is the queen of diamonds
4) the first card is a spade, but not the queen and the second card is the queen of hearts, spades, or clubs.

The probability for each case is:

1) $(1/52)(1/51)(12/50) = 12/132600$
2) $(1/52)(2/51)(13/50) = 26/132600$
3) $(12/52)(1/51)(12/50) = 144/132600$
4) $(12/52)(3/51)(13/50) = 468/132600$

In total, the probability is $650/132600 = 1/204$. (Because the four events above are disjoint, we add the probabilities.)

7. Consider two concentric circles with radii x and 1 where $x < 1$. In the annulus, the region between the concentric circles, 6 circles are constructed in the following way. Each of these circles is tangent to the inner circle, the outer circle, and the two adjacent circles. For which value of x is this construction possible?

(A) 1/6

(B) 1/4

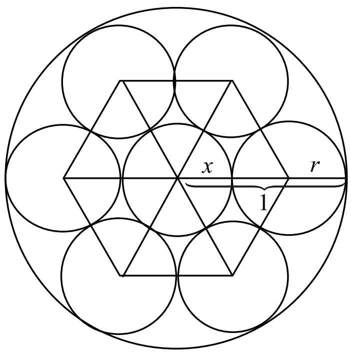
(C) 1/3

(D) 1/2

Answer: 1/3

Solution:

Connect the centers of the six circles which are in the annulus, thus obtaining a regular hexagon. Connect these six centres to the common centre of the circles or radii 1 and x which form the annulus.



This way we obtained six equilateral triangles. The sides of each triangle are $x + r$, $x + r$, and $2r$. Thus $x + r = 2r$ and $x = r$. Since $x + 2r = 1$, it follows that $3x = 1$ and $x = 1/3$.

8. Consider the following table of numbers. Though not obvious, there is a definite pattern.

1	3	7	3	11	13	13	19	19	55	
2	5	2	7	11	8	17	17	34	29	
3	1	5	9	5	13	15	21	23	21	

What are the numbers in the last column (from top to bottom) that complete the pattern?

(A) 33, 21, 75

(B) 21, 37, 39

(C) 31, 23, 89

(D) 37, 47, 59

Answer: 31, 23, 89

Solution:

There are three separate diagonal sequences. The first one is the Fibonacci sequence. It starts with the 1 in the first column, then goes to the 1 at the bottom of the next column and proceeds diagonally. This creates the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55; therefore, the bottom number in the last column must be $34 + 55 = 89$.

1	3	7	3	11	13	13	19	19	55	
2	5	2	7	11	8	17	17	34	29	
3	1	5	9	5	13	15	21	23	21	89

The second sequence, starting with 2 in the first column, is a sequence of primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29; therefore, the top number in the last column must be 31.

1	3	7	3	11	13	13	19	19	55	31
2	5	2	7	11	8	17	17	34	29	
3	1	5	9	5	13	15	21	23	21	

The third sequence, starting with 3 in the first column, is a sequence of odd numbers: 3, 5, 7, 9, 11, 13, 15, 17, 19, 21; therefore, the middle number in the last column must be 23.

1	3	7	3	11	13	13	19	19	55	
2	5	2	7	11	8	17	17	34	29	23
3	1	5	9	5	13	15	21	23	21	

9. In a convex pentagon of perimeter 10, each diagonal is parallel to one of the sides. Find the sum of the lengths of the diagonals.

(A) $5\left(1+\sqrt{5}\right)$

(B) $\frac{5}{2}\left(1+\sqrt{5}\right)$

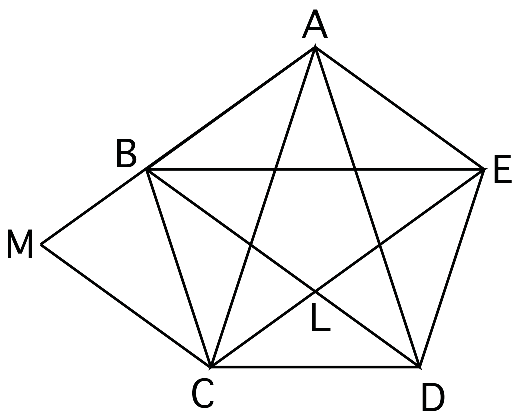
(C) $\frac{5}{2}\left(2\sqrt{5}-1\right)$

(D) $5\left(2\sqrt{5}-1\right)$

Answer: $5\left(1+\sqrt{5}\right)$

Solution:

Denote by L the point of intersection of EC and DB. Extend the line segment ab to the point M so that MC is parallel to AE. By assumption, CE (diagonal) is parallel to AB (side), and thus, by construction, AECM is a parallelogram. Again, CE (diagonal) is parallel to AB (side) and since BD (diagonal) is parallel to AE (side), we conclude that AELB is a parallelogram.



Note that the triangles CLD and BAE are similar (corresponding angles are equal, due to the assumption on the parallelism of diagonals and sides). For the same reason, the triangles CMA and DLE are similar. Thus

$$\frac{\mathrm{EC}}{\mathrm{AB}}=\frac{\mathrm{EL}+\mathrm{LC}}{\mathrm{AB}}=1+\frac{\mathrm{LC}}{\mathrm{AB}}=1+\frac{\mathrm{DL}}{\mathrm{EA}}=1+\frac{\mathrm{DL}}{\mathrm{MC}}=1+\frac{\mathrm{AB}}{\mathrm{EC}}$$

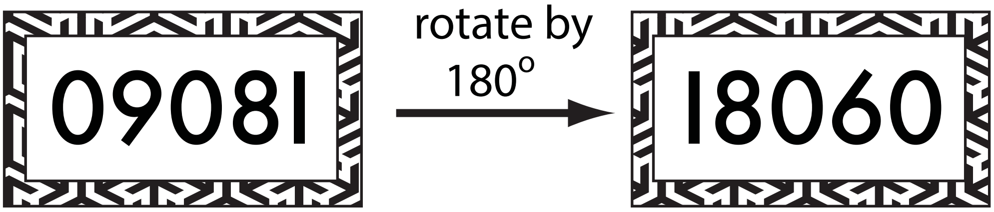
The last equals sign is true for the following reason. Note that $\frac{\mathrm{AB}}{\mathrm{EC}} = \frac{\mathrm{LE}}{\mathrm{AM}}$ because the triangles CMA and DLE are similar. Because AECM is a parallelogram, LE = AB and AM = CE and thus $\frac{\mathrm{LE}}{\mathrm{AM}} = \frac{\mathrm{AB}}{\mathrm{EC}}$.

Let $x = \frac{\mathrm{EC}}{\mathrm{AB}}$. Then the above equation implies that $x = 1 + \frac{1}{x}$, $x^2 - x - 1 = 0$ and (since $x > 0$, we obtain $x = \frac{1}{2}\left(1 + \sqrt{5}\right)$. Thus, we proved that CE $= x$AB.

Using analogous arguments, we obtain AD $= x$BC, BE $= x$CD, AC $= x$DE, and BD $= x$EA. Adding up the five equalities, we obtain

$$\mathrm{CE} + \mathrm{AD} + \mathrm{BE} + \mathrm{AC} + \mathrm{BD} = x\,(\mathrm{AB} + \mathrm{BC} + \mathrm{CD} + \mathrm{DE} + \mathrm{EA}) = 10x = 5\left(1 + \sqrt{5}\right)$$

10. A total of 100,000 raffle tickets have been printed for a fundraiser, each one with a distinct 5-digit number ranging from 00000 to 99999. But some tickets are ambiguous. An ambiguous ticket is defined to be one that shows two different numbers depending on the ticket's orientation. For example, the ticket below is ambiguous since it could be either 09081 or 18060, depending on its orientation. On the other hand, the ticket number 80008 is not ambiguous.



How many of the 100,000 tickets are ambiguous? Ticket number 09081 and its corresponding rotated ticket 18060 count as two ambiguous tickets.

Note: Digits used on the tickets appear as shown:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

(A) less than 2800

(B) between 2800 and 3000

(C) between 3001 and 3200

(D) more than 3200

Answer: 3050

Solution:

The ambiguous ticket numbers must include only the numbers that represent some number when rotated by 180 degrees, i.e., 0, 1, 6, 8, and 9.

Denote a ticket number by ABCDE. We know that there are only 5 choices for A, and these are the same for choices for B, C, D, and E as well. Therefore, there are $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 5^5 = 3125$ ticket numbers, which are also ticket numbers when the ticket is rotated by 180 degrees. Some of these numbers, such as 09081, constitute an ambiguous ticket, since when rotated, the ticket number is different–it is 18060. However, ticket number 80008 is not ambiguous, since, after rotation, we again obtain 80008.

So, exactly how many of the 3125 ticket numbers are not ambiguous? There are 5 possible choices for A; and once A is picked, E is uniquely determined (if A=0 then E=0; if A=1 then E=1; if A=6 then E=9; if A=8 then E=8; if A=9 then E=6). Likewise, there are 5 possible choices for B, and once B is picked, D is uniquely determined. There are 3 possible choices for C (since the digit for C must look the same as its rotated image; so C can be 0, 1, or 8). Therefore, $5 \cdot 5 \cdot 3 \cdot 1 \cdot 1 = 75$ of the 3125 are not ambiguous, which leaves us with $3125 - 75 = 3050$ ambiguous tickets.

The End