Math@Mac Online Mathematics Competition

Wednesday, November 30, 2016

SOLUTIONS

---

1. Suppose that a bag contains the nine letters of the word AOXOMOXOA. If you take one letter out of the bag at a time and line them up left to right, what is the probability that you will spell the word AOXOMOXOA?

(A) between 0.01 and 0.1

(B) between 0.001 and 0.01

(C) between 0.0001 and 0.001

(D) between 0.00001 and 0.0001

Answer: (C) between 0.0001 **and** 0.001

Solution: Note that we can consider the placement of two As, two Xs and one M only, as those will uniquely determine where the four Os are placed. The word is nine letters long. We can place a pair of As in two locations in $\binom{9}{2}$ ways. We can place the two Xs somewhere in the remaining 7 locations in $\binom{7}{2}$ ways. Finally we can place M in any of the remaining 5 locations. Thus, there are $\binom{9}{2} \cdot \binom{9}{2} \cdot 5 = 3780$ possible words we can make. Of those, only 1 will spell AOXOMOXOA, thus the chance is $\frac{1}{3780} \approx 0.00026$.

Note that we can count in different ways: we can place M in any of 9 locations. After M is placed somewhere, a pair of As can be placed in remaining 8 locations in $\binom{8}{2}$ ways and a pair of Xs in the remaining 6 locations in $\binom{6}{2}$ ways. Again, $9 \cdot \binom{8}{2} \cdot \binom{6}{2} = 3780$.

2. A motorist travels the first 10 kilometers of a trip at 30km/hour. How fast would he have to drive for the next 10 kilometers if the total trip has an average speed of 50 km/hour?

(A) 70 km/h

(B) 80 km/h

(C) 110 km/h

(D) 150 km/h

Answer: (D) 150 km/h

Solution: Let x be the speed over the second 10km. Then the total time for the 20 km trip is

$$\frac{10}{30} + \frac{10}{x} = \frac{1}{3} + \frac{10}{x}$$

hours.

For the average speed to be 50, we must have that

$$\frac{20}{\frac{1}{3} + \frac{10}{x}} = 50$$

Solving for x, we obtain $x = 150$.

3. For how many integers $n \geq 1$ does the expression $3^{2n+1} - 4^{n+1} + 6^n$ yield a prime number?

(A) 1

(B) 2

(C) 4

(D) infinitely many

Answer: (A) 1

Solution: The idea is to factor:

$$\begin{aligned}
3^{2n+1} - 4^{n+1} + 6^n &= 3 \cdot 3^{2n} - 4 \cdot 2^{2n} + 2^n \cdot 3^n \\
&= 3 \cdot 3^{2n} - 3 \cdot 2^{2n} - 2^{2n} + 2^n \cdot 3^n \\
&= 3\left(3^{2n} - 2^{2n}\right) + 2^n\left(3^n - 2^n\right) \\
&= 3\left(3^n - 2^n\right)\left(3^n + 2^n\right) + 2^n\left(3^n - 2^n\right) \\
&= \left(3^n - 2^n\right)\left(3^{n+1} + 2^{n+2}\right)
\end{aligned}$$

For the expression $3^{2n+1} - 4^{n+1} + 6^n$ to be prime, one of the two factors must be 1. Since $3^{n+1} + 2^{n+2} > 1$ for all $n \geq 1$, we conclude that $3^n - 2^n = 1$.

If $n \geq 2$, then $3^n - 2^n > 1$, and if $n = 1$, then $3^n - 2^n = 1$.

Thus, $n = 1$ is the only solution; i.e., $3^{2n+1} - 4^{n+1} + 6^n$ is prime only when $n = 1$ (in which case it is equal to 17).

4. A set C of positive integers is called *cool* if any two numbers in C are relatively prime. Bob wants to build a cool set from numbers between 1 and 30 (inclusive), in such a way that his set contains as many numbers as possible. How many different cool sets can he build?

(A) 12

(B) 16

(C) 24

(D) 30

Answer: (C) 24

Solution: Note that a largest cool set must contain the number 1 (if not, we can add 1 to obtain a larger cool set). As well, it cannot contain a number which is the product of two prime numbers, as otherwise we can replace that number with the two primes and obtain a larger set. Thus, every number in a cool set must be a prime, or a power of a prime.

In other words, every cool set that Bob can build must be of the form

$$\{1, 2^{n_1}, 3^{n_2}, 5^{n_3}, 7^{n_4}, 11^{n_5}, 13^{n_6}, 17^{n_7}, 19^{n_8}\}$$

where $n_i \geq 1$.

Since all numbers in C must be smaller than or equal to 30, we conclude that

$n_1 = 1, 2, 3, 4$
$n_2 = 1, 2, 3$
$n_3 = 1, 2$

and

$n_4 = n_5 = \cdots = n_8 = 1$.

Thus, there is a total of $4 \cdot 3 \cdot 2 = 24$ cool sets.

5. Carly plots a point A, and then starts drawing rays starting at A, so that all angles she gets (i.e., between any two rays) are integer multiples of 10^{o}. What is the largest number of rays she can draw so that all the angles at A between *any two* rays (not just adjacent rays) are distinct?

(A) 5

(B) 6

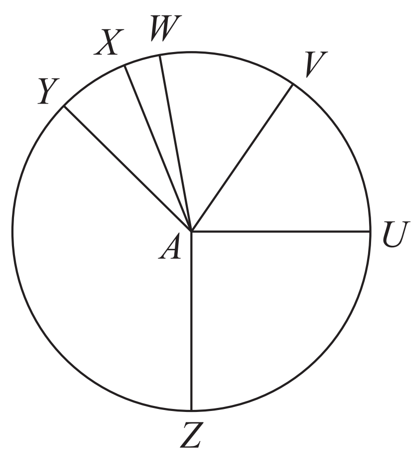
(C) 7

(D) 8

Answer: (B) 6

Solution: Denote by $n \geq 2$ the number of rays drawn by Carly. Then there are $\binom{n}{2} = \frac{n(n-1)}{2}$ pairs of rays, and each pair of rays determines two angles which add up to 360^{o}. Hence the total number of angles between all pairs of the n rays is exactly $n(n-1)$. Each of these angles is smaller than 360^{o}. Since all angles are supposed to be integer multiples of 10^{o}, there are at most 35 values for the measures of these angles. Since they are distinct, $n(n-1) \leq 35$. If $n = 6$, then $n(n-1) = 30$, and if $n = 7$, then $n(n-1) = 42$. Thus, $n \leq 6$.

It is possible to draw 6 rays, determining 30 distinct angles. For instance, $\angle UAV = 60^{\text{o}}$, $\angle VAW = 40^{\text{o}}$, $\angle WAX = 10^{\text{o}}$, $\angle XAY = 20^{\text{o}}$, $\angle YAZ = 140^{\text{o}}$, $\angle ZAU = 90^{\text{o}}$.



Next, we find $\angle WAY = 30^{\text{o}}$, $\angle VAX = 50^{\text{o}}$, $\angle VAY = 70^{\text{o}}$, $\angle UAW = 100^{\text{o}}$, $\angle UAX = 110^{\text{o}}$, $\angle UAY = 130^{\text{o}}$, $\angle ZAV = 150^{\text{o}}$, $\angle XAZ = 160^{\text{o}}$, $\angle WAZ = 170^{\text{o}}$.

All of these angles are distinct, and smaller than 180^{o}. Corresponding to these 15 angles, there are 15 angles greater than 180^{o} (all distinct!), yielding a total of 30 distinct angles.

6. The diagonals of square $ABCD$ meet at the point O. The bisector of the angle OAB meets the segment BO at N, and meets the segment BC at P. The length of NO is x. What is the exact length of PC?

(A) $x\left(\sqrt{2}+\frac{1}{2}\right)$

(B) $2x$

(C) $x\sqrt{5}$

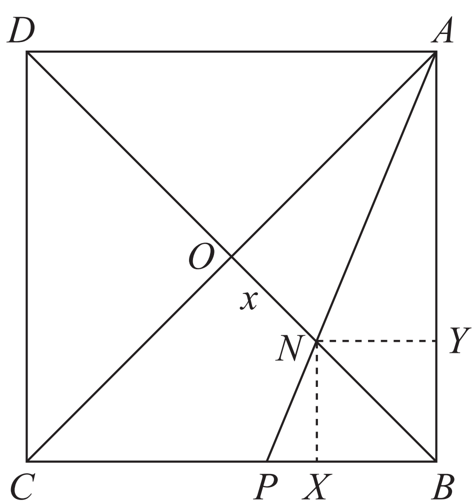
(D) $x\left(\frac{5}{4}+\frac{\sqrt{2}}{2}\right)$

Answer: (B) 2x

Solution: Look at the triangle ABC: the segments $\overline{OB}$ and $\overline{AP}$ are bisectors of the angles $\angle ABC$ and $\angle BAC$, respectively. Thus, the point N is the incentre of the triangle ABC, and hence equidistant from all three sides.

We conclude that $NX = NY = NO = x$.

Since BN is the diagonal of the square $NXBY$, $BN = x\sqrt{2}$.



Triangles APC and ABN are similar because their angles are equal. Thus, $AB/AC = BN/PC$. Let $AB = a$. then $AC = a\sqrt{2}$, and $AB/AC = BN/PC$ implies

$$\frac{a}{a\sqrt{2}} = \frac{x\sqrt{2}}{PC},$$

and so $PC = x\sqrt{2}\sqrt{2} = 2x$.

7. Alice walks two-thirds across a railroad bridge from point A to point B when she sees a train approaching at 45 km/h. She does a very quick calculation and realizes that if she runs at a certain speed r, she can make it to *either* end of the bridge and avoid the train. What is the *smallest* value of r, i.e., what is the slowest speed at which she can do it?

(A) 11 km/h

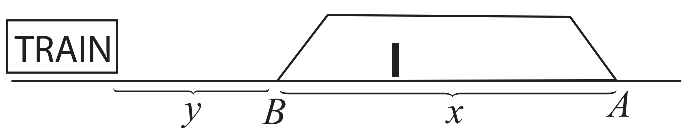
(B) 12 km/h

(C) 15 km/h

(D) 16 km/h

Answer: (C) 15 km/h

Solution: By x we denote the length of the bridge, and by y the distance between the train and the bridge. If Alice runs to the point A, then $45 = \frac{x+y}{t_1}$ and $r = \frac{2x/3}{t_1}$, where t_1 is the time for Alice and the train to reach A.



If Alice runs to B, then $45 = \frac{y}{t_2}$ and $r = \frac{x/3}{t_2}$, where t_2 is the time for Alice and the train to reach B.
From $r = \frac{2x/3}{t_1}$ we find $t_1 = \frac{2x/3}{r}$, and so

$$45 = \frac{x+y}{t_1} = \frac{x+y}{\frac{2x/3}{r}} = \frac{(x+y)r}{\frac{2x}{3}}$$

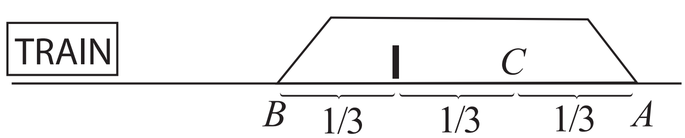
Since $t_2 = \frac{x/3}{r}$, then $45 = \frac{yr}{x/3}$. Setting these equal, we get

$$\frac{yr}{\frac{x}{3}} = \frac{(x+y)r}{\frac{2x}{3}}$$

and thus $y = \frac{x+y}{2}$, i.e., $x = y$.
Now from $45 = \frac{x+y}{t_1}$ we get $45 = \frac{2x}{t_1}$ and $t_1 = \frac{2x}{45}$. Combining with $t_1 = \frac{2x/3}{r}$, we obtain $2xr = 45 \cdot \frac{2x}{3}$ and $r = 15$ (km/h).

Alternative solution: The time that it takes the train to reach point B is $\frac{y}{45}$. The time that it takes Alice to reach pont B is $\frac{x/3}{r} = \frac{x}{3r}$.
So we need
$$\frac{y}{45} = \frac{x}{3r}$$
and thus $y = \frac{15x}{r}$.
The time that it takes the train to reach point A is $\frac{x+y}{45}$. The time that it takes Alice to reach pont A is $\frac{2x/3}{r} = \frac{2x}{3r}$. So we need
$$\frac{y+x}{45} = \frac{2x}{3r}$$
Substituting $y = \frac{15x}{r}$ gives
$$\frac{\frac{15x}{r} + x}{45} = \frac{2x}{3r}$$
and thus $r = 15$.

Alternative solution: If we know that Alice and the train can get to the point B at the same time, then if she runs in the opposite direction at the same speed she will be at C when the train reaches B. We know that Alice and the train will arrive at A at the same time, which means that Alice can run across 1/3 of the length of the bridge in the time it takes the train to cross the entire bridge. Thus, Alice runs at 1/3 of the speed of the train, i.e., 15 km/h.



8. Triangle ABC is an isosceles triangle with two inscribed circles. The larger circle has radius $2r$, and the smaller circle with radius r is tangent to the larger circle and to the two equal sides of the triangle. The area of the triangle ABC is xr^2. What is x?

(A) $16\sqrt{2}$

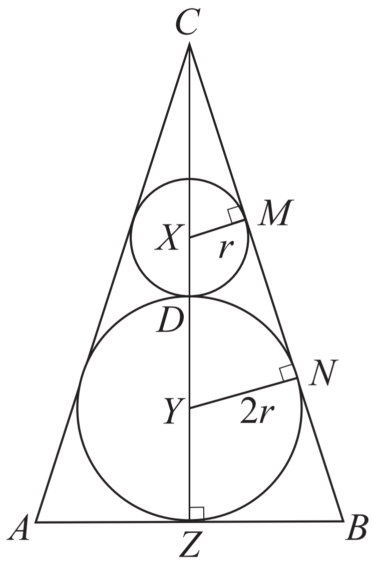
(B) $8\sqrt{2}$

(C) $8\sqrt{2}+4$

(D) $8\left(\sqrt{2}+1\right)$

Answer: (A) $16\sqrt{2}$

Solution: Label the cetres of the two circles by X and Y; M and N are the feet of the altitudes to BC, D is the point of tangency of the two circles. Z is the foot of the altitude from C. See the figure.



The triangles ΔXCM and ΔYCN are similar. Note that $YC = YX + XC = 2r + r + XC = 3r + XC$. Thus, from $XC/YC = 2r/r$ we obtain

$$\frac{3r + XC}{XC} = 2$$

and so $XC = 3r$.

Using the Pythagorean Theorem, we find

$$MC = \sqrt{XC^2 - XM^2} = \sqrt{9r^2 - r^2} = 2\sqrt{2}r$$

We know that $\angle BZC = 90^{\text{o}}$, and therefore the triangles ΔBZC and ΔXMC are similar. As well, $\text{area}(ABC) = 2\text{area}(BZC)$.

Using the similarity of triangles ΔBZC and ΔXMC and the fact that the ratio of the areas of similar triangles equals the ratio of their side lengths squared, we find:

$$\begin{aligned}
\text{area}(ABC) &= 2\text{area}(BZC) \\
&= 2\left[\text{area}(XMC)\left(\frac{ZC}{MC}\right)^2\right] \\
&= 2\left[\left(\frac{1}{2}\cdot XM\cdot MC\right)\cdot\left(\frac{8r}{2\sqrt{2}r}\right)^2\right] \\
&= 2\left[\left(\frac{1}{2}\cdot r\cdot 2\sqrt{2}r\right)\cdot 8\right] \\
&= 16\sqrt{2}r^2
\end{aligned}$$

9. The sum

$$\frac{3}{1!+2!+3!}+\frac{4}{2!+3!+4!}+\frac{5}{3!+4!+5!}+\cdots+\frac{2016}{2014!+2015!+2016!}$$

is equal to

(A) $\dfrac{2016!+2}{2\cdot 2016!}$

(B) $\dfrac{2016!+1}{2\cdot 2016!}$

(C) $\dfrac{2016!-1}{2\cdot 2016!}$

(D) $\dfrac{2016!-2}{2\cdot 2016!}$

Answer: (D) $\dfrac{2016!-2}{2\cdot 2016!}$

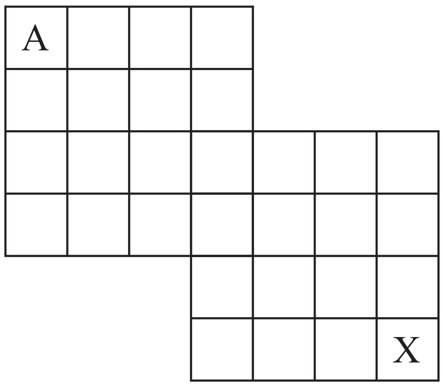
Solution: Observe that

$$\begin{aligned}\frac{n}{(n-2)!+(n-1)!+n!} &= \frac{1}{(n-2)!}\cdot\frac{n}{1+(n-1)+n(n-1)}\\ &= \frac{1}{(n-2)!}\cdot\frac{1}{n}\\ &= \frac{1}{(n-1)!}\cdot\frac{n-1}{n}\\ &= \frac{1}{(n-1)!}\cdot\left(1-\frac{1}{n}\right)\\ &= \frac{1}{(n-1)!}-\frac{1}{n!}\end{aligned}$$

Now we are done:

$$\begin{aligned}&\frac{3}{1!+2!+3!}+\frac{4}{2!+3!+4!}+\frac{5}{3!+4!+5!}+\cdots+\frac{2016}{2014!+2015!+2016!}\\ &= \frac{1}{2!}-\frac{1}{3!}+\frac{1}{3!}-\frac{1}{4!}+\frac{1}{4!}-\frac{1}{5!}+\cdots+\frac{1}{2015!}-\frac{1}{2016!}\\ &= \frac{1}{2!}-\frac{1}{2016!}\\ &= \frac{2016!-2}{2\cdot 2016!}\end{aligned}$$

10. Consider a game board shown below. You are to move a piece from A to X by moving it to an adjacent square either to the right or down. In how many different ways can you do it?



(A) 245

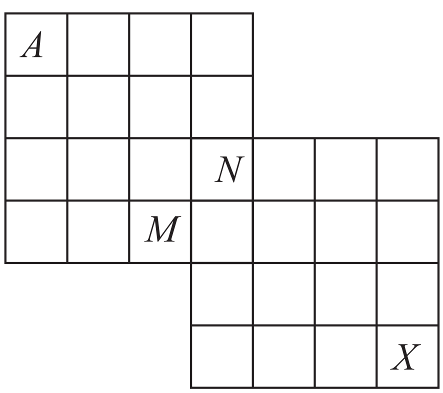
(B) 280

(C) 300

(D) 320

Answer:(C) 300

Solution: Label squares M and N as shown. Note that a path from A to X must pass through one and only one of M or N.



To reach N from A, one must make 5 moves, of which 3 are to the right and 2 are down. This can be done in $\binom{5}{2} = \binom{5}{3}$ ways. Then to reach X from N, one must make 6 moves, three to the right amd three down. This can be done in $\binom{6}{3}$ ways.
Thus, the number of paths from A to X that pass thought N is $\binom{5}{2}\binom{6}{3} = 200$
The number of paths from A to M is $\binom{5}{2} = \binom{5}{3}$. The first move from M must be to the right, and from there are $\binom{5}{2} = \binom{5}{3}$ possible paths to X. Thus, the number of paths from A

to X that pass thought M is $\binom{5}{2}\binom{5}{2} = 100$.
Thus the total number of paths from A to X is

$$\binom{5}{2}\binom{6}{3} + \binom{5}{2}\binom{5}{2} = 200 + 100 = 300$$

Alternative solution: Brute force. Starting from A count in how many ways one can arrive at each square. Note that the number in each square is equal to the sum of the number in the square above it and to the left of it (assuming that the the squares outside the board carry the value zero). Thus, the answer is $140 + 160 = 300$.

A	1	1	1			
1	2	3	4			
1	3	6	10	10	10	10
1	4	10	20	30	40	50
			20	50	90	140
			20	70	160	X

The End