Math@Mac Online Mathematics Competition

Wednesday, November 29, 2017

SOLUTIONS

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1. Assume that x_1 and x_2 are distinct solutions of the equation $x^2 + 2017x + 1 = 0$ and y_1 and y_2 are distinct solutions of the equation $y^2 + 2018y + 1 = 0$. Find the value of the expression

$$(x_1 - y_1)(x_2 - y_2)(x_1 - y_2)(x_2 - y_1)$$

Answer: 7921

Solution:
NOTE: THERE WAS A TYPO IN THE QUESTION AS IT APPEARED ONLINE. The equations were given as $x^2 + 2107x + 1 = 0$ and $y^2 + 2018y + 1 = 0$.

Using Vieta's formulas, we obtain

$$x_1 + x_2 = -2107, \;\; x_1x_2 = 1, \;\; y_1 + y_2 = -2018, \;\; y_1y_2 = 1$$

Now

$$\begin{aligned}
&(x_1 - y_1)(x_2 - y_2)(x_1 - y_2)(x_2 - y_1) \\
&\quad = (x_1 - y_1)(x_1 - y_2)(x_2 - y_2)(x_2 - y_1) \\
&\quad = \left(x_1^2 - x_1y_2 - x_1y_1 + y_1y_2\right)\left(x_2^2 - x_2y_1 - x_2y_2 + y_1y_2\right) \\
&\quad = \left[x_1^2 - x_1(y_1 + y_2) + y_1y_2\right]\left[x_2^2 - x_2(y_1 + y_2) + y_1y_2\right] \\
&\quad = \left(x_1^2 + 2018x_1 + 1\right)\left(x_2^2 + 2018x_2 + 1\right) \\
&\quad = \left(x_1^2 + 2107x_1 + 1 - 89x_1\right)\left(x_2^2 + 2107x_2 + 1 - 89x_2\right) \\
&\quad = (-89x_1)(-89x_2) = 7921x_1x_2 = 7921
\end{aligned}$$

We will mark the above solution as correct.

Without the typo, i.e., with equations given as $x^2 + 2017x + 1 = 0$ and $y^2 + 2018y + 1 = 0$, the solution is 1.

2. Given are two distinct points in the plane, P and Q. A bug crawls in such a way that its distance from P is twice its distance from Q. What curve is the bug tracing?

(A) line

(B) parabola

(C) hyperbola

(D) circle

(E) none of the above

Answer: (D) circle

Solution: Let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$, and denote the location of the bug by $B = (x, y)$. Then

$$\begin{aligned} d(B,P) &= 2d(B,Q) \\ \sqrt{(x-p_1)^2+(y-p_2)^2} &= 2\sqrt{(x-q_1)^2+(y-q_2)^2} \\ x^2-2xp_1+p_1^2+y^2-2yp_2+p_2^2 &= 4\left(x^2-2xq_1+q_1^2+y^2-2yq_2+q_2^2\right) \\ 3x^2+3y^2-2x(4q_1-p_1)-2y(4q_2-p_2)+4\left(q_1^2+q_2^2\right)-p_1^2-p_2^2 &= 0 \end{aligned}$$

Since the coefficients of x^2 and y^2 are equal, this equation represents a circle. (To figure out the centre and the radius, we would complete the square.)

3. Emily and Johnny are playing a really exciting game, called "throw a coin ten times." They flip a coin, and if it's heads, Emily gets a point. If it's tails, Johnny gets a point. After ten throws, they compare their scores. What is the probability that the game is a tie? Express your answer as a fraction.

Answer: 63/256

Solution: As usual we use T and H for tails and heads.

After one throw, it's H or T, each with 50 percent chance. After two throws, possible outcomes are HH, HT, TH, and TT; a tie occurs in two of $2^2 = 4$ outcomes (so the probability of a tie is $1/2$).

After three throws, there are $2^3 = 8$ outcomes: HHH, HHT, HTH, THH, HTT, THT, TTH, TTT. We can represent these as

$$1 \quad 3 \quad 3 \quad 1$$

where (say, from left to right) 1 represents one way of obtaining three H's, 3 represents three ways of obtaining two H's and a T, the next 3 represents three ways of obtaining one H and two T's and 1 represents one way of obtaining three T's.

After four throws, we obtain (keeping the previous row)

$$1 \quad 3 \quad 3 \quad 1$$

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

The middle 6 represents 6 ways (out of $2^4 = 16$ possible outcomes) to obtain 2 H's and 2 T's (i.e., a tie). Thus, the probability of a tie after 4 throws is $6/16 = 3/8$.

Clearly, the pattern is Pascal's triangle:

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

We could continue for four more rows, to identify the middle term. Alternatively, notice that that term is

$$\binom{10}{5} = 252$$

and thus the required probability is

$$\frac{\binom{10}{5}}{2^{10}} = \frac{63}{256}$$

4. Denote by x_n the solution of the equation

$$\sqrt{x+1}-\sqrt{x}=\left(\sqrt{2}-1\right)^n$$

where $n \geq 1$. Assume that m represents a certain integer. Then $x_{2018}-x_{2017}$ is equal to

(A) $\sqrt{2}+m$

(B) $2\sqrt{2}+m$

(C) m

(D) $m\sqrt{2}+1$

(E) $m\sqrt{2}-1$

(F) $m\left(\sqrt{2}+1\right)$

Answer: (C) m

Solution: Note that

$$\left(\sqrt{x+1}+\sqrt{x}\right)\left(\sqrt{x+1}-\sqrt{x}\right)=1$$

and since

$$\frac{1}{\sqrt{2}-1}=\frac{1}{\sqrt{2}-1}\frac{\sqrt{2}+1}{\sqrt{2}+1}=\sqrt{2}+1$$

we obtain

$$\sqrt{x+1}+\sqrt{x}=\frac{1}{\sqrt{x+1}-\sqrt{x}}=\frac{1}{\left(\sqrt{2}-1\right)^n}=\left(\sqrt{2}+1\right)^n$$

Subtracting the given equation from this one, we obtain

$$\sqrt{x}=\frac{1}{2}\left[\left(\sqrt{2}+1\right)^n-\left(\sqrt{2}-1\right)^n\right]$$

The binomial theorem implies that $\left(\sqrt{2}+1\right)^n=A+B\sqrt{2}$, where A and B are positive integers.

If n is even, then $\left(\sqrt{2}-1\right)^n=A-B\sqrt{2}$, and if n is odd, then $\left(\sqrt{2}-1\right)^n=-A+B\sqrt{2}$.

Thus, if n is even, then $\sqrt{x}=B\sqrt{2}$, and $x=2B^2$. If n is odd, then $\sqrt{x}=A$, and $x=A^2$.

In either case, x is a positive integer, and thus the difference $x_{2018}-x_{2017}$ must be an integer.

5. Find all real numbers x which satisfy the equation

$$\left[(x-1)^2\right] = [x]$$

where [ ] denotes the greatest integer function. Recall the definition of $[x]$: If a real number x is written as $x = n + \alpha$, where n is an integer and $0 \leq \alpha < 1$, then $[x] = n$.

The solution is of the form $(a, b) \cup [c, d)$. Identify a, b, c, and d.

Answer: $a = 0,\ b = 1,\ c = 1 + \sqrt{2},\ d = 1 + \sqrt{3}$

Solution: Since the greatest integers of $(x-1)^2$ and x are equal, we conclude that

$$\left|(x-1)^2 - x\right| < 1$$

i.e., $-1 < x^2 - 3x + 1 < 1$. Solving, we obtain that $x \in (0, 1) \cup (2, 3)$.

If $x \in (0, 1)$, then $\left[(x-1)^2\right] = 0$ and $[x] = 0$, so all $x \in (0, 1)$ are solutions.

If $x \in (2, 3)$, then $x = 2 + \alpha$, with $0 \leq \alpha < 1$. Then $[x] = 2$, and thus the given equation reads

$$\left[\alpha^2 + 2\alpha + 1\right] = 2$$

Consequently,

$$2 \leq \alpha^2 + 2\alpha + 1 < 3.$$

Solving this inequality we obtain

$$\alpha \in (-1 - \sqrt{3}, -1 - \sqrt{2}] \cup [-1 + \sqrt{2}, -1 + \sqrt{3})$$

which, combined with $0 \leq \alpha < 1$, gives

$$\alpha \in [-1 + \sqrt{2}, -1 + \sqrt{3})$$

Thus,

$$x = 2 + \alpha \in [1 + \sqrt{2}, 1 + \sqrt{3})$$

and the solution is

$$x \in (0, 1) \cup [1 + \sqrt{2}, 1 + \sqrt{3})$$

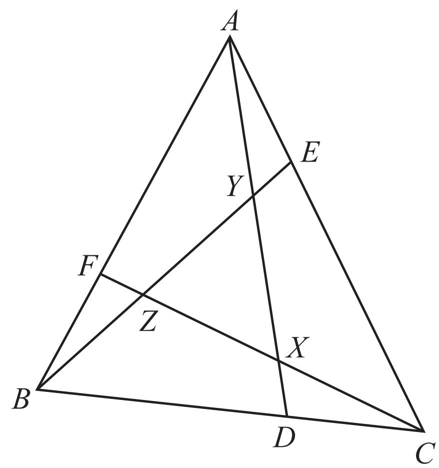
6. In the figure, CD, AE and BF are one-third of their respective sides. It can be shown that

$$AY : YX : XD = 3 : 3 : 1$$

with similar ratios for the segments BE and CF. If

$$\text{area } \Delta XYZ = x \cdot \text{area } \Delta ABC$$

what is x? Answer in the form of a fraction (i.e., do not convert into a decimal number).



Answer: 1/7

Solution: We use ΔMNP to denote the area of the triangle with vertices M, N, and P. By construction $\Delta CFB = \frac{1}{3}\Delta ABC$, and similarly for the other two triangles, i.e.,

$$\Delta CFB = \Delta ABE = \Delta CAD = \frac{1}{3}\Delta ABC$$

As well,

$$\Delta CXD = \frac{1}{7}\Delta CAD = \frac{1}{7}\frac{1}{3}\Delta ABC = \frac{1}{21}\Delta ABC$$

Analogously, we obtain

$$\Delta CXD = \Delta AYE = \Delta BZF = \frac{1}{21}\Delta ABC$$

Thus

$$\begin{aligned}\Delta XYZ =& \Delta ABC - \Delta CFB - \Delta ABE - \Delta CAD + \Delta CXD + \Delta AYE + \Delta BZF \\ =& \frac{1}{7}\Delta ABC\end{aligned}$$

7. A jar contains at least two red balls and at least two white balls. If two balls are randomly drawn from the jar, the probability of drawing two red balls is five times the probability of drawing two white balls. Also, the probability of drawing one red and one white ball is six times the probability of drawing two white balls. How many red balls and how many white balls are in the jar?

Answer: 6 red and 3 white

Solution: Denote by r the number of red balls, by w the number of white balls, and by t the total number of balls in the jar. Then

$$\begin{aligned}
&P(\text{two red balls}) = \frac{r}{t} \cdot \frac{r-1}{t-1} \\
&P(\text{two white balls}) = \frac{w}{t} \cdot \frac{w-1}{t-1} \\
&P(\text{one red ball and one white ball}) = \frac{r}{t} \cdot \frac{w}{t-1} + \frac{w}{t} \cdot \frac{r}{t-1}
\end{aligned}$$

Then $P(\text{two red balls}) = 5P(\text{two white balls})$ implies

$$\begin{aligned}
\frac{r}{t} \cdot \frac{r-1}{t-1} &= 5 \cdot \frac{w}{t} \cdot \frac{w-1}{t-1} \\
r(r-1) &= 5w(w-1)
\end{aligned}$$

As well, from $P(\text{one red ball and one white ball}) = 6P(\text{two white balls})$ we conclude

$$\begin{aligned}
\frac{r}{t} \cdot \frac{w}{t-1} + \frac{w}{t} \cdot \frac{r}{t-1} &= 6 \cdot \frac{w}{t} \cdot \frac{w-1}{t-1} \\
2wr &= 6w(w-1)
\end{aligned}$$

and $r = 3w - 3$. Combining with $r(r-1) = 5w(w-1)$, we obtain

$$\begin{aligned}
(3w-3)(3w-4) &= 5w(w-1) \\
4w^2 - 16w + 12 &= 0 \\
4(w-1)(w-3) &= 0
\end{aligned}$$

Since $w \geq 2$, we conclude that $w = 3$. then $r = 3w - 3 = 6$.

8. Let $P(x)$ be a polynomial of degree four such that $P(2) = P(-2) = P(-3) = -1$ and $P(1) = P(-1) = 1$. What is $P(0)$?

Answer: $P(0) = 2$

Solution: Since $P(2) = P(-2) = P(-3) = -1$, each of the numbers $2, -2$, and -3 is a root of the polynomial $P(x) + 1$. Thus, $P(x) + 1 = (x-2)(x+2)(x+3)(ax+b)$, and

$$P(x) = (x-2)(x+2)(x+3)(ax+b) - 1$$

Now using $P(1) = P(-1) = 1$ we obtain $-12a - 12b - 1 = 1$ and $6a - 6b - 1 = 1$. Solving this system gives $a = 1/12$ and $b = -1/4$. Thus

$$P(x) = (x-2)(x+2)(x+3)\left(\frac{1}{12}x - \frac{1}{4}\right) - 1$$

and $P(0) = 2$.

9. Write

$$\frac{1}{196} = 0.d_1 d_2 d_3 d_4 \ldots$$

i.e., d_j is the j-th digit after the decimal point. What is d_{21}?

Answer: $d_{21} = 2$

Solution: Note that

$$10^{20} \cdot \frac{1}{196} = d_1 d_2 d_3 \ldots d_{20}.d_{21} d_{22} d_{23} \ldots$$

The idea is to determine the remainder when 10^{20} is divided by 196. Then

$$\frac{10^{20}}{196} = \text{quotient} + \frac{\text{remainder}}{196}$$

and so quotient $= d_1 d_2 d_3 \ldots d_{20}$ and the first decimal of remainder/196 is d_{21}.
We use notation $a \equiv b \pmod{m}$ to say that a and b have the same remainder when divided by m.
Since $196 \cdot 5 = 980$, we conclude that $1000 = 10^3 \equiv 20 \pmod{196}$. Then

$$\left(10^3\right)^2 = 10^6 \equiv 20^2 = 400 \pmod{196}$$

and because $400 \equiv 8 \pmod{196}$ we conclude that

$$10^6 \equiv 8 \pmod{196}$$

We continue in the similar way:

$$\left(10^6\right)^3 = 10^{18} \equiv 8^3 = 512 \pmod{196}$$

and because $512 \equiv 120 \pmod{196}$ we get

$$10^{18} \equiv 120 \pmod{196}.$$

Thus

$$10^{19} \equiv 1200 \pmod{196}$$

$$10^{19} \equiv 24 \pmod{196}$$

and

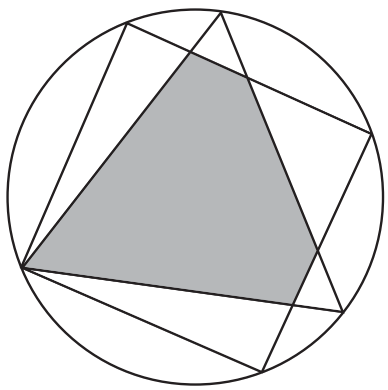
$$10^{20} \equiv 240 \pmod{196}$$

So, finally

$$10^{20} \equiv 44 \pmod{196}$$

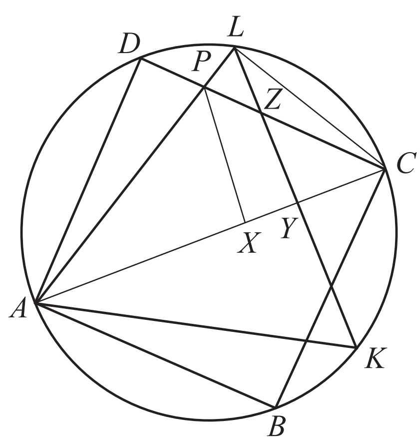
Since $44/196 = 0.2244\ldots$, we conclude that $d_{21} = 2$.

10. An equilateral triangle and a square are inscribed into a circle of radius 1, so that they have a common vertex. Find the area of the shaded region (i.e., the region common to both the triangle and the square). The answer is in the form $a\sqrt{b}+c$. Identify a, b, and c.



Answer: $a = 2,\ b = 3,\ c = -9/4$

Solution: Label the vertices of the triangle and the square as shown below. By $|MN|$ we denote the length of the line segment MN, and we use $a(\ldots)$ for the area.



The sides of the square are $|AD| = |DC| = |CB| = |BA| = \sqrt{2}$, and the sides of the triangle are $|AL| = |LK| = |KA| = \sqrt{3}$.

The triangle ACL is the right triangle with $\angle ALC = 90^{\text{o}}$. From Pythagorean theorem, $|LC| = 1$. The triangles ACL and CLY are similar (identical angles), and thus

$$\frac{|YC|}{|LC|} = \frac{|LC|}{|AC|} = \frac{1}{2}$$

which implies that $|YC| = 1/2$.

Now $\angle PAX = 30^{\text{o}}$ (so that the triangle PAX is 60-30-90) and $|AX| = \sqrt{3}|PX|$.

As well $\angle PCX = 45^{\mathrm{o}}$, which implies that $|CX| = |PX|$ and

$$\left(1+\sqrt{3}\right)|PX| = |CX| + |AX| = 2$$

i.e.,

$$|PX| = \frac{2}{1+\sqrt{3}} = \sqrt{3}-1$$

Now

$$\begin{aligned} a(\text{shaded pentagon}) &= 2a(APZY) = 2\left[a(APC) - a(YZC)\right] \\ &= |AC| \cdot |PX| - |YC| \cdot |ZY| \\ &= 2\left(\sqrt{3}-1\right) - \left(\frac{1}{2}\right)^2 = 2\sqrt{3} - \frac{9}{4} \end{aligned}$$